ANNALES

DE

L'INSTITUT FOURIER

Chiara CAMERE

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Tome 66, nº 2 (2016), p. 687-709.

<http://aif.cedram.org/item?id=AIF_2016__66_2_687_0>

Ann. Inst. Fourier, Grenoble
66, 2 (2016) 687-709

LATTICE POLARIZED IRREDUCIBLE HOLOMORPHIC SYMPLECTIC MANIFOLDS

by Chiara CAMERE (*)

Abstract. — We generalize lattice-theoretical mirror symmetry for K3 surfaces to lattice polarized higher dimensional irreducible holomorphic symplectic manifolds. In the case of fourfolds of $K3^{[2]}$-type we then describe mirror families of polarized fourfolds and we give an example with mirror non-symplectic involutions.

Résumé. — On généralise la construction de la symétrie miroir des surfaces K3 aux variétés irréductibles holomorphes symplectiques X polarisées par un réseau. Dans le cas des variétés de type $K3^{[2]}$ on étudie la famille miroir des variétés polarisées et on généralise la notion de couple d'involutions non-symplectiques miroirs.

1. Introduction

One striking prediction about geometrical objects coming from physics is the mirror conjecture. Mirror symmetry for holomorphic symplectic manifolds has already been studied by Verbitsky in [24], where he shows that general non-projective holomorphic symplectic manifolds are mirror self-dual; nothing is known about projective holomorphic symplectic manifolds apart for the two-dimensional case of K3 surfaces.

In [7] Dolgachev, based on former work by Pinkham [22] and Nikulin [19], develops a mirror construction for lattice polarized projective K3 surfaces. First of all, he defines a moduli space $\mathfrak{M}_M$ parametrizing M-polarized K3 surfaces, i.e. those S for which M is primitively embedded in $\mathrm{Pic}(S)$.

Keywords: lattice polarized irreducible holomorphic symplectic manifold, mirror symmetry, lattice polarized hyperkähler manifold, mirror involution.
Math. classification: 14J15, 32G13, 14J33, 14J35.
(*) This work was developed while the author was a member of the DFG Research Training Group "Analysis, Geometry and String Theory" and of the Institute of Algebraic Geometry at Leibniz University Hannover, whose support the author gratefully acknowledges.

Then he shows that, whenever there is a decomposition $M^\perp \cap H^2(S,\mathbb{Z}) = U(m)\oplus \check{M}$, where U is the standard hyperbolic lattice and m is an integer, then $\mathfrak{M}_{\check{M}}$ is a mirror moduli space: its dimension equals the Picard number of the very general member of $\mathfrak{M}_M$ and vice versa. Moreover, the Griffiths–Yukawa coupling $\mathrm{Y} : S^2(H^1(S,T_S)) \to H^{0,2}(S)^{\otimes 2}$ is a symmetric pairing and for some open subset $\mathcal{U}$ of a compactification of $\mathfrak{M}_M$ near a boundary point, it can be identified with the quadratic form on $\check{M}\otimes\mathbb{C}$. Finally, the period map of K3 surfaces induces a holomorphic multivalued map, the *mirror map*, from the open set $\mathcal{U}$ above to the tube domain $\mathrm{Pic}(X')_\mathbb{R}+i\mathcal{K}_{X'}$, where $X'\in\mathfrak{M}_{\check{M}}$ and $\mathcal{K}_{X'}$ is its Kähler cone.

Interesting examples of such a duality are given by Dolgachev, e.g. mirror partners of polarized K3s and Arnold's Strange Duality, and also by Borcea [5] and Voisin [26], who introduced the notion of mirror non-symplectic involutions. Later Gross and Wilson, in [12], related mirror symmetry for K3 surfaces to Strominger–Yau–Zaslow's conjectural construction of T-duality for Calabi–Yau threefolds.

In this paper we generalize the definition of this lattice-theoretical mirror construction to higher dimensional irreducible holomorphic symplectic manifolds. After reviewing the basic notions of lattice theory and of the theory of hyperkähler manifolds, in Section 3 we construct moduli spaces of marked lattice polarized irreducible holomorphic symplectic manifolds and study their period domains. Given a hyperkähler manifold X *of type* L, i.e. $H^2(X,\mathbb{Z}) = L$, and a primitive embedding $j : M \subset L$ with M of signature $(1,t)$, we define a coarse moduli space $\mathcal{M}_{M,j}$ of irreducible holomorphic symplectic (M,j)-polarized manifolds of type L; the main result of the paper is the following

Theorem 1.1. — *Let $\mathcal{M}^+_{M,j}$ be a connected component of $\mathcal{M}_{M,j}$; the period map restricts surjectively to $\mathcal{P}_{M,j} : \mathcal{M}^+_{M,j} \to D^+_M$ where D^+_M is a symmetric homogenous domain of type IV.*

Then in Section 4 we show how the theory in [7] carries through to higher dimensions: we define mirror moduli spaces so that they share the same properties mentioned above; roughly speaking, this duality exchanges the complex and the Kähler structure of the manifolds. In the case of fourfolds of $K3^{[2]}$-type we then describe mirror families of polarized fourfolds and we generalize also the notion of mirror non-symplectic involutions in Section 5.

Acknowledgements. — The author wants to thank Klaus Hulek for suggesting this problem and for many enlightening discussions. She is also grateful to Samuel Boissière and Alessandra Sarti for their precious comments and to Igor Dolgachev for his kind interest and for his remarks. She also wants to express her gratitude to the referee for the careful reading and the suggestions.

2. Preliminary notions

2.1. Lattices

A *lattice* L is a free $\mathbb{Z}$-module equipped with a nondegenerate symmetric bilinear form $(\cdot,\cdot)$ with integer values. Its *dual lattice* is $L^* := \mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})$ and can also be described as $L^* \cong \{x \in L \otimes \mathbb{Q} \mid (x,v) \in \mathbb{Z} \quad \forall v \in L\}$. Since L is a sublattice of L^* of the same rank, the quotient $A_L := L^*/L$ is a finite abelian group, so-called *discriminant group*, of order $\mathrm{discr}(L)$, the *discriminant of* L. We denote by $\ell(A_L)$ the minimal number of generators of A_L (i.e. the *length* of A_L). In a basis $\{e_i\}_i$ of L, if $M := ((e_i,e_j))_{i,j}$ is a Gram matrix, one has $\mathrm{discr}(L) = |\det(M)|$.

A lattice L is called *even* if $(x,x) \in 2\mathbb{Z}$ for all $x \in L$. In this case, the bilinear form induces a finite quadratic form $q_L : A_L \longrightarrow \mathbb{Q}/2\mathbb{Z}$. If $(t_{(+)}, t_{(-)})$ is the signature of $L \otimes \mathbb{R}$, the triple of invariants $(t_{(+)}, t_{(-)}, q_L)$ characterizes the *genus* of the even lattice L (see [6, §7, Ch. 15], [20, Corollary 1.9.4]).

A lattice L is called *unimodular* if $A_L = \{0\}$. An embedding of a sublattice $i : M \subset L$ is called *primitive* if $L/i(M)$ is a free $\mathbb{Z}$-module. If L is unimodular and $M \subset L$ is a primitive sublattice, then M and its orthogonal $M^\perp$ in L have isomorphic discriminant groups and $q_M = -q_{M^\perp}$. When L is no longer unimodular, the picture becomes more complicated, and the following result helps with finding all non-isomorphic primitive embeddings of M.

Theorem 2.1 ([20, Proposition 1.15.1]). — *The primitive embeddings of M with invariants $(m_{(+)}, m_{(-)}, q_M)$ into an even lattice L with invariants $(t_{(+)}, t_{(-)}, q_L)$ are determined by the sets $(H_M, H_L, \gamma; K, \gamma_K)$ satisfying the following conditions:*

- *H_M is a subgroup of A_M, H_L is a subgroup of A_L and $\gamma\colon H_M \to H_L$ is an isomorphism of groups such that for any $x \in H_M$, $q_L(\gamma(x)) = q_M(x)$.*
- *K is a lattice of invariants $(t_{(+)} - m_{(+)}, t_{(-)} - m_{(-)}, q_K)$ with $q_K = ((-q_M) \oplus q_L)|_{\Gamma^\perp/\Gamma}$, where Γ is the graph of γ in $A_M \oplus A_L$, $\Gamma^\perp$ is the orthogonal complement of Γ in $A_M \oplus A_L$ with respect to the*

bilinear form induced on $A_M \oplus A_L$ and with values in $\mathbb{Q}/\mathbb{Z}$; finally γ_K is an automorphism of A_K that preserves q_K. Moreover K is the orthogonal complement of M in L.

