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Ursula LUDWIG

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A COMPLEX IN MORSE THEORY COMPUTING INTERSECTION HOMOLOGY

by Ursula LUDWIG

ABSTRACT. — Let X be a space with isolated conical singularities. The aim of this article is to establish, using anti-radial Morse functions on X , a combinatorial complex which computes the intersection homology of X . The complex constructed here, is generated by the smooth critical points of the Morse function and representatives of the de Rham cohomology (in low degree) of the link manifolds of the singularities of X . It can be seen as an analogue of the famous Thom-Smale complex for smooth Morse functions and singular homology on a compact manifold. The article also discusses the homotopy principle familiar in smooth Morse homology in this singular context.

RÉSUMÉ. — Dans cet article on associe à une fonction de Morse f anti-radiale sur un espace singulier X à singularités coniques un complexe généré par les points critiques de f et par certaines formes sur le link de la singularité. Ce complexe calcule de façon canonique l'homologie d'intersection. Également on discute le comportement de ce complexe par rapport aux homotopies. Le complexe construit dans cet article est un analogue du complexe de Thom-Smale sur une variété lisse pour une fonction de Morse lisse et l'homologie singulière.

1. Introduction

Let M be a smooth compact manifold and $f : M \rightarrow \mathbb{R}$ a smooth Morse function. The famous Morse inequalities state that there is a relation between the number of critical points of index i of the Morse function f , denoted by $c_i(f)$ and the i -th Betti number (for singular homology) of M , denoted by $b_i(M)$. More precisely in their strong form the Morse inequalities state, that for all k , $0 \leq k \leq \dim M$:

$$(1.1) \quad \sum_{i=0}^k (-1)^{k-i} c_i(f) \geq \sum_{i=0}^k (-1)^{k-i} b_i(M).$$

Keywords: intersection homology, Morse theory, radial vector fields, Thom-Smale complex.

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A way to prove these Morse inequalities is to show the existence of a combinatorial complex (C_*, ∂_*) , which is generated by the critical points of f and whose homology is isomorphic to the singular homology $H_*(M)$ (see e.g. [7], pg. 106). Let g be a Riemannian metric on M , such that the pair (f, g) satisfies the Morse-Smale transversality condition, i.e. all intersections of stable and unstable manifolds of critical points of f (and the flow associated to the vector field $-\nabla_g f$) are transverse. Then, such a complex has first been established by Thom [44] and Smale [43], using the unstable cell decomposition of M for the negative gradient flow, i.e. the flow associated to the vector field $-\nabla_g f$.

The Thom-Smale complex has seen a revival in the 1990s, where the idea of counting trajectories between critical points, has been exploited in an infinite dimensional context by Floer [13]. Generalisations to Morse-Bott functions (see [3]), invariant Morse functions (see [5], [3]), to manifolds with boundaries (see [1, 26, 28]) and to stratified spaces (see [31]), have been studied since.

In the 1980s, inspired by ideas in quantum field theory, Witten [51] defined another complex, generated by the critical points of a Morse function, which this time computes the singular cohomology of M . The Witten complex is produced in an analytic way, using a deformation of the de Rham complex by means of the Morse function. It was conjectured by Witten, that for big deformation parameter the Witten complex converges to the dual Thom-Smale complex. A rigorous proof of Witten's approach has been given by Helffer and Sjöstrand using semi-classical analysis in [23]. Another proof of the comparison theorem between the Witten complex and the Thom-Smale complex has been given by Bismut and Zhang in [5]. The approach in [5] is based on a result of Laudенbach [27], who gave a reinterpretation of the Thom-Smale complex in terms of currents. The comparison result of these two complexes in Morse theory, was used in [4] and [5] to generalise the Cheeger-Müller theorem on the comparison of Reidemeister and analytic torsion.

For singular spaces, an important topological invariant is the intersection homology introduced by Goresky and MacPherson in the 1980s (see [16, 17], see also the book [25] for an introduction to intersection homology; the definition of intersection homology will be recalled in Section 5.1). Intersection homology on singular spaces does satisfy essentially all nice properties (Poincaré duality, Lefschetz hyperplane theorem etc.), singular homology satisfies on smooth manifolds.

The aim of this article is to define an analogue of the Thom-Smale complex on singular spaces with isolated singularities, which computes the intersection homology of the space. Let us point out, that the complex constructed here is a complex of \mathbb{R} -vector spaces. This is in contrast to the smooth situation, where the Thom-Smale complex can also be defined over \mathbb{Z} .

Let us explain the setting and the main result of this paper: In the whole article, the space X will be a space with *isolated conical singularities* of dimension $\dim X = n$. Let us recall the main properties of such a space (see Definition 2.1 for full details): Outside a finite set of points $\text{Sing}(X)$, called the singular set, X is a smooth manifold. Moreover, for each point $p \in \text{Sing}(X)$ there exists an open neighbourhood $U(p) \subset X$, which is homeomorphic to a cone

$$(1.2) \quad U(p) \simeq cL_p := ([0, \infty) \times L_p) / \sim,$$

where L_p is a smooth compact connected manifold, called the link of X at p . The top stratum $X \setminus \text{Sing}(X)$ is equipped with a conical Riemannian metric g .

Let $f : X \rightarrow \mathbb{R}$ be an *anti-radial Morse function* (see Definition 2.6). We denote by $\text{Crit}(f) := \text{Crit}(f|_{X \setminus \text{Sing}(X)}) \cup \text{Sing}(X)$ the set of critical points of f . Singular points of X are local maxima of the anti-radial Morse function. After possibly perturbing the conical metric g outside a neighbourhood of $\text{Crit}(f)$, we can assume that the negative gradient flow satisfies the Morse-Smale transversality condition (see Definition 3.1 and the discussion thereafter). We will moreover assume that the gradient vector field is standard near critical points (see Definition 2.7).

For $p \in \text{Sing}(X)$, let us denote by $H^*(L_p)$ the de Rham cohomology of the link manifold L_p . Let

$$(1.3) \quad \Xi_p^{n-k} := \{\xi_{p,l}^{n-k} \mid l = 1, \dots, \dim H^{n-k}(L_p)\} \subset \Omega^{n-k}(L_p)$$

be a set of closed forms, such that $\{[\xi_{p,l}^{n-k}] \mid l = 1, \dots, \dim H^{n-k}(L_p)\} \subset H^{n-k}(L_p)$ is a basis of $H^{n-k}(L_p)$. Let us denote by

$$(1.4) \quad \Xi_p := \bigcup_{k \geq \frac{n}{2} + 1} \Xi_p^{n-k} \text{ and by } \Xi := \bigcup_{p \in \text{Sing}(X)} \Xi_p.$$

We equip each unstable cell $W^u(p)$, $p \in \text{Crit}(f)$, with an orientation. Using the negative gradient flow, this gives a way of “counting with signs” the trajectories of the negative gradient flow between two smooth critical points $p, q \in \text{Crit}(f|_{X \setminus \text{Sing}(X)})$, with $\text{ind}(p) - \text{ind}(q) = 1$. We denote this number by $n(p, q)$ (see Definition 3.3). We moreover assume in the whole

paper that the top stratum $X \setminus \text{Sing}(X)$ is oriented. Hence, as explained in Section 3.2, all stable cells $W^s(p)$ as well as all intersections $W^s(p) \cap L_q$, $q \in \text{Sing}(X)$, $p \in \text{Crit}(f|_{X \setminus \text{Sing}(X)})$, inherit orientations.

We denote by $f_{sm} := f|_{X \setminus \text{Sing}(X)}$ and by $\text{Crit}_k(f_{sm})$ the set of smooth critical points of index k .

The main idea of the article consists in the construction of the following complex:

DEFINITION. — *To the anti-radial, standard Morse-Smale pair (f, g) and the set Ξ we associate a complex $(C_*^u(f, g, \Xi), \partial_*)$ as follows:*

$$C_k^u = C_k^u(f, g, \Xi)$$

$$:= \begin{cases} \bigoplus_{p \in \text{Crit}_k(f_{sm})} \mathbb{R} \cdot [W^u(p)] \oplus \bigoplus_{\substack{p \in \text{Sing}(X), \\ \xi_{p,l}^{n-k} \in \Xi_p^{n-k}}} \mathbb{R} \cdot [\xi_{p,l}^{n-k}] & \text{if } k \geq \frac{n}{2} + 1, \\ \bigoplus_{p \in \text{Crit}_k(f_{sm})} \mathbb{R} \cdot [W^u(p)] & \text{if } k < \frac{n}{2} + 1. \end{cases}$$

The boundary operator ∂_* is defined as follows:

$$(1.5) \quad \partial[W^u(p)] = \sum_{q \in \text{Crit}_{k-1}(f_{sm})} n(p, q) \cdot [W^u(q)] \text{ for } p \in \text{Crit}_k(f_{sm});$$

and

$$(1.6) \quad \partial[\xi_{p,l}^{n-k}] = \sum_{q \in \text{Crit}_{k-1}(f_{sm})} \left(\int_{W^s(q) \cap L_p} \xi_{p,l}^{n-k} \right) \cdot [W^u(q)]$$

for $p \in \text{Sing}(X)$, $\xi_{p,l}^{n-k} \in \Xi_p^{n-k}$.

We denote by $IH_*(X)$ the intersection homology with lower middle perversity of X and real coefficients. The main result of this article is the following theorem:

MAIN THEOREM. — *Let X be a singular space with isolated conical singularities. Let (f, g) be an anti-radial, standard Morse-Smale pair and let Ξ be a set of representatives of the de Rham cohomology of the link manifolds of $\text{Sing}(X)$ as in (1.3) and (1.4).*

Then the complex $(C_^u(f, g, \Xi), \partial_*)$ is well-defined, i.e. $\partial_*^2 = 0$, and computes the intersection homology with lower middle perversity of X ,*

$$(1.7) \quad H_*((C_*^u(f, g, \Xi), \partial_*)) \simeq IH_*(X).$$

Moreover, let (f^α, g^α) and (f^β, g^β) be two anti-radial Morse-Smale pairs. Then there is a canonical isomorphism of homologies:

$$(1.8) \quad H_*((C_*^u(f^\alpha, g^\alpha, \Xi), \partial_*)) \longrightarrow H_*((C_*^u(f^\beta, g^\beta, \Xi), \partial_*)).$$

For $p \in \text{Sing}(X)$, let us denote by $IH_*(cL_p, L_p)$ the relative intersection homology with lower middle perversity of the cone cL_p . Set

$$(1.9) \quad c_i(f) := c_i(f_{sm}) + \sum_{p \in \text{Sing}(X)} IH_i(cL_p, L_p).$$

As a corollary of the Main Theorem, one gets the following Morse inequalities for the intersection Betti numbers $Ib_i(X) := \dim IH_i(X)$:

$$(1.10) \quad \sum_{i=0}^k (-1)^{k-i} c_i(f) \geq \sum_{i=0}^k (-1)^{k-i} Ib_i(X), \quad 0 \leq k \leq n.$$

In the definition of the complex $(C_*^u(f, g, \Xi), \partial_*)$ we have used a set of representatives Ξ of the de Rham cohomology of the link manifolds. The set Ξ contains only forms of “low degree”, the degree at which we truncate is related to the lower middle perversity. For any other perversity \underline{p} in the sense of the theory of Goresky and MacPherson [16], one can define in a completely analogous way a combinatorial complex computing the intersection homology of X with perversity \underline{p} . For this, one has to choose, in the definition of Ξ a truncation at a different degree which only depends on the perversity \underline{p} (see Example 8.2.2).

The Main Theorem gives a way of computing the intersection homology of a singular space using Morse theory. While this is interesting in itself, the initial motivation of the present article originated from a different mathematical problem: In [30] the author has studied the Witten deformation of the complex of L^2 -forms on a singular space with isolated conical singularities using anti-radial Morse functions. The combinatorial complex has been defined in a special situation (for so-called special anti-radial Morse functions) and it has been shown (Theorem II in [30]), that for special anti-radial Morse functions the Witten complex converges to the dual combinatorial complex. Using the results of the present article and combining them with the analytic result of [30], one can show that for an *arbitrary* anti-radial Morse function, the Witten complex in [30] converges to the dual of $(C_*^u(f, g, \Xi), \partial_*)$.

The motivation for [30] as well as for the present article, comes from a topic in global analysis of singular spaces, which has achieved some interest in recent years, namely the study of analytic torsion for spaces with conical singularities [12, 47, 21, 22]. Comparison theorems between analytic and topological torsion on smooth manifolds, *aka* Cheeger-Müller type theorems, have been an object of intensive study during the last 40 years in global analysis (see [39, 9, 34, 35]). As mentioned before, the most general

comparison result of torsions on smooth compact manifolds is due to Bismut and Zhang [4], who approached the question using Morse theory and the Witten deformation, as well as local index techniques. It is therefore natural to try to generalise the approach in [4] to the singular context. The present article is a step in this direction.

