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INTERSECTION OF CURVES ON SURFACES AND THEIR APPLICATIONS TO MAPPING CLASS GROUPS

by Nariya KAWAZUMI & Yusuke KUNO (*)

ABSTRACT. — We introduce an operation which measures the self intersections of paths on an oriented surface. As applications, we give a criterion of the realizability of a generalized Dehn twist, and derive a geometric constraint on the image of the Johnson homomorphisms.

RÉSUMÉ. — Nous introduisons une opération qui mesure l'auto-intersection des chemins sur une surface orientée. Comme applications, nous donnons un critère de la réalisabilité d'un twist de Dehn généralisé, et nous obtenons une contrainte géométrique sur l'image des homomorphismes de Johnson.

1. Introduction

In the study of the mapping class group of a surface, it is sometimes convenient and important to work with curves, i.e., loops and paths, on the surface. Since the mapping class group is the group of isotopy classes of diffeomorphisms of the surface, it acts on the homotopy classes of curves. As is illustrated in the classical theorem of Dehn-Nielsen, this action distinguishes elements of the mapping class group well. Moreover, this action preserves intersections of curves.

In an earlier part of this paper (§3 and §4) we introduce an operation which measures the self intersections of paths on an oriented surface. Let S

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be an oriented surface, and $o, \iota \in \partial S$ points on the boundary. We denote by $\Pi S(o, \iota)$ the set of homotopy classes of paths from o to ι , and by $\hat{\pi}(S)$ the set of homotopy classes of non-trivial free loops on S . In §3.2 we define a \mathbb{Q} -linear map

$$\mu: \mathbb{Q}\Pi S(o, \iota) \rightarrow \mathbb{Q}\Pi S(o, \iota) \otimes \mathbb{Q}\hat{\pi}(S),$$

by using the self intersections of a path from o to ι . This map is inspired by Turaev's self intersection [32], and is actually a refinement of it.

The operation μ is closely related to the Goldman-Turaev Lie bialgebra. The free \mathbb{Q} -vector space $\mathbb{Q}\hat{\pi}(S)$ spanned by the set $\hat{\pi}(S)$ is an involutive Lie bialgebra with respect to the Goldman bracket [11] and the Turaev cobracket [33]. In [16] [15], we showed that the \mathbb{Q} -vector space $\mathbb{Q}\Pi S(o, \iota)$ is a (left) $\mathbb{Q}\hat{\pi}(S)$ -module with respect to a structure map $\sigma: \mathbb{Q}\hat{\pi}(S) \otimes \mathbb{Q}\Pi S(o, \iota) \rightarrow \mathbb{Q}\Pi S(o, \iota)$. In fact, by investigating the properties of μ we arrive at the notion of a *comodule of a Lie coalgebra*, and that of a *bimodule of a Lie bialgebra*. To our knowledge, these algebraic structures are new. Thus in §2 we record their definitions after recalling the definition of a Lie bialgebra. In §3, we show that $\mathbb{Q}\Pi S(o, \iota)$ is a $\mathbb{Q}\hat{\pi}(S)$ -bimodule with respect to σ and μ .

The \mathbb{Q} -vector spaces $\mathbb{Q}\Pi S(o, \iota)$ and $\mathbb{Q}\hat{\pi}(S)$ have decreasing filtrations coming from the augmentation ideal of the group ring of $\pi_1(S)$. Using these filtrations we can construct completions of $\mathbb{Q}\Pi S(o, \iota)$ and $\mathbb{Q}\hat{\pi}(S)$. In [15] we showed that the Goldman bracket and the operation σ extend naturally to completions. In §4, using a certain product formula for μ , we show that the Turaev cobracket and the operation μ extends also to completions. Working with filtrations and completions seems complicated at first sight, but we need it for applications to the generalized Dehn twists and the Johnson homomorphism.

In a later part of this paper (§5 and §6), assuming that S is compact with non-empty boundary, we discuss applications of the $\mathbb{Q}\hat{\pi}(S)$ -bimodule structure on $\mathbb{Q}\Pi S(o, \iota)$ to the study of the mapping class group $\mathcal{M}(S, \partial S)$ of S . Here $\mathcal{M}(S, \partial S)$ means that we consider only diffeomorphisms and isotopies fixing the boundary pointwise. The key ingredients for our discussion are the compatibility condition (2.2.2) of σ and μ , and the fact that μ is compatible with the action of $\mathcal{M}(S, \partial S)$.

In §5, we discuss an application to generalized Dehn twists. Generalized Dehn twists were first introduced in [19] for a once bordered surface based on a result in [16] which describes the Dehn-Nielsen image of the right handed Dehn twist in terms of σ . Later the construction was generalized for any oriented surface in [15]. For each unoriented loop C on

S , one can construct the generalized Dehn twist along C , denoted by t_C , as an element of a certain enlargement of $\mathcal{M}(S, \partial S)$. When C is simple, t_C is the usual right handed Dehn twist along C . In general, we can ask whether t_C is realized by a diffeomorphism of the surface, i.e., whether t_C lies in $\mathcal{M}(S, \partial S)$. We give a criterion of the realizability of t_C in terms of μ (Proposition 5.2.1). Using this we can extend results about a figure eight [19] [15] to loops in wider classes (Theorem 5.3.1).

In §6, we discuss an application to the Johnson homomorphisms. The higher Johnson homomorphisms, introduced by Johnson [13] and improved by Morita [26], are important algebraic tools to study the mapping class group and the Torelli group for a once bordered surface. Let $E \subset \partial S$ be a subset such that each connected component of ∂S has a unique point of E . The “smallest” Torelli group (in the sense of Putman [30]), denoted by $I(S, E)$, is the subgroup of $\mathcal{M}(S, \partial S)$ consisting of mapping classes acting on $H_1(S, E; \mathbb{Z})$ trivially. In the previous paper [15], the authors introduced a Lie subalgebra $L^+(S, E)$ of the completion of $\mathbb{Q}\hat{\pi}(S)$ and an injective group homomorphism $\tau: I(S, E) \rightarrow L^+(S, E)$. Here the Hausdorff series equips $L^+(S, E)$ with a pro-nilpotent group structure. When S is once bordered, τ is essentially the same as Massuyeau’s improvement [20] of the Johnson map introduced by Kawazumi [14], and thus the graded quotients of τ recovers all the higher Johnson homomorphisms. In this way we can generalize the Johnson homomorphisms to all compact oriented surfaces with non-empty boundary. Recently there are approaches to generalize the first Johnson homomorphism for compact oriented surfaces with non-connected boundary, see Church [8], and Putman [29]. It would be interesting to compare their constructions with ours.

The fundamental problem in the study of the Johnson homomorphisms is to determine their image. In the once bordered case, the graded quotient of $L^+(S, E)$ with respect to the filtration we consider in this paper is canonically isomorphic to a graded Lie algebra which Morita introduced as a target of the Johnson homomorphisms in [26]. Moreover he introduced the Morita trace in a purely algebraic way to prove that the Johnson image is a proper Lie subalgebra of the Lie algebra. In our geometric formulation of the Johnson homomorphisms, we give a constraint on the image of τ (Theorem 6.2.1) even for the non-connected boundary case. Namely, we show that the image of τ is contained in the kernel of the Turaev cobracket. We can see that this constraint is non-trivial using ideas in §5. Further, based on a tensorial description of the homotopy intersection form given

by Massuyeau and Turaev [22], we study some lower terms of a tensorial description of the Turaev cobracket (Theorem 6.3.2). In particular we show that the Morita trace [26] is recovered from the graded quotient of the Turaev cobracket, or equivalently, the lowest term of our geometric constraint (Theorem 6.4.1). In particular, the fact proved by Morita that the Johnson image is included in the kernel of the Morita trace follows from the basic fact that any diffeomorphism of the surface S preserves the self-intersections of any curve on S . At the present we do not know much about higher terms of the Turaev cobracket. However, we hope that the image of τ is characterized by the Turaev cobracket (Conjecture 6.2.3).

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2. Lie bialgebras and their bimodules

For the sake of the reader we collect the definitions of a Lie bialgebra and its bimodules.

We work over the rationals \mathbb{Q} . Let V be a \mathbb{Q} -vector space and let $T = T_V: V^2 \rightarrow V^2$ and $N = N_V: V^3 \rightarrow V^3$ be the linear maps defined by $T(X \ Y) = Y \ X$ and $N(X \ Y \ Z) = X \ Y \ Z + Y \ Z \ X + Z \ X \ Y$ for $X, Y, Z \in V$.

2.1. Lie bialgebras

Let \mathfrak{g} be a \mathbb{Q} -vector space equipped with \mathbb{Q} -linear maps $\mu: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ and $\delta: \mathfrak{g} \rightarrow \mathfrak{g} \times \mathfrak{g}$. Recall that \mathfrak{g} is called a *Lie bialgebra* with respect to μ and δ , if

- (1) the pair (\mathfrak{g}, μ) is a Lie algebra, i.e., μ satisfies *the skew condition* and *the Jacobi identity*

$$T = -\mu \circ \delta: \mathfrak{g}^2 \rightarrow \mathfrak{g}, \quad (\mu \circ \delta)N = 0: \mathfrak{g}^3 \rightarrow \mathfrak{g},$$

(2) the pair (\mathfrak{g}, δ) is a Lie coalgebra, i.e., δ satisfies *the coskew condition* and *the coJacobi identity*

$$T\delta = -\delta: \mathfrak{g} \rightarrow \mathfrak{g}^2, \quad N(\delta - 1)\delta = 0: \mathfrak{g} \rightarrow \mathfrak{g}^3,$$

(3) the maps σ and δ satisfy *the compatibility*

$$X, Y \in \mathfrak{g}, \quad \delta[X, Y] = \sigma(X)(\delta Y) - \sigma(Y)(\delta X).$$

Here we denote $[X, Y] := (X \ Y)$ and $\sigma(X)(Y \ Z) = [X, Y] \ Z + Y \ [X, Z]$ for $X, Y, Z \in \mathfrak{g}$. The map σ is called the *bracket*, and the map δ is called the *cobacket*.

Moreover if *the involutivity*

$$\delta = 0: \mathfrak{g} \rightarrow \mathfrak{g}$$

holds, we say \mathfrak{g} is *involutive*.

2.2. Lie comodules and bimodules

Let \mathfrak{g} be a Lie algebra. Recall that a *left \mathfrak{g} -module* is a pair (M, σ) where M is a \mathbb{Q} -vector space and σ is a \mathbb{Q} -linear map $\sigma: \mathfrak{g} \times M \rightarrow M$, $X \ m \rightarrow Xm$, satisfying

$$X, Y \in \mathfrak{g}, \ m \in M, \quad [X, Y]m = X(Ym) - Y(Xm).$$

This condition is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} \mathfrak{g} \times \mathfrak{g} \times M & \xrightarrow{(1_{\mathfrak{g}} \ \sigma)((1-T) \ 1_M)} & \mathfrak{g} \times M \\ 1_M \downarrow & & \sigma \downarrow \\ \mathfrak{g} \times M & \xrightarrow{\sigma} & M. \end{array}$$

If we define $\bar{\sigma}: M \times \mathfrak{g} \rightarrow M$ by $\bar{\sigma}(m \ X) := -\sigma(X \ m) = -Xm$ for $m \in M$ and $X \in \mathfrak{g}$, then the pair $(M, \bar{\sigma})$ is a *right \mathfrak{g} -module*, i.e., the following diagram commutes:

$$\begin{array}{ccc} M \times \mathfrak{g} \times \mathfrak{g} & \xrightarrow{(\bar{\sigma} \ 1_{\mathfrak{g}})(1_M \ (1-T))} & M \times \mathfrak{g} \\ 1_M \downarrow & & \bar{\sigma} \downarrow \\ M \times \mathfrak{g} & \xrightarrow{\bar{\sigma}} & M. \end{array}$$

Next let (\mathfrak{g}, δ) be a Lie coalgebra and M a \mathbb{Q} -vector space equipped with a \mathbb{Q} -linear map $\mu: M \times M \rightarrow \mathfrak{g}$. We say the pair (M, μ) is a *right*

\mathfrak{g} -comodule if the following diagram commutes:

$$(2.2.1) \quad \begin{array}{ccc} M & \xrightarrow{\mu} & M \otimes \mathfrak{g} \\ \mu \downarrow & & 1_M \otimes \delta \downarrow \\ M \otimes \mathfrak{g} & \xrightarrow{(1_M \otimes (1-T))(\mu \otimes 1_{\mathfrak{g}})} & M \otimes \mathfrak{g} \otimes \mathfrak{g}. \end{array}$$

Similarly, we say a pair $(M, \bar{\mu})$ is a *left \mathfrak{g} -comodule* if $\bar{\mu}$ is a \mathbb{Q} -linear map $\bar{\mu}: M \otimes \mathfrak{g} \rightarrow M$ and the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\bar{\mu}} & \mathfrak{g} \otimes M \\ \bar{\mu} \downarrow & & \delta \otimes 1_M \downarrow \\ \mathfrak{g} \otimes M & \xrightarrow{((1-T) \otimes 1_M)(1_{\mathfrak{g}} \otimes \bar{\mu})} & \mathfrak{g} \otimes \mathfrak{g} \otimes M. \end{array}$$

If we denote the switch map by $T_{\mathfrak{g},M}: \mathfrak{g} \otimes M \rightarrow M \otimes \mathfrak{g}$, $X \otimes m \rightarrow m \otimes X$, then it is easy to see that $(M, \bar{\mu})$ is a left \mathfrak{g} -comodule if and only if $(M, -T_{\mathfrak{g},M}\bar{\mu})$ is a right \mathfrak{g} -comodule.

Finally let \mathfrak{g} be a Lie bialgebra with $[\cdot, \cdot]$ the bracket and δ the cobracket, (M, σ) a left \mathfrak{g} -module, and (M, μ) a right \mathfrak{g} -comodule with the same underlying vector space M . We define $\bar{\sigma}: M \otimes \mathfrak{g} \rightarrow M$ by $\bar{\sigma}(m \otimes X) := -Xm$, as before. Then $(M, \bar{\sigma})$ is a right \mathfrak{g} -module. We say the triple $(M, \bar{\sigma}, \mu)$ is a *right \mathfrak{g} -bimodule* if σ and μ satisfy *the compatibility*

$$(2.2.2) \quad m \otimes M, Y \in \mathfrak{g}, \quad \sigma(Y)\mu(m) - \mu(Ym) - (\bar{\sigma} \otimes 1_{\mathfrak{g}})(1_M \otimes \delta)(m \otimes Y) = 0.$$

Here $\sigma(Y)\mu(m) = (\sigma \otimes 1_M)(Y \otimes \mu(m)) + (1_M \otimes \text{ad}(Y))\mu(m)$ and $\text{ad}(Y)(Z) = [Y, Z]$ for $Z \in \mathfrak{g}$. Then we also call the triple $(M, \sigma, \bar{\mu})$ defined by $\bar{\mu} := -T_{M,\mathfrak{g}}\mu: M \otimes \mathfrak{g} \rightarrow M$ a *left \mathfrak{g} -bimodule*. Moreover, if \mathfrak{g} is involutive and the condition

$$(2.2.3) \quad \bar{\sigma}\mu = 0: M \rightarrow M$$

holds, we say M is *involutive*.

3. The Goldman-Turaev Lie bialgebra and its bimodule

Let S be a connected oriented surface. We denote by $\hat{\pi}(S) = [S^1, S]$ the homotopy set of oriented free loops on S . In other words, $\hat{\pi}(S)$ is the set of conjugacy classes of $\pi_1(S)$. We denote by $\pi_1(S) \rightarrow \hat{\pi}(S)$ the natural projection, and we also denote by $\mathbb{Q}\pi_1(S) \rightarrow \mathbb{Q}\hat{\pi}(S)$ its \mathbb{Q} -linear extension.