Two such sets, $(H_M, H_L, \gamma; K, \gamma_K)$ and $(H'_M, H'_L, \gamma'; K', \gamma_{K'})$, determine isomorphic primitive embeddings if and only if

(1) *$H_M = H'_M$;*
(2) *there exist $\xi \in O(q_L)$ and $\psi : K \to K'$ isomorphism for which $\gamma' = \xi \circ \gamma$ and $\bar{\psi} \circ \gamma_K = \gamma_{K'} \circ \bar{\psi}$, where $\bar{\psi}$ is the isomorphism of the discriminant forms q_K and $q_{K'}$ induced by ψ.*

In this paper U will be the unique even unimodular hyperbolic lattice of rank two and A_k, D_h, E_l will be the even, negative definite lattices associated to the Dynkin diagrams of the corresponding type ($k \geqslant 1$, $h \geqslant 4$, $l = 6, 7, 8$). For $d \equiv -1$ (4), the following negative definite lattice will be used in the sequel

$$K_d := \begin{pmatrix} -(d+1)/2 & 1 \\ 1 & -2 \end{pmatrix}$$

Moreover, $L(t)$ denotes the lattice whose bilinear form is the one on L multiplied by $t \in \mathbb{N}^*$.

We recall the following result by Nikulin on splitting of lattices.

Theorem 2.2 ([20, Theorem 1.13.5]). — *Let L be an even indefinite lattice of signature $(t_{(+)}, t_{(-)})$ and assume that $t_{(+)} > 0$ and $t_{(-)} > 0$. Then:*

(1) *If $t_{(+)} + t_{(-)} \geqslant 3 + \ell(A_L)$, then $L \cong U \oplus W$ for a certain even lattice W.*
(2) *If $t_{(-)} \geqslant 8$ and $t_{(+)} + t_{(-)} \geqslant 9 + \ell(A_L)$, then $L \cong E_8 \oplus W'$ for a certain even lattice W'.*

Finally, recall that the *divisor* $\operatorname{div} f$ of a primitive element $f \in L$ is the generator of the ideal (f, L) in $\mathbb{Z}$.

Theorem 2.3 ([23, Prop. 3.7.3, Eichler's criterion]). — *If L contains $U \oplus U$, then, given two primitive elements $f, f' \in L$ such that $f^2 = (f')^2$ and $\operatorname{div} f = \operatorname{div} f'$, there is an isometry $\sigma \in O(L)$ such that $\sigma(f) = f'$.*

2.2. Irreducible holomorphic symplectic manifolds

Irreducible holomorphic symplectic manifolds, also called hyperkähler manifolds, have received a growing interest since it is known that if X is a compact simply connected Kähler manifold with $c_1(X)_{\mathbb{R}} = 0$, then there is a finite étale cover of X that is a product of manifolds of three different types, namely complex tori, Calabi–Yau's and irreducible holomorphic symplectic ones (see [3]).

A compact Kähler manifold X is *irreducible holomorphic symplectic* if it is simply connected and admits a symplectic two-form $\omega_X \in H^{2,0}(X)$, unique up to multiplication by a nonzero scalar. The existence of such a symplectic form ω_X immediately implies that $\dim X$ is an even integer. Moreover, K_X is trivial, in particular $c_1(X) = 0$, and $T_X \cong \Omega^1_X$. For a complete survey of this topic we refer the reader to the nice book [11] and references therein.

The group $H^2(X, \mathbb{Z})$ carries a natural structure of lattice; the quadratic form on it is the so-called Beauville–Bogomolov quadratic form q, which is even in all known examples. We briefly recall here the deformation types of irreducible holomorphic symplectic manifolds that will appear in the sequel.

- **K3 surfaces** These are compact complex connected surfaces S with $b_1(S) = 0$ and trivial canonical bundle. There is a lattice isomorphism between $H^2(S, \mathbb{Z})$ endowed with the cup-product and the lattice $U^{\oplus 3} \oplus E_8^{\oplus 2}$.
- **The Hilbert scheme of a K3 surface** Let S be a smooth K3 surface and let $X = S^{[2]}$ be the Hilbert scheme of S of 0-dimensional subschemes of length 2; X can be constructed also as the blow-up along the image of the diagonal Δ of the symmetric product $S^{(2)}$. In particular, $b_2(X) = 23$ and $h^{1,1}(X) = 21$. The Beauville–Bogomolov lattice $(H^2(X, \mathbb{Z}), q)$ is $L = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2 \rangle$. Often, irreducible holomorphic symplectic manifolds that are deformation equivalent to X are said to be *of* $K3^{[2]}$*- type*.

  If S is projective, then so is X, and $\mathrm{Pic}(X) \cong \mathrm{Pic}(S) \oplus \mathbb{Z}e$, where $2e$ is the class of the exceptional divisor and $e^2 = -2$.

  The construction can be generalized in dimension $2n$, taking the Hilbert scheme of S of 0-dimensional subschemes of length n (see [3]).

The only other known deformation types are generalized Kummer manifolds and O'Grady's examples in dimension 6 and 10.

A *marking* of an irreducible holomorphic symplectic manifold is an isometry $\phi : H^2(X, \mathbb{Z}) \to L$, where L is a fixed even non-degenerate lattice of signature $(3, b_2(X) - 3)$; a pair (X, ϕ) is then said to be *marked*. Similarly to what happens for K3 surfaces, there exists a moduli space $\mathcal{M}_L$ of marked irreducible holomorphic symplectic manifolds of type L, and one can define a *period map* $\mathcal{P}_0 : \mathcal{M}_L \to D_L$ such that $\mathcal{P}_0(X,\phi) = \left[\phi(H^{2,0}(X))\right]$ in the *period domain* $D_L := \{[\omega] \in \mathbb{P}(L_{\mathbb{C}}) \mid (\omega,\omega) = 0, (\omega,\bar{\omega}) > 0\}$.

Already in [3] it was shown that the period map is a local isomorphism; later Huybrechts in [14] showed that $\mathcal{P}_0$ is surjective, even when restricted to a connected component. Finally, Verbitsky in [25] proved the global Torelli theorem, which we recall here, following Markman (see also Huybrechts's Bourbaki talk [15]).

Theorem 2.4 ([17, Theorem 1.3]). — *Let X and Y be two irreducible holomorphic symplectic manifolds, which are deformation equivalent to each other. Then:*

(1) *X and Y are bimeromorphic if and only if there exists a parallel transport operator $f : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ that is an isomorphism of integral Hodge structures;*
(2) *if this is the case, there exists an isomorphism $\tilde{f} : X \to Y$ inducing f if and only if f sends some Kähler class on X to a Kähler class on Y.*

For the definition of a parallel transport operator we refer to [17, Definition 1.1], where also monodromy operators are defined.

Definition 2.5. — *Given a marked pair (X,ϕ) of type L, we define the* monodromy group *as* $\mathrm{Mo}^2(L) := \phi \circ \mathrm{Mo}^2(X) \circ \phi^{-1}$, *where* $\mathrm{Mo}^2(X) \subset GL(H^2(X,\mathbb{Z}))$ *is the group of monodromy operators of X restricted to the second cohomology group.*

It was proven by Verbitsky in [25] that $\mathrm{Mo}^2(L)$ is an arithmetic subgroup of $O(L)$; on the other hand we do not have an explicit description of this group in all known examples.

Moduli spaces of marked irreducible holomorphic symplectic manifolds are not Hausdorff, but we know exactly how to describe non-separated points.

Theorem 2.6 ([17, Theorem 2.2]). — *Let $\mathcal{M}_L^0$ be a fixed connected component of $\mathcal{M}_L$.*

(1) *The period map $\mathcal{P}_0$ restricted to $\mathcal{M}_L^0$ is surjective.*
(2) *For any $p \in D_L$, the fibre $\mathcal{P}_0^{-1}(p)$ consists of pairwise non-separated points.*
(3) *The marked pair (X,ϕ) is a Hausdorff point of $\mathcal{M}_L$ if and only if the positive cone and the Kähler cone coincide.*

Things behave better when one restricts oneself to moduli of *polarized* marked irreducible holomorphic symplectic manifolds, which have been studied in [9] (see in particular Theorem 1.5). We denote with $\mathcal{A}_X$ the ample cone of X. Furthermore, given a primitive element $h \in L$, let $\mathrm{Mo}^2(h) := \{g \in \mathrm{Mo}^2(L) \mid g(h) = h\}$ be the subgroup of h-polarized monodromy operators and let Γ_h be the image of $\mathrm{Mo}^2(h)$ via the restriction map $\alpha : O(L) \to O(h^\perp)$.