Let us give some indications on the ideas used in the proof of the Main Theorem and how they are related to existing literature: For the proof of the first part of the theorem, the main idea is to study the compactification of the unstable manifolds of critical points and to refine the unstable cell decomposition. This is done by adapting to the singular setting the result of Laudenbach [27] (see also the book [29] for more detailed proofs). Let us underline, that throughout the article, we will use the compactification of stable and unstable manifolds à la Laudenbach (see Remark 3.5 for its relation to the compactification in the topology of “broken trajectories”).

For discussing how the geometric complex changes, when passing from the anti-radial Morse-Smale pair (f^α, g^α) to another anti-radial Morse-Smale pair (f^β, g^β) , we use Morse theory on the product space $\tilde{X} = X \times S^1$. This approach is close to the approach used in smooth Morse homology inspired from Floer homology (see Section 4.1.3 in [42], also Section 4.2 in [49]). Thus, at this stage, we do not proceed as in Section (f) and (g) in [27], where this passage is explained using ideas from bifurcation theory (in particular in [27] the phenomenon of birth-death points is discussed in detail). However, still our compactification of stable and unstable cells is the compactification à la Laudenbach.

Let us mention, that the complex constructed here, using the smooth critical points of the anti-radial Morse function and forms on the links of singular points, is related to the definition of the complex associated to a Morse-Bott function on a smooth manifold (see Section 3 in [3]). With the right interpretations, the complex $(C_*^u(f, g, \Xi), \partial_*)$ can be seen as a subcomplex of the Morse-Bott complex associated to the “blow-up” of X with the “blow-up” function of f on it.

The present article uses anti-radial Morse functions. They are inspired from radial vector fields as introduced by Marie-Hélène Schwartz in [40, 41] in her study of characteristic classes on singular spaces via obstruction theory. Another powerful tool on singular spaces are stratified Morse functions as introduced by Goresky and MacPherson in [19]. While one can get the Morse inequalities (1.10) as well using methods in [19], the dynamical system point of view in Morse theory is much easier adapted to anti-radial

Morse functions. Note that, as has been seen in [20], even to define stable/unstable manifolds in the context of stratified Morse theory is not an easy task.

The article is organised as follows: In Section 2 we recall some basic notions, in particular we recall the notion of a singular space with conical singularities and we define anti-radial Morse functions on a singular space. Moreover we recall the notion of an *smcs* in the sense of Laudenbach [27]. In Section 3 we study the decomposition of X into stable resp. unstable cells of an anti-radial Morse function. This is a straightforward generalisation of smooth Morse theory. In Section 4 we give a refinement of the unstable cell decomposition. In Section 5 we use the construction of Section 4 to define a subcomplex $(D^*(f, g, \Xi, T), \partial_*)$ of the intersection chain complex of X . In Section 6 we prove the first part of the Main Theorem, i.e. we prove that the abstract complex $(C_*^u(f, g, \Xi), \partial_*)$ is well-defined. Moreover, using the constructions of Section 5, we prove that the complex $(C_*^u(f, g, \Xi), \partial_*)$ computes the intersection homology of X . Moreover we define a pairing of $(C_*^u(f, g, \Xi), \partial_*)$ with the complex of L^2 -forms on X , which induces the canonical pairing between intersection homology and L^2 -cohomology of X .

In Section 7 we consider two anti-radial Morse-Smale pairs (f^α, g^α) and (f^β, g^β) and construct the quasi-isomorphism of complexes leading to (1.8). To this purpose we generalise some of the concepts from Section 3 and Section 4 to the singular space $\tilde{X} = X \times S^1$, which is a singular space with an one-dimensional singular stratum. Finally, in Section 8 we illustrate our construction by several examples.

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2. Some basic definitions

2.1. Singular spaces with isolated conical singularities.

In this subsection we recall the definition of a space with isolated conical singularities.

In the whole article the following notations will be used: For Z a topological space, we will denote by

$$(2.1) \quad cZ := ([0, \infty) \times Z) /_{(0,x) \sim (0,y)},$$

the infinite cone over Z . We will denote by $r \in [0, \infty)$ the radial coordinate in cZ and by 0 the cone point. For $\delta > 0$ we denote by

$$(2.2) \quad c_\delta Z := ([0, \delta) \times Z) /_{(0,x) \sim (0,y)},$$

the open cone truncated at $r = \delta$.

DEFINITION 2.1. — *Let X be a topological space, $\text{Sing}(X) \subset X$ a finite set of points, such that $X_{sm} := X \setminus \text{Sing}(X)$ is a smooth oriented manifold of dimension n . Let g be a Riemannian metric on X_{sm} . The pair (X, g) is called a space with isolated conical singularities if it admits a disjoint decomposition*

$$(2.3) \quad X = M \cup \left(\bigcup_{p \in \text{Sing}(X)} U_\delta(p) \right),$$

with the following properties:

- (1) M is a smooth compact manifold with boundary of dimension $\dim M = n$.
- (2) For $p \in \text{Sing}(X)$, $U_\delta(p)$ denotes an open neighbourhood of p . There exists a diffeomorphism

$$(2.4) \quad \varphi : U_\delta(p) \setminus \{p\} \simeq c_\delta L_p \setminus \{0\},$$

where L_p is a smooth compact connected manifold of dimension $\dim L_p = n - 1 =: m$, called the link of X at p . Moreover

$$(2.5) \quad g|_{U_\delta(p) \setminus \{p\}} = \varphi^* (dr^2 + r^2 g_{L_p}),$$

where g_{L_p} is a fixed metric on the manifold L_p (not depending on r); and the diffeomorphism φ extends to a homeomorphism, still denoted by φ ,

$$(2.6) \quad \varphi : U_\delta(p) \simeq c_\delta L_p.$$

(3) The boundary of M is the disjoint union of the link manifolds L_p :

$$(2.7) \quad \partial M = \bigcup_{p \in \text{Sing}(X)} L_p.$$

The set $\text{Sing}(X)$ is called the singular set of X .

For $p \in \text{Sing}(X)$ the open neighbourhood $U_\delta(p)$ appearing in part (2) of Definition 2.1 is the δ -neighbourhood of p , with respect to the inner metric induced from g . Note that, for $0 < \epsilon < \delta$, the ϵ -neighbourhood of p , $U_\epsilon(p)$ can be identified via the chart φ with $c_\epsilon L_p$. Moreover we identify $\partial U_\epsilon(p)$ with $L_{p,\epsilon} := \{\epsilon\} \times L_p \subset cL_p$.

2.2. Submanifolds with conical singularities (smcs) in the sense of Laudenbach

In this subsection, for convenience of the reader, we will recall the notion of a submanifold with conical singularities of dimension k of a smooth manifold N as defined by Laudenbach in [27], Section (a). In the rest of the article we will refer to it as *smcs in the sense of Laudenbach* or more shortly as *smcs*. In Definition 2.4 we extend the notion of an *smcs* to closed subsets of the singular space X . The reader is warned that the meaning of “conical” is slightly different in Definition 2.1 and Definition 2.2.

Let N be a smooth manifold of dimension n . Let $\Sigma \subset N$ be a closed subset. A stratification of Σ is a filtration by closed subsets

$$(2.8) \quad \Sigma = \Sigma_k \supseteq \Sigma_{k-1} \supseteq \dots \supseteq \Sigma_0.$$

The following definition is inductive.

DEFINITION 2.2 (Section (a) in [27]). — *Let N be a smooth manifold of dimension n . An smcs of N of dimension 0 is a discrete finite set of points in N . A stratified subset $\Sigma = (\Sigma_k, \Sigma_{k-1}, \dots, \Sigma_0)$ of N is an smcs of dimension k if the following conditions are satisfied:*

- (1) *For any $i \leq k$ the set $\Sigma_{(i)} := \Sigma_i \setminus \Sigma_{i-1}$ is either empty or a smooth submanifold of N of dimension i . The sets $\Sigma_{(i)}$ are called the strata of Σ .*
- (2) *For any point $x \in \Sigma_{(i)}$, there exist a neighbourhood V in N , a diffeomorphism $\varphi : V \simeq D^i \times D^{n-i}$ from V into a product of discs, and an smcs $T = (T_{k-i}, \dots, T_0, \emptyset, \dots, \emptyset)$ of dimension $k - i$ in D^{n-i} such that:*

$$(2.9) \quad \varphi(V \cap (\Sigma_k, \dots, \Sigma_0)) = D^i \times (T_{k-i}, \dots, T_0, \emptyset, \dots, \emptyset).$$

- (3) If $x \in \Sigma_0 = \Sigma_{(0)}$, there is an n -dimensional C^1 -ball B in N centred at x such that

$$\Sigma' := \Sigma \cap \partial B$$

is an smcs of dimension $(k - 1)$ in the $(n - 1)$ -sphere ∂B , and

$$(2.10) \quad (B, B \cap \Sigma_k, \dots, B \cap \Sigma_1) = (B, c\Sigma'_{k-1}, \dots, c\Sigma'_0),$$

where $c\Sigma'_i$ denotes the cone over Σ'_i with respect to the linear structure of the C^1 -parametrised ball B .

Let N be a smooth manifold. A submanifold S of N is said to be transverse to an smcs Σ of N , if S is transverse to each stratum $\Sigma_{(i)}$ of Σ .

PROPOSITION 2.3 (Lemma 1 in [27]). — *If a submanifold $S \subset N$ of codimension q is transverse to an smcs $\Sigma = (\Sigma_k, \Sigma_{k-1}, \dots, \Sigma_0)$ of N of dimension k , then the intersection $\Sigma \cap S = (\Sigma_k \cap S, \Sigma_{k-1} \cap S, \dots, \Sigma_0 \cap S)$ is an smcs of dimension $k - q$ of S .*

We extend the notion of an smcs to closed subsets Σ of a singular space with conical singularities (X, g) as follows:

DEFINITION 2.4. — *Let (X, g) be a space with isolated conical singularities and let $\Sigma \subset X$ be a closed subset of X with a stratification*

$$(2.11) \quad \Sigma = \Sigma_k \supseteq \Sigma_{k-1} \supseteq \dots \supseteq \Sigma_0.$$

We call $\Sigma = (\Sigma_k, \dots, \Sigma_0)$ an smcs of X of dimension k if the following conditions hold:

- (1) Singular points of X are contained in Σ_0 ,

$$(2.12) \quad \text{Sing}(X) \cap \Sigma \subset \Sigma_0.$$

- (2) The closed subset $\Sigma \cap X_{sm}$ of X_{sm} with the stratification

$$(2.13) \quad \Sigma_k \cap X_{sm} \supseteq \Sigma_{k-1} \cap X_{sm} \supseteq \dots \supseteq \Sigma_0 \cap X_{sm}$$

is an smcs of dimension k of the smooth manifold X_{sm} .

- (3) For any $p \in \Sigma \cap \text{Sing}(X)$ there exists an $\epsilon > 0$ such that the intersection of Σ with $\partial U_\epsilon(p)$ is transverse and in the chart (2.6) we have

$$(2.14) \quad \varphi|_{U_\epsilon(p) \cap \Sigma} : U_\epsilon(p) \cap \Sigma \simeq c_\epsilon(L_{p,\epsilon} \cap \varphi(\Sigma)) \subset c_\epsilon(L_{p,\epsilon}).$$

In the following, we will often denote an smcs of dimension k simply by Σ or by (Σ, Σ_{k-1}) .

In Section 4, Section 5 and Section 6 we will use the following result, which follows from existing literature on stratified spaces:

PROPOSITION 2.5. — *Let X be a singular space with conical singularities, $\dim X = n$. Let $\Sigma = (\Sigma_n, \dots, \Sigma_0)$ be an smcs of X of dimension n , with $\Sigma_n = X$. Then X admits a triangulation compatible with the stratification Σ , i.e. each closed subset Σ_i of the filtration is a union of closed simplices of the triangulation.*

Sketch of proof. — Let us first explain the result for the case where X is smooth. One of the most prominent notions for stratified spaces is the so-called Whitney-(b) condition introduced by Whitney in [50] (see also e.g. Section 1.4.3 in the book [38] for the definition). The Whitney-(b) condition is a C^1 -invariant (see Corollary 3.3 in [45]). The charts appearing in Definition 2.2 are C^1 -charts in which the Whitney-(b) condition holds. Therefore Σ is a Whitney-(b) stratification of X and Whitney stratified sets admit triangulations compatible with the stratification (see [14], [24], [46] and the references therein). By inspecting the proof of [14] and [15] one can check that in the case of an smcs the triangulation constructed in [14] is a smooth triangulation of the smooth manifold X .