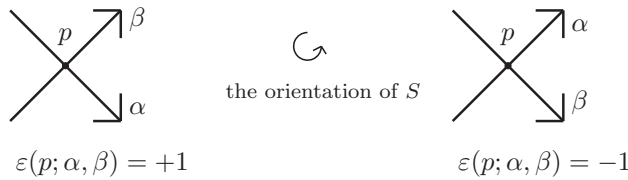


Figure 1. local intersection number

3.1. The Goldman-Turaev Lie bialgebra

We recall the definition of the Goldman-Turaev Lie bialgebra [11] [33].

Let α and β be oriented immersed loops on S such that their intersections consist of transverse double points. For each $p \in \alpha \cap \beta$, let $\alpha_p \beta_p \in \pi_1(S, p)$ be the loop going first along the loop α based at p , then going along β based at p . Also, let $\varepsilon(p; \alpha, \beta) \in \{\pm 1\}$ be the local intersection number of α and β at p . See Figure 1. The Goldman bracket of α and β is defined as

$$(3.1.1) \quad [\alpha, \beta] := \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta) [\alpha_p \beta_p] \in \mathbb{Q}\hat{\pi}(S).$$

The free \mathbb{Q} -vector space $\mathbb{Q}\hat{\pi}(S)$ spanned by the set $\hat{\pi}(S)$ equipped with this bracket is a Lie algebra. See [11]. Let $1 \in \hat{\pi}(S)$ be the class of a constant loop, then its linear span $\mathbb{Q}1$ is an ideal of $\mathbb{Q}\hat{\pi}(S)$. We denote by $\mathbb{Q}\hat{\pi}(S)$ the quotient Lie algebra $\mathbb{Q}\hat{\pi}(S)/\mathbb{Q}1$, and let $\varpi: \mathbb{Q}\hat{\pi}(S) \rightarrow \mathbb{Q}\hat{\pi}(S)$ be the projection. We write $|| := \varpi \circ /: \mathbb{Q}\pi_1(S) \rightarrow \mathbb{Q}\hat{\pi}(S)$.

Let $\alpha: S^1 \rightarrow S$ be an oriented immersed loop such that its self intersections consist of transverse double points. Set $D = D_\alpha := \{(t_1, t_2) \in S^1 \times S^1; t_1 \neq t_2, \alpha(t_1) = \alpha(t_2)\}$. For $(t_1, t_2) \in D$, let $\alpha_{t_1 t_2}$ (resp. $\alpha_{t_2 t_1}$) be the restriction of α to the interval $[t_1, t_2]$ (resp. $[t_2, t_1]$) $\subset S^1$ (they are indeed loops since $\alpha(t_1) = \alpha(t_2)$). Also, let $\dot{\alpha}(t_i) \in T_{\alpha(t_i)}S$ be the velocity vectors of α at t_i , and set $\varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2)) = +1$ if $(\dot{\alpha}(t_1), \dot{\alpha}(t_2))$ gives the orientation of S , and $\varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2)) = -1$ otherwise. The Turaev cobracket of α is defined as

$$(3.1.2) \quad \delta(\alpha) := \sum_{(t_1, t_2) \in D} \varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2)) [\alpha_{t_1 t_2}] \wedge [\alpha_{t_2 t_1}] \in \mathbb{Q}\hat{\pi}(S) \wedge \mathbb{Q}\hat{\pi}(S).$$

This gives rise to a well-defined Lie cobracket $\delta: \mathbb{Q}\hat{\pi}(S) \wedge \mathbb{Q}\hat{\pi}(S) \rightarrow \mathbb{Q}\hat{\pi}(S)$ (note that $\delta(1) = 0$). Moreover, $\mathbb{Q}\hat{\pi}(S)$ is an involutive Lie bialgebra with respect to the Goldman bracket and the Turaev cobracket. See [33]. The involutivity is due to Chas [3]. We call $\mathbb{Q}\hat{\pi}(S)$ the *Goldman-Turaev Lie bialgebra*.

Remark 3.1.3. — Finding the minimal number of (self) intersections of curves is a classical problem in low-dimensional topology. There are several studies of this problem using the Goldman bracket and the Turaev cobracket (and their generalizations). For these, we refer to Chas [3] [4], Chas and Kronigold [6] [5], Chernov [7], Cahn [1], and Chernov and Cahn [2].

3.2. A $\mathbb{Q}\hat{\pi}(S)$ -bimodule

Hereafter we assume that the boundary of S is not empty. Take distinct points $o, \iota \in \partial S$, and let $\Pi S(o, \iota)$ be the homotopy set $[[([0, 1], 0, 1), (S, o, \iota)]$. In this subsection we show that the free \mathbb{Q} -vector space $\mathbb{Q}\Pi S(o, \iota)$ spanned by the set $\Pi S(o, \iota)$ has the structure of an involutive right $\mathbb{Q}\hat{\pi}(S)$ -bimodule in the sense of §2.2. In §3.3 we discuss the case $o = \iota$.

A left $\mathbb{Q}\hat{\pi}(S)$ -module structure. Let α be an oriented immersed loop on S , and $\beta: [0, 1] \rightarrow S$ an immersed path from o to ι such that their intersections consist of transverse double points. Then the formula

$$(3.2.1) \quad \sigma(\alpha \cdot \beta) := \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta) \beta|_{op} \alpha_p \beta_p|_{p\iota} \in \mathbb{Q}\Pi S(o, \iota)$$

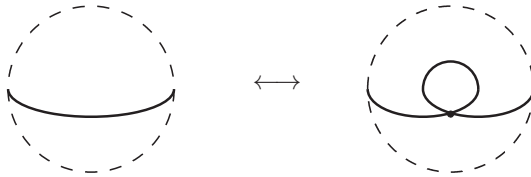
gives rise to a well-defined \mathbb{Q} -linear map $\sigma: \mathbb{Q}\hat{\pi}(S) \rightarrow \mathbb{Q}\Pi S(o, \iota)$. Here, $\varepsilon(p; \alpha, \beta) \in \{\pm 1\}$ has the same meaning as before, and $\beta|_{op} \alpha_p \beta_p|_{p\iota}$ means the path going first from o to p along β , then going along α based at p , and finally going from p to ι along β . By the same proof as that of [16] Proposition 3.2.2, we see that $\mathbb{Q}\Pi S(o, \iota)$ is a left $\mathbb{Q}\hat{\pi}(S)$ -module with respect to σ . See also [15] §4.

A right $\mathbb{Q}\hat{\pi}(S)$ -comodule structure. Let $\gamma: [0, 1] \rightarrow S$ be an immersed path from o to ι such that its self intersections consist of transverse double points. Let $\Gamma = \Gamma_\gamma \cap \text{Int}(S)$ be the set of double points of γ . For $p \in \Gamma$, we denote $\gamma^{-1}(p) = \{t_1^p, t_2^p\}$, so that $t_1^p < t_2^p$. Set

$$(3.2.2) \quad \mu(\gamma) := - \sum_p \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p)) (\gamma_{0t_1^p} \gamma_{t_2^p\iota}) \cdot |\gamma_{t_1^p t_2^p}| \in \mathbb{Q}\Pi S(o, \iota) \otimes \mathbb{Q}\hat{\pi}(S).$$

Here $\varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p)) \in \{\pm 1\}$ has the same meaning as before, $\gamma_{0t_1^p} \gamma_{t_2^p\iota}$ is the conjunction of the restrictions of γ to $[0, t_1^p]$ and $[t_2^p, 1]$, and $\gamma_{t_1^p t_2^p}$ is the restriction of γ to $[t_1^p, t_2^p]$. The map μ is closely related to Turaev’s self intersection [32]. See Remark 3.3.3.

(ω_1) *birth-death of monogons*



(ω_2) *birth-death of bigons*



(ω_3) *jumping over a double point*

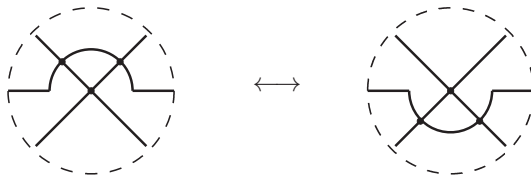


Figure 2. local moves (ω_1), (ω_2), and (ω_3)

PROPOSITION 3.2.3. — *The formula (3.2.2) gives rise to a well-defined \mathbb{Q} -linear map*

$$\mu: \mathbb{Q}\text{IIS}(0, 1) \rightarrow \mathbb{Q}\text{IIS}(0, 1) \otimes \mathbb{Q}\hat{\pi}(S).$$

Moreover, $\mathbb{Q}\text{IIS}(0, 1)$ is a right $\mathbb{Q}\hat{\pi}(S)$ -comodule with respect to μ .

Proof. — Any immersions γ and γ' with $\gamma(0) = \gamma'(0) = 0$ and $\gamma(1) = \gamma'(1) = 1$, homotopic to each other relative to $\{0, 1\}$, such that their self intersections consist of transverse double points, are related by a sequence of three local moves (ω_1), (ω_2), (ω_3), and an ambient isotopy of S . See Goldman [11] §5 and Figure 2. To prove that μ is well-defined, it is sufficient to verify that $\mu(\gamma) = \mu(\gamma')$ if γ and γ' are related by one of the three moves.

Suppose γ and γ' are related by the move (ω_1). The contribution of the double point in the right picture of the move (ω_1) is zero, since the class of a null-homotopic loop is zero in $\mathbb{Q}\hat{\pi}(S)$. Hence $\mu(\gamma) = \mu(\gamma')$.

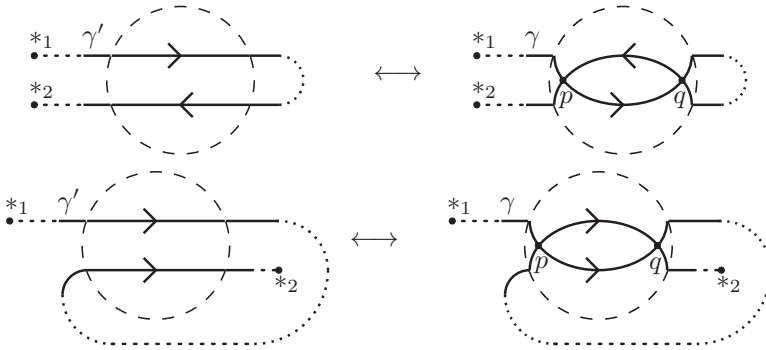


Figure 3. invariance under the move (ω_2)

Suppose γ and γ' are related by the move (ω_2) . We may assume that the left picture corresponds to γ' . Then γ has two more double points than γ' . We write them by p and q so that $t_1^p < t_1^q$. As in Figure 3, there are two possibilities: $t_2^q < t_2^p$ or $t_2^p < t_2^q$, but in any case, $\gamma_{0t_1^p}\gamma_{t_2^p1}$ is homotopic to $\gamma_{0t_1^q}\gamma_{t_2^q1}$ relative to $\{0, 1\}$, $|\gamma_{t_1^p t_2^p}| = |\gamma_{t_1^q t_2^q}|$, and $\varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p)) = -\varepsilon(\dot{\gamma}(t_1^q), \dot{\gamma}(t_2^q))$. Hence the contributions from p and q cancel and $\mu(\gamma) = \mu(\gamma')$.

Suppose γ and γ' are related by the move (ω_3) . Similarly to the case of (ω_2) , we see that a cancellation happens and $\mu(\gamma) = \mu(\gamma')$. Typical cases are illustrated in Figure 4, where the contributions from p (resp. q) and p (resp. q) cancel.

We next show that $\mathbb{Q}\Pi S(0, 1)$ is a right $\mathbb{Q}\hat{\pi}(S)$ -comodule, i.e., $(1_{\mathbb{Q} S(0, 1)}(1-T))(\mu \ 1_{\mathbb{Q}\hat{\pi}(S)}) \mu = (1_{\mathbb{Q} S(0, 1)} \ \delta) \mu$. Let $\gamma: [0, 1] \rightarrow S$ be an immersed path from 0 to 1 such that its self intersections consist of transverse double points. To compute $(1_{\mathbb{Q} S(0, 1)}(1-T))(\mu \ 1_{\mathbb{Q}\hat{\pi}(S)})\mu(\gamma)$, we need to compute $\mu(\gamma_{0t_1^p}\gamma_{t_2^p1})$ where $p \in \Gamma$. The double points of $\gamma_{0t_1^p}\gamma_{t_2^p1}$ come from those of γ . Let q be a double point of $\gamma_{0t_1^p}\gamma_{t_2^p1}$ and denote $\gamma^{-1}(q) = \{t_1^q, t_2^q\}$, so that $t_1^q < t_2^q$. There are three possibilities: (i) $t_1^q < t_2^q < t_1^p < t_2^p$, (ii) $t_1^p < t_2^p < t_1^q < t_2^q$, (iii) $t_1^q < t_1^p < t_2^p < t_2^q$. In cases (i) and (ii), the contribution to $(\mu \ 1_{\mathbb{Q}\hat{\pi}(S)})\mu(\gamma)$ from (p, q) is

$$\begin{aligned} &\varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p))\varepsilon(\dot{\gamma}(t_1^q), \dot{\gamma}(t_2^q)) \left(\gamma_{0t_1^q}\gamma_{t_2^q t_1^p}\gamma_{t_2^p1} \right) \left| \gamma_{t_1^q t_2^q} \right| \left| \gamma_{t_1^p t_2^p} \right| \quad \text{and} \\ &\varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p))\varepsilon(\dot{\gamma}(t_1^q), \dot{\gamma}(t_2^q)) \left(\gamma_{0t_1^p}\gamma_{t_2^p t_1^q}\gamma_{t_2^q1} \right) \left| \gamma_{t_1^q t_2^q} \right| \left| \gamma_{t_1^p t_2^p} \right|, \end{aligned}$$

respectively. Here $\gamma_{t_2^p t_1^p}$ means the restriction of γ to $[t_2^p, t_1^p]$ and $\gamma_{0t_1^q}\gamma_{t_2^q t_1^p}\gamma_{t_2^p1}$ means the conjunction. Therefore the contributions to $(\mu \ 1_{\mathbb{Q}\hat{\pi}(S)})\mu(\gamma)$

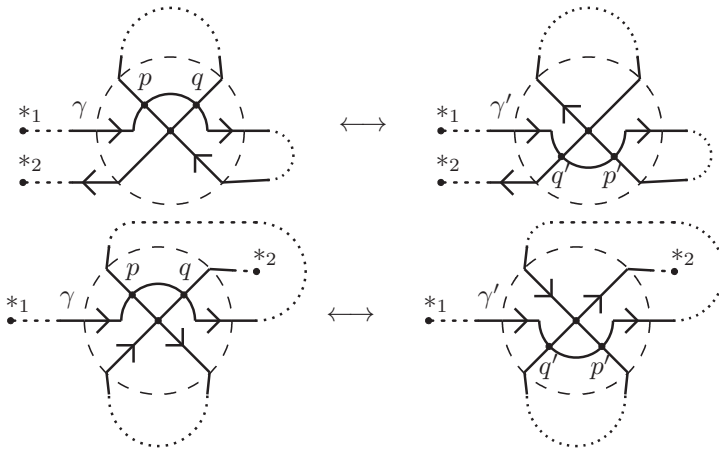


Figure 4. invariance under the move (ω_3)

from (p, q) of type (i) or (ii) are written as a linear combination of tensors of the form $u \ (v \ w + w \ v)$. Since $(1 - T)(v \ w + w \ v) = 0$, these contributions vanish on $\mathbb{Q}\Pi S(0, 1) \ \mathbb{Q}\hat{\pi}(S) \ \mathbb{Q}\hat{\pi}(S)$. Hence we only need to consider the contributions from (iii), and

$$(3.2.4) \quad (1_{\mathbb{Q} S(0, 1)} (1 - T))(\mu \ 1_{\mathbb{Q}\hat{\pi}(S)})\mu(\gamma) = \sum_{\substack{p, q \\ t_1^q < t_1^p < t_2^q < t_2^p}} \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p))\varepsilon(\dot{\gamma}(t_1^q), \dot{\gamma}(t_2^q))z(\gamma, p, q),$$

where

$$z(\gamma, p, q) = (\gamma_{0t_1^q} \gamma_{t_2^q 1}) \left(|h_{t_1^q t_1^p} \gamma_{t_2^p t_2^q}| \ |h_{t_1^p t_2^p}| - |h_{t_1^p t_2^p}| \ |h_{t_1^q t_1^p} \gamma_{t_2^p t_2^q}| \right).$$