Theorem 2.7 ([17, Theorem 8.4]). — *Let $h \in L$ be a primitive element and let D_h^+ be one of the two connected components of $D_L \cap \mathbb{P}(h^\perp)$. Let $\mathcal{M}_h^+$ be a connected component of the moduli space of polarized marked pairs $\{(X, \phi) \in \mathcal{P}_0^{-1}(D_h^+) \mid \phi^{-1}(h) \in \mathcal{A}_X\}$. Then the period map restricts to an open embedding with dense image*

$$\mathcal{P}_h : \mathcal{M}_h^+ / \mathrm{Mo}^2(h) \to D_h^+/\Gamma_h$$

3. Moduli spaces of lattice polarized irreducible holomorphic symplectic manifolds

This construction aims to generalize the one by Gritsenko, Hulek and Sankaran in [9] for polarized irreducible holomorphic symplectic manifolds. Here we treat the subject in full generality, and we will then specialize it to the case of fourfolds of $K3^{[2]}$-type in Section 5.

Let X be an irreducible holomorphic symplectic manifold of type L and let $j : M \subset L$ be a fixed primitive embedding of a sublattice M of signature $(1, t)$; we will freely identify M with $j(M)$ whenever confusion is not possible.

Definition 3.1. — *An M-polarization of an irreducible holomorphic symplectic manifold X is a lattice embedding $i : M \to \mathrm{Pic}(X)$.*

A j-marking of an M-polarized manifold X is a marking $\phi : H^2(X, \mathbb{Z}) \to L$ such that $\phi \circ i = j$; a pair (X, ϕ) with X an M-polarized irreducible holomorphic symplectic manifold of given deformation type and ϕ a j-marking is said to be (M, j)-polarized.

If $\mathrm{Pic}(X) = i(M)$, we say that (X, ϕ) is strictly (M, j)-polarized.

Since $i(M_\mathbb{C}) \subset H^{1,1}(X)$, we have $\mathcal{P}_0(X, \phi) \in \mathbb{P}(M_\mathbb{C}^\perp)$; hence, we can consider a restricted period domain

$$D_M = \{[\omega] \in \mathbb{P}(N_\mathbb{C}) \mid (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0\}$$

where $N = j(M)^\perp$. This has two connected components and each one is a symmetric homogeneous domain of type IV (see [10]). Since $N_\mathbb{C}$ depends only on the signature of N, the period domain D_M depends only on M and not on j.

We need a notion of an ample polarization, and to introduce it, we make the following assumption:

Assumption. — *There exists a set $\Delta(L) \subset L$ such that the Kähler cone $\mathcal{K}_X$ of a marked (X,ϕ) can be described as*

$$\mathcal{K}_X = \left\{h \in H^{1,1}(X,\mathbb{R}) \mid (h,h) > 0, (h,\delta) > 0\ \forall \delta \in \Delta(X)^+\right\}$$

where $\Delta(X)^+ := \left\{\delta \in \phi^{-1}(\Delta(L)) \cap \mathrm{Pic}(X) \mid (\delta,\kappa) > 0\right\}$ for $\kappa \in \mathcal{K}_X$ a fixed Kähler class.

As we will see more in detail in Section 5, Assumption 3 is satisfied in the case of fourfolds of $K3^{[2]}$-type. Given such an embedding $j : M \subset L$, define the positive cone $C(M) = \{x \in M_{\mathbb{R}} \mid (x,x) > 0\}$ and pick one of the two connected components $C^+(M)$. Given $\Delta(M) := \Delta(L) \cap M$ and $H_\delta = \{x \in M_{\mathbb{R}} \mid (x,\delta) = 0\}$, we fix a connected component of $C^+(M) \setminus (\cup_{\delta\in\Delta(M)} H_\delta)$ and call it $K(M)$. This choice induces the choice of a set $\Delta(M)^+ = \{\delta \in \Delta(M) \mid (x,\delta) > 0\ \forall x \in K(M)\}$ such that

$$\Delta(M) = \Delta(M)^+ \amalg (-\Delta(M)^+)\,.$$

Definition 3.2. — *We say that (X,ϕ) as above is ample (strictly) (M,j)-polarized if $i(K(M))$ contains a Kähler class.*

Lemma 3.3. — *If (X,ϕ) is ample strictly (M,j)-polarized, then:*

1. *$i(\Delta(M)^+) = \Delta(X)^+$;*
2. *$i(K(M)) = \mathcal{K}_X$.*

Proof. — (1) Take $\kappa \in i(K(M)) \cap \mathcal{K}_X$; given $\delta \in i(\Delta(M)^+)$, we have $(\delta,\kappa) = (\tilde{\delta},\tilde{k}) > 0$ for $\tilde{\delta} = \phi(\delta) \in \Delta(M)^+$ and $\tilde{k} = \phi(\kappa) \in K(M)$. On the other hand, suppose that there exists $\delta \in \Delta(X)^+ \setminus i(\Delta(M)^+)$; we have $\delta \in i(\Delta(M)^-)$ and hence $(\delta,\kappa) = (\tilde{\delta},\tilde{k}) < 0$, in contradiction with our assumption.

(2) This follows immediately from (1) and the definitions. □

Given a smooth family $f : \mathcal{X} \to \mathcal{U}$ of irreducible holomorphic symplectic manifolds of given deformation type, an M-polarization of f is an injection $i_{\mathcal{U}} : M_{\mathcal{U}} \to \mathcal{P}ic_{\mathcal{X}/S} \subset R^2 f_*\mathbb{Z}$, from the constant sheaf $M_{\mathcal{U}}$ to the relative Picard sheaf $\mathcal{P}ic_{\mathcal{X}/S}$, such that for every $t \in \mathcal{U}$ the map i_t defines an M-polarization of $\mathcal{X}_t$. A j-marking of the family is then defined (see [7]) as an isomorphism of local systems $\phi_{\mathcal{U}} : R^2 f_*\mathbb{Z} \to L_{\mathcal{U}}$ such that $\phi_t \circ i_t = j$ for all $t \in \mathcal{U}$. Such a marking allows us to define the *period map* of f as $\mathcal{P}_f : t \in \mathcal{U} \mapsto [\phi_t(\omega_{\mathcal{X}_t})] \in D_M$, that is holomorphic by the local

Torelli Theorem [3, Théorème 5]. Local moduli spaces and period maps are then glued together and give a coarse moduli space $\mathcal{M}_{M,j}$ of (M,j)-polarized irreducible holomorphic symplectic manifolds of fixed type and a holomorphic map $\mathcal{P}_{M,j} : \mathcal{M}_{M,j} \to D_M$ that is the restriction of the period map $\mathcal{P}_0$.

The group $O(L,M) = \{g \in O(L) \mid g(m) = m \ \forall m \in M\}$ acts properly and discontinuously on D_M; choose a connected component D_M^+ of D_M and a connected component $\mathcal{M}_{M,j}^+$ of $\mathcal{P}_{M,j}^{-1}(D_M^+)$; it is a connected component of $\mathcal{M}_{M,j}$, and the period map restricts to a surjective holomorphic map $\mathcal{P}_{M,j}^+ : \mathcal{M}_{M,j}^+ \to D_M^+$, which is a local isomorphism.

As defined in Markman [17], consider $\mathrm{Mo}^2(L) := \phi \circ \mathrm{Mo}^2(X) \circ \phi^{-1}$ where $(X,\phi) \in \mathcal{M}_L^+$. We define *$(M,j)$-polarized monodromy operators*

$$\mathrm{Mo}^2(M,j) := \{g \in \mathrm{Mo}^2(L) \mid g(m) = m \ \forall m \in M\} = \mathrm{Mo}^2(L) \cap O(L,M)$$

In other words, an element $g \in \mathrm{Mo}^2(M,j)$ satisfies $g \circ j = j$. This group acts on $\mathcal{M}_{M,j}^+$ via $(X,\phi) \mapsto (X, g\circ\phi)$ for $g \in \mathrm{Mo}^2(M,j)$.