Let now X be a space with isolated conical singularities. One gets the claim, using condition (3) in Definition 2.4, the above arguments in the smooth case and the construction in [14]. □

2.3. Anti-radial Morse functions. Standard Morse functions

DEFINITION 2.6. — *Let (X, g) be a space with isolated conical singularities. A continuous function $f : X \rightarrow \mathbb{R}$ is called an anti-radial Morse function, if the following two conditions hold:*

- (1) *The restriction $f_{sm} := f|_{X_{sm}}$ is a smooth Morse function.*
- (2) *Near a singular point $p \in \text{Sing}(X)$ the function f has the following normal form in the local coordinates $(r, y) \in U_\delta(p) \simeq c_\delta L_p$ in (2.6):*

$$(2.15) \quad f(r, y) = f(p) - \frac{1}{2}r^2.$$

Condition (2) in Definition 2.6 implies in particular that every singular point $p \in \text{Sing}(X)$ is a local maximum for the anti-radial Morse function. We will use the following convention for $p \in \text{Sing}(X)$,

$$(2.16) \quad \text{ind}(p) := \dim X = n.$$

We denote by $\text{Crit}(f_{sm})$ the set of critical points of f_{sm} , by $\text{Crit}_i(f_{sm})$ the set of critical points of f_{sm} of index i . We denote by

$$(2.17) \quad \text{Crit}(f) := \text{Crit}(f_{sm}) \cup \text{Sing}(X).$$

Let $p \in \text{Crit}_i(f_{sm})$. By Morse Lemma (see e.g. Lemma 2.2 in [32]) there exists an open neighbourhood $U(p)$ of p and local coordinates x_1, \dots, x_n near p such that for $x = (x_1, \dots, x_n) \in U(p)$:

$$(2.18) \quad f(x) = f(p) + \frac{1}{2} (-x_1^2 \dots - x_i^2 + x_{i+1}^2 + \dots + x_n^2).$$

DEFINITION 2.7. — Let (X, g) be a space with isolated conical singularities and let $f : X \rightarrow \mathbb{R}$ be an anti-radial Morse function. We say that the pair (f, g) is Standard Morse, shortly (SM), if for all $p \in \text{Crit}_i(f_{sm})$ we have

$$(2.19) \quad -\nabla_g f = x_1 \frac{\partial}{\partial x_1} + \dots + x_i \frac{\partial}{\partial x_i} - x_{i+1} \frac{\partial}{\partial x_{i+1}} - \dots - x_n \frac{\partial}{\partial x_n}$$

in a Morse chart (2.18) near p .

3. The negative gradient flow. Stable and unstable cell decomposition

This section discusses stable and unstable manifolds, trajectory spaces, as well as the Morse-Smale transversality condition in our singular setting. The results are straightforward generalisations of smooth Morse theory.

3.1. The negative gradient flow and the Morse-Smale condition

In this subsection (X, g) is a singular space with conical isolated singularities and $f : X \rightarrow \mathbb{R}$ an anti-radial Morse function. For a set $A \subset X$, we will denote by \overline{A} the closure of A in X .

The negative gradient vector field $-\nabla_g f$ on X_{sm} induces a smooth flow on X_{sm} , which extends continuously to a flow on X

$$(3.1) \quad \Phi : \mathbb{R} \times X \longrightarrow X.$$

For $p \in \text{Crit}(f)$ the stable resp. unstable set of p is defined as follows

$$(3.2) \quad W^{s/u}(p) = W^{s/u}(p, (f, g)) := \{x \in X \mid \lim_{t \rightarrow \pm\infty} \Phi(t, x) = p\}.$$

For $p \in \text{Crit}(f_{sm})$, by smooth Morse theory (see e.g. Theorem 2.7 in [49]), both the stable and unstable set $W^{s/u}(p)$ are submanifolds of X_{sm} .

For $p \in \text{Sing}(X)$, using the anti-radiality of the Morse function, it is easy to see that $\overline{W^s(p)} = \{p\}$. Moreover $W^u(p) \cap X_{sm}$ is a submanifold of X_{sm} with $\overline{W^u(p)} \cap \text{Sing}(X) = \{p\}$.

Note that one has a (disjoint) decomposition of X into unstable resp. stable cells

$$(3.3) \quad X = \bigcup_{p \in \text{Crit}(f)} W^{u/s}(p).$$

Note that by the anti-radiality condition, for $p \in \text{Sing}(X)$, $q \in \text{Crit}(f_{sm})$ the intersection $W^u(p) \cap W^s(q)$ lies in X_{sm} and $W^s(p) \cap W^u(q) = \emptyset$. Moreover for $p, p' \in \text{Sing}(X)$, $p \neq p'$,

$$(3.4) \quad \overline{W^u(p) \cap W^s(p')} = \emptyset.$$

Hence, one can generalise the Morse-Smale transversality condition to this singular setting:

DEFINITION 3.1. — *Let (X, g) be a space with isolated conical singularities and let $f : X \rightarrow \mathbb{R}$ be an anti-radial Morse function. The pair (f, g) satisfies the Morse-Smale transversality condition, if*

for all $p, q \in \text{Crit}(f)$, the intersection $W^u(p) \cap W^s(q)$ is transverse. (T)

The Morse-Smale condition (T) implies that the intersection $W^u(p) \cap W^s(q)$ is a smooth manifold of dimension $\text{ind}(p) - \text{ind}(q)$. Note that, it is easily seen by adapting the arguments in [43] (in particular Lemma 1.2 in [43]) that the Morse-Smale transversality condition (T) can always be achieved by a small perturbation of the metric g outside a small neighbourhood of $\text{Crit}(f)$ (in Proposition 6.4 of [31] a proof of this fact has been given in a more general situation).

Remark 3.2. — For a smooth manifold equipped with a smooth Morse function the stable and unstable cell decomposition (3.3) has been noticed already by Thom in [44]. The Morse-Smale condition for smooth Morse functions on smooth manifolds is not yet present in Thom’s note [44]. It has been introduced by Smale [43].

3.2. Orientation

For the rest of the article we will assume that (X, g) is a space with isolated conical singularities, $f : X \rightarrow \mathbb{R}$ is an anti-radial Morse function such that the pair (f, g) satisfies the conditions (T) and (SM).

In this subsection we will shortly explain our conventions on orientation.

Stable and unstable cells of critical points are contractible and therefore orientable. By choosing an orientation on all unstable cells one gets induced orientations on all intersections $W^u(p) \cap W^s(q)$, $p, q \in \text{Crit}(f)$

(for more details we refer the reader to e.g. [49], Section 3.4). The orientation of $W^u(p) \cap W^s(q)$ together with the negative gradient flow induce an orientation of the unparametrised trajectory space

$$(3.5) \quad \widetilde{\mathcal{M}}(p, q) := W^u(p) \cap W^s(q) \cap f^{-1}(a),$$

where $a \in]f(q), f(p)[$ is a regular level. In particular, if $\text{ind}(p) = \text{ind}(q) + 1$, the unparametrised trajectory space $\widetilde{\mathcal{M}}(p, q)$ is a finite set of points equipped with signs. Hence, we can define, precisely as in smooth Morse theory:

DEFINITION 3.3. — *Let $p, q \in \text{Crit}(f)$, $\text{ind}(p) - \text{ind}(q) = 1$. We define*

$$(3.6) \quad n(p, q) := \text{number of points in } \widetilde{\mathcal{M}}(p, q) \text{ counted with signs.}$$

Recall that in this article X_{sm} was assumed to be oriented (see Definition 2.1). For $p \in \text{Sing}(X)$ the orientation of the unstable cell $W^u(p)$ will be chosen to be compatible with that of X_{sm} . Moreover we will orient the stable cells compatibly, i.e. we have for each $p \in \text{Crit}(f_{sm})$ that $T_p W^u(p) \oplus T_p W^s(p) = T_p X_{sm}$ and the orientations are such that the orientation of $W^u(p)$ followed by that of $W^s(p)$ gives the orientation of X_{sm} .

Let L be the link manifold of a singularity of X . Using the gradient flow, we have induced orientations on the intersection $L \cap W^s(p)$, $p \in \text{Crit}(f_{sm})$. The orientation of $L \cap W^s(p)$ is used in the definition (1.6) of the boundary operator of the complex $(C_*(f, g, \Xi), \partial_*)$: the closed forms in Ξ will be integrated over the oriented *smcs* $L \cap W^s(p)$.

The orientability of X (more precisely the orientability of the link manifold L) is crucial in this article: Poincaré duality on the oriented link manifold is used in the proof of Theorem 6.2 (more precisely in (6.10)).

For a non-orientable space X the constructions of this article can be adapted by using a set Ξ of closed forms with values in the orientation bundle of L instead (see Example 8.3).

3.3. The stable/unstable cell decomposition

For $p \in \text{Crit}(f)$, we denote by $\overline{W^{s/u}(p)}$ the closure of the stable/unstable set $W^{s/u}(p)$ in X . Recall from Section 2.1 that for $p \in \text{Sing}(X)$ and $\epsilon > 0$ small enough, we can identify a small neighbourhood $U_\epsilon(p)$ with $c_\epsilon L_p$ and the boundary $\partial U_\epsilon(p)$ with L_p . We omit the subscript p in the following for simplicity.

The next proposition generalises Proposition 2 in [27] to our setting:

PROPOSITION 3.4.

- (a) Let $r \in \text{Crit}(f)$. Then $\overline{W^u(r)}$ is an *smcs* of X . The strata of $\overline{W^u(r)} \setminus W^u(r)$ are unstable manifolds $W^u(q)$, where $q \in \text{Crit}(f_{sm})$ with $\text{ind}(q) < \text{ind}(r)$. Let $q \in \text{Crit}_{\text{ind}(r)-1}(f_{sm})$ with $W^u(q) \subset \overline{W^u(r)}$. Then, near $W^u(q)$, $W^u(r)$ consists of $n_+(r, q) + n_-(r, q)$ connected components, $W^u(q)$ being the oriented boundary of $n_+(r, q)$ of these. Moreover $n(r, q) = n_+(r, q) - n_-(r, q)$.
- (b) For $r \in \text{Crit}(f_{sm})$, the set $\overline{W^s(r)}$ is an *smcs* of X . The strata of $\overline{W^s(r)} \setminus W^s(r)$ are stable manifolds $W^s(q)$, where $q \in \text{Crit}(f)$ with $\text{ind}(q) > \text{ind}(r)$. Moreover, if $\text{ind}(r) = \text{ind}(q) - 1$, near $W^s(q)$, $W^s(r)$ consists of $n_+(q, r) + n_-(q, r)$ connected components, $W^s(q)$ being the oriented boundary of $n_+(q, r)$ (resp. $n_-(q, r)$) of these.

Proof. — By anti-radiality, the negative gradient flow does only leave singular points of X in positive time. Hence the *smcs*-property in (a) and (b) follow, as in smooth Morse theory, by a repeated application of Lemma 4 in [27] (see also Section A.7 in [29]). Note that, if $p \in \overline{W^s(r)} \cap \text{Sing}(X)$ by the anti-radiality condition the intersection $\overline{W^s(r)} \cap L$ is transverse and therefore by Proposition 2.3 also an *smcs*. Since the negative gradient vector field has the form $r \frac{\partial}{\partial r}$ near a singular point, we have

$$(3.7) \quad \overline{W^s(r)} \cap U_\epsilon(p) = c_\epsilon(\overline{W^s(r)} \cap L).$$

This shows that $\overline{W^s(r)}$ satisfies the condition (3) in the Definition 2.4 of an *smcs* in X . □

Remark 3.5. — Let us mention, that the literature inspired from Floer theory usually considers the closure of stable and unstable cells with respect to the topology of broken trajectories $\overline{W^u(p)}^{broken}$. This compactification takes place outside the manifold. The space $\overline{W^u(p)}^{broken}$ is homeomorphic to a manifold with corners, and there is a natural surjective map $\overline{W^u(p)}^{broken} \rightarrow \overline{W^u(p)}$. The interested reader is referred to Section 4.4 of the book [37] where the two ways of compactifying trajectory spaces in smooth Morse theory, as well as the relation between them, is discussed in detail.

COROLLARY 3.6. — Let $p \in \text{Sing}(X)$ be a singular point of X with link manifold L . Then the decomposition of L induced from the stable cell decomposition (3.3) of X induces a stratification Σ_L^s ,

$$(3.8) \quad L = \bigcup_{q \in \text{Crit}_0(f_{sm})} \overline{W^s(q)} \cap L \supseteq \dots \supseteq \bigcup_{q \in \text{Crit}_{n-1}(f_{sm})} \overline{W^s(q)} \cap L,$$

which makes L into an smcs. Moreover, if $\text{ind}(r) = \text{ind}(q) - 1$, near $W^s(q) \cap L$, $W^s(r) \cap L$ consists of $n_+(q, r) + n_-(q, r)$ connected components, $W^s(q) \cap L$ being the oriented boundary of $n_+(q, r)$ (resp. $n_-(q, r)$) of these.