On the other hand, to compute $(1_{\mathbb{Q} S(0, 1)} \ \delta)\mu(\gamma)$ we need to compute $\delta(|h_{t_1^p t_2^p}|)$ where $p \in \Gamma$. Each double point of the loop $\gamma_{t_1^p t_2^p}$ comes from $q \in \Gamma$ such that $t_1^p < t_1^q < t_2^q < t_2^p$. Thus $\delta(|h_{t_1^p t_2^p}|)$ is equal to

$$\sum_{q: t_1^p < t_1^q < t_2^q < t_2^p} \varepsilon(\dot{\gamma}(t_1^q), \dot{\gamma}(t_2^q)) \left(|h_{t_1^q t_2^q}| \ |h_{t_2^q t_2^p} \gamma_{t_1^p t_1^q}| - |h_{t_2^q t_2^p} \gamma_{t_1^p t_1^q}| \ |h_{t_1^q t_2^q}| \right),$$

and

$$(3.2.5) \quad (1_{\mathbb{Q} S(0, 1)} \ \delta)\mu(\gamma) = - \sum_{\substack{p, q \\ t_1^p < t_1^q < t_2^q < t_2^p}} \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p))\varepsilon(\dot{\gamma}(t_1^q), \dot{\gamma}(t_2^q))w(\gamma, p, q),$$

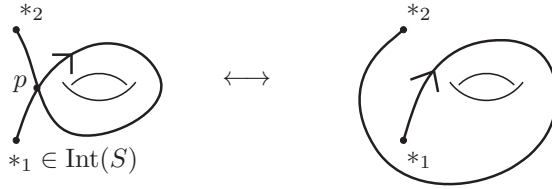


Figure 5. the formula does not work if a base point lies in $\text{Int}(S)$

where

$$w(\gamma, p, q) = (\gamma_{0t_1^p} \gamma_{t_2^p 1}) \left(|\gamma_{t_1^q t_2^q}| \quad |\gamma_{t_2^q t_2^p} \gamma_{t_1^p t_1^q}| - |\gamma_{t_2^q t_2^p} \gamma_{t_1^p t_1^q}| \quad \gamma_{t_1^q t_2^q} \right).$$

Since $z(\gamma, p, q) = -w(\gamma, q, p)$, the right hand sides of (3.2.4) and (3.2.5) are equal. This completes the proof. \square

Remark 3.2.6. — We have taken γ_0 and γ_1 from ∂S . If at least one of γ_0 and γ_1 lies in $\text{Int}(S)$, we need to consider another kind of local move illustrated in Figure 5. In this case the formula (3.2.2) does not work. For example, in Figure 5, the contribution from p in the left picture is non-trivial.

We show that σ and μ satisfy the compatibility (2.2.2) and the involutivity (2.2.3).

PROPOSITION 3.2.7. — *The \mathbb{Q} -vector space $\mathbb{Q}\text{IIS}(\gamma_0, \gamma_1)$ is an involutive right $\mathbb{Q}\hat{\pi}(S)$ -bimodule with respect to $\sigma: \mathbb{Q}\hat{\pi}(S) \rightarrow \mathbb{Q}\text{IIS}(\gamma_0, \gamma_1)$ and $\mu: \mathbb{Q}\text{IIS}(\gamma_0, \gamma_1) \rightarrow \mathbb{Q}\text{IIS}(\gamma_0, \gamma_1) \rightarrow \mathbb{Q}\hat{\pi}(S)$.*

Proof. — We first prove the involutivity. Let $\gamma: [0, 1] \rightarrow S$ be an immersed path from γ_0 to γ_1 such that its self intersections consist of transverse double points. Then

$$\bar{\sigma}\mu(\gamma) = \sum_p \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p)) \sigma \left(|\gamma_{t_1^p t_2^p}| \quad (\gamma_{0t_1^p} \gamma_{t_2^p 1}) \right).$$

Let q be an intersection of the loop $\gamma_{t_1^p t_2^p}$ and the path $\gamma_{0t_1^p} \gamma_{t_2^p 1}$ and we denote $\gamma^{-1}(q) = \{t_1^q, t_2^q\}$ so that $t_1^q < t_2^q$. There are two possibilities: (i) $t_1^q < t_1^p < t_2^q < t_2^p$, (ii) $t_1^p < t_1^q < t_2^p < t_2^q$. The contribution to $\bar{\sigma}\mu(\gamma)$ from (p, q) are

$$(3.2.8) \quad \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p)) \varepsilon(\dot{\gamma}(t_2^q), \dot{\gamma}(t_1^q)) \gamma_{0t_1^p} \gamma_{t_2^q t_2^p} \gamma_{t_1^p t_2^q} \gamma_{t_1^q t_1^p} \gamma_{t_2^p 1}$$

in case (i), and

$$(3.2.9) \quad \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p)) \varepsilon(\dot{\gamma}(t_1^q), \dot{\gamma}(t_2^q)) \gamma_{0t_1^p} \gamma_{t_2^q t_2^p} \gamma_{t_1^q t_2^q} \gamma_{t_1^p t_1^q} \gamma_{t_2^p 1}$$

in case (ii). If we interchange p and q in (3.2.9), we get the minus of (3.2.8). Therefore the contributions from (p, q) in case (i) and those in case (ii) cancel and $\bar{\sigma}\mu(\gamma) = 0$.

We next show the compatibility. Let α be an immersed loop on S and $\gamma: [0, 1] \rightarrow S$ an immersed path from γ_0 to γ_1 such that their intersections and self intersections consist of transverse double points. The compatibility is equivalent to the following.

$$(3.2.10) \quad \mu(\sigma(\alpha \cap \gamma)) = \sigma(\alpha)\mu(\gamma) - (\bar{\sigma} \cdot 1_{\mathbb{Q}\hat{\pi}(S)})(\gamma \cap \delta(\alpha)).$$

Here, $\sigma(\alpha)\mu(\gamma) = (\sigma \cdot 1_{\mathbb{Q}\hat{\pi}(S)})(\alpha \cap \mu(\gamma)) + (1_{\mathbb{Q}S(\gamma_0, \gamma_1)} \cdot \text{ad}(\alpha))\mu(\gamma)$. We compute the left hand side of (3.2.10). First of all, we have

$$\mu(\sigma(\alpha \cap \gamma)) = \sum_{p \in \alpha \cap \gamma} \varepsilon(p; \alpha, \gamma)\mu(\gamma_{\sigma_p \alpha_p \gamma_p \cdot 1}).$$

Let q be a double point of $\gamma_{\sigma_p \alpha_p \gamma_p \cdot 1}$. There are three possibilities: (i) q comes from a double point of α , (ii) q comes from a double point of γ , (iii) q comes from an intersection of α and γ , which is different from p .

Suppose q comes from a double point of α . We denote $\alpha^{-1}(q) = \{t_1^q, t_2^q\} \subset S^1$, so that $t_2^q, \alpha^{-1}(p), t_1^q$ are arranged in this order according to the orientation of S^1 . (Since p is a simple point of α , the preimage $\alpha^{-1}(p)$ consists of one point. For simplicity, we write $\alpha^{-1}(p)$ for the unique point in the preimage.) The contribution to $\mu(\gamma_{\sigma_p \alpha_p \gamma_p \cdot 1})$ from such q is

$$-\varepsilon(\dot{\alpha}(t_1^q), \dot{\alpha}(t_2^q))\gamma_{\sigma_p \alpha_p q \alpha_{qp} \gamma_p \cdot 1} \cdot |\alpha_{t_1^q t_2^q}|.$$

Here, α_{pq} (resp. α_{qp}) means the restriction of α to the interval $[\alpha^{-1}(p), t_1^q]$ (resp. $[t_2^q, \alpha^{-1}(p)]$). Thus the contributions to $\mu(\sigma(\alpha \cap \gamma))$ from (p, q) such that q is of type (i) is

$$(3.2.11) \quad - \sum_p \sum_{\gamma(t_1^q, t_2^q)} \varepsilon(p; \alpha, \gamma)\varepsilon(\dot{\alpha}(t_1^q), \dot{\alpha}(t_2^q))\gamma_{\sigma_p \alpha_p q \alpha_{qp} \gamma_p \cdot 1} \cdot |\alpha_{t_1^q t_2^q}|,$$

where the second sum is taken over ordered pairs (t_1^q, t_2^q) such that $\alpha(t_1^q) = \alpha(t_2^q)$ and $\alpha^{-1}(p) \in [t_2^q, t_1^q]$. On the other hand, we have

$$\delta(\alpha) = \sum_{(t_1^q, t_2^q)} \varepsilon(\dot{\alpha}(t_2^q), \dot{\alpha}(t_1^q))|\alpha_{t_2^q t_1^q}| \cdot |\alpha_{t_1^q t_2^q}|,$$

where the sum is taken over ordered pairs (t_1^q, t_2^q) such that $\alpha(t_1^q) = \alpha(t_2^q)$, $t_1^q = t_2^q$, and

$$\sigma(|\alpha_{t_2^q t_1^q}| \cap \gamma) = \sum_p \varepsilon(p; \alpha, \gamma)\gamma_{\sigma_p \alpha_p q \alpha_{qp} \gamma_p \cdot 1},$$

where the sum is taken over $p \in \alpha \cap \gamma$ such that $\alpha^{-1}(p) \in [t_2^q, t_1^q]$. Therefore, (3.2.11) is equal to $-(\bar{\sigma} \cdot 1_{\mathbb{Q}\hat{\pi}(S)})(\gamma \cap \delta(\alpha))$.

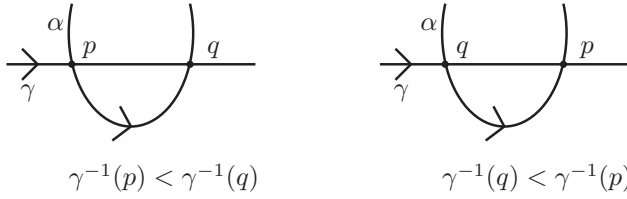


Figure 6. the type (iii)

Suppose q comes from a double point of γ . We denote $\gamma^{-1}(q) = \{s_1^q, s_2^q\}$, so that $s_1^q < s_2^q$. There are three possibilities: (ii-a) $\gamma^{-1}(p) < s_1^q < s_2^q$, (ii-b) $s_1^q < \gamma^{-1}(p) < s_2^q$, (ii-c) $s_1^q < s_2^q < \gamma^{-1}(p)$. The contributions to $\mu(\sigma(\alpha \ \gamma))$ from (p, q) of type (ii-a) are

$$\sum_p \sum_{\alpha \ \gamma} \sum_q \varepsilon(p; \alpha, \gamma) \varepsilon(\dot{\gamma}(s_1^q), \dot{\gamma}(s_2^q)) \gamma_{op} \alpha_p \gamma_{pq} \gamma_{q \ 1} \ \gamma_{s_1^q s_2^q},$$

and those from (p, q) of type (ii-c) are

$$\sum_p \sum_{\alpha \ \gamma} \sum_q \varepsilon(p; \alpha, \gamma) \varepsilon(\dot{\gamma}(s_1^q), \dot{\gamma}(s_2^q)) \gamma_{oq} \gamma_{qp} \alpha_p \gamma_{p \ 1} \ \gamma_{s_1^q s_2^q}.$$

The sum of these two is equal to $(\sigma \ 1_{\mathbb{Q}\hat{\pi}}(s))(\alpha \ \mu(\gamma))$. The contributions to $\mu(\sigma(\alpha \ \gamma))$ from (p, q) of type (ii-b) are

$$\sum_p \sum_{\alpha \ \gamma} \sum_q \varepsilon(p; \alpha, \gamma) \varepsilon(\dot{\gamma}(s_1^q), \dot{\gamma}(s_2^q)) \gamma_{oq} \gamma_{q \ 1} \ \gamma_{qp} \alpha_p \gamma_{pq}.$$

This is equal to $(1_{\mathbb{Q} \ S(0, 1)} \ \text{ad}(\alpha))\mu(\gamma)$. Therefore, the contributions from (p, q) such that q is of type (ii) is $(\sigma \ 1_{\mathbb{Q}\hat{\pi}}(s))(\alpha \ \mu(\gamma)) + (1_{\mathbb{Q} \ S(0, 1)} \ \text{ad}(\alpha))\mu(\gamma) = \sigma(\alpha)\mu(\gamma)$.

Suppose q comes from an intersection of α and γ , which is different from p . If $\gamma^{-1}(p) < \gamma^{-1}(q)$, the contribution is

$$(3.2.12) \quad - \varepsilon(p; \alpha, \gamma) \varepsilon(q; \alpha, \gamma) \gamma_{op} \alpha_{pq} \gamma_{q \ 1} \ \alpha_{qp} \gamma_{pq},$$

and if $\gamma^{-1}(q) < \gamma^{-1}(p)$, the contribution is

$$(3.2.13) \quad - \varepsilon(p; \alpha, \gamma) \varepsilon(q; \gamma, \alpha) \gamma_{oq} \alpha_{qp} \gamma_{p \ 1} \ \gamma_{qp} \alpha_{pq}.$$

See Figure 6. If we interchange p and q in (3.2.13), we get the minus of (3.2.12). Therefore the sum of the contributions from (p, q) of type (iii) is zero. We have established the formula (3.2.10). \square

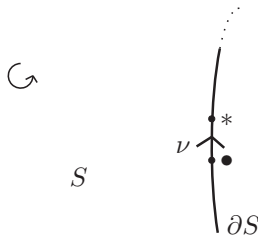


Figure 7. ν , \bullet , and ν

3.3. The case $\sigma_0 = 1$

Fix ∂S . In this subsection we give a structure of an involutive right $\mathbb{Q}\hat{\pi}(S)$ -bimodule on the \mathbb{Q} -vector space $\mathbb{Q}\pi_1(S, \bullet) = \mathbb{Q}\text{IIS}(\bullet, \bullet)$.

Definition of σ . The \mathbb{Q} -linear map $\sigma: \mathbb{Q}\hat{\pi}(S) \rightarrow \mathbb{Q}\pi_1(S, \bullet) \rightarrow \mathbb{Q}\pi_1(S, \bullet)$ is defined by setting $\sigma_0 = \sigma_1 = 1$ and applying the formula (3.2.1).