The restriction map induces an injective map $\alpha : g \in O(L,M) \mapsto g_{|N} \in O(N)$; we define the subgroup $\Gamma_{M,j} := \alpha(\mathrm{Mo}^2(M,j))$ in $O(N)$.

Proposition 3.4. — *The set $\mathcal{M}_{M,j}^+$ is invariant under the action of $\mathrm{Mo}^2(M,j)$ and the restriction of the period map is $\mathrm{Mo}^2(M,j)$-equivariant, so that we get a surjective map*

$$\mathcal{M}_{M,j}^+/\mathrm{Mo}^2(M,j) \xrightarrow{\mathcal{P}_{M,j}^+} D_M^+/\Gamma_{M,j}$$

Proof. — Given $(X,\phi) \in \mathcal{M}_{M,j}^+$ and $g \in \mathrm{Mo}^2(M,j)$, there is an embedding $i : M \subset \mathrm{Pic}(X)$ such that $\phi \circ i = j$; then $g \circ \phi$ is again a j-marking since $g \circ \phi \circ i = g \circ j = j$.

The equivariance of the restricted period map is trivial. □

To obtain a quasi-projective variety we need to show that $\Gamma_{M,j}$ is of finite index inside $O(N)$. By a result of Markman combined with work of Kneser (see also [10]), it follows that if X is of $K3^{[2]}$-type, then $\mathrm{Mo}^2(L)$ is related to the so-called *stable orthogonal group*,

$$\tilde{O}^+(L) = \left\{g \in O(L) \mid g_{|A_L} = \mathrm{id}, \mathrm{sn}_{\mathbb{R}}^L(g) = 1\right\}$$

where the *real spinor norm* $\mathrm{sn}_{\mathbb{R}}^L : O(L_{\mathbb{R}}) \to \mathbb{R}^*/(\mathbb{R}^*)^2 \cong \{\pm 1\}$ is defined as

$$\mathrm{sn}_{\mathbb{R}}^L(g) = \left(-\frac{v_1^2}{2}\right) \cdots \left(-\frac{v_m^2}{2}\right)(\mathbb{R}^*)^2$$

for $g \in O(L_{\mathbb{R}})$ factored as a product of reflections $g = \rho_{v_1} \circ \cdots \circ \rho_{v_m}$ with $v_i \in L_{\mathbb{R}}$.

Proposition 3.5. — *If* $\mathrm{Mo}^2(L) \supset \tilde{O}^+(L)$, *the group* $\Gamma_{M,j}$ *is an arithmetic subgroup of* $O(N)$.

Proof. — As $\mathrm{Aut}(A_N)$ is finite, $\tilde{O}^+(N)$ is of finite index in $O(N)$. Hence to see that $\Gamma_{M,j}$ is also of finite index in $O(N)$, it suffices to show that $\tilde{O}^+(N) \subset \Gamma_{M,j}$.

Given $g \in \tilde{O}^+(N)$ we want to prove that there exists $f \in \mathrm{Mo}^2(M)$ such that $\alpha(f) = g$. Take $f \in O(L)$ to be the map induced on L by $\mathrm{id}_M \oplus g$; then by definition $f \in O(L, M)$. Moreover, $f_{|(A_M \oplus A_N)} = \mathrm{id}_{A_M \oplus A_N}$, since $g \in \tilde{O}(N)$, and $A_L \subset A_M \oplus A_N$ (from $M \oplus N \subset L \subset L^* \subset M^* \oplus N^*$), hence $f_{|A_L} = \mathrm{id}_{A_L}$ and $f \in \tilde{O}(L)$.

Next, consider the extension of g by linearity to $N_{\mathbb{R}}$; we know that there are $v_1, \dots, v_m \in N_{\mathbb{R}}$ such that $g = \rho_{v_1} \circ \dots \circ \rho_{v_m}$ in $O(N_{\mathbb{R}})$ and $\mathrm{sn}^N_{\mathbb{R}}(g) = 1$. We will still denote by ρ_{v_i} the reflection of $L_{\mathbb{R}}$ with respect to $v_i \in N_{\mathbb{R}} \subset L_{\mathbb{R}}$; for all v_i with $i = 1, \dots, m$ we have $(\rho_{v_i})_{|M_{\mathbb{R}}} = \mathrm{id}_{M_{\mathbb{R}}}$ since $(v_i, m) = 0$ for all $m \in M_{\mathbb{R}}$, hence also $f = \rho_{v_1} \circ \dots \circ \rho_{v_m}$ in $O(L_{\mathbb{R}})$ and $\mathrm{sn}^L_{\mathbb{R}}(f) = \mathrm{sn}^N_{\mathbb{R}}(g) = 1$, i.e. $f \in O^+(L)$.

So indeed, $f \in \mathrm{Mo}^2(M, j)$ and, by construction, $\alpha(f) = g$. □

Corollary 3.6. — *If* $\mathrm{Mo}^2(L) \supset \tilde{O}^+(L)$, *the quotient* $D^+_{M,j}/\Gamma_{M,j}$ *is a quasi-projective variety of dimension* $\mathrm{rk}\, L - 2 - \mathrm{rk}\, M$.

Proof. — This follows from Proposition 3.5 and from Baily–Borel's theorem [1]. □

Next we restrict to a connected component $\mathcal{M}^a_{M,j} \subset \mathcal{M}^+_{M,j}$ of the moduli space of ample (M, j)-polarized irreducible holomorphic symplectic manifolds and we denote by $\mathcal{M}^{sa}_{M,j}$ the subset of ample strictly (M, j)-polarized ones.

Lemma 3.7. — *The set* $\mathcal{M}^a_{M,j}$ *is open in the analytic topology of* $\mathcal{M}^+_{M,j}$.

Proof. — We have $\mathcal{M}^a_{M,j} = \cup(\mathcal{M}^a_{M,j} \cap \mathcal{M}_h)$, and once a polarization h is fixed, ampleness is an open condition due to the stability of Kähler manifolds (see [27, §9.3.3]). □

Remark 3.8. — A priori $\mathcal{M}^a_{M,j}$ is not algebraic: a necessary condition for this is the injectivity of the restriction of the period map to $\mathcal{M}^a_{M,j}$, since otherwise this moduli space would be non-separated, and this is not always the case.

Theorem 3.9. — *The subset* $\mathcal{M}^{sa}_{M,j}$ *is Hausdorff in* $\mathcal{M}^+_{M,j}$ *and is invariant under the action of* $\mathrm{Mo}^2(M, j)$. *Moreover, its image via the period map is connected and dense.*

Proof. — First we show that

$$\mathcal{P}^+_{M,j}(\mathcal{M}^{sa}_{M,j}) = D^\circ_M := D^+_M \setminus \left(\bigcup_{\nu \in N\setminus\{0\}} H_\nu \right),$$

where $H_\nu = \{\lambda \in N_{\mathbb{C}} \mid (\lambda, \nu) = 0\}$.

Indeed, given $(X, \phi) \in \mathcal{M}^{sa}_{M,j}$ and $\pi = \mathcal{P}^+_{M,j}(X, \phi) \in D_M$, we see that $\pi \notin H_\nu$ for any $\nu \in N\setminus\{0\}$: otherwise, $\nu \in \pi^\perp$ and $\phi^{-1}(\nu) \in \mathrm{Pic}(X)\setminus i(M)$, contradicting our assumption.

Given $\pi \in D^\circ_M$ and $(X, \phi) \in \mathcal{P}^{-1}_{M,j}(\pi)$, the pair (X, ϕ) is strictly (M, j)-polarized; moreover, there is a bijection $\rho : \mathcal{P}^{-1}_{M,j}(\pi) \to \mathcal{KT}(X)$ via $\rho(Y, \eta) = \eta^{-1}(\phi(\mathcal{K}_X))$ by [17, Proposition 5.14], where $\mathcal{KT}(X)$ is the set of Kähler-type chambers of X. If $i(K(M)) \cap \mathcal{K}_X \neq \emptyset$, there is nothing to prove; otherwise, $i(K(M))$ meets a different Kähler-type chamber since $\Delta(X) = i(\Delta(M))$. Hence, there exists $(Y, \eta) \in \mathcal{M}^{sa}_{M,j} \cap \mathcal{P}^{-1}_{M,j}(\pi)$; in fact, it follows easily from Theorem 2.6 and Lemma 3.3 that there exists a unique such (Y, η), so that $\mathcal{M}^{sa}_{M,j}$ is Hausdorff.