Proof. — From Proposition 3.4 (b) we deduce, that the stratification of X_{sm}

$$X_{sm} = \bigcup_{q \in \text{Crit}_0(f_{sm})} \overline{W^s(q)} \setminus \text{Sing}(X) \supseteq \dots \supseteq \bigcup_{q \in \text{Crit}_n(f_{sm})} \overline{W^s(q)} \setminus \text{Sing}(X)$$

makes X_{sm} into an smcs. Note that

$$\bigcup_{q \in \text{Crit}_n(f_{sm})} \overline{W^s(q)} \setminus \text{Sing}(X) = \text{Crit}_n(f_{sm}).$$

The claim follows by applying Proposition 2.3. □

Remark 3.7. — In Section 8 of [31] it has been proved that, as in smooth Morse theory, the stable manifolds of the anti-radial Morse function f generate a Thom-Smale complex for the singular space X . For $p \in \text{Crit}(f)$ we denote by $[W^s(p)]$ the corresponding generator. As a consequence of Proposition 3.4 (b) one gets, that the boundary operator in this complex is defined by

$$(3.9) \quad \partial[W^s(r)] = \pm \sum_{\text{ind}(q)=\text{ind}(r)+1} n(q, r) \cdot [W^s(q)].$$

The main result in [31] (Theorem 8.2) states, that the complex generated by the stable manifolds of an anti-radial Morse function does compute the singular homology of X . Actually, the theory in [31] is developed in a more general setting, namely on general Thom-Mather stratified spaces.

4. A refinement of the unstable cell decomposition of X

In this section we will define a refinement of the unstable cell decomposition $X = \bigcup_{q \in \text{Crit}(f)} W^u(q)$, by decomposing $W^u(p)$ for each singular point $p \in \text{Sing}(X)$. Since f is anti-radial, there are no flow lines between two points $p, p' \in \text{Sing}(X)$. It is therefore enough to explain the construction for the case where $\text{Sing}(X) = \{p\}$. We denote by L the link of X at p .

4.1. Compatible triangulation T of the link manifold L

Let Σ_L^s be the stratification of L induced from the stable cell decomposition (see Corollary 3.6). By Corollary 3.6 and Proposition 2.5, there exists

a smooth triangulation T of L , compatible with the stratification Σ_L^s . By a result in [36], we can moreover assume that the dual cell decomposition is a smooth cell decomposition of L . Let D_{n-k}^s be an open $(n - k)$ -cell of T . Since the triangulation T is compatible with the stratification Σ_L^s of L , there exists exactly one $q \in \text{Crit}(f_{sm})$ with

$$(4.1) \quad D_{n-k}^s \subset W^s(q) \cap L$$

and we write

$$(4.2) \quad D_{n-k}^s \sim W^s(q).$$

Note that, since $n - k = \dim D_{n-k}^s \leq \dim W^s(q) - 1 = n - 1 - \text{ind}(q)$,

$$(4.3) \quad \text{ind}(q) \leq k - 1.$$

Recall that X is oriented (see Definition 2.1), all unstable and all stable cells are oriented according to the conventions in Section 3.2. We choose orientations on all cells of T as follows: All $(n - k)$ - cells D_{n-k}^s of T with

$$(4.4) \quad D_{n-k}^s \sim W^s(q) \text{ for some } q \in \text{Crit}_{k-1}(f_{sm})$$

inherit an orientation from the orientation of $W^s(q) \cap L$. All other cells are oriented arbitrarily.

Let us denote by (T_*^s, ∂_*^s) the complex generated by the closed cells of the triangulation T . We denote the generator corresponding to the closed cell \overline{D}_i^s by $[D_i^s] \in T_i^s$. The boundary map $\partial_i^s : T_i^s \rightarrow T_{i-1}^s$ is the unique \mathbb{R} -linear map such that $\partial^s[D_i^s] = \sum_{[D_{i-1}^s] \in T_{i-1}^s} \pm [D_{i-1}^s]$; the sign in the above definition is $+$ if the orientations of $[D_i^s]$ and $[D_{i-1}^s]$ are compatible and $-$ else. Up to signs the complex (T_*^s, ∂_*^s) can be identified with the complex of simplicial chains of L with respect to the triangulation T .

Let T' be the barycentric subdivision of T . We denote by D_{k-1}^u the (open) dual cell of D_{n-k}^s in the barycentric subdivision T' . We orient D_{k-1}^u such that the orientation of D_{k-1}^u followed by the negative gradient flow and the orientation of D_{n-k}^s yields the orientation of X . We denote by (T_*^u, ∂_*^u) the complex dual to (T_*^s, ∂_*^s) . It is generated by the (closed) dual cells $[D^u]$ of cells $[D^s] \in T_*^s$. Note that with the orientations chosen above, we have that

$$(4.5) \quad \langle \partial^u[D^u], [D^s] \rangle = \pm \langle [D^u], \partial^s[D^s] \rangle \text{ for all } [D^u] \in T^u, [D^s] \in T^s;$$

the sign in (4.5) depends only on the dimension of the cells.

4.2. The cone $\tilde{c}D^u$

We denote by $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$.

DEFINITION 4.1. — For a dual cell $[D_{k-1}^u] \in T_{k-1}^u$, we define the “cone over D_{k-1}^u with respect to the flow Φ ” by

$$(4.6) \quad \tilde{c}D_{k-1}^u := \{\Phi_t(x) \mid x \in D_{k-1}^u, t \in \overline{\mathbb{R}}\}.$$

Note that, from (4.6) and the fact that the flow Φ is continuous and Morse-Smale, we have $\overline{\tilde{c}D_{k-1}^u} = \tilde{c}\overline{D_{k-1}^u}$. Moreover, $p \in \overline{\tilde{c}D_{k-1}^u}$. The cone $\tilde{c}D_{k-1}^u$ intersects the open cell D_{n-k}^s in a single point and the intersection is transverse. The flow Φ and the orientation of D_{k-1}^u induce an orientation of the interior of $\tilde{c}D_{k-1}^u$.

The next proposition gives a description of the boundary of $\overline{\tilde{c}D_{k-1}^u}$.

PROPOSITION 4.2. — Let D_{n-k}^s be an $(n - k)$ -dimensional open cell in T , with $D_{n-k}^s \sim W^s(q)$ for some $q \in \text{Crit}(f_{sm})$. Denote by D_{k-1}^u the dual cell of D_{n-k}^s . Then $\overline{\tilde{c}D_{k-1}^u}$ is an smcs of X . More precisely

(a) Let $\text{ind}(q) = k - 1$. Then,

$$(4.7) \quad \overline{W^u(q)} \subset \overline{\tilde{c}D_{k-1}^u}.$$

Near $W^u(q)$ the cone $\tilde{c}D_{k-1}^u$ is diffeomorphic to a (single) half-space $\mathbb{R}^{k-1} \times \mathbb{R}^{>0}$ with boundary $W^u(q) \simeq \mathbb{R}^{k-1}$ and the orientations of $W^u(q)$ and of $\tilde{c}D_{k-1}^u$ are compatible.

Moreover

$$(4.8) \quad W^u(q') \cap \overline{\tilde{c}D_{k-1}^u} \neq \emptyset \text{ for all } q' \in \left(\bigcup_{l \geq k-1} \text{Crit}_l(f_{sm}) \right) \setminus \{q\}.$$

(b) Let $\text{ind}(q) < k - 1$. Then,

$$(4.9) \quad \overline{W^u(q)} \subset \overline{\tilde{c}D_{k-1}^u}.$$

Moreover

$$(4.10) \quad W^u(q') \cap \overline{\tilde{c}D_{k-1}^u} \neq \emptyset \text{ for all } q' \in \bigcup_{l \geq k-1} \text{Crit}_l(f_{sm}).$$

Proof. — By Morse-Smale transversality we can assume that f is self-indexing. Note first, that since the triangulation T is compatible with the stratification Σ_L^s of L , and since D_{k-1}^u is a dual cell in the barycentric subdivision of T , $\overline{D_{k-1}^u}$ is an smcs of L transverse to every stable manifold $W^s(q')$, $q' \in \text{Crit}(f_{sm})$. Let $U_\epsilon(p)$ be a small neighbourhood of the singularity p of X . Then, by anti-radiality, obviously

$$(4.11) \quad \overline{\tilde{c}D_{k-1}^u} \cap U_\epsilon(p) = c_\epsilon \overline{D_{k-1}^u}.$$

Therefore $\widetilde{cD}_{k-1}^u \cap f^{-1}([n-1+\epsilon, n])$ is an *smcs* of $f^{-1}([n-1+\epsilon, n])$. By downward induction on $k \in \{0, \dots, n-1\}$ and applying Lemma 4 in [27] (see also Section A.7 in [29]), $\widetilde{cD}_{k-1}^u \cap f^{-1}([k-\epsilon, n])$ is an *smcs* of $f^{-1}([k-\epsilon, n])$. In particular, $\widetilde{cD}_{k-1}^u = \widetilde{cD}_{k-1}^u$ is an *smcs* of X .

The claims (4.7)-(4.10) follow from the following observation: \overline{D}_{k-1}^u intersects only cells \widetilde{D}^s of the triangulation T , which are adjacent to D_{n-k}^s , i.e. $D_{n-k}^s \subset \widetilde{D}^s$. Since the triangulation T is compatible with the stratification Σ_L^s of L , we have that

$$(4.12) \quad \widetilde{D}^s \sim W^s(r) \text{ for some } r \in \text{Crit}(f) \text{ with } \text{ind}(r) \leq \text{ind}(q). \quad \square$$

4.3. The map τ^T from forms on L to currents on L

We denote by $(C_*^{T'}(L), \partial_*)$ the complex of simplicial chains of L with respect to the triangulation T' . Note that $(C_*^{T'}(L), \partial_*)$ can be seen as a sub-complex of the complex of de Rham currents on L , by associating to each closed simplex the current of integration over it.

The constructions in this section are adapted from [27], [29].

DEFINITION 4.3. — For $\xi \in \Omega^{n-k}(L)$ we define

$$(4.13) \quad \tau^T(\xi) := \sum_{[D_{n-k}^s] \in T_{n-k}^s} \left(\int_{D_{n-k}^s} \xi \right) \cdot [D_{k-1}^u] \in C_{k-1}^{T'}(L).$$

We denote by $(\Omega^*(L), \widetilde{d})$ the de Rham complex of smooth forms on L .

PROPOSITION 4.4. — Let $\xi \in \Omega^*(L)$.

- (a) We have $\tau^T(\widetilde{d}\xi) = \pm \partial \tau^T(\xi)$. In particular $\partial \tau^T(\xi) = 0$, when ξ is closed.
- (b) For a closed form $\xi \in \Omega^*(L)$ the regular current ξ and the integration current $\pm \tau^T(\xi)$ are homologous.

Proof. — (a) For a cell $[D_{n-k}^s] \in T_{n-k}^s$, we denote by $[D_{k-1}^u] \in T_{k-1}^u$ the dual cell. For cells $[D_{n-k+1}^s] \in T_{n-k+1}^s$ and $[D_{n-k}^s] \in T_{n-k}^s$ let us denote by $\alpha(D_{n-k+1}^s, D_{n-k}^s) \in \{\pm 1\}$ the coefficients defining the boundary operator in the complex (T_*^s, ∂_*^s) :

$$(4.14) \quad \partial[D_{n-k+1}^s] = \sum_{[D_{n-k}^s] \in T_{n-k}^s} \alpha(D_{n-k+1}^s, D_{n-k}^s) \cdot [D_{n-k}^s].$$

The boundary operator in the dual complex (T_*^u, ∂_*^u) is then given by

$$(4.15) \quad \partial[D_{k-1}^u] = \pm \sum_{[D_{k-2}^u] \in T_{k-2}^u} \alpha(D_{n-k+1}^s, D_{n-k}^s) \cdot [D_{k-2}^u].$$

Therefore:

$$\begin{aligned}
 \partial\tau^T(\xi) &= \sum_{[D_{n-k}^s] \in T_{n-k}^s} \left(\int_{D_{n-k}^s} \xi \right) \cdot \partial[D_{k-1}^u] \\
 (4.16) \quad &= \pm \sum_{[D_{n-k+1}^s] \in T_{n-k+1}^s} \left(\int_{\partial[D_{n-k+1}^s]} \xi \right) \cdot [D_{k-2}^u] \\
 &= \pm \sum_{[D_{n-k+1}^s] \in T_{n-k+1}^s} \left(\int_{D_{n-k+1}^s} \tilde{d}\xi \right) \cdot [D_{k-2}^u] = \pm\tau^T(\tilde{d}\xi)
 \end{aligned}$$

(b) To proof the claim one can proceed exactly as in [29], Proposition 6.6.4. □

5. A subcomplex of the intersection chain complex

The aim of this section is to construct a subcomplex $(D_*^u(f, g, \Xi, T), \partial_*)$ of the intersection chain complex, associated to the anti-radial standard Morse-Smale pair (f, g) , the set of representatives Ξ of the cohomologies of the link manifolds (see (1.3) and (1.4)) and the compatible triangulations T of the link manifolds. The construction uses the refinement of the unstable cell decomposition done in Section 4.

The subcomplex $(D_*^u(f, g, \Xi, T), \partial_*)$ will be used in Section 6 to prove the first part of the Main Theorem.