Definition of μ . We regard that the orientation of ∂S is induced from that of S . Pick an orientation preserving embedding $\nu: [0, 1] \rightarrow \partial S$ such that $\nu(1) = \bullet$, and set $\nu(0) = \ast$. See Figure 7. Notice that we have an isomorphism $\nu: \mathbb{Q}\pi_1(S, \bullet) = \mathbb{Q}\text{IIS}(\bullet, \bullet) = \mathbb{Q}\text{IIS}(\ast, \bullet), u \mapsto \nu u$. We define $\mu: \mathbb{Q}\pi_1(S, \bullet) \rightarrow \mathbb{Q}\pi_1(S, \bullet) \rightarrow \mathbb{Q}\hat{\pi}(S)$ to be $(\nu^{-1})_{\ast} \circ 1_{\mathbb{Q}\hat{\pi}(S)} \circ \mu \circ \nu$. Namely, if $\sigma_0 = \sigma_1 = 1$ we define μ so that the following diagram commutes:

$$(3.3.1) \quad \begin{array}{ccc} \mathbb{Q}\pi_1(S, \bullet) & \xrightarrow{\mu} & \mathbb{Q}\pi_1(S, \bullet) \rightarrow \mathbb{Q}\hat{\pi}(S) \\ \nu \downarrow & & \downarrow \nu^{-1}_{\mathbb{Q}\hat{\pi}(S)} \\ \mathbb{Q}\text{IIS}(\bullet, \bullet) & \xrightarrow{\mu} & \mathbb{Q}\text{IIS}(\ast, \bullet) \rightarrow \mathbb{Q}\hat{\pi}(S). \end{array}$$

Let $\gamma: [0, 1] \rightarrow S$ be an immersed path with $\gamma(0) = \gamma(1) = \bullet$ such that its self intersections consist of transverse double points and the velocity vectors $\dot{\gamma}(0)$ and $\dot{\gamma}(1)$ are linearly independent. Let $\Gamma \subset \text{Int}(S)$ be the set of double points of γ except \bullet . Then we have

$$(3.3.2) \quad \mu(\gamma) = \begin{cases} - \sum_p \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p)) (\gamma_{0t_1^p} \gamma_{t_2^p 1}) / |\gamma_{t_1^p t_2^p}| & \text{if } \varepsilon(\dot{\gamma}(0), \dot{\gamma}(1)) = +1 \\ 1 / |\gamma| & \\ - \sum_p \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p)) (\gamma_{0t_1^p} \gamma_{t_2^p 1}) / |\gamma_{t_1^p t_2^p}| & \text{if } \varepsilon(\dot{\gamma}(0), \dot{\gamma}(1)) = -1. \end{cases}$$

Remark 3.3.3. — Our construction is inspired by Turaev’s self intersection $\mu = \mu^T: \pi_1(S, \bullet) \rightarrow \mathbb{Z}\pi_1(S, \bullet)$ introduced in [32] §1.4. Actually, for

any $\gamma \in \pi_1(S, \cdot)$, we have

$$(3.3.4) \quad \mu^T(\gamma)\gamma = -(1_{\mathbb{Q}\pi_1(S, \cdot)} - \varepsilon)\mu(\gamma).$$

Here $\varepsilon: \mathbb{Q}\hat{\pi}(S) \rightarrow \mathbb{Q}$ is the \mathbb{Q} -linear map given by $\varepsilon(\alpha) = 1$ for $\alpha \in \hat{\pi}(S)$.

Remark 3.3.5. — To define μ for the case $\partial_0 = \partial_1 = \emptyset$, we have moved the start point of paths slightly along the *negatively* oriented boundary of S . It is also possible to move the start point slightly along the *positively* oriented boundary of S , and to define another operation which is similar to but different from what we have defined. If we denote by μ_- the former operation and μ_+ the latter, we have

$$\mu_-(\gamma) - \mu_+(\gamma) = -1 \cdot |\gamma|$$

for any $\gamma \in \mathbb{Q}\pi_1(S, \cdot)$. Our choice of convention matches with that of the homotopy intersection form in [22]. See also §6.3.

PROPOSITION 3.3.6. — *If $\partial_0 = \partial_1 = \emptyset$, the pair (σ, μ) defined above gives an involutive right $\mathbb{Q}\hat{\pi}(S)$ -bimodule structure on the \mathbb{Q} -vector space $\mathbb{Q}\pi_1(S, \cdot)$.*

Proof. — By the commutativity of the diagram (3.3.1) and Proposition 3.2.3, it follows that μ defines a right $\mathbb{Q}\hat{\pi}(S)$ -comodule structure on $\mathbb{Q}\pi_1(S, \cdot)$. Since ν is compatible with σ , i.e., the action of $\mathbb{Q}\hat{\pi}(S)$, Proposition 3.2.7 implies the compatibility and the involutivity of (σ, μ) for $\partial_0 = \partial_1 = \emptyset$. □

4. Completion of the Turaev cobracket

The \mathbb{Q} -vector space $\mathbb{Q}\hat{\pi}(S)$ has a natural decreasing filtration and we can consider the completion $\widehat{\mathbb{Q}\hat{\pi}(S)}$. As is shown in [15] §4 and is recalled in §4.1, the Goldman bracket and the operation σ extend to completions. In this section we show that μ is compatible with the filtrations of $\mathbb{Q}\Pi S(\partial_0, \partial_1)$ and $\mathbb{Q}\hat{\pi}(S)$, and also show that the Turaev cobracket extends to a complete Lie cobracket $\delta: \widehat{\mathbb{Q}\hat{\pi}(S)} \rightarrow \widehat{\mathbb{Q}\hat{\pi}(S)} \hat{\wedge} \widehat{\mathbb{Q}\hat{\pi}(S)}$.

4.1. Completion of the Goldman Lie algebra

We make a few remarks on filtered vector spaces. Let $V = F_0V \supset F_1V \supset \dots$ be a filtered \mathbb{Q} -vector space. The projective limit $\widehat{V} := \varprojlim_n V/F_nV$ is

again a filtered \mathbb{Q} -vector space with the filter $F_n \widehat{V} := \text{Ker}(\widehat{V} \rightarrow V/F_n V)$. We say V is complete if the natural map $V \rightarrow \widehat{V}$ is isomorphic. If V and W are filtered \mathbb{Q} -vector spaces, the tensor product $V \otimes W$ is naturally filtered by $F_n(V \otimes W) = \sum_{p+q=n} F_p V \otimes F_q W$. The complete tensor product $\widehat{V \otimes W}$ is defined as $\widehat{V \otimes W} := \widehat{V \otimes W} = \varinjlim_n (V \otimes W)/F_n(V \otimes W)$. Note that we have a natural isomorphism $\widehat{V} \otimes \widehat{W} = \widehat{V \otimes W}$.

DEFINITION 4.1.1. — *A complete Lie algebra is a pair (V, \cdot) , where V is a complete filtered \mathbb{Q} -vector space and $\cdot : V \widehat{\otimes} V \rightarrow V$ is a \mathbb{Q} -linear map continuous with respect to the topologies coming from the filtrations, and satisfies the skew condition $T = - : V \widehat{\otimes} V \rightarrow V$ and the Jacobi identity $(\widehat{\cdot} 1)N = 0 : V \widehat{\otimes} V \widehat{\otimes} V \rightarrow V$. Here, $T : V \widehat{\otimes} V \rightarrow V \widehat{\otimes} V$ is that induced from the switch map $T : V \otimes V \rightarrow V \otimes V$, etc. We call \cdot a complete Lie bracket. Similarly, we define a complete Lie coalgebra, bialgebra, and a complete V -module, comodule, and bimodule.*

Let S be a connected oriented surface. Take some base point $x \in S$ and set

$$\mathbb{Q}\hat{\pi}(S)(n) := \mathbb{Q}[1 + (I\pi_1(S, x))^n] / \mathbb{Q}\hat{\pi}(S), \text{ for } n \geq 0.$$

Here, $I\pi_1(S, x) := \text{Ker}(\mathbb{Q}\pi_1(S, x) \rightarrow \mathbb{Q}, \pi \mapsto x \cdot 1)$ is the augmentation ideal of the group ring $\mathbb{Q}\pi_1(S, x)$. We regard $(I\pi_1(S, x))^0 = \mathbb{Q}\pi_1(S, x)$. The space $\mathbb{Q}\hat{\pi}(S)(n)$ is independent of the choice of x . Moreover, the Goldman bracket satisfies

$$(4.1.2) \quad [\mathbb{Q}\hat{\pi}(S)(n_1), \mathbb{Q}\hat{\pi}(S)(n_2)] \subset \mathbb{Q}\hat{\pi}(S)(n_1 + n_2 - 2), \text{ for } n_1, n_2 \geq 1$$

(see [15, §4.1]). This implies that the Goldman bracket induces a complete Lie bracket on the projective limit

$$\widehat{\mathbb{Q}\hat{\pi}(S)} := \varprojlim_n \mathbb{Q}\hat{\pi}(S) / \mathbb{Q}\hat{\pi}(S)(n).$$

We call $\widehat{\mathbb{Q}\hat{\pi}(S)}$ the *completed Goldman Lie algebra* of S . We denote

$$\widehat{\mathbb{Q}\hat{\pi}(S)}(n) := F_n \widehat{\mathbb{Q}\hat{\pi}(S)} = \text{Ker}(\widehat{\mathbb{Q}\hat{\pi}(S)} \rightarrow \mathbb{Q}\hat{\pi}(S) / \mathbb{Q}\hat{\pi}(S)(n)), \text{ } n \geq 0.$$

For $n \geq 0$, let

$$\mathbb{Q}\hat{\pi}(S)(n) := \varpi(\mathbb{Q}\hat{\pi}(S)(n)) = \mathbb{Q}[1 + (I\pi_1(S, x))^n] / \mathbb{Q}\hat{\pi}(S).$$

Since $1/0 = 0$, $\mathbb{Q}\hat{\pi}(S)(0) = \mathbb{Q}\hat{\pi}(S)(1) = \mathbb{Q}\hat{\pi}(S)$. The natural map $\mathbb{Q}\hat{\pi}(S) / \mathbb{Q}\hat{\pi}(S)(n) \rightarrow \mathbb{Q}\hat{\pi}(S) / \mathbb{Q}\hat{\pi}(S)(n)$ is a \mathbb{Q} -linear isomorphism for any n . Hence $\widehat{\mathbb{Q}\hat{\pi}(S)}$ is also written as

$$(4.1.3) \quad \widehat{\mathbb{Q}\hat{\pi}(S)} = \varprojlim_n \mathbb{Q}\hat{\pi}(S) / \mathbb{Q}\hat{\pi}(S)(n).$$

Let $\gamma : [0, 1] \rightarrow \partial S$. We make $\mathbb{Q}\Pi S(\gamma)$ filtered by taking some path $\gamma : [0, 1] \rightarrow \Pi S(\gamma)$ and setting

$$(4.1.4) \quad F_n \mathbb{Q}\Pi S(\gamma) := \gamma(I\pi_1(S, \gamma))^n, \quad \text{for } n \geq 0.$$

Then this is independent of the choice of γ (see [15, §2.1]). In particular we can consider the completion $\widehat{\mathbb{Q}\Pi S}(\gamma)$. By [15, §4.1], we see that σ induces a \mathbb{Q} -linear map $\sigma : \widehat{\mathbb{Q}\hat{\pi}(S)} \rightarrow \widehat{\mathbb{Q}\Pi S}(\gamma)$, and the complete vector space $\widehat{\mathbb{Q}\Pi S}(\gamma)$ has a structure of a complete right $\widehat{\mathbb{Q}\hat{\pi}(S)}$ -module. As a special case, the completed group ring $\widehat{\mathbb{Q}\pi_1(S, \gamma)} := \varprojlim_n \mathbb{Q}\pi_1(S, \gamma)/(I\pi_1(S, \gamma))^n$, where $\gamma : [0, 1] \rightarrow \partial S$, has a structure of a complete right $\widehat{\mathbb{Q}\hat{\pi}(S)}$ -module.

4.2. Intersection of paths

Take points $p_1, p_2, p_3, p_4 \in \partial S$. We introduce a \mathbb{Q} -linear map

$$\kappa : \mathbb{Q}\Pi S(p_1, p_2) \otimes \mathbb{Q}\Pi S(p_3, p_4) \rightarrow \mathbb{Q}\Pi S(p_1, p_4) \otimes \mathbb{Q}\Pi S(p_3, p_2)$$

using the intersections of two based paths.

The generic case. First we consider the case $\{p_1, p_2\} \cap \{p_3, p_4\} = \emptyset$. Let $x, y : [0, 1] \rightarrow S$ be immersed paths such that $x(0) = p_1, x(1) = p_2, y(0) = p_3, y(1) = p_4$ and their intersections consist of transverse double points. Set

$$(4.2.1) \quad \kappa(x, y) := - \sum_{p \in x \cap y} \varepsilon(p; x, y) (x_{1p} y_{p4}) \otimes (y_{3p} x_{p2}) \in \mathbb{Q}\Pi S(p_1, p_4) \otimes \mathbb{Q}\Pi S(p_3, p_2).$$

By an argument similar to the proof of Proposition 3.2.3, we see that (4.2.1) gives rise to a well-defined \mathbb{Q} -linear map $\kappa : \mathbb{Q}\Pi S(p_1, p_2) \otimes \mathbb{Q}\Pi S(p_3, p_4) \rightarrow \mathbb{Q}\Pi S(p_1, p_4) \otimes \mathbb{Q}\Pi S(p_3, p_2)$.

The degenerate case. Next we consider the case $\{p_1, p_2\} \cap \{p_3, p_4\} \neq \emptyset$. The idea is, as was the case of μ , to move the points p_1, p_2 slightly along the negatively oriented boundary of S to achieve $\{p_1, p_2\} \cap \{p_3, p_4\} = \emptyset$, then to apply the formula (4.2.1). Using two examples we explain this procedure more precisely.

For the first example, suppose p_1, p_2, p_4 are distinct and $p_3 = p_2$. Set $p_3 = p_2$ and let \bullet and ν be as in Figure 7. We assume that the image of ν is so small that it does not contain p_1 and p_4 . Notice that we have isomorphisms $\nu : \mathbb{Q}\Pi S(p_1, p_2) \rightarrow \mathbb{Q}\Pi S(p_1, \bullet), x \mapsto x\nu$, and $\nu : \mathbb{Q}\pi_1(S, p_2) = \mathbb{Q}\Pi S(p_2, p_2) \rightarrow \mathbb{Q}\Pi S(p_2, \bullet), x \mapsto x\nu$. Here and throughout this paper $\bar{\nu} : \Pi S(p_2, \bullet) \rightarrow \Pi S(p_2, p_2)$ is the



Figure 8. $\varepsilon(\dot{x}(\delta_1), \dot{y}(\delta_2)) = (-1)^{\delta_1 + \delta_2 + 1}$

inverse of ν . We define $\kappa: \mathbb{Q}\Pi(\cdot, \cdot) \rightarrow \mathbb{Q}\Pi(\cdot, \cdot) \rightarrow \mathbb{Q}\Pi(\cdot, \cdot) \rightarrow \mathbb{Q}\Pi_1(S, \cdot)$ so that the following diagram commutes:

$$\begin{CD} \mathbb{Q}\Pi S(\cdot, \cdot) @>\kappa>> \mathbb{Q}\Pi S(\cdot, \cdot) @>\kappa>> \mathbb{Q}\Pi S(\cdot, \cdot) @>\kappa>> \mathbb{Q}\Pi_1(S, \cdot) \\ @V\nu \downarrow V_{\mathbb{Q}(S, \cdot)}VV @VV\nu \downarrow V_{\mathbb{Q}(S, \cdot)}V @VV\nu \downarrow V_{\mathbb{Q}(S, \cdot)}V @VV\nu \downarrow V_{\mathbb{Q}(S, \cdot)}V \\ \mathbb{Q}\Pi S(\cdot, \bullet) @>\kappa>> \mathbb{Q}\Pi S(\cdot, \bullet) @>\kappa>> \mathbb{Q}\Pi S(\cdot, \bullet) @>\kappa>> \mathbb{Q}\Pi S(\cdot, \bullet) \end{CD}$$

For the second example, suppose $\nu_1 = \nu_2 = \nu_3 = \nu_4$. (This is the most extreme case.) Set $\nu = \nu_1$ and let \bullet and ν be as in Figure 7. Notice that we have three isomorphisms $\mathbb{Q}\Pi_1(S, \cdot) = \mathbb{Q}\Pi_1(S, \bullet), x \rightarrow \nu x \bar{\nu}, \mathbb{Q}\Pi_1(S, \cdot) = \mathbb{Q}\Pi S(\bullet, \cdot), x \rightarrow \nu x$, and $\mathbb{Q}\Pi_1(S, \cdot) = \mathbb{Q}\Pi S(\cdot, \bullet), x \rightarrow x \bar{\nu}$, for all of which we use the letter ν . We define $\kappa: \mathbb{Q}\Pi_1(S, \cdot) \rightarrow \mathbb{Q}\Pi_1(S, \cdot) \rightarrow \mathbb{Q}\Pi_1(S, \cdot) \rightarrow \mathbb{Q}\Pi_1(S, \cdot)$ so that the following diagram commutes:

$$\begin{CD} \mathbb{Q}\Pi_1(S, \cdot) @>\kappa>> \mathbb{Q}\Pi_1(S, \cdot) @>\kappa>> \mathbb{Q}\Pi_1(S, \cdot) @>\kappa>> \mathbb{Q}\Pi_1(S, \cdot) \\ @V\nu \downarrow V_{\mathbb{Q}\Pi_1(S, \cdot)}VV @VV\nu \downarrow V_{\mathbb{Q}\Pi_1(S, \cdot)}V @VV\nu \downarrow V_{\mathbb{Q}\Pi_1(S, \cdot)}V @VV\nu \downarrow V_{\mathbb{Q}\Pi_1(S, \cdot)}V \\ \mathbb{Q}\Pi_1(S, \bullet) @>\kappa>> \mathbb{Q}\Pi_1(S, \bullet) @>\kappa>> \mathbb{Q}\Pi S(\bullet, \cdot) @>\kappa>> \mathbb{Q}\Pi S(\cdot, \bullet) \end{CD}$$

An explicit formula for κ is given as follows. Let $x, y: [0, 1] \rightarrow S$ be immersed paths with $x(0) = \nu_1, x(1) = \nu_2, y(0) = \nu_3, y(1) = \nu_4$, such that their intersections in the interior of S consist of transverse double points. Furthermore, if $x(\delta_1) = y(\delta_2)$ for some $\delta_1, \delta_2 \in \{0, 1\}$, we assume that $\varepsilon(\dot{x}(\delta_1), \dot{y}(\delta_2)) = (-1)^{\delta_1 + \delta_2 + 1}$. See Figure 8. Note that this condition is always satisfied by a suitable homotopy. Then we have

$$(4.2.2) \quad \kappa(x, y) = - \sum_{p \in x \cap y \setminus \partial S} \varepsilon(p; x, y)(x_{-1p} y_p \nu_4) \nu_3 (y_{-3p} x_p \nu_2).$$

Remark 4.2.3. — To define κ , we have taken ν_i from ∂S . By the same reason as is explained in Remark 3.2.6, the formula (4.2.1) does not work if at least one of ν_i lies in $\text{Int}(S)$.