Finally remark that D°_M is connected and dense by Baire's category theorem. □

Corollary 3.10. — *The period map induces a bijection*

$$\mathcal{P}^{sa}_{M,j} : \mathcal{M}^{sa}_{M,j} / \mathrm{Mo}^2(M, j) \longrightarrow D^\circ_M / \Gamma_{M,j}$$

Proof. — It follows from Proposition 3.4 and Theorem 3.9 that the period map restricts to a bijection $\mathcal{P}^{sa}_{M,j} : \mathcal{M}^{sa}_{M,j} \to D^\circ_M$ and that the restriction is equivariant with respect to the action of $\mathrm{Mo}^2(M, j)$. □

Remark 3.11. — If the primitive embedding $j : M \subset L$ is unique up to isometry of L, then $\mathcal{M}_M$ can be seen as the moduli space of M-polarized irreducible holomorphic symplectic manifolds, getting rid of markings as done by Dolgachev in [7]. On the other hand, since L is no longer unimodular, this is a stronger condition to require with respect to the case of K3 surfaces as it is not always satisfied even in the case of polarizations (see [9]). Proposition 5.2 describes some cases in which this happens.

4. Mirror symmetry

4.1. Griffiths–Yukawa coupling

In this section we limit ourselves to recalling some notations and facts from §4 in [7] and we focus our attention on the few modifications needed in higher dimensions.

From now on suppose that $\operatorname{rk} M \leqslant 20$, so that its orthogonal N in L (which is unique up to isometry once we fix the embedding $j : M \hookrightarrow L$ by Theorem 2.1) is indefinite. Fix an isotropic vector $f \in N_{\mathbb{R}}$, so that $(f, f) = 0$, and set

$$N_f = \{x \in N_{\mathbb{R}} \mid (x, f) = 1\}, \; V_f = \{x \in N_{\mathbb{R}} \mid (x, f) = 0\} / \mathbb{R}f;$$

let C_f^+ be a connected component of the cone

$$C_f = \{x \in V_f \mid (x, x) > 0\}$$

The corresponding *tube domain*, which is the complexification of C_f^+, is

$$\mathcal{H}_f = N_f + iC_f^+$$

Proposition 4.1 ([7, Corollary 4.3]). — *The choice of an isotropic $f \in N_{\mathbb{R}}$ determines an isomorphism $D_M^+ \cong \mathcal{H}_f$.*

The tube domain realization explained above allows us to relate the Griffiths–Yukawa coupling on $(X, \phi) \in \mathcal{M}_{M,j}^+$ to the intersection product on $(V_f)_{\mathbb{C}}$, which is induced by the quadratic form on N. The period domain D_M^+ parametrizes weight-two Hodge structures, hence we are interested in looking at the Griffiths–Yukawa quadratic form

$$\begin{gathered} Y : S^2 H^1(X, T_X)_\phi \longrightarrow H^{0,2}(X)^{\otimes 2} \\ (\theta_1, \theta_2) \mapsto \varphi^{1,1}(\theta_1) \circ \varphi^{2,0}(\theta_2) \end{gathered}$$

where $H^1(X, T_X)_\phi$ is the tangent space of $\mathcal{M}_{M,j}^+$ at the point (X, ϕ), defined as the orthogonal in $H^1(X, T_X)$ of $i(M)$ with respect to the pairing

$$H^1(X, T_X) \otimes H^{1,1}(X) \longrightarrow H^{0,2}(X),$$

and $\varphi^{i,j} : H^1(X, T_X) \to \operatorname{Hom}(H^{i,j}(X), H^{i-1,j+1}(X))$, for $1 \leqslant i \leqslant 2n$ and $0 \leqslant j \leqslant 2n-1$, is given by the interior product with a tangent vector (see [8]).

Proposition 4.2 ([7, Corollary 4.4]). — *For any $\mu \in D_M^+$ the choice of a representative $l \in L$ of μ such that $(l, f) = 1$ defines a canonical isomorphism*

$$\alpha_\mu : T_\mu D_M^+ \to (V_f)_{\mathbb{C}}$$

Moreover, given $(X, \phi) \in \mathcal{M}_{M,j}^+$, the quadratic form on $(V_f)_{\mathbb{C}}$ coincides with the Griffiths–Yukawa pairing with respect to the normalization $H^{0,2}(X) \cong \mathbb{C}$ defined by $\phi^{-1}(l) \in H^{2,0}(X)$.

4.2. The mirror map

Again the theoretical construction contained in §5 and §6 of [7] carries over to higher dimensions with very little modification. First of all, let us recall some definitions.

Definition 4.3. — *Let S be an even indefinite lattice and m a positive integer; an isotropic vector $f \in S$ is m-admissible if* $\operatorname{div} f = m$ *and there exists another isotropic vector $g \in S$ such that $(f,g) = m$,* $\operatorname{div} g = m$.

Due to [7, Lemma 5.4], this is equivalent to the existence of a primitive embedding $U(m) \to S$ such that $f \in U(m)$.

Suppose that there exists an m-admissible $f \in N$; then $N = U(m) \oplus \check{M}$, with $\check{M}$ a primitive sublattice of signature $(1, h^{1,1} - \operatorname{rk} M - 1)$, and we have $\check{M} \cong (\mathbb{Z}f)^{\perp}_{N}/\mathbb{Z}f$. The definition of $\check{M}$ depends only on the choice of f if $U(m)$ admits a unique primitive embedding in N: this happens in particular if $(m, \det N) = 1$, $\check{M}$ is unique in its genus and there is a surjection $O(\check{M}) \to O(q_{\check{M}})$. Once this holds, we can define $\check{\jmath}$ to be the composition of the embedding $\check{M} \subset N$ and of the embedding $j^{\perp} : N \subset L$, and this depends only on f and j.

Definition 4.4. — *The moduli space $\mathcal{M}^{+}_{\check{M},\check{\jmath}}$ is the mirror moduli space of $\mathcal{M}^{+}_{M,j}$.*

Proposition 4.5. — *We have* $\dim \mathcal{M}^{+}_{\check{M},\check{\jmath}} = \operatorname{rk} M$ *and* $\dim \mathcal{M}^{+}_{\check{M},\check{\jmath}} + \dim \mathcal{M}^{+}_{M,j} = h^{1,1}$.

We now introduce the Baily–Borel compactification of the period domain, defined as its closure in the Harish–Chandra embedding (see [10] for a nice survey of the topic), contained in the obvious compactification given by the closure D^{*}_{M} of D^{+}_{M} inside the quadric

$$\mathcal{Q}_M = \{[\omega] \in \mathbb{P}(N_{\mathbb{C}}) \mid (\omega, \omega) = 0\}.$$

A *boundary component* is a subset of the form $\mathbb{P}(I_{\mathbb{C}}) \cap D^{*}_{M}$ for some isotropic subspace $I \subset N_{\mathbb{R}}$ of dimension 1 or 2; such a component is called *rational* if the corresponding I can be defined over $\mathbb{Q}$. In particular, 0-dimensional rational boundary components of D^{+}_{M} are in bijection with primitive isotropic elements of N.

When $\Gamma_{M,j}$ is an arithmetic subgroup of $O(N_{\mathbb{Q}})$, it acts on the set of rational boundary components of D^{+}_{M}, which we denote $\mathcal{RB}$, and for each rational boundary component F its stabilizer $N(F) = \{g \in \Gamma_{M,j} \mid g(F) = F\}$

acts discretely on F. Then the Baily–Borel compactification is the union

$$\overline{D_M^+/\Gamma_{M,j}} = D_M^+/\Gamma_{M,j} \coprod \left(\coprod_{F\in\mathcal{RB}/\Gamma_{M,j}} F/N(F) \right)$$

endowed with a structure of a normal projective algebraic variety.