5.1. Definition of the subcomplex $(D_*^u(f, g, \Xi, T), \partial_*)$ of the intersection chain complex

The intersection homology of a singular space has been defined by Goresky and MacPherson in [16] and [17]; we recall the definition (of simplicial intersection homology) for convenience of the reader (see [16], see also [25] Section 4.2): Let \tilde{S} be a triangulation of X which is compatible with the stratification $(X, \text{Sing}(X))$. We denote by $C_i^{\tilde{S}}(X)$ the space of simplicial i -chains in \tilde{S} (with coefficients in \mathbb{R}). A simplicial chain $\sigma \in C_i^{\tilde{S}}(X)$ is allowed for the lower middle perversity \underline{m} if

$$(5.1) \quad \dim(\text{Sing}(X) \cap \sigma) \leq \dim \sigma - n + \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

By convention, we set $\dim \emptyset = -\infty$. We denote by $IC_i^{\tilde{S}}(X)$ the subspace of the space of simplicial i -chains in \tilde{S} , consisting of allowable i -chains σ ,

such that $\partial\sigma$ is an allowable $(i - 1)$ -chain. We denote by $IC_i(X)$ the limit of $IC_i^{\tilde{S}}(X)$ over all triangulations \tilde{S} of X compatible with the stratification $(X, \text{Sing}(X))$. The complex $(IC_*(X), \partial_*)$ is the intersection chain complex of X with lower middle perversity \underline{m} . Its homology is called the intersection homology with lower middle perversity \underline{m} :

$$(5.2) \quad IH_*(X) := IH_*^{\underline{m}}(X) := H_*((IC_*(X), \partial_*)).$$

Since X has isolated singularities only, every triangulation \tilde{S} compatible with the stratification $(X, \text{Sing}(X))$ is flag-like. Hence, by a result in [18], the natural map $(IC_*^{\tilde{S}}(X), \partial_*) \rightarrow (IC_*(X), \partial_*)$ is a quasi-isomorphism

$$(5.3) \quad H_*((IC_*^{\tilde{S}}(X), \partial_*)) \simeq IH_*(X).$$

We have a decomposition of X given by all unstable cells of $\text{Crit}(f)$. As explained in Section 4, for each $p \in \text{Sing}(X)$ we choose a triangulation T_p of the link L_p , compatible with the stable cell decomposition. As in Section 4 we can then decompose $W^u(p)$ into cones $\tilde{c}D_p^u$. Recall that by Proposition 3.4 and Proposition 4.2 the closures of all unstable cells and each cone $\overline{\tilde{c}D_p^u}$ is an *smcs*. Thus we get a decomposition of X into all unstable cells of $\text{Crit}(f_{sm})$, all singular points in $\text{Sing}(X)$ and all interiors of the cones $\tilde{c}D_p^u$. This decomposition induces a stratification Σ of X , which gives X a structure of an *smcs*. Thus, by Proposition 2.5, X admits a triangulation S compatible with the stratification Σ of X . Note that S is therefore also compatible with the stratification $(X, \text{Sing}(X))$ of X . We denote by $(IC_*^S(X), \partial_*)$ the complex of intersection chains of X in S .

DEFINITION 5.1. — Let $p \in \text{Sing}(X)$. For $\xi \in \Omega^{n-k}(L_p)$ we define

$$(5.4) \quad \tilde{c}\tau^{T_p}(\xi) := \sum_{[D_{p,n-k}^s] \in T_{p,n-k}^s} \left(\int_{D_{p,n-k}^s} \xi \right) \cdot [\tilde{c}D_{p,k-1}^u] \in C_k^S(X).$$

LEMMA 5.2. — Let $p \in \text{Sing}(X)$. Let $k \geq \frac{n}{2} + 1$ and $\xi \in \Omega^{n-k}(L_p)$ a closed form. Then

$$(5.5) \quad \tilde{c}\tau^{T_p}(\xi) \in IC_k^S(X)$$

with boundary

$$(5.6) \quad \partial\tilde{c}\tau^{T_p}(\xi) = \sum_{q \in \text{Crit}_{k-1}(f_{sm})} \left(\int_{W^s(q) \cap L_p} \xi \right) \cdot [W^u(q)] \in IC_{k-1}^S(X).$$

Proof. — For simplicity we omit the subscript p in the following. For $k \geq \frac{n}{2} + 1$ the chain $\tilde{c}\tau^T(\xi)$, which contains the singular point of X , satisfies the allowability condition (5.1) for lower middle perversity. Using

Proposition 4.2 and Proposition 4.4 (a) we get

$$\begin{aligned}
 \partial \tilde{c}\tau^T(\xi) &= \sum_{q \in \text{Crit}_{k-1}(f_{sm})} \sum_{D_{n-k}^s \sim W^s(q)} \left(\int_{D_{n-k}^s} \xi \right) \cdot [W^u(q)] \\
 (5.7) \qquad &= \sum_{q \in \text{Crit}_{k-1}(f_{sm})} \left(\int_{W^s(q) \cap L} \xi \right) \cdot [W^u(q)] \in C^S(X).
 \end{aligned}$$

Therefore $\partial \tilde{c}\tau^T(\xi)$ is allowed, since it does not contain the singular point. We conclude that $\tilde{c}\tau^T(\xi), \partial \tilde{c}\tau^T(\xi) \in IC_*^S(X)$. \square

Let Ξ be the set of representatives of $\bigoplus_{\substack{p \in \text{Sing}(X), \\ k \geq \frac{n}{2} + 1}} H^{n-k}(L_p)$ as defined in (1.3) and (1.4).

DEFINITION/LEMMA 5.3. — We denote by $(D_*^u(f, g, \Xi, T), \partial_*)$ the following subcomplex of $(IC_*^S(X), \partial_*)$:

$$\begin{aligned}
 D_k^u &= D_k^u(f, g, \Xi, T) \\
 := \begin{cases} \bigoplus_{p \in \text{Crit}_k(f_{sm})} \mathbb{R} \cdot [W^u(p)] \oplus \bigoplus_{\substack{p \in \text{Sing}(X), \\ \xi_{p,l}^{n-k} \in \Xi_p^{n-k}}} \mathbb{R} \cdot [\tilde{c}\tau^T(\xi_{p,l}^{n-k})] & \text{if } k \geq \frac{n}{2} + 1, \\ \bigoplus_{p \in \text{Crit}_k(f_{sm})} \mathbb{R} \cdot [W^u(p)] & \text{if } k < \frac{n}{2} + 1. \end{cases}
 \end{aligned}$$

Proof. — From (5.6) we have that $\partial \tilde{c}\tau^T(\xi_{p,l}^{n-k}) \in D_{k-1}^u(f, g, \Xi, T)$. Moreover, by Lemma 5.2, all chains in $D_k^u(f, g, \Xi, T)$ are intersection chains in $IC_k^S(X)$. \square

5.2. Dependence of the subcomplex $(D_*^u(f, g, \Xi, T), \partial_*)$ on the choice of representatives Ξ and on the choice of the compatible triangulation T

PROPOSITION 5.4. — Let $k \geq \frac{n}{2} + 1$. Let $\xi, \xi' \in \Omega^{n-k}(L)$ be two closed forms representing the same cohomology class $[\xi] = [\xi'] \in H^{n-k}(L)$, i.e. for some form $\alpha \in \Omega^{n-k-1}(L)$:

$$(5.8) \qquad \qquad \qquad \xi' = \xi + d\alpha.$$

Then we have

$$(5.9) \quad \tilde{c}\tau^T(\xi') = \tilde{c}\tau^T(\xi) \pm \partial \tilde{c}\tau^T(\alpha) \pm \left(\sum_{r \in \text{Crit}_k(f_{sm})} \left(\int_{W^s(r) \cap L} \alpha \right) \cdot [W^u(r)] \right)$$

and

$$(5.10) \quad \partial \tilde{c}\tau^T(\xi') = \partial \tilde{c}\tau^T(\xi) \pm \partial \left(\sum_{r \in \text{Crit}_k(f_{sm})} \left(\int_{W^s(r) \cap L} \alpha \right) \cdot [W^u(r)] \right).$$

Proof. — The claim in (5.9) follows using Proposition 4.2 and Proposition 4.4 and arguing as in Lemma 5.2. The claim in (5.10) is a direct consequence of (5.9). \square

PROPOSITION 5.5. — *Let T and \tilde{T} be two triangulations of L compatible with the stratification Σ_L^s . Let $k \geq \frac{n}{2} + 1$ and let $\xi \in \Omega^{n-k}(L)$ be a closed form. Then*

$$(5.11) \quad \tilde{c}\tau^T(\xi) - \tilde{c}\tau^{\tilde{T}}(\xi) \in \partial(IC_*(X)).$$

Proof. — Note that, by Proposition 4.4 the regular current ξ (in L) is homologous to both integration currents $\tau^T(\xi)$ and $\tau^{\tilde{T}}(\xi)$. We deduce that $[\tau^T(\xi)] = [\tau^{\tilde{T}}(\xi)] \in H_*(L)$, i.e. for some k -chain σ in L ,

$$(5.12) \quad \tau^T(\xi) - \tau^{\tilde{T}}(\xi) = \partial\sigma.$$

The chain σ can be moved to be transverse to all cells $W^s(q)$, $q \in \text{Crit}(f_{sm})$. Set

$$(5.13) \quad \tilde{c}\sigma := \{ \Phi_t(x) \mid x \in \sigma, t \in \overline{\mathbb{R}} \}.$$

Since by (5.6) we have $\partial(\tilde{c}\tau^T(\xi)) = \partial(\tilde{c}\tau^{\tilde{T}}(\xi))$, we get

$$(5.14) \quad \partial \tilde{c}\sigma = \tilde{c}\tau^T(\xi) - \tilde{c}\tau^{\tilde{T}}(\xi) \in IC_k(X).$$

Note that

$$(5.15) \quad \dim(\tilde{c}\sigma) = k + 1 \geq \frac{n}{2} + 2,$$

and therefore $\tilde{c}\sigma \in IC_{k+1}(X)$. This proves the claim. \square

6. The geometric complex. Main Theorem (first part)

Section 6.1 and Section 6.2 give a proof of the first part of the Main Theorem.

6.1. Well-definedness of the abstract geometric complex
 $(C_*^u(f, g, \Xi), \partial_*)$

We denote by $(C_*^u(f, g, \Xi), \partial_*)$ the complex defined in the introduction.

LEMMA 6.1. — *The boundary operator ∂_* is well-defined and $\partial_*^2 = 0$.*

Proof. — Let us denote by $i_\epsilon : L_{p,\epsilon} = L_p \times \{\epsilon\} \rightarrow X$ the inclusion and by $\pi : cL_p \setminus \{0\} \simeq (0, \infty) \times L_p \rightarrow L_p$ the projection. By abuse of notation we will denote the pull-back $\pi^*\xi \in \Omega^*(cL_p \setminus \{p\})$ of a form $\xi \in \Omega^*(L_p)$ still by ξ .

Recall first that for $q \in \text{Crit}_{k-1}(f_{sm})$ by Proposition 3.4 and Proposition 2.3, $\overline{W^s}(q) \cap L$ is an *smcs* of L of dimension $m - (k - 1) = n - k$. By A.2 and A.3 of [29] integration of smooth forms on an *smcs* is well-defined. Therefore the integral

$$(6.1) \quad \int_{\overline{W^s}(q) \cap L_p} \xi := \int_{\overline{W^s}(q) \cap L_{p,\epsilon}} i_\epsilon^* \pi^* \xi$$

is well-defined. Using Stokes' theorem, one sees that the right hand side of (6.1) does not depend on $\epsilon > 0$ chosen small enough.

We now prove that $\partial_*^2 = 0$. By the anti-radiality of f , the fact that $\partial^2[W^u(p)] = 0$ for $p \in \text{Crit}(f_{sm})$ can be proved using smooth Morse theory.