Remark 4.2.4. — The map κ is inspired by Turaev’s intersection $\lambda: \mathbb{Z}\Pi_1(S, \cdot) \rightarrow \mathbb{Z}\Pi_1(S, \cdot) \rightarrow \mathbb{Z}\Pi_1(S, \cdot)$ introduced in [32, §1.4]. Fix $\nu \in \partial S$

and consider the \mathbb{Q} -linear map $\kappa: \mathbb{Q}\pi_1(S, \) \rightarrow \mathbb{Q}\pi_1(S, \) \rightarrow \mathbb{Q}\pi_1(S, \)$
 $\mathbb{Q}\pi_1(S, \)$ introduced above. Then for any $x, y \in \pi_1(S, \)$, we have

$$(4.2.5) \quad \lambda(x, y) = -(1_{\mathbb{Q}\pi_1(S, \)} \circ \varepsilon)\kappa(x, y^{-1}).$$

Here, ε is the augmentation map of the group ring $\mathbb{Q}\pi_1(S, \)$.

4.3. Product formulas

In this subsection we show product formulas for κ and μ .

LEMMA 4.3.1. — *Let $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \partial S$.*

(1) *For any $x \in \mathbb{Q}\Pi S(\gamma_1, \gamma_2)$, $y \in \mathbb{Q}\Pi S(\gamma_2, \gamma_2)$, and $z \in \mathbb{Q}\Pi S(\gamma_3, \gamma_4)$, we have*

$$\kappa(xy, z) = \kappa(x, z)(1 - y) + (x - 1)\kappa(y, z) \in \mathbb{Q}\Pi S(\gamma_1, \gamma_4) \oplus \mathbb{Q}\Pi S(\gamma_3, \gamma_2).$$

(2) *For any $x \in \mathbb{Q}\Pi S(\gamma_1, \gamma_2)$, $y \in \mathbb{Q}\Pi S(\gamma_3, \gamma_4)$, and $z \in \mathbb{Q}\Pi S(\gamma_4, \gamma_4)$, we have*

$$\kappa(x, yz) = \kappa(x, y)(z - 1) + (1 - y)\kappa(x, z) \in \mathbb{Q}\Pi S(\gamma_1, \gamma_4) \oplus \mathbb{Q}\Pi S(\gamma_3, \gamma_2).$$

Here, $\kappa(x, z)(1 - y)$ means the image of $\kappa(x, z) - y$ by the map $\mathbb{Q}\Pi S(\gamma_1, \gamma_4) \oplus \mathbb{Q}\Pi S(\gamma_3, \gamma_2) \rightarrow \mathbb{Q}\Pi S(\gamma_1, \gamma_4) \oplus \mathbb{Q}\Pi S(\gamma_3, \gamma_2)$, $u - v = w$, etc.

Proof. — We only prove the formula $\kappa(xy, z) = \kappa(x, z)(1 - y) + (x - 1)\kappa(y, z)$. The other formula is proved similarly. Let $x, y, z: [0, 1] \rightarrow S$ be immersed paths with $x(0) = \gamma_1, x(1) = \gamma_2, y(0) = \gamma_2, y(1) = \gamma_2, z(0) = \gamma_3, z(1) = \gamma_4$, such that their intersections in the interior of S consist of transverse double points. Moreover, we assume that $\varepsilon(\dot{x}(\delta_1), \dot{z}(\delta_2)) = (-1)^{\delta_1 + \delta_2 + 1}$ (resp. $\varepsilon(\dot{y}(\delta_1), \dot{z}(\delta_2)) = (-1)^{\delta_1 + \delta_2 + 1}$) for any $\delta_1, \delta_2 \in \{0, 1\}$ with $x(\delta_1) = z(\delta_2)$ (resp. $y(\delta_1) = z(\delta_2)$).

Applying the formula (4.2.2), we compute

$$\begin{aligned} \kappa(xy, z) &= - \sum_{p \in (xy) \cap z \cap \partial S} \varepsilon(p; xy, z)(xy)_{1p} z_{p4} - z_{3p}(xy)_{p3} \\ &= - \sum_{p \in x \cap z \cap \partial S} \varepsilon(p; x, z)(x)_{1p} z_{p4} - (z)_{3p} x_{p2} y \\ &\quad - \sum_{p \in y \cap z \cap \partial S} \varepsilon(p; y, z)(xy)_{2p} z_{p4} - (z)_{3p} y_{p3}. \end{aligned}$$

The first and the second terms are equal to $\kappa(x, z)(1 - y)$ and $(x - 1)\kappa(y, z)$, respectively. □

COROLLARY 4.3.2. — *Let $n \geq 2$ and let $x_1, \dots, x_{n+2} \in \partial S$ be $n + 2$ points. For any $x_1, \dots, x_{n+1}, x_i \in \text{QHS}(x_i, x_{i+1})$, we have*

$$\begin{aligned} \kappa(x_1 \cdots x_n, x_{n+1}) &= \sum_{i=1}^n ((x_1 \cdots x_{i-1}) \cap (x_{i+1} \cdots x_n)) \kappa(x_i, x_{n+1}) \\ \kappa(x_1, x_2 \cdots x_n, x_{n+1}) &= \sum_{i=2}^{n+1} ((x_2 \cdots x_{i-1}) \cap (x_{i+1} \cdots x_n)) \kappa(x_1, x_i) \end{aligned}$$

Proof. — Note that we have $\kappa(x_i, x_{n+2}) \cap (x_{i+1} \cdots x_n) \cap (x_{n+1}) = \kappa(x_i, x_{n+2}) \cap (x_{i+1} \cdots x_n x_{n+1})$, etc. The assertion follows from Lemma 4.3.1 and induction on n . □

LEMMA 4.3.3. — *Let $x_1, x_2, x_3 \in \partial S$ be three points. For any $x \in \text{QHS}(x_1, x_2)$ and $y \in \text{QHS}(x_2, x_3)$, we have*

$$\mu(xy) = \mu(x)(y \cap x_1) + (x \cap x_1)\mu(y) + (1_{\text{QHS}(x_1, x_3)} \setminus \setminus) \kappa(x, y) \in \text{QHS}(x_1, x_3) \cap \hat{\pi}(S).$$

Here, $\mu(x)(y \cap x_1)$ means the image of $\mu(x) \cap y$ by the map $\text{QHS}(x_1, x_2) \cap \hat{\pi}(S) \rightarrow \text{QHS}(x_2, x_3) \cap \text{QHS}(x_1, x_3) \cap \hat{\pi}(S)$, $u \cap v \cap w = uw \cap v$, etc.

Proof. — Let $x, y: [0, 1] \rightarrow S$ be immersed paths with $x(0) = x_1, x(1) = x_2, y(0) = x_2, y(1) = x_3$, such that their intersections and self intersections in the interior of S consist of transverse double points. Moreover, we assume that $\varepsilon(\dot{x}(\delta_1), \dot{y}(\delta_2)) = (-1)^{\delta_1 + \delta_2 + 1}$ for any $\delta_1, \delta_2 \in \{0, 1\}$ with $x(\delta_1) = y(\delta_2)$, and $\varepsilon(\dot{x}(0), \dot{x}(1)) = +1$ (resp. $\varepsilon(\dot{y}(0), \dot{y}(1)) = +1$) if $x(0) = x(1)$ (resp. $y(0) = y(1)$). Note that it is always possible to achieve this by a suitable homotopy. Let Γ_x and Γ_y be the set of double points of x and y , respectively. Then the set of double points of xy is $\Gamma_x \cup \Gamma_y \cup (x \cap y \setminus \partial S)$. We have

$$\begin{aligned} \mu(xy) &= - \sum_{p \in \Gamma_x} \varepsilon(x(t_1^p), x(t_2^p))(x \cap x_p x_p \cap y) \cap |x_{t_1^p} t_2^p| \\ &\quad - \sum_{p \in \Gamma_y} \varepsilon(y(t_1^p), y(t_2^p))(xy \cap y_p y_p \cap x) \cap |y_{t_1^p} t_2^p| \\ &\quad - \sum_{p \in x \cap y \setminus \partial S} \varepsilon(x(p), y(p))(x \cap y_p y_p \cap x) \cap |x_p \cap y \cap y_p|. \end{aligned}$$

The first and the second terms are equal to $\mu(x)(y \cap x_1)$ and $(x \cap x_1)\mu(y)$, respectively. Since $|x_p \cap y \cap y_p| = |y \cap x_p x_p \cap x|$, the third term is equal to $(1_{\text{QHS}(x_1, x_3)} \setminus \setminus) \kappa(x, y)$. Hence $\mu(xy) = \mu(x)(y \cap x_1) + (x \cap x_1)\mu(y) + (1_{\text{QHS}(x_1, x_3)} \setminus \setminus) \kappa(x, y)$. □

By Corollary 4.3.2, Lemma 4.3.3, and induction on n , we have the following.

COROLLARY 4.3.4. — *Let $n \geq 2$ and let $x_1, \dots, x_{n+1} \in \partial S$ be $n + 1$ points. For any $x_1, \dots, x_n, x_i \in \text{QHS}(x_i, x_{i+1})$, we have*

$$\begin{aligned} \mu(x_1 \cdots x_n) &= \sum_{i=1}^n ((x_1 \cdots x_{i-1}) \cup (x_{i+1} \cdots x_n)) \mu(x_i) \\ &\quad + \sum_{i < j} ((x_1 \cdots x_{i-1}) \cup (x_{j+1} \cdots x_n)) K_{i,j}, \end{aligned}$$

where $K_{i,j} = (1_{\text{QHS}(x_i, x_{j+1})} \cup (x_{i+1} \cdots x_{j-1})) (\kappa(x_i, x_j) (1 - (x_{i+1} \cdots x_{j-1})))$.

4.4. μ and the filtrations of $\text{QHS}(x_0, x_1)$, $\text{QH}(S)$

We assume that the boundary of S is not empty. Take $x_0, x_1 \in \partial S$.

LEMMA 4.4.1. — *The following diagram is commutative:*

$$\begin{CD} \text{QH}(S, x_0, x_1) @>\mu>> \text{QH}(S, x_0, x_1) @>\text{QH}(S) @>>> \text{QH}(S) \\ @V \cup VV @. @VV (1-T) \cup 1_{\text{QH}(S)} V @. \\ \text{QH}(S) @>\delta>> \text{QH}(S) @>\text{QH}(S) @>>> \text{QH}(S). \end{CD}$$

Proof. — Let $\gamma: [0, 1] \rightarrow S$ be an immersed path with $\gamma(0) = x_0, \gamma(1) = x_1$ such that its self intersections consist of transverse double points and $\varepsilon(\dot{\gamma}(0), \dot{\gamma}(1)) = +1$. By (3.3.2),

$$\mu(\gamma) = - \sum_p \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p)) (\gamma_{0t_1^p} \gamma_{t_2^p 1}) / |\gamma_{t_1^p t_2^p}|.$$

Using $|\gamma_{0t_1^p} \gamma_{t_2^p 1}| = |\gamma_{t_2^p 1} \gamma_{0t_1^p}|$, we obtain

$$\begin{aligned} &(1 - T) \cup 1_{\text{QH}(S)} \mu(\gamma) \\ &= \sum_p \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p)) (|\gamma_{t_1^p t_2^p}| / |\gamma_{t_2^p 1} \gamma_{0t_1^p}| - |\gamma_{t_2^p 1} \gamma_{0t_1^p}| / |\gamma_{t_1^p t_2^p}|). \end{aligned}$$

This coincides with $\delta(\gamma)$. □

PROPOSITION 4.4.2. — *Let $x_0, x_1 \in \partial S$. For $n \geq 2$, we have*

$$\begin{aligned} \mu(F_n \text{QHS}(x_0, x_1)) &= \sum_{p+q=n-2} F_p \text{QHS}(x_0, x_1) \text{QH}(S)(q) \\ &\quad \text{QHS}(x_0, x_1) \text{QH}(S). \end{aligned}$$

In particular, for ∂S and $n \geq 2$, we have

$$\mu(I\pi_1(S, \gamma)^n) = \sum_{p+q=n-2} I\pi_1(S, \gamma)^p \cdot \mathbb{Q}\hat{\pi}(S)(q).$$

Proof. — If $\gamma_0 = \gamma_1 = \gamma$, the assertion follows from Corollary 4.3.4 by setting $x_i = \gamma$ and $x_i \in I\pi_1(S, \gamma)$. In the general case, take $\gamma \in \Pi S(\gamma_0, \gamma_1)$ and $x_i \in I\pi_1(S, \gamma_1)$, $1 \leq i \leq n$. Again by Corollary 4.3.4, we have $\mu(\gamma x_1 \cdots x_n) = \sum_{p+q=n-2} F_p \mathbb{Q}\Pi S(\gamma_0, \gamma_1) \cdot \mathbb{Q}\hat{\pi}(S)(q)$. \square

By Lemma 4.4.1 and (4.1.3), we obtain

COROLLARY 4.4.3. — *The Turaev cobracket $\delta: \mathbb{Q}\hat{\pi}(S) \rightarrow \mathbb{Q}\hat{\pi}(S)$ satisfies*

$$\delta(\mathbb{Q}\hat{\pi}(S)(n)) = \sum_{p+q=n-2} \mathbb{Q}\hat{\pi}(S)(p) \cdot \mathbb{Q}\hat{\pi}(S)(q)$$

for any $n \geq 2$. Moreover, δ induces a complete Lie cobracket $\delta: \widehat{\mathbb{Q}\hat{\pi}(S)} \rightarrow \widehat{\mathbb{Q}\hat{\pi}(S)} \wedge \widehat{\mathbb{Q}\hat{\pi}(S)}$.