Now we choose an m-admissible primitive isotropic $f \in N$, and consequently we fix a splitting $N = U(m) \oplus \check{M}$ and an isotropic $g \in U(m)$ such that $(f,g) = m$. For a rational boundary component F, we define $Z_{M,j}(f) = \{h \in N(F) \mid h(f) = f\}$ and $Z_{M,j}(f)^+$ as the subgroup of elements preserving $K(\check{M})$; thus we have an action of $Z_{M,j}(f)$ on $\mathcal{H}_f = V_f + iC_f^+$, and we can identify $Z_{M,j}(f)^+$ with the subgroup preserving $\mathcal{H}_f^+ = V_f + iK(\check{M})$.

Let F be the 0-dimensional rational boundary component corresponding to f. The theory in [7] holds also in our situation, hence there exist open neighborhoods $\tilde{\mathcal{U}}^*$ and $\mathcal{U}$ respectively of F in D_M^* and of $F/N(F)$ in $\overline{D_M^+/\Gamma_{M,j}}$, and an analytic isomorphism $\alpha : \tilde{\mathcal{U}}^*/Z_{M,j}(f)^+ \to \mathcal{U}$, which restricts to an isomorphism $\alpha : \tilde{\mathcal{U}}/Z_{M,j}(f)^+ \to \mathcal{U}_F$ for $\tilde{\mathcal{U}} = \tilde{\mathcal{U}}^* \cap \mathcal{H}_f^+$.

Theorem 4.6. — *The period map induces the mirror map $\alpha^{-1} : \mathcal{U}_F \to \tilde{\mathcal{U}} \subset \mathcal{H}_f^+$, which is multi-valued with monodromy group $Z_{M,j}(f)^+$, sending a neighborhood of F to the tube domain $\mathcal{H}_f^+ \cong \mathrm{Pic}(X') + i\mathcal{K}_{X'}$ for $(X',\phi') \in \mathcal{M}^{sa}_{\check{M},\check{j}}$.*

Remark 4.7. — By the construction, we have $(V_f)_{\mathbb{C}} \cong \check{M}_{\mathbb{C}}$, and consequently $(V_f)_{\mathbb{C}} \cong \mathrm{Pic}(X') \otimes \mathbb{C}$ for the very general member $(X',\phi') \in \mathcal{M}^+_{\check{M},\check{j}}$. By Theorem 4.6 and Proposition 4.2, it then follows that in an open neighborhood of a point at the boundary, after the choice of a normalization, the Griffiths–Yukawa coupling on the tangent space of D_M^+ coincides with the intersection form on the $H^{1,1}(X')$. This is analogous to what happens in the case of Calabi–Yau threefolds, where the Yukawa coupling coincides with the quantum intersection product on the mirror.

5. The $K3^{[2]}$-type case

From now on let X be a fourfold of $K3^{[2]}$-type, so that $b_2(X) = 23$ and $L = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2\rangle$. In this case, by works of Hassett and Tschinkel [13] and of Mongardi [18] (see also [2]), Assumption 3 is satisfied with

$$\Delta(L) = \left\{\delta \in L \mid \delta^2 = -2\right\} \cup \left\{\delta \in L \mid \delta^2 = -10, \mathrm{div}(\delta) = 2\right\}.$$

By a result of Markman combined with work of Kneser (see also [10]), it follows that

$$\mathrm{Mo}^2(L) = \mathrm{Ref}(L) = \tilde{O}^+(L) = \left\{ g \in O(L) \mid g_{|A_L} = \mathrm{id}, \mathrm{sn}^L_{\mathbb{R}}(g) = 1 \right\}$$

Hence the hypothesis of Proposition 3.5 is satisfied, and $\Gamma_{M,j}$ is an arithmetic subgroup of $O(N)$.

Theorem 5.1. — *For $L = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2 \rangle$ we have $\Gamma_{M,j} = \tilde{O}^+(N)$.*

Proof. — Proposition 3.5 tells us that $\Gamma_{M,j} \supset \tilde{O}^+(N)$. Vice versa, in this case $\mathrm{Mo}^2(M,j) = \tilde{O}^+(L) \cap O(L,M)$. Given $f \in \mathrm{Mo}^2(M,j)$, then $f_{|A_M \oplus A_L} = \mathrm{id}$ and hence $f_{|A_N} = \mathrm{id}$, since by Nikulin's Theorem 2.1 $\Gamma, \Gamma^\perp \subset A_M \oplus A_L$. On the other hand, $\Gamma_{M,j} \subset O^+(N)$ because it preserves the connected component D_M^+, hence we have equality. □

In this case we can find some criteria for the unicity of the embedding j.

Proposition 5.2. — *If there is no subgroup $H \subset A_M$ such that $H \cong \mathbb{Z}/2\mathbb{Z}$, the orthogonal N is unique in its genus $(2, 20-t, q_N)$, where $q_N = (-q_M) \oplus q_L$ on $A_M \oplus A_L$, and the projection $O(N) \to O(q_N)$ is surjective, then M admits a unique primitive embedding $j : M \hookrightarrow L$ up to isometry.*

Proof. — The proof is exactly the same of the proof of [4, Proposition 2.7]; we briefly sketch it here. The primitive embeddings $M \subset L$, up to isometry, are in one-to-one correspondence with the sets of quintuples $(H_M, H_L, \gamma; N, \gamma_N)$ as in Nikulin's Theorem 2.1. Under the hypotheses, the only possibility is $H_M \cong H_L \cong \{0\}$ and $\gamma = \mathrm{id}$. Moreover, the orthogonal N is in the genus $(2, 20-t, q_N)$ for $q_N = (-q_M) \oplus q_L$ on $A_M \oplus A_L$, hence it is isomorphic to N by assumption, and the surjectivity of $O(N) \to O(q_N)$ implies that different choices of γ_N induce isometric embeddings in L. □

In particular this is true if M is unimodular of rank $\mathrm{rk}\, M \leqslant 20$ or if $A_M = \bigoplus_{p_i > 2 \mathrm{prime}} (\mathbb{Z}/p_i\mathbb{Z})^{\oplus a_i}$ and $\mathrm{rk}\, M \leqslant 21 - \max(a_i)$. It is important to stress though that the orthogonal N will not in general have a unique embedding, so that $\check{j}$ will not be the only possible embedding of $\check{M}$.

Remark 5.3. — Consider now a primitive embedding $j_{K3} : M \subset L_{K3}$ and take $j = j_{K3} \oplus \mathrm{id}_{\langle -2 \rangle}$ to be the induced primitive embedding $M \subset L$. We can then find a mirror lattice either in L_{K3} or in L, obtaining respectively $\check{M}_{K3}$ and $\check{M} = \check{M}_{K3} \oplus \langle -2 \rangle$. Take an M-polarized K3 surface S and an $\check{M}_{K3}$-polarized K3 surface S' in the mirror family; then $S^{[2]}$ and $(S')^{[2]}$ will be respectively M-polarized and $\check{M}$-polarized mirror partners. On the other hand, since for any M-polarized K3 surface S, $S^{[2]}$ is also $(M \oplus \langle -2 \rangle)$-polarized, the family of Hilbert schemes of M-polarized K3 surfaces has codimension 1 inside $\mathcal{M}^+_{M,j}$, whereas the mirror moduli space has the same dimension as in the K3 case.

5.1. The polarized case

Let $M \subset L$ be the rank 1 sublattice $\langle 2d\rangle$ for a positive integer d. In the $K3^{[2]}$-type case, the following result is known:

Theorem 5.4 ([9, Prop. 3.6 and 3.12]). — *The sublattice $M = \langle 2d\rangle$ admits up to two non-isometric primitive embeddings in L. Let h be a generator of M; then the following holds:*

1. *there is always a* split *embedding j_s, corresponding to* $\operatorname{div} h = 1$, *such that $N_s = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle -2\rangle \oplus \langle -2d\rangle$, $\det N_s = 4d$ and $A_{N_s} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2d\mathbb{Z}$;*
2. *if $d \equiv 3$ modulo 4, then M admits a second embedding j_{ns}, called* non-split, *corresponding to* $\operatorname{div} h = 2$, *such that $N_{ns} = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus K_d$, $\det N_{ns} = d$ and $A_{N_{ns}} = \mathbb{Z}/d\mathbb{Z}$.*

In both cases, $\Gamma_{M,j} \cong \tilde{O}(N)^+$.

Fix $j : M \subset L$ as above and let $\mathcal{M}^+_{2d}$ be the subset of $\mathcal{M}^+_{M,j}$, where h corresponds to an ample class; as already shown in [9], we get an open algebraic embedding of $\mathcal{M}^+_{2d}/\operatorname{Mo}^2(M,j)$ into $D^+_M/\tilde{O}(N)^+$.