In the following, we write simply ξ for $\xi_{p,l}^{n-k} \in \Xi$. Using Corollary 3.6 (for the last equality in (6.2)) we have:

$$\begin{aligned} \partial^2[\xi] &= \partial \left(\sum_{q \in \text{Crit}_{k-1}(f_{sm})} \left(\int_{\overline{W^s}(q) \cap L} \xi \right) \cdot [W^u(q)] \right) \\ &= \sum_{q \in \text{Crit}_{k-1}(f_{sm})} \left(\int_{\overline{W^s}(q) \cap L} \xi \right) \cdot \partial[W^u(q)] \\ (6.2) \quad &= \sum_{q \in \text{Crit}_{k-1}(f_{sm})} \left(\int_{\overline{W^s}(q) \cap L} \xi \right) \sum_{r \in \text{Crit}_{k-2}(f_{sm})} n(q, r) \cdot [W^u(r)] \\ &= \sum_{r \in \text{Crit}_{k-2}(f_{sm})} \left(\int_{\sum_{q \in \text{Crit}_{k-1}(f_{sm})} n(q,r) \cdot [\overline{W^s}(q) \cap L]} \xi \right) [W^u(r)] \\ &= \pm \sum_{r \in \text{Crit}_{k-2}(f_{sm})} \left(\int_{\partial[W^s(r) \cap L]} \xi \right) [W^u(r)]. \end{aligned}$$

Using Stokes' formula we get from (6.2) that $\partial^2[\xi] = 0$. □

6.2. Embedding of the geometric complex into the intersection chain complex. Proof of the isomorphism (1.7) of the Main Theorem

By (1.6) and (5.6) the map

$$(6.3) \quad (C_*^u(f, g, \Xi), \partial_*) \longrightarrow (D_*^u(f, g, \Xi, T), \partial_*)$$

defined by

$$(6.4) \quad \begin{aligned} W^u(p) &\mapsto W^u(p), \quad p \in \text{Crit}(f_{sm}) \\ \xi &\mapsto \tilde{c}\tau(\xi), \quad \xi \in \Xi, \end{aligned}$$

is a well-defined isomorphism of chain complexes between the complex $(C_*^u(f, g, \Xi), \partial_*)$ and the complex $(D_*^u(f, g, \Xi, T), \partial_*)$. By composition with the natural map

$$(6.5) \quad (D_*^u(f, g, \Xi, T), \partial_*) \subset (IC_*^S(X), \partial_*) \longrightarrow (IC_*(X), \partial_*),$$

we get a chain map

$$(6.6) \quad h_{(\Xi, T)} : (C_*^u(f, g, \Xi), \partial_*) \longrightarrow (IC_*(X), \partial_*).$$

The next theorem proves the isomorphism (1.7) of the Main Theorem:

THEOREM 6.2. — *The chain map (6.6) induces an isomorphism of homologies:*

$$(6.7) \quad H_*((C_*^u(f, g, \Xi), \partial_*) \simeq IH_*(X).$$

Proof. — Let us denote by $(C_*^u(f_{sm}), \partial_*) \subset (C_*^u(f, g, \Xi), \partial_*)$ the subcomplex generated by all unstable cells of points in $\text{Crit}(f_{sm})$. From smooth Morse theory (see Theorem 7.4 in [33]) we know that the homology of this complex computes canonically the absolute homology of the manifold with boundary M (see Definition 2.1),

$$(6.8) \quad H_*((C_*^u(f_{sm}), \partial_*) \simeq H_*(M).$$

Let $p \in \text{Sing}(X)$. Let us denote by $IH_*(cL_p, L_p)$ the relative intersection homology with lower middle perversity of the cone cL_p . Recall that the local calculation of intersection homology gives (see Section 2.4 in [17], also Proposition 4.7.2 in [25])

$$(6.9) \quad IH_i(cL_p, L_p) = \begin{cases} H_{i-1}(L_p) & \text{for } i \geq \frac{n}{2} + 1, \\ 0 & \text{otherwise.} \end{cases}$$

The quotient complex $((C^u(f, g, \Xi)/C^u(f_{sm}))_*, \partial_*)$ has boundary operator $\partial_* = 0$ and we have using de Rham's Theorem and Poincaré Duality on the link:

$$(6.10) \quad H_i(((C^u(f, g, \Xi)/C^u(f_{sm}))_*, \partial_*)) = \begin{cases} \bigoplus_{p \in \text{Sing}(X)} H_{dR}^{n-i}(L_p) & \text{for } i \geq \frac{n}{2} + 1, \\ 0 & \text{otherwise,} \end{cases} \\ = \begin{cases} \bigoplus_{p \in \text{Sing}(X)} H_{i-1}(L_p) & \text{for } i \geq \frac{n}{2} + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, by (6.9) and (6.10) we get

$$(6.11) \quad H_*((C^u(f, g, \Xi)/C^u(f_{sm}))_*, \partial_*) \simeq \bigoplus_{p \in \text{Sing}(X)} IH_*(cL_p, L_p).$$

We have a commutative diagram of chain complexes, with exact rows (6.12)

$$(6.12) \quad \begin{array}{ccccccc} 0 \rightarrow (C_*^u(f_{sm}), \partial_*) & \longrightarrow & (C_*^u(f, g, \Xi), \partial_*) & \longrightarrow & (C_*^u(f, g, \Xi)/C_*^u(f_{sm}), \partial_*) & \rightarrow & 0 \\ & & \downarrow h_{\Xi, T} & & \downarrow h_{cL, L, \Xi, T} & & \\ \iota_M \downarrow & & & & & & \\ 0 \rightarrow (C_*(M), \partial_*) & \longrightarrow & (IC_*(X), \partial_*) & \longrightarrow & (IC_*(X, M), \partial_*) & \rightarrow & 0. \end{array}$$

From (6.8) and (6.11) we deduce that the maps ι_M resp. $h_{cL, L, \Xi, T}$ are quasi-isomorphisms. Using the long exact homology sequences associated to the short exact sequences in (6.12) and the 5-Lemma we deduce the isomorphism (6.7). □

6.3. The pairing with L^2 -cohomology

The conical Riemannian metric g on X induces an L^2 -metric on forms on X_{sm} . We denote by (\mathcal{C}^∞, d) the complex of smooth L^2 -forms on X_{sm} , i.e.

$$(6.13) \quad \mathcal{C}^{\infty, i} = \{\omega \in \Omega^i(X_{sm}) \mid \omega \in L^2, d\omega \in L^2\}.$$

The cohomology of the complex of L^2 -forms is the so-called L^2 -cohomology of X , first introduced by Cheeger [10]:

$$(6.14) \quad H_{(2)}^*(X) := H^*((\mathcal{C}^\infty, d)).$$

By the result of Cheeger-Goresky-MacPherson [11], for spaces with isolated conical singularities, the integration map

$$(6.15) \quad \int : (IC_*(X), \partial_*) \times (\mathcal{C}^\infty, d) \longrightarrow \mathbb{R}$$

induces an isomorphism between intersection cohomology with lower middle perversity and L^2 -cohomology.

The integration map (6.15) induces a pairing

$$(6.16) \quad P_{(\Xi, T)} = \int \circ (h_{\Xi, T}, \text{id}) : (C_*^u(f, g, \Xi), \partial_*) \times (C^\infty, d) \longrightarrow \mathbb{R}.$$

THEOREM 6.3. — *The restriction of the integration map*

$$(6.17) \quad P_{(\Xi, T)} : (C^u(f, g, \Xi), \partial) \times (C^\infty, d) \longrightarrow \mathbb{R}$$

to closed L^2 -forms does not depend on the choice of the triangulation T of the links of points in $\text{Sing}(X)$ used in the construction of $h_{\Xi, T}$. The isomorphism induced from integration

$$(6.18) \quad H_{(2)}^*(X) \simeq H^*(\text{Hom}(C_*^u(f, g, \Xi), \partial_*)),$$

is the canonical isomorphism between intersection cohomology and L^2 -cohomology.

Proof. — Let T and \tilde{T} be two triangulations of L compatible with the stratification Σ_L^s . Let $k \geq \frac{n}{2} + 1$ and $\xi \in \Omega^{n-k}(L) \cap \Xi$ a closed form. Then from Proposition 5.5 and Stokes' theorem we get that $P_{(\Xi, T)}(\xi, \omega) = P_{(\Xi, \tilde{T})}(\xi, \omega)$. □

7. Homotopy. Main Theorem (second part)

7.1. Radial Morse function on $X \times S^1$

The space $\tilde{X} := X \times S^1$ is a stratified space of dimension $n + 1$, with two strata: the singular 1-dimensional stratum $\tilde{X}_{(1)} = \text{Sing}(X) \times S^1$ and the top stratum $\tilde{X}_{(n+1)} = (X \setminus \text{Sing}(X)) \times S^1 = \tilde{X} \setminus \tilde{X}_{(1)}$.

Let (f^α, g^α) and (f^β, g^β) be two anti-radial Morse-Smale pairs on X . Let us fix a homotopy $\{g_s\}_{s \in [0, 1]}$ of conical Riemannian metrics on X : For $p \in \text{Sing}(X)$ there exists an open neighbourhood $U(p)$ as well as a homotopy $\{g_{L_p, s}\}$ of Riemannian metrics on the link manifold L_p , such that

$$(7.1) \quad g_{s|U(p)} = dr^2 + r^2 g_{L_p, s}.$$

We also fix a homotopy $\{f_s\}_{s \in [0, 1]}$ of anti-radial Morse functions: For $p \in \text{Sing}(X)$, $U(p)$ as above and $s \in [0, 1]$, we have

$$(7.2) \quad f_{s|U(p)}(r) = f_s(p) - \frac{1}{2}r^2.$$

We will moreover assume that the homotopies g_s and f_s are constant near the end points, i.e. that for some fixed $\delta \in [0, 1/4]$ we have

$$(7.3) \quad \begin{aligned} (f_s, g_s) &= (f^\alpha, g^\alpha) \text{ for } s \in [0, \delta], \\ (f_s, g_s) &= (f^\beta, g^\beta) \text{ for } s \in [1 - \delta, 1]. \end{aligned}$$

We parametrise the circle S^1 by $[-1, 1]$, with the endpoints being identified.

PROPOSITION 7.1. — *The function $F = F_\kappa : X \times S^1 \rightarrow \mathbb{R}$ defined by*

$$(7.4) \quad F(x, s) = \frac{\kappa}{2}(1 + \cos \pi s) + f_{|s|}(x)$$

has the following properties:

- (a) *The function F is continuous and strata-wise smooth.*
- (b) *For $\kappa \gg 0$ the set of critical points of the restriction $F|_{\tilde{X}_{(n+1)}}$ is given by*

$$(7.5) \quad \text{Crit}(F|_{\tilde{X}_{(n+1)}}) = (\text{Crit} f_{sm}^\alpha \times \{0\}) \cup (\text{Crit} f_{sm}^\beta \times \{1\}).$$

All critical points of $F|_{\tilde{X}_{(n+1)}}$ are non-degenerate and have Morse index

$$(7.6) \quad \text{ind}_F(p, 0) = \text{ind}_{f^\alpha}(p) + 1 \text{ and } \text{ind}_F(p, 1) = \text{ind}_{f^\beta}(p).$$

- (c) *The set of critical points of the restriction $F|_{\tilde{X}_{(1)}} : \text{Sing}(X) \times S^1 \rightarrow \mathbb{R}$ is precisely given by $(\text{Sing}(X) \times \{0\}) \cup (\text{Sing}(X) \times \{1\})$; the points in $\text{Sing}(X) \times \{0\}$ are maxima, the points in $\text{Sing}(X) \times \{1\}$ are minima.*

Let g_{S^1} denote the standard metric on the circle. Set $G := g_{x,|s|} + g_{S^1}$, which is a Riemannian metric on $\tilde{X}_{(n+1)}$. The negative gradient vector field $-\nabla_G F$ satisfies the following conditions:

- (i) Let $p \in \text{Sing}(X)$. In $U(p) \times S^1$ the negative gradient vector field $-\nabla_G F$ is of the form

$$(7.7) \quad r \frac{\partial}{\partial r} - \frac{\pi \kappa}{2} \sin \pi s \frac{\partial}{\partial s}.$$

- (ii) The flow induced from $-\nabla_G F$ is well defined for all times and yields a continuous, strata-preserving, strata-wise smooth map $\tilde{\Phi} : \mathbb{R} \times \tilde{X} \rightarrow \tilde{X}$.
- (iii) The sets $X \times \{0\}$ and $X \times \{1\}$ are invariant under the flow $\tilde{\Phi}$. The flow $\tilde{\Phi}$ restricted to $X \times \{0\}$ coincides with the flow on X induced by (f^α, g^α) and similarly for $X \times \{1\}$ and (f^β, g^β) .
- (iv) For κ chosen big enough, there are no flow lines from $X \times \{1\}$ to $X \times \{0\}$.

- (v) For $p \in \text{Sing}(X)$ the point $(p, 0)$ is a local maximum of F . The only trajectories ending in positive infinite time in $(p, 1)$ are trajectories coming from $(p, 0)$.

Note that from (7.7), one sees that the gradient $-\nabla_G F$ is “tangential” to the singular stratum $\tilde{X}_{(1)}$ and is “radial”. For a point $y \in \text{Crit}(F)$ we denote by $\widetilde{W^{s/u}}(y)$ the stable resp. unstable set with respect to the flow $\tilde{\Phi}$. For $q \in \text{Crit}(f_{sm}^\alpha)$, we write $W^{s/u}((q, 0)) := W^{s/u}(q, (f^\alpha, g^\alpha), X \times \{0\})$ for the stable/unstable manifold of $q \in X \times \{0\}$ with respect to the negative gradient flow associated to the pair (f^α, g^α) on $X \times \{0\}$. Similarly, for $q \in \text{Crit}(f_{sm}^\beta)$ and $W^{s/u}((q, 1)) := W^{s/u}(q, (f^\beta, g^\beta), X \times \{1\})$.

PROPOSITION 7.2.