The \mathbb{Q} -vector space $\widehat{\mathbb{Q}\hat{\pi}(S)}$ has a structure of a involutive complete Lie bialgebra with respect to the complete Lie bracket in §4.1 and the complete Lie cobracket above. We call this *the completed Goldman-Turaev Lie bialgebra*. We define

$$\widehat{\mathbb{Q}\Pi S}(\gamma_0, \gamma_1) := \varinjlim_n \mathbb{Q}\Pi S(\gamma_0, \gamma_1) / F_n \mathbb{Q}\Pi S(\gamma_0, \gamma_1),$$

which is a $\widehat{\mathbb{Q}\hat{\pi}(S)}$ -module by means of σ , as was stated in §4.1. As another consequence of Proposition 4.4.2, μ induces a \mathbb{Q} -linear map

$$(4.4.4) \quad \mu: \widehat{\mathbb{Q}\Pi S}(\gamma_0, \gamma_1) \rightarrow \widehat{\mathbb{Q}\Pi S}(\gamma_0, \gamma_1) \wedge \widehat{\mathbb{Q}\hat{\pi}(S)},$$

which makes $\widehat{\mathbb{Q}\Pi S}(\gamma_0, \gamma_1)$ a complete involutive right $\widehat{\mathbb{Q}\hat{\pi}(S)}$ -bimodule. In §5 we will use this bimodule structure to prove that some generalized Dehn twists are not realized by diffeomorphisms.

5. Application to generalized Dehn twists

In this and the next section we discuss applications of our consideration of the (self) intersections of curves to the study of the mapping class groups. In this section we study *generalized Dehn twists*, which were first introduced in [19] for a once bordered surface and were generalized in [15] for any oriented surface.

5.1. Generalized Dehn twists

Generalized Dehn twists are associated with not necessarily simple loops on a surface, and are defined as elements of a certain enlargement of the mapping class group of the surface. We recall the definition of a generalized Dehn twist following [15, §5]. For another treatment, see [22].

Let S be a compact connected oriented surface with non-empty boundary, or a surface obtained from such a surface by removing finitely many points in the interior. We denote by $\mathcal{M}(S, \partial S)$ the mapping class group of the pair $(S, \partial S)$, i.e., the group of orientation preserving diffeomorphisms of S fixing ∂S pointwise, modulo isotopies relative to ∂S . The group $\mathcal{M}(S, \partial S)$ naturally acts on each $\Pi S(p_0, p_1)$, $p_0, p_1 \in \partial S$.

Let $E \subset \partial S$ be a subset which contains at least one point of any connected component of ∂S . Then we can construct a small additive category $\mathbb{Q}C(S, E)$, whose set of objects is E , and whose set of morphisms from $p_0 \in E$ to $p_1 \in E$ is $\mathbb{Q}\Pi S(p_0, p_1)$. As we mentioned in §4.1, $\mathbb{Q}\Pi S(p_0, p_1)$ is filtered and its completion $\widehat{\mathbb{Q}\Pi S}(p_0, p_1)$ is defined. Let $\widehat{\mathbb{Q}C}(S, E)$ be a small additive category whose set of objects is E , and whose set of morphisms from $p_0 \in E$ to $p_1 \in E$ is $\widehat{\mathbb{Q}\Pi S}(p_0, p_1)$. In [15], $\widehat{\mathbb{Q}C}(S, E)$ is called the completion of $\mathbb{Q}C(S, E)$.

The action of $\mathcal{M}(S, \partial S)$ on $\Pi S(p_0, p_1)$ naturally induces a \mathbb{Q} -linear automorphism of $\mathbb{Q}\Pi S(p_0, p_1)$, as well as a \mathbb{Q} -linear automorphism of $\widehat{\mathbb{Q}\Pi S}(p_0, p_1)$. In this way we obtain a group homomorphism of Dehn-Nielsen type

$$(5.1.1) \quad \widehat{DN}: \mathcal{M}(S, \partial S) \rightarrow \text{Aut}(\widehat{\mathbb{Q}C}(S, E)),$$

where $\text{Aut}(\widehat{\mathbb{Q}C}(S, E))$ is the group of covariant functors from $\widehat{\mathbb{Q}C}(S, E)$ to itself, which act on the set of objects as the identity, and act on each set of morphisms as \mathbb{Q} -linear automorphisms. This group homomorphism is injective ([15] §3.1).

Let $C \subset S \setminus \partial S$ be an unoriented loop. Take $q \in S$ and let $x = \pi_1(S, q)$ be a based loop which is homotopic to C as an unoriented loop. We set

$$L(C) := \left| \frac{1}{2}(\log x)^2 \right| \in \widehat{\mathbb{Q}\hat{\pi}}(S)(2),$$

where

$$\log x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x - 1)^n \in \widehat{\mathbb{Q}\pi_1}(S, q)$$

and $/: \widehat{\mathbb{Q}\pi_1}(S, q) \rightarrow \widehat{\mathbb{Q}\hat{\pi}}(S)$ is the map induced by $/: \mathbb{Q}\pi_1(S, q) \rightarrow \mathbb{Q}\hat{\pi}(S)$. The quantity $L(C)$ is independent of the choice of q and x .

A family of \mathbb{Q} -linear homomorphisms $D = D^{(p_0, p_1)} : \widehat{\mathbb{Q}\Pi S}(p_0, p_1) \rightarrow \widehat{\mathbb{Q}\Pi S}(p_0, p_1)$, $p_0, p_1 \in E$, is called a derivation of $\widehat{\mathbb{Q}C}(S, E)$, if it satisfies the Leibniz rule

$$D(uv) = (Du)v + u(Dv)$$

for any $p_0, p_1, p_2 \in E$, $u \in \widehat{\mathbb{Q}\Pi S}(p_0, p_1)$, and $v \in \widehat{\mathbb{Q}\Pi S}(p_1, p_2)$. The set of derivations of $\widehat{\mathbb{Q}C}(S, E)$ naturally has a structure of a Lie algebra, which we denote by $\text{Der}(\widehat{\mathbb{Q}C}(S, E))$. Then we obtain a Lie algebra homomorphism $\sigma : \widehat{\mathbb{Q}\hat{\pi}}(S) \rightarrow \text{Der}(\widehat{\mathbb{Q}C}(S, E))$, by collecting the structure morphisms $\sigma : \widehat{\mathbb{Q}\hat{\pi}}(S) \rightarrow \widehat{\mathbb{Q}\Pi S}(p_0, p_1) \rightarrow \widehat{\mathbb{Q}\Pi S}(p_0, p_1)$, $p_0, p_1 \in E$ (see [15, §4.1]). For $p_0, p_1 \in E$, the exponential of the derivation $\sigma(L(C)) \in \text{End}(\widehat{\mathbb{Q}\Pi S}(p_0, p_1))$ converges and we obtain an automorphism

$$\exp(\sigma(L(C))) = \sum_{n=0}^{\infty} \frac{1}{n!} \sigma(L(C))^n \in \text{Aut}(\widehat{\mathbb{Q}C}(S, E)),$$

which we call the *generalized Dehn twist along C* ([15] §5.3). If C is simple, then this is (the $\widehat{\text{DN}}$ -image of) the usual right handed Dehn twist along C ([15] §5.2).

Remark 5.1.2. — Actually $\exp(\sigma(L(C)))$ lies in a subgroup $A(S, E) \subset \text{Aut}(\widehat{\mathbb{Q}C}(S, E))$, which was introduced in [15, §3.3].

5.2. A criterion of the realizability

A natural question is whether $\exp(\sigma(L(C)))$ is realized by a diffeomorphism, i.e., whether $\exp(\sigma(L(C)))$ is in the $\widehat{\text{DN}}$ -image. In [15] [19] we showed that if C is a figure eight, then $\exp(\sigma(L(C)))$ is not in the $\widehat{\text{DN}}$ -image. To extend this result for curves in wider classes, we consider the self intersections of curves.

Let $C \subset S \setminus \partial S$ be an unoriented free loop, and $N \subset S \setminus \partial S$ a connected compact subsurface which is a neighborhood of C , and not diffeomorphic to D^2 . If the generalized Dehn twist $\exp(\sigma(L(C)))$ is the $\widehat{\text{DN}}$ -image of a mapping class $\varphi \in \mathcal{M}(S, \partial S)$, the support of (a representative of) φ is included in the subsurface N , by the localization theorem [15, §5.3].

Using the fact that μ maps simple paths to zero and a diffeomorphism preserves the simplicity of curves, together with cut and paste techniques developed in [15], we have the following.

PROPOSITION 5.2.1. — *Suppose that the inclusion homomorphism $\pi_1(N) \rightarrow \pi_1(S)$ is injective. If the generalized Dehn twist $\exp(\sigma(L(C)))$ is realized by a diffeomorphism, we have*

$$\mu(\sigma(L(C))(\gamma)) = 0 \quad \widehat{\mathbb{Q}\Pi N}(\nu_0, \nu_1) \widehat{\mathbb{Q}\hat{\pi}(N)}$$

for any distinct points $\nu_0, \nu_1 \in \partial N$ and any simple path $\gamma \in \Pi N(\nu_0, \nu_1)$. When $\nu_0 = \nu_1 = \nu$, the same conclusion holds if $\varepsilon(\dot{\gamma}(0), \dot{\gamma}(1)) = +1$.

Proof. — Let $\varphi \in \text{Diff}(S, \partial S)$ be a representative of $\exp(\sigma(L(C)))$. By the remark above, we may assume that the support of φ is included in N . We denote by the same letter φ the restriction of φ to N , which we can regard as an element of the mapping class group $\mathcal{M}(N, \partial N)$. Also we regard C as an unoriented free loop on N and $L(C)$ as an element of $\widehat{\mathbb{Q}\hat{\pi}(N)}$.

Let $\partial N = \coprod_i \partial_i N$ be the decomposition into connected components. Then, by [15, §3.3] (see also [15, §5.4]), there exist some $a_i \in \mathbb{Q}$ such that

$$\varphi \exp(-\sigma(L(C))) = \exp\left(\sigma\left(\sum_i a_i L(\partial_i N)\right)\right) \in \text{Aut}(\widehat{\mathbb{Q}C(N, \partial N)}).$$

Since C and $\partial_i N$ are disjoint, the derivations $\sigma(L(C))$ and $\sigma(L(\partial_i N))$ commute with each other. This implies $\varphi^n = \exp(n\sigma(L(C) + \sum_i a_i L(\partial_i N)))$ for any $n \in \mathbb{Z}$. Since $\varphi^n(\gamma)$ is a simple path, we have $\mu(\varphi^n(\gamma)) = 0$. Hence we obtain

$$\mu\left(\sigma\left(L(C) + \sum_i a_i L(\partial_i N)\right)(\gamma)\right) = 0.$$

On the other hand, by [15, §5.2], $\exp(\sigma(L(\partial_i N)))$ is realized by the Dehn twist along the simple closed curve $\partial_i N$. This implies $\mu(\sigma(L(\partial_i N))(\gamma)) = 0$. Hence we obtain $\mu(\sigma(L(C))(\gamma)) = 0$. This completes the proof. \square

In the case S is compact, i.e., has no punctures, we have another criterion for the realizability of generalized Dehn twists.

PROPOSITION 5.2.2. — *Assume that S is compact, and let $C \subset S \setminus \partial S$ be an unoriented loop whose generalized Dehn twist $\exp(\sigma(L(C)))$ is realized by a diffeomorphism. Then we have*

$$\delta L(C) = 0 \quad \widehat{\mathbb{Q}\hat{\pi}(S)} \widehat{\mathbb{Q}\hat{\pi}(S)}.$$

Proof. — Take $\nu_0, \nu_1 \in E$. Any orientation-preserving diffeomorphism φ of S fixing the boundary pointwise preserves the comodule structure map $\mu: \widehat{\mathbb{Q}\Pi S}(\nu_0, \nu_1) \rightarrow \widehat{\mathbb{Q}\Pi S}(\nu_0, \nu_1) \widehat{\mathbb{Q}\hat{\pi}(S)}$. Hence, for any $n \in \mathbb{Z}$, we have

$$\mu \exp(n\sigma(L(C))) = \exp(n\sigma(L(C)))\mu,$$

and so

$$\begin{aligned}
 &(\sigma(L(C)) \widehat{1} + 1 \widehat{\sigma(L(C))})\mu \\
 &= \mu\sigma(L(C)): \widehat{\mathbb{Q}\Pi S}(0, 1) \rightarrow \widehat{\mathbb{Q}\Pi S}(0, 1) \widehat{\mathbb{Q}\hat{\pi}(S)}.
 \end{aligned}$$

From (2.2.2) this is equivalent to

$$(\widehat{\sigma} \widehat{1}_{\widehat{\mathbb{Q}\hat{\pi}(S)}})(1_{\widehat{\mathbb{Q}\Pi S}(0, 1)} \widehat{\delta})(v \widehat{L(C)}) = 0 \in \widehat{\mathbb{Q}\Pi S}(0, 1) \widehat{\mathbb{Q}\hat{\pi}(S)}$$

for any $v \in \widehat{\mathbb{Q}\Pi S}(0, 1)$. By [15, §6.2], the intersection of the kernels of the structure map $\sigma: \widehat{\mathbb{Q}\hat{\pi}(S)} \rightarrow \text{End}(\widehat{\mathbb{Q}\Pi S}(0, 1))$ for all $0, 1 \in E$, is zero. Hence we have $\delta L(C) = 0$. This proves the proposition. \square

In [19] the second-named author posed the following question.

QUESTION 5.2.3 ([19] Question 5.11). — *Let C be an unoriented loop on $\Sigma_{g,1}$, a surface of genus g with one boundary component, and suppose the generalized Dehn twist along C is realized by a diffeomorphism. Is C homotopic to a power of a simple closed curve?*

In view of Proposition 5.2.2 we come to the following conjecture.

CONJECTURE 5.2.4. — *Let C be an unoriented loop that satisfies $\delta L(C) = 0$. Then C would be homotopic to a power of a simple closed curve.*

If the conjecture is true, then the answer to the question is also affirmative. But the conjecture looks like a question which was posed by Turaev [33] and whose counter-examples was given by Chas [3].

5.3. New examples not realized by a diffeomorphism

In this subsection we prove the following.

THEOREM 5.3.1. — *Let S and $E = \partial S$ be as in §5.1 and $C \subset S \setminus \partial S$ an unoriented immersed loop whose self intersections consist of transverse double points. Assume that C is non-simple and the inclusion homomorphism $\pi_1(N(C)) \rightarrow \pi_1(S)$ is injective, where $N(C)$ is a closed regular neighborhood of C . Then the generalized Dehn twist $\exp(\sigma(L(C)))$ is not in the image of $\widehat{DN}: \mathcal{M}(S, \partial S) \rightarrow \text{Aut}(\widehat{\mathbb{Q}\mathcal{C}(S, E)})$.*

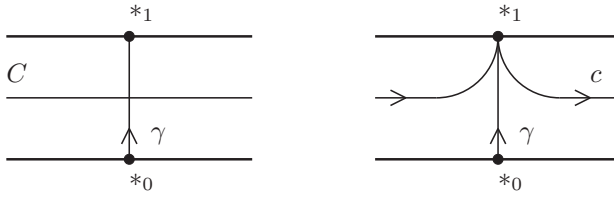


Figure 9. $\pi_1(S, \{0, 1\})$ -representative of C

Let S be an oriented surface and $0, 1 \in \partial S$ distinct points. Using μ and the augmentation $\mathbb{Q}\Pi S(\{0, 1\}) \rightarrow \mathbb{Q}, \Pi S(\{0, 1\}) \xrightarrow{x \mapsto 1}$, we define a \mathbb{Q} -linear map $\hat{\mu}: \mathbb{Q}\Pi S(\{0, 1\}) \rightarrow \mathbb{Q}\hat{\pi}(S)$ as the composite

$$\hat{\mu}: \mathbb{Q}\Pi S(\{0, 1\}) \xrightarrow{\mu} \mathbb{Q}\Pi S(\{0, 1\}) \xrightarrow{\mathbb{Q}} \mathbb{Q}\hat{\pi}(S) = \mathbb{Q}\hat{\pi}(S).$$

By Proposition 4.4.2, $\hat{\mu}$ extends to a \mathbb{Q} -linear map $\hat{\mu}: \widehat{\mathbb{Q}\Pi S}(\{0, 1\}) \rightarrow \widehat{\mathbb{Q}\hat{\pi}(S)}$. We denote by $\widehat{\mathbb{Q}H_1(S; \mathbb{Z})}$ the completed group ring of the integral first homology group $H_1(S; \mathbb{Z})$. There is a natural projection $\hat{\pi}(S) \rightarrow H_1(S; \mathbb{Z})$, which induces a \mathbb{Q} -linear map $\varpi: \widehat{\mathbb{Q}\hat{\pi}(S)} \rightarrow \widehat{\mathbb{Q}H_1(S; \mathbb{Z})}/\mathbb{Q}1$. Here $\mathbb{Q}1$ is the 1-dimensional subspace spanned by the identity element of $H_1(S; \mathbb{Z})$.