Proposition 5.5. — *The period map $\mathcal{P}^+_{M,j}$ restricts to an isomorphism*

$$\mathcal{M}^+_{2d}/\operatorname{Mo}^2(M,j) \to \left(D^+_M \setminus \coprod_{\delta\in\Delta(N)} (H_\delta \cap D^+_M)\right) /\tilde{O}(N)^+$$

Proof. — This is a straightforward consequence of [17, Theorem 8.4] and of [18, Proposition 2.12]. □

Now we want to compute $\tilde{O}(N)$-orbits of m-admissible isotropic vectors f in N for an integer $m|\det N$. In both cases we can apply Eichler's criterion 2.3: orbits are classified by $\operatorname{div} f|\det N$. By Scattone's work[23, Proposition 4.1.3], there is a bijection between $\tilde{O}(N)$-orbits of isotropic vectors in N and the set of isotropic elements in A_N modulo multiplication by ± 1, induced by the map $f \in N \mapsto f/\operatorname{div} f + N \in A_N$.

The split case. Let e and t denote respectively the generators of $\langle -2\rangle$ and $\langle -2d\rangle$ in N_s. Then the discriminant group A_{N_s} is generated by $e/2$ and $t/2d$ and the discriminant quadratic form q_s is given by

$$q_s(\alpha\frac{t}{2d} + \beta\frac{e}{2}) = -\frac{\alpha^2+\beta^2 d}{2d} \in \mathbb{Q}/2\mathbb{Z}$$

for $\alpha = 0, \dots, 2d-1$ and $\beta = 0, 1$. Let u and v denote a standard basis of one of the two orthogonal summands U inside N_s, and write $d = d'k^2$ with d' square-free.

Lemma 5.6. — *The m-admissible isotropic vectors f in N_s, up to the action of $\tilde{O}(N_s)$, are of the form*

$$f = \begin{cases} t + m(u + \frac{d}{m^2}v) & \text{if } m|k \\ t + m^2 e + m(u + \frac{d+m^2}{m^2}v) & \text{if } m \nmid k,\ \frac{m}{2}|k \text{ and } d' \equiv 3(4) \end{cases} \tag{5.1}$$

Proof. — Let $I(q_s)$ be the set of isotropic elements in A_{N_s}; it is clear that it is the union of isotropic elements of $\mathbb{Z}/2d\mathbb{Z}$ with respect to the restricted form $q_s(\alpha\frac{t}{2d}) = -\frac{\alpha^2}{2d}$ and of isotropic elements $y = \alpha\frac{t}{2d} + \frac{e}{2}$, since e is not isotropic. Moreover, it is easy to remark that $\operatorname{ord}(f/\operatorname{div} f) = \operatorname{div} f$ and, since we are interested in classification up to the action of $\tilde{O}(N_s)$, we need to find only one isotropic element for each possible order m.

The computation of isotropic elements of $\mathbb{Z}/2d\mathbb{Z}$ has been done by Scattone [23, Theorem 4.0.1] in the K3 case; they are in bijection with elements in the cyclic subgroup of order k. Hence m has to divide k, and an isotropic element $x \in \mathbb{Z}/2d\mathbb{Z}$ of order m is $x_m = \frac{t}{m}$.

Now take $y = \alpha\frac{t}{2d} + \frac{e}{2}$; then $q_s(y) = -\frac{\alpha^2+d}{2d} \in 2\mathbb{Z}$ if and only if $\alpha^2 + d = 4dl$ for an integer l. By reducing this equality modulo 4 we see that the only possible case is $d' \equiv 3\ (4)$ and $\alpha = d'kh$ with $h = 1, \dots, k-1$ odd. In this case we observe that $m = \operatorname{ord}(y)|2k$, and we are interested in finding elements of order not dividing k; in particular, new possible orders m are those such that $\frac{m}{2}|k$ and $m \nmid k$. For example, an isotropic element $x \in A_{N_s}$ of order m is $x_m = \frac{t}{m} + \frac{e}{2}$.

Up to isometry, the corresponding isotropic vectors in N_s are precisely of the forms given in (5.1). Indeed, given such an $f \in N_s$, we have $f^2 = 0$, $(f, v) = m$,

$$(f, u) = \begin{cases} \frac{d}{m} \in m\mathbb{Z} & \text{if } m|k \\ \frac{d+m^2}{m} \in m\mathbb{Z} & \text{if } m \nmid k,\ \frac{m}{2}|k \text{ and } d' \equiv 3(4) \end{cases}$$

$$(f, e) = \begin{cases} 0 & \text{if } m|k \\ -2m^2 \in m\mathbb{Z} & \text{if } m \nmid k,\ \frac{m}{2}|k \text{ and } d' \equiv 3(4) \end{cases}$$

Hence $\operatorname{div} f = m$. □

Now we restrict to the case $m = 1$ and consider the isotropic primitive vector $f = t + u + dv$. Since it is unimodular, U admits a unique primitive embedding in N up to isometry; hence its orthogonal is $\check{M} = U \oplus E_8^{\oplus 2} \oplus$

$\langle -2\rangle \oplus \langle -2d\rangle$ and we can assume that $U = \mathbb{Z}f + \mathbb{Z}v$. The sublattice $\check{M}$ admits two non-isometric primitive embeddings in L; it follows from the definition that $\check{\jmath}$ satisfies $\check{\jmath}(\check{M})^{\perp} = U \oplus \langle 2d\rangle$. Hence the period domain $D^{+}_{\check{M}}$ is exactly the one described by Dolgachev in [7], with tube domain realization isomorphic to the upper half-plane $\mathbb{H}$.

By [7, Theorem 7.1] and Theorem 5.1, the global monodromy group $\Gamma_{\check{M},\check{\jmath}}$ is conjugate in $PSL(2,\mathbb{R})$ to the subgroup $\Gamma_0(d)^+$ generated by

$$\Gamma_0(d) = \{(a_{ij}) \in SL(2,\mathbb{Z}) \mid d|a_{21}\}$$

and by the Fricke involution

$$F = \begin{pmatrix} 0 & -\frac{1}{\sqrt{d}} \\ \sqrt{d} & 0 \end{pmatrix} \in PSL(2,\mathbb{R})$$

The main difference with what happens in the case of polarized K3 surfaces is that here we only get a local isomorphism from our moduli space to the modular curve

$$\mathcal{M}^{+}_{\check{M},\check{\jmath}}/\operatorname{Mo}^2(\check{M},\check{\jmath}) \xrightarrow{\quad \mathcal{P} \quad} \mathbb{H}/\Gamma_0(d)^+$$

The non-split case. In this case $d \equiv 3\ (4)$. Let e and w_1, w_2 denote respectively the generators of $\langle -2\rangle$ and of a copy of U in L, so that $M = \langle h\rangle \subset U \oplus \langle -2\rangle$ with $h = 2w_1 + \frac{d+1}{2}w_2 + e$. The orthogonal K_d is generated by $b_1 = w_1 - \frac{d+1}{4}w_2$ and $b_2 = w_2 + e$. The discriminant group $A_{N_{ns}}$ is generated by $t = \frac{1}{d}h - w_2$, and the discriminant quadratic form q_{ns} is given by

$$q_{ns}(\alpha t) = -\frac{2\alpha^2}{d} \in \mathbb{Q}/2\mathbb{Z}$$

for $\alpha = 0, \dots, d-1$. Let u and v denote a standard basis of one of the two orthogonal summands U inside N_{ns}, and write $d = d'k^2$ with $d' \equiv 3\ (4)$ square-free.

Lemma 5.7. — *The m-admissible isotropic vectors f in N_{ns}, up to the action of $\tilde{O}(N_{ns})$, are of the form*

$$f = 2b_1 + b_2 + m\left(u + \frac{d}{m^2}v\right) \text{ if } m|k \tag{5.2}$$

Proof. — Computations similar to the ones in the proof of Lemma 5.6 show that the order m of an isotropic element $\alpha t \in \mathbb{Z}/d\mathbb{Z}$ has to divide k. Given f as in (5.2), we have $f^2 = 0$, $(f, b_1) = -d \in m\mathbb{Z}$, $(f, b_2) = 0$, $(f, u) = \frac{d}{m} \in m\mathbb{Z}$ and $(f, v) = m$. Hence $\operatorname{div} f = m$. □

Now we restrict to the case $m = 1$ and consider the hyperbolic lattice $U = \mathbb{Z}f \oplus \mathbb{Z}v$. Since it is unimodular, U admits a primitive embedding into N, unique up to isometry, and its orthogonal is $\check{M} = U \oplus E_8^{\oplus 2} \oplus K_d$. The sublattice $\check{M}$ admits a unique primitive embedding into L and $\check{\jmath}(\check{M})^\perp = U \oplus \langle 2d\rangle$; the period domain $D^+_{\check{M}}$ is exactly as in the split case and everything remarked above holds again.