- (a) For $y \in \text{Crit}(F|_{\tilde{X}_{(n+1)}})$, the stable set $\widetilde{W^s}(y)$ is a smooth submanifold of $\tilde{X}_{(n+1)}$. The unstable set $\widetilde{W^u}(y)$ is a smooth submanifold of $\tilde{X}_{(n+1)}$ and $\widetilde{W^u}(y) \cap \tilde{X}_{(1)} = \emptyset$.
- (b) For $q \in \text{Crit}(f_{sm}^\alpha)$ we have $\widetilde{W^s}((q, 0)) = W^s((q, 0)) \subset X \times \{0\}$. For $q \in \text{Crit}(f_{sm}^\beta)$ we have $\widetilde{W^u}((q, 1)) = W^u((q, 1)) \subset X \times \{1\}$.
- (c) Let $p \in \text{Sing}(X)$. The stable sets of $(p, 0)$ resp. $(p, 1)$ are submanifolds of dimension 0 resp. 1 of $\tilde{X}_{(1)}$. The unstable sets are stratified spaces with two strata.

Proof. — As noted before $-\nabla_G F$ is tangential and radial. One therefore gets the claim by adapting Proposition 5.1 and Proposition 5.2 in [31] to the present situation. □

Note that from Proposition 7.2 one has that, for $x, y \in \text{Crit}(F)$, either $\widetilde{W^u}(x) \cap \widetilde{W^s}(y) \subset \tilde{X}_{(1)}$ or $\widetilde{W^u}(x) \cap \widetilde{W^s}(y) \subset \tilde{X}_{(n+1)}$. We say that the pair (F, G) satisfies the Morse-Smale condition if the intersection $\widetilde{W^u}(x) \cap \widetilde{W^s}(y)$ is transverse (in $\tilde{X}_{(1)}$ resp. in $\tilde{X}_{(n+1)}$). Applying the Morse theory for stratified spaces with tangential conditions developed in [31] (more precisely, see the proof of Proposition 6.4 therein) one can prove that, after possibly perturbing, one can in addition to (i)-(v) assume:

- (vi) The pair (F, G) satisfies the Morse-Smale condition.

Analogously to Proposition 3.4 one can study the closure of the stable and unstable sets of critical points of F . In Proposition 7.3 we will only give the result related to the closure of $\widetilde{W^s}((q, 1))$, $q \in \text{Crit}(f_{sm}^\beta)$, which will be needed in Lemma 7.4.

We orient the unstable cells of (F, G) as follows: For $p \in \text{Crit}(f^\beta)$, the unstable cells $\widetilde{W^u}((p, 1)) = W^u((p, 1)) \subset X \times \{1\}$ inherit orientations from

the orientations of the unstable cells of (f^β, g^β) . For $p \in \text{Crit}(f^\alpha)$, we orient the unstable cells $\widetilde{W}^u((p, 0))$ by the orientation induced by $\frac{\partial}{\partial s}$ followed by the orientation of the cell $W^u(p, 0)$. We orient \widetilde{X} by $\frac{\partial}{\partial s}$ followed by the orientation of X . All stable cells \widetilde{W}^s are oriented according to our convention explained in Section 3.2.

Let $p \in \text{Sing}(X)$. We will identify a tubular neighbourhood of $\{p\} \times S^1 \subset \widetilde{X}$ with $c_\epsilon L_p \times S^1$ and its boundary with $L_p \times S^1$. By abuse of notation, for a form $\xi \in \Omega^*(L_p)$, we still denote by ξ the pull-back form $\pi^*\xi \in \Omega^*(L_p \times S^1)$, where $\pi : L_p \times S^1 \rightarrow L_p$. For $r, q \in \text{Crit}(f^\beta)$ with $\text{ind}_{f^\beta}(r) - \text{ind}_{f^\beta}(q) = 1$, we denote by $n^\beta(r, q)$ the number of trajectories, counted with signs, between r and q for the flow associated to (f^β, g^β) on $X \simeq X \times \{1\}$. Note that with our convention for the orientation of unstable cells, this is as well the number of trajectories, counted with signs, for the flow $\widetilde{\Phi}$ between the points $(r, 1)$ and $(q, 1)$. For $r \in \text{Crit}_{k-1}(f_{sm}^\alpha)$ and $q \in \text{Crit}_{k-1}(f_{sm}^\beta)$ we denote by

$$(7.8) \quad \widetilde{n}(r, q) = \begin{cases} \text{number of trajectories of the flow } \widetilde{\Phi} \text{ between } (r, 0) \text{ and } (q, 1) \\ \text{which pass through } X \times \{1/2\} \text{ (counted with signs).} \end{cases}$$

PROPOSITION 7.3. — *Let the pair (F, G) be as constructed above, such that (i)-(vi) hold. Let $q \in \text{Crit}_{k-1}(f_{sm}^\beta)$.*

- (a) *The closure of $\widetilde{W}^s((q, 1))$ is an smcs in $\widetilde{X}_{(n+1)}$.*
- (b) *For $p \in \text{Sing}(X)$, the transverse intersection $\widetilde{W}^s((q, 1)) \cap (L_p \times S^1)$ is an smcs of $L_p \times S^1$. Moreover the boundary of $\widetilde{W}^s((q, 1)) \cap (L_p \times [0, 1])$ is given by:*

$$\begin{aligned} \partial \left[\widetilde{W}^s(q, 1) \cap (L_p \times [0, 1]) \right] = & \pm \left(\sum_{r \in \text{Crit}_k(f_{sm}^\beta)} n^\beta(r, q) \left[\widetilde{W}^s(r, 1) \cap (L_p \times [0, 1]) \right] \right. \\ & - \left[\widetilde{W}^s(q, 1) \cap (L_p \times \{1\}) \right] \\ & \left. + \sum_{r \in \text{Crit}_{k-1}(f_{sm}^\alpha)} \widetilde{n}(r, q) \left[\widetilde{W}^s(r, 0) \cap (L_p \times \{0\}) \right] \right). \end{aligned}$$

Proof. — The notation in the part (b) has to be understood in the sense of Remark 3.7. The proof is a direct generalisation of Proposition 3.4 and Corollary 3.6. □

7.2. The geometric complex associated to (F, G)

For $p \in \text{Sing}(X)$ let Ξ_p^{n-k} be the set of representatives of $H^{n-k}(L_p)$ as defined in (1.3). As in (1.4), set $\Xi_p := \bigoplus_{k \geq \frac{n}{2} + 1} \Xi_p^{n-k}$ and $\Xi = \bigoplus_{p \in \text{Sing}(X)} \Xi_p$.

Let us denote by $(C_*^u(f^\alpha, g^\alpha, \Xi), \partial_*^\alpha)$ resp. $(C_*^u(f^\beta, g^\beta, \Xi), \partial_*^\beta)$ the abstract geometric complex (as defined in the introduction) associated to the pair (f^α, g^α) resp. (f^β, g^β) and the set Ξ . In the next definition, for $q \in \text{Crit}(f_{sm}^\beta)$, we will simply write $[q]$ for the generator $[W^u(q)]$ of $(C_*^u(f^\beta, g^\beta, \Xi), \partial_*^\beta)$; similarly for $q \in \text{Crit}(f_{sm}^\alpha)$ seen as generator of $(C_*^u(f^\alpha, g^\alpha, \Xi), \partial_*^\alpha)$.

DEFINITION/LEMMA 7.4. — *The map*

$$(7.9) \quad \Psi : (C_*^u(f^\alpha, g^\alpha, \Xi), \partial_*^\alpha) \longrightarrow (C_*^u(f^\beta, g^\beta, \Xi), \partial_*^\beta)$$

defined by

$$\xi_p \mapsto \xi_p - \sum_{q \in \text{Crit}_k(f_{sm}^\beta)} \left(\int_{\tilde{W}^s(q,1) \cap (L \times [0,1])} \xi \right) \cdot [q],$$

for $p \in \text{Sing}(X), \xi_p \in \Xi_p^{n-k}$, and

$$[r] \mapsto \sum_{q \in \text{Crit}_k(f_{sm}^\beta)} \tilde{n}(r, q) \cdot [q], \text{ for } r \in \text{Crit}(f_{sm}^\alpha).$$

is a map of chain complexes.

Proof. — To show that the map Ψ is a map of chain complexes, we follow an idea from smooth Morse theory (see e.g. Section 4.2.1 in [49]) and study a geometric complex associated to the Morse-Smale pair (F, G) and the set Ξ . In the following we will use two copies of Ξ , one for the Morse pair (f^α, g^α) and one for the Morse pair (f^β, g^β) . We denote these two copies by Ξ^α and Ξ^β respectively. For $\xi_p^{n-k} \in \Xi_p^{n-k}$, we denote by $\xi_p^{n-k, \alpha} \in \Xi_p^{n-k, \alpha}$ resp. $\xi_p^{n-k, \beta} \in \Xi_p^{n-k, \beta}$ the two copies. In the next definition, we identify the point $p \in \text{Crit}(f^\beta)$ with $(p, 1) \in \text{Crit}(F)$. We identify the point $p \in \text{Crit}(f^\alpha)$ with $(p, 0) \in \text{Crit}(F)$. We denote by $(C_*(F, G, \Xi), \Delta_*)$ the following abstract complex:

$$(7.10) \quad C_k(F, G, \Xi) := C_{k-1}(f^\alpha, g^\alpha) \oplus C_k(f^\beta, g^\beta).$$

The boundary operator Δ_* is defined as:

$$(7.11) \quad \Delta_k = \begin{pmatrix} -\partial_{k-1}^\alpha & 0 \\ \Psi_{k-1} & \partial_k^\beta \end{pmatrix}.$$

Note that only “half” of the trajectories between $(r, 0)$ and $(q, 1)$, $r \in \text{Crit}(f^\alpha)$, $q \in \text{Crit}(f^\beta)$, are counted in the definition of the boundary operator Δ_* .

By definition of the boundary operator Δ_* and the fact that $(\partial_*^\alpha)^2 = (\partial_*^\beta)^2 = 0$, one has the following equivalence

$$(7.12) \quad \Delta^2 = 0 \iff -\Psi\partial^\alpha + \partial^\beta\Psi = 0.$$

Again one can use smooth Morse theory (see e.g. [49], Section 4.2.1) to prove that for $y \in \text{Crit}(F|_{\widetilde{X}_{(n+1)}})$: $\Delta^2 y = 0$. To show that for $p \in \text{Sing}(X)$ and $\xi_p^{n-k, \alpha} \in \Xi_p^{n-k, \alpha}$:

$$(7.13) \quad \Delta^2 \xi^\alpha = (-\Psi\partial^\alpha + \partial^\beta\Psi) \xi^\alpha = 0,$$

one uses the definition of Ψ , ∂^α , ∂^β and part (b) of Proposition 7.3. □

7.3. The canonical isomorphism for two anti-radial Morse-Smale pairs (f^α, g^α) and (f^β, g^β)

THEOREM 7.5. — *Let (f^α, g^α) and (f^β, g^β) be anti-radial Morse-Smale pairs. Then there is a canonical isomorphism between the associated homologies*

$$(7.14) \quad \Psi^{\beta\alpha} : H_*((C_*^u(f^\alpha, g^\alpha, \Xi), \partial_*^\alpha)) \rightarrow H_*((C_*^u(f^\beta, g^\beta, \Xi), \partial_*^\beta)).$$

Moreover the following functorial relations are fulfilled for the family $\{\Psi^{\alpha\beta} \mid (f^\alpha, g^\alpha), (f^\beta, g^\beta) \text{ anti-radial Morse-Smale pairs}\}$:

- $\Psi_*^{\gamma\beta} \Psi_*^{\beta\alpha} = \Psi_*^{\gamma\alpha}$
- $\Psi_*^{\alpha\alpha} = \text{id}$.

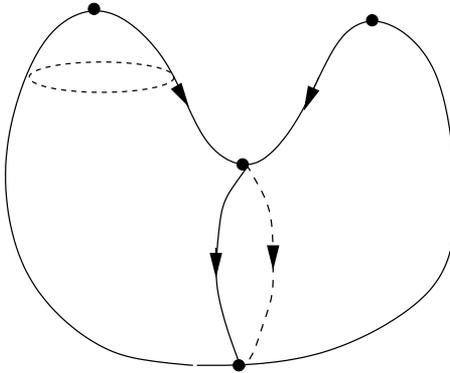
Sketch of proof. — In addition to the considerations done in Section 7.2, one has to treat homotopies of homotopies similarly to Section 4.2.2 in [49] (see also Section 4.3.1 in [42]). This can be done considering Morse theory (and an associated complex) for an appropriate function $X \times S^1 \times S^1 \rightarrow \mathbb{R}$. In Section 7.2, we have shown how to generalise the construction of Section 4.2.1 in [49] to anti-radial Morse-Smale pairs on the singular space X . Along the same lines, also the construction of homotopies of homotopies in Section 4.2.2. in [49] can be generalised to anti-radial Morse-Smale pairs on X . □

8. Examples

8.1. The case of a smooth manifold

Let M be a smooth manifold of dimension n and let (f, g) be a smooth standard Morse-Smale pair. Let $p \in \text{Crit}_n(f)$. We now artificially see $X := M$ as a singular space with a conical singularity at p and link manifold $L_p \simeq S^{n-1}$. The Morse function f is an anti-radial Morse function on the “singular space” X . The complex $(C_*^u(f, g, \Xi), \partial_*)$ is the usual Thom-Smale complex (with real coefficients). For a smooth manifold, the intersection homology for an arbitrary perversity \underline{p} is isomorphic to the singular homology of M .