Let $C \subset S \setminus \partial S$ be an unoriented immersed loop such that its self intersections consist of transverse double points, and let $\gamma \in \Pi S(\{0, 1\})$ be a simple path meeting C transversally in a single point. In this situation, we shall compute the quantity $\varpi\hat{\mu}(\sigma(L(C))\gamma)$.

Let c be a $\pi_1(S, \{0, 1\})$ -representative of C , as in Figure 9. Then we have $\sigma(L(C))\gamma = \gamma \log c \in \widehat{\mathbb{Q}\Pi S}(\{0, 1\})$, since $\sigma(|c^n|)\gamma = n\gamma c^n$ for $n \geq 0$. Now fix a parametrization $c: ([0, 1], \{0, 1\}) \rightarrow (S, \{0, 1\})$. When $p \in S$ is a double point of C , we denote $c^{-1}(p) = \{t_1^p, t_2^p\}$ so that $t_1^p < t_2^p$. Set $x_p := c_{t_2^p} c_{0t_1^p}$, and $y_p := c_{t_1^p} t_2^p$. By abuse of notation, we use the same letter x_p and y_p for the homology classes represented by these loops. Finally, let $h(x)$ be the formal power series defined by

$$h(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n+2} (x-1)^n.$$

PROPOSITION 5.3.2. — *Keep the notations as above. Then*

$$\begin{aligned} \varpi \hat{\mu}(\sigma(L(C))\gamma) \\ = - \sum_p \varepsilon(\dot{c}(t_1^p), \dot{c}(t_2^p))(y_p + x_p(y_p^2 - 1)h(c)) \quad \widehat{\mathbb{Q}H_1(S; \mathbb{Z})}/\mathbb{Q}1. \end{aligned}$$

Here we write the product of the group ring $\mathbb{Q}H_1(S; \mathbb{Z})$ multiplicatively.

To prove this proposition, we need a lemma.

LEMMA 5.3.3. — *In the polynomial ring $\mathbb{Q}[x]$, the following equalities hold.*

(1) For $n \geq 1$,

$$\sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \sum_{j=0}^{k-1} x^j = (x - 1)^{n-1}.$$

(2) For $n \geq 1$, set

$$f_n(x) := \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \sum_{j=0}^{k-1} (k - j)x^j.$$

Then $f_1(x) = 1$ and $f_n(x) = x(x - 1)^{n-2}$ for $n \geq 2$.

Proof.

(1) Since $\sum_{j=0}^{k-1} x^j = (x^k - 1)/(x - 1)$, the left hand side is equal to

$$\begin{aligned} \frac{1}{x - 1} \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (x^k - 1) &= \frac{1}{x - 1} \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} x^k \\ &= \frac{1}{x - 1} (x - 1)^n \\ &= (x - 1)^{n-1}. \end{aligned}$$

(2) The case $n = 1$ is clear. Let $n \geq 2$. By the first part, we compute

$$\begin{aligned}
 f_n(x) - (x - 1)^{n-1} &= \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \sum_{j=0}^{k-1} (k - 1 - j)x^j \\
 &= \sum_{k=0}^n \left(\binom{n-1}{k-1} + \binom{n-1}{k} \right) (-1)^{n-k} \sum_{j=0}^{k-1} (k - 1 - j)x^j \\
 &= \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^{n-k-1} \sum_{j=0}^k (k - j)x^j \\
 &\quad + \sum_{k=0}^n \binom{n-1}{k} (-1)^{n-k} \sum_{j=0}^{k-1} (k - 1 - j)x^j \\
 &= \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^{n-1-k} \sum_{j=0}^{k-1} x^j = (x - 1)^{n-2}.
 \end{aligned}$$

Therefore $f_n(x) = (x - 1)^{n-1} + (x - 1)^{n-2} = x(x - 1)^{n-2}$. □

Proof of Proposition 5.3.2. — We first compute $\hat{\mu}(\gamma c^k)$ for $k \geq 0$. We choose a representative of γc^k by sliding c into the left. See Figure 10. Each self intersection p of C creates k^2 self intersections of γc^k . These k^2 points are classified into $k + (k - 1)$ classes, according to their contributions to $\hat{\mu}(\gamma c^k)$. Namely, if $\varepsilon(\dot{c}(t_1^p), \dot{c}(t_2^p)) = 1$, for $0 \leq j \leq k - 1$, there are $k - j$ self intersections whose contributions are $-|y_p(x_p y_p)^j|$, and for $1 \leq j \leq k - 1$, there are $k - j$ self intersections whose contributions are $+|x_p(y_p x_p)^{j-1}|$. This is illustrated in Figure 11. The points in the box j^+ ($0 \leq j \leq k - 1$) contribute as $-|y_p(x_p y_p)^j|$, and the points in the box j^- ($1 \leq j \leq k - 1$) contribute as $+|x_p(y_p x_p)^{j-1}|$. If $\varepsilon(\dot{c}(t_1^p), \dot{c}(t_2^p)) = -1$, the contributions are the minus of the case of $\varepsilon(\dot{c}(t_1^p), \dot{c}(t_2^p)) = 1$. Therefore, we obtain

$$\begin{aligned}
 \hat{\mu}(\gamma c^k) &= - \sum_p \varepsilon(\dot{c}(t_1^p), \dot{c}(t_2^p)) \\
 (5.3.4) \quad &\times \left(\sum_{j=0}^{k-1} (k - j) |y_p(x_p y_p)^j| - \sum_{j=1}^{k-1} (k - j) |x_p(y_p x_p)^{j-1}| \right)
 \end{aligned}$$

We next compute $\hat{\mu}(\gamma(c - 1)^n)$ for $n \geq 1$. We claim that the contribution from p to $\hat{\mu}(\gamma(c - 1)^n)$ is $-\varepsilon(\dot{c}(t_1^p), \dot{c}(t_2^p))|y_p|$ if $n = 1$, and

$$-\varepsilon(\dot{c}(t_1^p), \dot{c}(t_2^p)) (|y_p x_p y_p (x_p y_p - 1)^{n-2}| - |x_p (y_p x_p - 1)^{n-2}|)$$

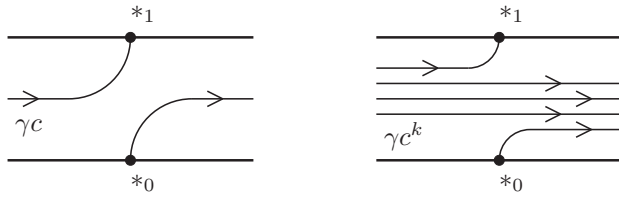


Figure 10. a representative of γc^k ($k = 4$)

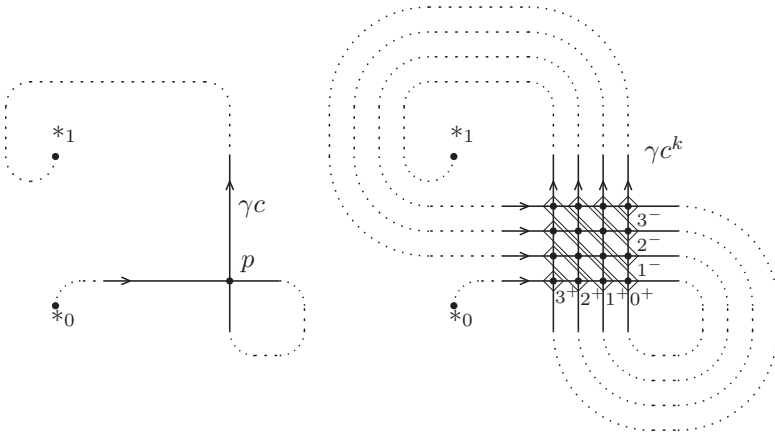


Figure 11. a picture near p ($\varepsilon(\dot{c}(t_1^p), \dot{c}(t_2^p)) = 1, k = 4$)

if $n \geq 2$. The case $n = 1$ is clear. If $n \geq 2$, by (5.3.4) the contribution is $-\varepsilon(\dot{c}(t_1^p), \dot{c}(t_2^p))$ times

$$(5.3.5) \quad \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \times \left(\sum_{j=0}^{k-1} (k-j) / y_p (x_p y_p)^j / - \sum_{j=1}^{k-1} (k-j) / x_p (y_p x_p)^{j-1} / \right).$$

By

$$\sum_{j=1}^{k-j} (k-j) / x_p (y_p x_p)^{j-1} / = \sum_{j=0}^{k-1} (k-j) / x_p (y_p x_p)^j / - \sum_{j=0}^{k-1} / x_p (y_p x_p)^j /$$

and Lemma 5.3.3, (5.3.5) is equal to

$$\begin{aligned} & |y_p x_p y_p (x_p y_p - 1)^{n-2}| - |x_p y_p x_p (y_p x_p - 1)^{n-2}| + |x_p (y_p x_p - 1)^{n-1}| \\ &= |y_p x_p y_p (x_p y_p - 1)^{n-2}| - |x_p (y_p x_p - 1)^{n-2}|. \end{aligned}$$

The claim is proved. Now we conclude

$$\hat{\mu}(\gamma \log c) = - \sum_p \varepsilon(\dot{c}(t_1^p), \dot{c}(t_2^p)) (|y_p| + |y_p x_p y_p h(x_p y_p)| - |x_p h(y_p x_p)|).$$

Applying ϖ and using $x_p y_p = c = y_p x_p \in H_1(S; \mathbb{Z})$, we obtain the desired formula. This completes the proof. \square

Proof of Theorem 5.3.1. — Assume that the generalized Dehn twist $\exp(\sigma(L(C)))$ is realized by a diffeomorphism. Let $N = N(C)$ be a closed regular neighborhood of C . Take a simple point $a \in S$ of C and let $\gamma: ([0, 1], \{0, 1\}) \rightarrow (N, \partial N)$ be a simple path in N meeting C transversally only at a . We denote $\gamma(0) = c_0$ and $\gamma(1) = c_1$. By Proposition 5.2.1, we have $\mu(\sigma(L(C))\gamma) = 0 \in \widehat{\mathbb{Q}\Pi N}(c_0, c_1) \cong \widehat{\mathbb{Q}\hat{\pi}(N)}$. In particular, we have $\varpi \hat{\mu}(\sigma(L(C))\gamma) = 0 \in \widehat{\mathbb{Q}H_1(N; \mathbb{Z})}/\mathbb{Q}1$.

We claim:

- (1) $\{x_p\}_p \cup \{c\}$ constitute a \mathbb{Z} -basis of $H_1(N; \mathbb{Z}) = H_1(C; \mathbb{Z})$, and
- (2) by an appropriate choice of a , we can arrange that $\sum_p \varepsilon(\dot{c}(t_1^p), \dot{c}(t_2^p)) = 0$.

To prove the first claim, note that only the underlying 4-valent graph structure of C , together with its (unoriented) parametrization matters. Let q be a double point of C and $f: \tilde{C} \rightarrow C$ a resolution of q . Namely, \tilde{C} is a 4-valent graph with a surjective map $S^1 \rightarrow \tilde{C}$, such that the composition $S^1 \rightarrow \tilde{C} \rightarrow C$ gives a parametrization of C , $f^{-1}(x)$ consist of a single point if $x = q$, and $f^{-1}(q)$ consist of two points, say q_+ and q_- . By the excision isomorphism, we have $H_1(C; \mathbb{Z}) = H_1(C, \{q\}; \mathbb{Z}) = H_1(\tilde{C}, \{q_+, q_-\}; \mathbb{Z})$. Consider the homology exact sequence of the pair

$$0 \rightarrow H_1(\tilde{C}; \mathbb{Z}) \rightarrow H_1(\tilde{C}, \{q_+, q_-\}; \mathbb{Z}) \xrightarrow{\partial} \tilde{H}_0(\{q_+, q_-\}; \mathbb{Z}) \rightarrow 0.$$

Then the ∂ -image of $x_q \in H_1(C; \mathbb{Z}) = H_1(\tilde{C}, \{q_+, q_-\}; \mathbb{Z})$ is $\pm(q_+ - q_-)$, which is a generator of $\tilde{H}_0(\{q_+, q_-\}; \mathbb{Z}) = \mathbb{Z}$.

Now we prove the first claim by induction on the number of double points of C . If C has only one double point, then \tilde{C} is a simple closed curve and $\{c\}$ is a \mathbb{Z} -basis of $H_1(\tilde{C}; \mathbb{Z})$. Therefore, the lifts of x_q and c to \tilde{C} are a \mathbb{Z} -basis of $H_1(\tilde{C}, \{q_+, q_-\}; \mathbb{Z})$. This settles a base case. Suppose that the number of double points of C is ≥ 2 . By the inductive assumption, the lifts of $\{x_p\}_{p \neq q} \cup \{c\}$ to \tilde{C} constitute a \mathbb{Z} -basis of $H_1(\tilde{C}; \mathbb{Z})$. Therefore the

lifts of $\{x_p\}_p \cup \{c\}$ to \tilde{C} constitute a \mathbb{Z} -basis of $H_1(\tilde{C}, \{q_+, q_-\}; \mathbb{Z})$, which completes the proof of the first claim.

To prove the second claim, let ℓ be the component of the set of simple points of C containing a , and ℓ' a component next to ℓ . Take a simple point $a \in \ell$ and let γ be a simple path meeting C transversally only at a . We arrange that $\varepsilon(\dot{c}(a), \dot{\gamma}(a)) = \varepsilon(\dot{c}(a'), \dot{\gamma}(a'))$. Let q be the double point of C between ℓ and ℓ' . We compare $\varepsilon(\dot{c}(t_1^p), \dot{c}(t_2^p))$'s with respect to a and a' . If $p = q$, then they are the same. If $p \neq q$, they are minus of each other. Hence the difference of the sums $\sum_p \varepsilon(\dot{c}(t_1^p), \dot{c}(t_2^p))$ for a and a' is two, in particular at least one of them is not zero. This proves the second claim.

Now choose a such that $\sum_p \varepsilon(\dot{c}(t_1^p), \dot{c}(t_2^p)) = 0$. By the first claim, we can define a group homomorphism $\Phi: H_1(N; \mathbb{Z}) \rightarrow \widehat{\mathbb{Q}t}$ to an infinite cyclic group generated by t , by $\Phi(x_p) = 1$ and $\Phi(c) = t$. This group homomorphism induces a \mathbb{Q} -linear map $\Phi: \widehat{\mathbb{Q}H_1(N; \mathbb{Z})} / \mathbb{Q}1 \rightarrow \widehat{\mathbb{Q}t} / \mathbb{Q}1$. Since $x_p y_p = c$, we have $\Phi(y_p) = t$. By Proposition 5.3.2,

$$\Phi(\varpi\mu(\sigma(L(C))\gamma)) = - \left(\sum_p \varepsilon(\dot{c}(t_1^p), \dot{c}(t_2^p)) \right) (t + (t^2 - 1)h(t)) \in \widehat{\mathbb{Q}t} / \mathbb{Q}1.$$

Finally, we claim $t + (t^2 - 1)h(t) = 0$. To prove this, consider an algebra homomorphism from $\widehat{\mathbb{Q}t}$ to $\mathbb{Q}[[s]]$, the ring of formal power series in s , given by $t \mapsto 1 + s$. This is a filter-preserving isomorphism, and the image of $t + (t^2 - 1)h(t)$ is $1 + 2s + (\text{higher term})$, which is not zero in $\mathbb{Q}[[s]] / \mathbb{Q}1$. This shows $t + (t^2 - 1)h(t) = 0$, which contradicts to $\varpi\mu(\sigma(L(C))\gamma) = 0$. This completes the proof. □

6. A geometric approach to the Johnson homomorphisms

In this section we study the Johnson homomorphisms following the treatments in [15], which is briefly recalled in §6.1. In §6.2, we show that the Turaev cobracket gives a geometric constraint on the Johnson image. As was shown in [16], in the case $S = \Sigma_{g,1}$, a once bordered surface of genus $g \geq 1$, the completed Goldman Lie algebra is isomorphic to the Lie algebra of symplectic derivations of the completed tensor algebra generated by the first rational homology group of the surface, \mathfrak{a}_g^- , through a symplectic expansion θ introduced by Massuyeau [20]. This isomorphism induces a complete Lie cobracket δ^θ on the Lie algebra \mathfrak{a}_g^- . We can consider the ‘‘Laurent expansion’’ of the cobracket δ^θ with respect to the natural degree on \mathfrak{a}_g^- . In §6.3, based on a theorem of Massuyeau and Turaev [22], we

prove that the principal term, which is of degree -2 , equals Schedler’s cobracket [31]. Moreover we prove that the (-1) -st and the 0 -th terms vanish. The latter term is computed in §6.5. These results are obtained independently by Massuyeau and Turaev [21]. In §6.4, we prove that all the Morita traces [26] factor through Schedler’s cobracket.