5.2. Non-symplectic involutions

In the recent paper [4] (see also [16]), the authors classify primitive embeddings of invariant sublattices T of non-symplectic involutions i of fourfolds X of $K3^{[2]}$-type, i.e. involutions such that $i^*\omega_X = -\omega_X$. The invariant sublattice T is known to be hyperbolic and two-elementary with a two-elementary orthogonal S. By the work of Nikulin [21], a two-elementary hyperbolic lattice T is completely determined by the triple (r, a_T, δ_T), where r is its rank, $a_T = l(A_T)$ is the length of its discriminant group and δ_T is the *parity* of the discriminant quadratic form q_T: $\delta_T = 0$ if $q_T(x) \in \mathbb{Z}/2\mathbb{Z}$ for all $x \in A_T$, 1 otherwise.

By [4, Proposition 6.1], the primitive embeddings j into L of a two-elementary hyperbolic sublattice T with invariants (r, a_T, δ_T) are in bijection with the couples $(a \pm 1, \delta_S)$, where S is the orthogonal complement of $j(T)$ in L, two-elementary with $a_S = l(A_S) = a \pm 1$ and parity δ_S.

Consider now $\mathcal{M}^+_{T,j}$ and look for the mirror family corresponding to the choice of a 1-admissible isotropic vector $f \in S$.

Lemma 5.8. — *There is a* 1*-admissible isotropic* $f \in S$ *and* $S \supset U$ *if and only if* $\operatorname{rk} T \leqslant 21 - l(A_T)$ *or* $\operatorname{rk} T = 22 - l(A_T)$ *and* $l(A_S) = l(A_T) + 1$ *except when* $(\operatorname{rk} T, l(A_T), \delta_T, l(A_S), \delta_S) = (15, 7, 1, 6, 0)$.

Proof. — If $\operatorname{rk} T \leqslant 20 - l(A_S)$, then this follows by Theorem 2.2. Otherwise, one of the following holds:

(1) $\operatorname{rk} T = 24 - l(A_T)$ and $l(A_S) = l(A_T) - 1 = \operatorname{rk} S$: we have $S = \langle 2\rangle^{\oplus 2} \oplus \langle -2\rangle^{\oplus l(A_T)-3}$ if $\delta_S = 1$, and if $\delta_S = 0$, then S is $U(2)^{\oplus 2}$. None contains a copy of U.

(2) $\operatorname{rk} T = 22 - l(A_T)$ and $l(A_S) = l(A_T) - 1 = \operatorname{rk} S - 2$: we have $S = U \oplus \langle 2\rangle \oplus \langle -2\rangle^{\oplus l(A_T)-2}$ if $\delta_S = 1$, and if $\delta_S = 0$, then S is either $U \oplus U(2)$, $U(2)^{\oplus 2} \oplus D_4$ or $U \oplus U(2) \oplus E_8(2)$. All contain a copy of U except $U(2)^{\oplus 2} \oplus D_4$.

(3) $\operatorname{rk} T = 22 - l(A_T)$ and $l(A_S) = l(A_T) + 1 = \operatorname{rk} S$: we have $S = \langle 2\rangle^{\oplus 2} \oplus \langle -2\rangle^{\oplus l(A_T)-1}$ with $\delta_S = 1$. None contains a copy of U.

(4) $\operatorname{rk} T = 20 - l(A_T)$ and $l(A_S) = l(A_T) + 1 = \operatorname{rk} S - 2$: we have $S = U \oplus \langle 2\rangle \oplus \langle -2\rangle^{\oplus l(A_T)}$ with $\delta_S = 1$, and this contains a copy of U. □

Once fixed such an $f \in S$ and a splitting $S = U \oplus \check{T}$, we get that $\check{T}$ is hyperbolic, two-elementary with invariants $(21-r, a_S, \delta_S)$ and the embedding $\check{\jmath}$ is the one corresponding to (a_T, δ_T). Moreover, by cancelling the points corresponding to non-admissible values of $(r, a_T, \delta_T, a_S, \delta_S)$, Figure 1 and Figure 2 in [4] can be combined in Figure 5.1, where every point denoted with $\bullet$ is mirror dual with the symmetric $\bullet$ with respect to the line G and symmetric $*$ and $\circ$ are mirrors, with the only exception of $(14, 6, 0, 7, 1)$.

Remark 5.9. — Since the generic member of each family carries a non-symplectic involution with prescribed invariant lattice, we thus get a notion of *mirror involution*. On the other hand, this does not agree with the notion of mirror involutions defined by the analogous construction on K3 surfaces, as described in [26], in the sense that pairs of natural involutions induced by mirror involutions on a K3 surface S are not mirror pairs on $S^{[2]}$.

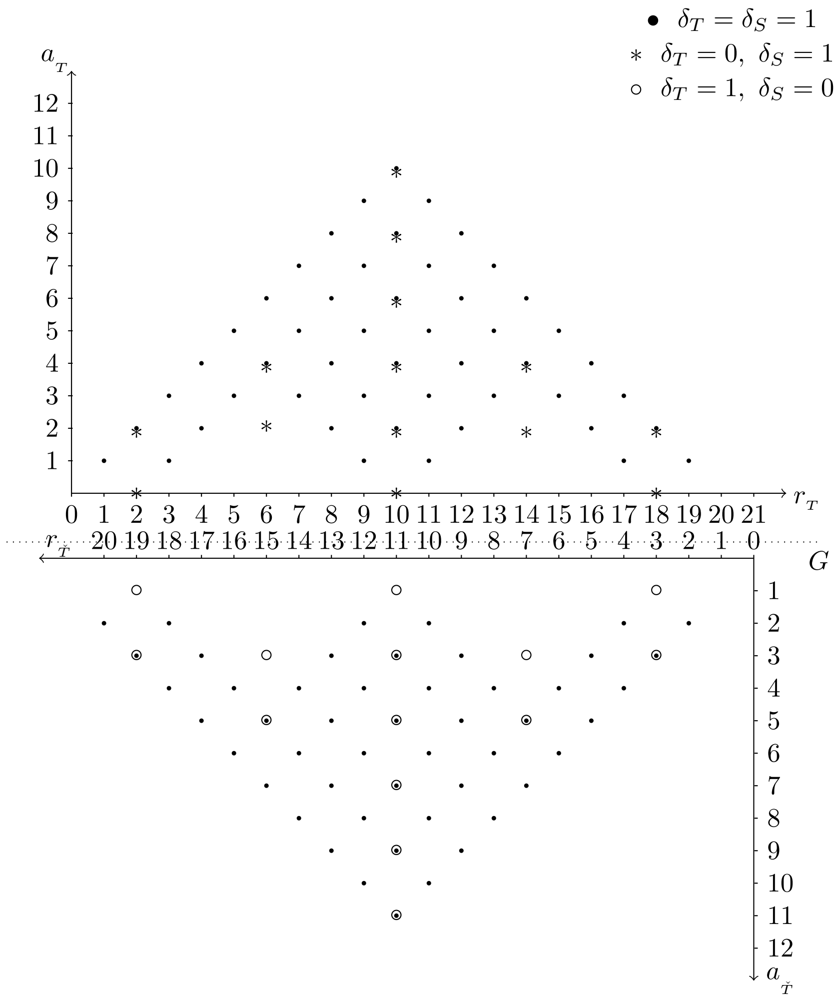

Figure 5.1. Mirror pairs of non-symplectic involutions

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Manuscrit reçu le 24 mars 2014,
révisé le 12 décembre 2014,
accepté le 10 septembre 2015.

Chiara CAMERE
Dipartimento di Matematica "Federigo Enriques"
Università degli Studi di Milano
Via Cesare Saldini 50
20133 Milano (Italy)
chiara.camere@unimi.it