Let us give a concrete, simple example: Let $X = S^2$ and let $f : X \rightarrow \mathbb{R}$ be the smooth Morse function pictured below with $\text{Crit}_2(f) = \{p_1, p_2\}$, $\text{Crit}_1(f) = \{q\}$, $\text{Crit}_0(f) = \{r\}$.



We will artificially see the local maximum p_1 as a singular point of X with link manifold $L = S^1$.

The constant function $\xi = a \in \mathbb{R}^*$ is a non-trivial closed 0-form on S^1 . Let

$$(8.1) \quad \Xi = \Xi_{p_1}^0 := \{\xi := \xi_{p_1}^0 := a\}.$$

The complex $(C_*^u(f, g, \Xi), \partial_*)$ is defined as follows: For the chain groups we have:

$$(8.2) \quad \begin{aligned} C_2^u &:= \mathbb{R} \cdot [W^u(p_2)] \oplus \mathbb{R} \cdot [\xi], \\ C_1^u &:= \mathbb{R} \cdot [W^u(q)], \\ C_0^u &:= \mathbb{R} \cdot [W^u(r)]. \end{aligned}$$

For the boundary operator we have:

$$\begin{aligned}
 (8.3) \quad \partial\xi &= \pm \left(\int_{\overline{W^s(q)} \cap L} a \right) \cdot [W^u(q)] = \pm a \cdot [W^u(q)], \\
 \partial[W^u(p_2)] &= \pm [W^u(q)], \\
 \partial[W^u(q)] &= 0 = \partial[W^u(r)];
 \end{aligned}$$

the signs above depend on the chosen orientations of the W^u 's. Note that, for $\xi = 1$, we recover the usual Thom-Smale complex for the smooth Morse function f and the manifold S^2 .

8.2. The suspension of the torus

Let $X = \Sigma T^2$ be the unreduced suspension of the torus. The space X has two isolated singularities, which we denote by P and Q . The link manifolds are $L_P = L_Q = T^2$. It is not difficult to see, that there exists an anti-radial Morse function f on X , with exactly 4 smooth critical points, which are all lying in $\{1/2\} \times T^2$. We denote them as follows: $\text{Crit}_2(f) = \{p\}$, $\text{Crit}_1(f) = \{q_1, q_2\}$, $\text{Crit}_0(f) = \{r\}$.

8.2.1. Lower middle perversity

Set $\Xi_P = \Xi_P^0 := \{\xi_P := \xi_P^0 := a\}$, and $\Xi_Q = \Xi_Q^0 := \{\xi_Q := \xi_Q^0 := b\}$, where $a, b \in \mathbb{R}^*$. The complex $(C_*^u(f, g, \Xi), \partial_*)$ is defined as follows:

For the chain groups we have:

$$\begin{aligned}
 (8.4) \quad C_3^u &:= \mathbb{R} \cdot [\xi_P] \oplus \mathbb{R} \cdot [\xi_Q], \\
 C_2^u &:= \mathbb{R} \cdot [W^u(p)], \\
 C_1^u &:= \mathbb{R} \cdot [W^u(q_1)] \oplus \mathbb{R} \cdot [W^u(q_2)], \\
 C_0^u &:= \mathbb{R} \cdot [W^u(r)].
 \end{aligned}$$

For the boundary operator we have:

$$\begin{aligned}
 (8.5) \quad \partial[\xi_P] &= \pm a \cdot [W^u(p)], \quad \partial[\xi_Q] = \pm b \cdot [W^u(p)], \\
 \partial[W^u(p)] &= \partial[W^u(q_{1,2})] = \partial[W^u(r)] = 0.
 \end{aligned}$$

One verifies easily that the complex $(C_*^u(f, g, \Xi), \partial_*)$ computes the intersection homology with lower middle perversity of X :

$$(8.6) \quad H_*((C_*^u(f, g, \Xi), \partial_*)) \simeq IH_*(X) = \begin{cases} \mathbb{R} & i = 0, 3, \\ 0 & i = 2, \\ \mathbb{R}^2 & i = 1. \end{cases}$$

8.2.2. Upper middle perversity \underline{n}

The suspension of the torus is an odd-dimensional singular space, which is not a Witt space. Therefore the intersection homology with lower middle perversity is not self-dual, but dual to the intersection homology with upper middle perversity \underline{n} . We have shortly mentioned in the introduction that the method explained in the paper can be adapted easily for any other perversity \underline{p} in the sense of the theory of Goresky and MacPherson. Let us illustrate this in the present example for the upper middle perversity \underline{n} . In this case the “truncation degree” in the definition of Ξ (see (1.4)) is $k \geq 2$, hence Ξ will be a set of representatives of the homology of the link manifold in degree 0 and 1. Let us denote by x, y the coordinates of the plane \mathbb{R}^2 , which covers $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. We have that $H^1(T^2) = \text{span}\{[dx], [dy]\}$.

Hence set:

$$(8.7) \quad \begin{aligned} \Xi_P^0 &:= \{\xi_P^0 := 1\}, & \Xi_Q^0 &:= \{\xi_Q^0 := 1\}, \\ \Xi_P^1 &:= \{\xi_{P,1}^1 := dx, \xi_{P,2}^1 := dy\}, & \Xi_Q^1 &:= \{\xi_{Q,1}^1 := dx, \xi_{Q,2}^1 := dy\}. \end{aligned}$$

The complex $(C_*^u(f, g, \Xi, \underline{n}), \partial_*)$ is defined as follows: For the chain groups we have:

$$(8.8) \quad \begin{aligned} C_3^u &:= \mathbb{R} \cdot [\xi_P^0] \oplus \mathbb{R} \cdot [\xi_Q^0], \\ C_2^u &:= \mathbb{R} \cdot [W^u(p)] \oplus \text{span } \Xi_P^1 \oplus \text{span } \Xi_Q^1, \\ C_1^u &:= \mathbb{R} \cdot [W^u(q_1)] \oplus \mathbb{R} \cdot [W^u(q_2)], \\ C_0^u &:= \mathbb{R} \cdot [W^u(r)]. \end{aligned}$$

Let us assume for simplicity, that the unstable manifold $\overline{W^u(q_1)}$ (resp. $\overline{W^u(q_2)}$) is the image under the quotient map $\mathbb{R}^2 \rightarrow \mathbb{R}^2/\mathbb{Z}^2 = T^2$ of the y - (resp. the x -axes). For the boundary operator we have:

$$(8.9) \quad \begin{aligned} \partial[\xi_P^0] &= \pm[W^u(p)], \quad \partial[\xi_Q^0] = \pm[W^u(p)], \\ \partial[\xi_{P/Q,1}^1] &= \pm[W^u(q_1)], \quad \partial[\xi_{P/Q,2}^1] = \pm[W^u(q_2)], \\ \partial[W^u(p)] &= \partial[W^u(q_{1,2})] = \partial[W^u(r)] = 0. \end{aligned}$$

This time the complex computes the intersection homology of X with upper middle perversity $IH_*^{\underline{n}}(X)$:

$$(8.10) \quad H_*((C_*^u(f, g, \Xi, \underline{n}), \partial_*)) \simeq IH_*^{\underline{n}}(X) = \begin{cases} \mathbb{R} & i = 0, 3, \\ \mathbb{R}^2 & i = 2, \\ 0 & i = 1. \end{cases}$$

8.3. The suspension of the real projective plane $\mathbb{R}\mathbb{P}^2$

Let us illustrate how to adapt the method explained in the article to the non-orientable case. Let X be the unreduced suspension of the real projective plane. The space X has two isolated singularities, which we denote by P and Q . The link manifolds are $L_P = L_Q = \mathbb{R}\mathbb{P}^2$. The space X is non-orientable, we denote by $o(X)$ the orientation bundle of X_{sm} . It is not difficult to see, that there exists an anti-radial Morse function f on X , with exactly 3 smooth critical points, which are all lying in $\{1/2\} \times \mathbb{R}\mathbb{P}^2$. We denote them as follows: $\text{Crit}_2(f) = \{p\}$, $\text{Crit}_1(f) = \{q\}$, $\text{Crit}_0(f) = \{r\}$. With other words the restriction $f|_{\{1/2\} \times \mathbb{R}\mathbb{P}^2}$ is a smooth Morse function on $\mathbb{R}\mathbb{P}^2$, which gives the usual decomposition of $\mathbb{R}\mathbb{P}^2$ into a CW-complex with one cell in dimension 0, 1 and 2.

In this case we have to choose $\Xi_P = \Xi_Q \subset H^0(\mathbb{R}\mathbb{P}^2, o(X)) = 0$. Hence, the complex $(C_*^u(f, g, \Xi), \partial_*)$ is defined as follows: For the chain groups we have:

$$(8.11) \quad \begin{aligned} C_3^u &:= 0, \\ C_2^u &:= \mathbb{R} \cdot [W^u(p)], \\ C_1^u &:= \mathbb{R} \cdot [W^u(q)], \\ C_0^u &:= \mathbb{R} \cdot [W^u(r)]. \end{aligned}$$

For the boundary operator we have:

$$(8.12) \quad \partial[W^u(p)] = 2 \cdot [W^u(q)], \quad \partial[W^u(q)] = 0, \quad \partial[W^u(r)] = 0.$$

The complex computes the intersection homology of X with lower middle perversity:

$$(8.13) \quad H_*((C_*^u(f, g, \Xi), \partial_*)) \simeq IH_*(X) = \begin{cases} \mathbb{R} & i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

8.4. Quotients under finite group action

Let G be a compact Lie group acting on a smooth manifold M . Let $\tilde{f} : M \rightarrow \mathbb{R}$ be a G -invariant Morse function on M , i.e. $\tilde{f}(hx) = \tilde{f}(x)$ for all $h \in G, x \in M$. In this situation one can study the G -equivariant Morse theory and establish Morse inequalities for the G -equivariant homology $H_*^G(M)$. The study of G -equivariant Morse theory has been initiated by Atiyah and Bott [2]. The reader is referred to [6], [8] for an account of the equivariant Morse inequalities, and to [3] for the Thom-Smale complex for G -invariant Morse functions and G -equivariant homology.

In the particular case of (certain) finite group actions, the present paper suggest yet another way of studying Morse theory: namely by studying Morse theory on the (singular) quotient space. We will shortly explain this point of view in this subsection.

Let M be an oriented manifold with an orientation preserving action of a finite group G . Let \tilde{g} be a G -invariant Riemannian metric on M . Let $\tilde{f} : M \rightarrow \mathbb{R}$ be a smooth G -invariant Morse function on M . For finite groups G -invariant Morse functions exist (see [48], Lemma 4.8).

The quotient space $X := M/G$ has a natural stratification by orbit types, which gives X the structure of a Whitney stratified space (see e.g. [38], Theorem 4.4.6).

Let us assume that X has only isolated singularities. The metric g induced from \tilde{g} is conical in the sense of Definition 2.1. Let us assume, that every fixed point of the action of G is a critical point of \tilde{f} of index $\dim M$. Then \tilde{f} descends to an anti-radial Morse function $f : X \rightarrow \mathbb{R}$.

For $p \in \text{Crit}(\tilde{f})$ let us denote by $\widetilde{W}^u(p)$ the unstable manifold of p with respect to the flow induced from the negative gradient flow $\nabla_{\tilde{g}}\tilde{f}$ on M . One has:

$$(8.14) \quad h(\widetilde{W}^u(p)) = \pm \widetilde{W}^u(h(p)) \text{ for all } h \in G, p \in \text{Crit}(\tilde{f}).$$

The sign in (8.14) depends on the orientation of the unstable cells, and the orientation can be chosen such that the sign is $+$. The unstable/stable cell decomposition of M for the Morse function \tilde{f} , descends into the unstable/stable cell decomposition of X for the Morse function f . If the gradient vector field $\nabla_{\tilde{g}}\tilde{f}$ on M is Morse-Smale, so is the gradient vector field $\nabla_g f$ on X .

The space X is a homology manifold, therefore the intersection homology $IH_*(X)$ is isomorphic to the singular homology of X (see [16], Section 6.4). For $p \in \text{Sing}(X)$ the link manifold L_p is a homology sphere. Hence $\Xi_p \subset H^0(L_p) \simeq \mathbb{R}$ does contain a single element. The complex $(C_*^u(f, g, \Xi), \partial_*)$ does compute the singular homology of X .

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Ursula LUDWIG
Universität Duisburg-Essen
Fakultät für Mathematik
Thea-Leymann-Str. 9
D-45127 Essen (Germany)
ursula.ludwig@uni-due.de