6.1. The Johnson homomorphisms

The higher Johnson homomorphisms on the higher Torelli groups for a once bordered surface are important tools to study the algebraic structure of the mapping class group. In [15], we gave a generalization of the classical construction of the Johnson homomorphisms to arbitrary compact oriented surfaces with non-empty boundary. In this subsection we briefly recall this construction.

Let S be a compact connected oriented surface of with non-empty boundary, and $E \subset \partial S$ a subset such that each connected component of ∂S has a unique point of E . We define the *Torelli group* $I(S, E)$ to be the kernel of the action of the mapping class group $\mathcal{M}(S, \partial S)$ on the first homology group $H_1(S, E; \mathbb{Z})$, which is the smallest Torelli group in the sense of Putman [30]. On the other hand, we define

$$L^+(S, E) := \left\{ u \in \widehat{\mathbb{Q}\hat{\pi}}(S)(3); \begin{array}{l} (\sigma(u) \hat{=} 1 + 1 \hat{=} \sigma(u)) \quad \Delta^{(0, 1)} = \Delta^{(0, 1)} \sigma(u) \\ \text{for any } (0, 1) \in E \end{array} \right\},$$

where Δ is the coproduct

$$\Delta = \Delta^{(0, 1)} : \widehat{\mathbb{Q}\hat{\Pi}S}(0, 1) \otimes \widehat{\mathbb{Q}\hat{\Pi}S}(0, 1) \hat{=} \widehat{\mathbb{Q}\hat{\Pi}S}(0, 1)$$

given by $\Delta x := x \hat{=} x$ for any $x \in \widehat{\mathbb{Q}\hat{\Pi}S}(0, 1)$, $(0, 1) \in E$.

Using the Hausdorff series, we can regard $L^+(S, E)$ as a pro-nilpotent group. In other words, using the injectivity of σ ([15] §6.2) and the exponential map, we have a bijection $L^+(S, E) \hat{=} \exp(\sigma(L^+(S, E))) \subset \text{Aut}(\widehat{\mathbb{Q}\hat{C}}(S, E))$, which endows $L^+(S, E)$ with a group structure. Here we use the fact that if D is an element of $\sigma(L^+(S, E))$, its exponential

$$\exp(D) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n$$

converges and is an element of $\text{Aut}(\widehat{\mathbb{Q}\hat{C}}(S, E))$. In [15, §6.3] we showed the inclusion

$$\widehat{\text{DN}}(I(S, E)) \subset \exp(\sigma(L^+(S, E))),$$

using a result of Putman [30] about generators of $I(S, E)$ and our formula for Dehn twists [16]. Hence we obtain a unique injective group homomorphism

$$(6.1.1) \quad \tau: I(S, E) \rightarrow L^+(S, E)$$

satisfying $\widehat{DN}/I(S, E) = \exp \circ \sigma \circ \tau$. We call it *the geometric Johnson homomorphism* of the Torelli group $I(S, E)$.

In the case $S = \Sigma_{g,1}$ and E consists of a single point ∂S , the Torelli group $I(S, E)$ is just the classical Torelli group $I_{g,1}$. As was shown in [15, §6.3], the map τ is essentially the same as Massuyeau’s improvement [20] of the total Johnson map [14]. In particular, the graded quotients of the geometric Johnson homomorphism $\tau: I_{g,1} \rightarrow L^+(\Sigma_{g,1}, \{ \})$ with respect to the Johnson filtration on $I_{g,1}$ and the filtration $\{L^+(\Sigma_{g,1}, \{ \})\}_{p \geq 3}$ are exactly the Johnson homomorphisms of all degrees.

6.2. A constraint on the Johnson image

Now we show that the Turaev cobracket gives an obstruction of the surjectivity of τ .

THEOREM 6.2.1.

$$\delta \circ \tau = 0: I(S, E) \rightarrow L^+(S, E) \quad \widehat{Q\Pi}(S) \xrightarrow{\delta} \widehat{Q\Pi}(S) \wedge \widehat{Q\Pi}(S).$$

Proof. — The proof is similar to that of Proposition 5.2.2. From the definition of τ , for any $\varphi \in I(S, E)$, there exists a unique $u \in L^+(S, E)$ such that $\varphi = \exp \sigma(u)$ on $\widehat{Q\Pi S}(0, 1)$ for any 0 and $1 \in E$. Then we have $\tau(\varphi) = u$ by definition. Let $\mu: \widehat{Q\Pi S}(0, 1) \rightarrow \widehat{Q\Pi S}(0, 1) \wedge \widehat{Q\Pi}(S)$ be the structure map of the comodule $\widehat{Q\Pi S}(0, 1)$. It is clear that μ is preserved by φ^n for any $n \in \mathbb{Z}$, namely, we have

$$(\exp \sigma(nu) \wedge \exp \sigma(nu))\mu(v) = \mu(\exp \sigma(nu)(v))$$

for any $n \in \mathbb{Z}$ and $v \in \widehat{Q\Pi S}(0, 1)$. Hence we have

$$(\sigma(u) \wedge 1 + 1 \wedge \sigma(u))\mu(v) = \mu(\sigma(u)(v))$$

which is equivalent to

$$(\bar{\sigma} - 1)(v \wedge \delta u) = 0 \quad \widehat{Q\Pi S}(0, 1) \wedge \widehat{Q\Pi}(S)$$

for any 0 and $1 \in E$, from (2.2.2). Again by the fact that the intersection of the kernels of the structure map σ ’s is zero, [17] §6.2, we conclude $\delta u = 0$. This proves the theorem. □

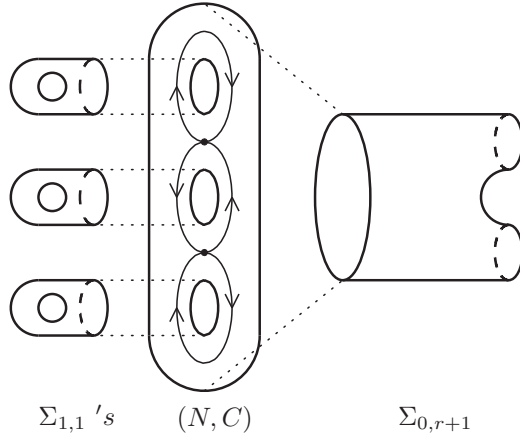


Figure 12. the case $g = 3, r = 2$

This constraint is non-trivial if the genus of the surface S is greater than 1.

PROPOSITION 6.2.2. — *If $g \geq 2$, we have $\delta/L^+(S, E) = 0$.*

Proof. — We denote by $\Sigma_{g,r}$ a compact connected oriented surface of genus g with r boundary components. Consider a spine C of the surface $N := \Sigma_{0,g+1}$ as in Figure 12. If $g \geq 2$, C has a self-intersection. We cap each of the g boundaries the curve C surrounds by a surface diffeomorphic to $\Sigma_{1,1}$ to obtain a compact surface S_0 diffeomorphic to $\Sigma_{g,1}$, and glue $\Sigma_{0,r+1}$ to the boundary of S_0 to get a compact surface S diffeomorphic to $\Sigma_{g,r}$. See Figure 12. Choose one point in each boundary component of S . We define E by the set of all these points. We may regard N as a regular neighborhood of C .

Consider the invariant $L(C) \in \widehat{\mathbb{Q}\hat{\pi}}(S)$. As was proved in [15, §5.1], the action of $L(C)$ stabilizes the coproduct Δ . Since $[C] = 0 \in H_1(S; \mathbb{Q})$, we have $L(C) \in L^+(S, E)$. From the proof of Theorem 5.3.1 and the compatibility of the comodule structure map and the cobracket (2.2.2), we have $\delta L(C) = 0 \in \widehat{\mathbb{Q}\hat{\pi}}(N) \widehat{\otimes} \widehat{\mathbb{Q}\hat{\pi}}(N)$. As was proved in [15, §6.2], the inclusion homomorphism $\widehat{\mathbb{Q}\hat{\pi}}(N) \rightarrow \widehat{\mathbb{Q}\hat{\pi}}(S_0)$ is injective. Since the inclusion homomorphism $\pi_1(S_0) \rightarrow \pi_1(S)$ has a right inverse coming from capping all the boundaries except one by $r - 1$ discs, the inclusion homomorphism $\widehat{\mathbb{Q}\hat{\pi}}(S_0) \rightarrow \widehat{\mathbb{Q}\hat{\pi}}(S)$ is injective. Hence we have $\delta L(C) = 0 \in \widehat{\mathbb{Q}\hat{\pi}}(S) \widehat{\otimes} \widehat{\mathbb{Q}\hat{\pi}}(S)$. This proves the proposition. \square

From Theorem 6.2.1 the Zariski closure of the subgroup $\tau(l(S, E))$ is included in the closed Lie subalgebra $\text{Ker}(\delta|_{L^+(S, E)})$. In view of this theorem we raise the following conjecture.

CONJECTURE 6.2.3. — *The Zariski closure of the subgroup $\tau(l(S, E))$ equals the closed Lie subalgebra $\text{Ker}(\delta|_{L^+(S, E)})$:*

$$\overline{\tau(l(S, E))} = \text{Ker}(\delta|_{L^+(S, E)}).$$

This conjecture questions the extensionality of the Johnson image. Hain [12] already described its comprehension. By Turaev’s theorem [32, p.234], Corollary 2, μ captures the simplicity of a based loop on a surface. This conjecture is its analogue in the mapping class group. It is closely related to Conjecture 5.2.4. But it seems quite optimistic even in the simplest case $S = \Sigma_{g,1}$. The cokernel of the Johnson homomorphisms in the case $S = \Sigma_{g,1}$ is known to have plenty of Sp -irreducible components including the Morita traces [26]. For details, see [10] and references therein. In the succeeding subsections we will prove that all the Morita traces are recovered from our constraint. Very recently Enomoto [9] proved that the Enomoto-Satoh traces [10] do not factor through the leading term δ^{alg} in the “Laurent expansion” of the Turaev cobracket δ in §6.3. But we do not know whether they are can be recovered from δ itself or not.

6.3. The graded quotient of the Turaev cobracket

For the rest of this section we suppose $S = \Sigma_{g,1}$, a compact connected oriented surface of genus g with one boundary component. Then the Torelli group $l(S, E) = l(\Sigma_{g,1}, \{ \cdot \})$ is classically denoted by $l_{g,1}$. Moreover as we will briefly recall below (for details, see [15, §6.3]), the Lie algebra $L^+(\Sigma_{g,1}, \{ \cdot \})$ is identified with (the completion of) the positive part of Kontsevich’s “Lie wor(1)d”, ℓ_g^+ [17]. Preceding Kontsevich, Morita [24] [25] introduced the Lie algebra $H_+ = \ell_g^+$ as a target of the higher Johnson homomorphisms.

Let $H := H_1(\Sigma_{g,1}; \mathbb{Q})$ be the first homology group of $\Sigma_{g,1}$, and consider $\widehat{T} := \prod_{m=0} H^m$, the completed tensor algebra generated by H , which has a natural filtration $\{\widehat{T}_p\}_{p>0}$ defined by $\widehat{T}_p := \prod_{m=p} H^m$. Via the intersetion pairing $(\cdot, \cdot) : H \times H \rightarrow \mathbb{Q}$, which is skew-symmetric and non-degenerate, we identify H and its dual $H^* = \text{Hom}(H, \mathbb{Q}) : H^* \cong H^*$, $X(Y, \cdot) = (Y, X)$. Let $\omega \in H^2$ be the two tensor corresponding to $-1_H \in \text{Hom}(H, H) = H \otimes H = H^2$. In other words, if $\{A_i, B_i\}_{i=1}^g \in H$ is

a symplectic basis of the first homology group H , then we have $\omega = \sum_{i=1}^g A_i B_i - B_i A_i \in H^2$. Here and for the rest of this paper, we omit the symbol \otimes because we regard it as the product operation on the completed tensor algebra \widehat{T} . By definition, the *Lie algebra of symplectic derivations* is $\mathfrak{a}_g^- = \text{Der}_\omega(\widehat{T})$, i.e., the Lie algebra of (continuous) derivations on the algebra \widehat{T} annihilating ω . The restriction

$$\mathfrak{a}_g^- \rightarrow \text{Hom}(H, \widehat{T}) = H \oplus \widehat{T} = H \oplus \widehat{T}_1, \quad D \rightarrow D/H$$

is injective. The image is described as follows. Define a \mathbb{Q} -linear map $N: \widehat{T} \rightarrow \widehat{T}$ by

$$N/H \circ m = \sum_{p=0}^{m-1} \nu^p, \quad \text{for } m \geq 1,$$

where ν is the cyclic permutation given by $X_1 X_2 \cdots X_m \rightarrow X_2 X_3 \cdots X_1$ ($X_i \in H$), and $N/H \circ 0 = 0$. As was shown in [16], we can identify

$$\mathfrak{a}_g^- = \prod_{m=1}^{\infty} N(H^{\otimes m}) \oplus H \oplus \widehat{T}$$

by the restriction map stated above. In particular, the Lie algebra \mathfrak{a}_g^- is naturally graded. We say $D \in \mathfrak{a}_g^-$ is of degree m if $D \in N(H^{\otimes m})$. Then D is of degree $m - 2$ as a derivation of the graded algebra \widehat{T} . Moreover the Lie bracket on \mathfrak{a}_g^- is homogeneous of degree -2 . Now the algebra \widehat{T} has the complete coproduct Δ given by $\Delta(X) = X \widehat{\otimes} 1 + 1 \widehat{\otimes} X$, $X \in H$. Let \mathfrak{l}_g^+ be the Lie subalgebra of \mathfrak{a}_g^- consisting of the derivations of degree ≥ 3 and stabilizing the coproduct on \widehat{T} . The Lie algebra \mathfrak{l}_g^+ is an ideal of (the completion of) Kontsevich’s “Lie wor(1)d” ℓ_g [17].

We denote $\pi := \pi_1(\Sigma_{g,1}, \cdot)$. Let $\theta: \pi \rightarrow \widehat{T}$ be a *symplectic expansion*. By definition, θ is a group homomorphism of π into the group-like elements of \widehat{T} , $\theta(x) = 1 + [x] \pmod{\widehat{T}_2}$ for any $x \in \pi$, and $\theta(\zeta) = \exp(\omega)$ [20]. Here $[x] \in H$ is the homology class of x and $\zeta \in \pi$ is a boundary loop on $\partial\Sigma_{g,1}$ in the opposite direction. See Figure 13.

Any symplectic expansion induces an isomorphism $\theta: \widehat{\mathbb{Q}\pi} \xrightarrow{\cong} \widehat{T}$ of complete Hopf algebras. Here $\widehat{\mathbb{Q}\pi}$ is the completed group ring of π (see §4.1). Moreover, in [15, §6.1], we showed that the map

$$(6.3.1) \quad -\lambda_\theta: \widehat{\mathbb{Q}\pi}(\Sigma_{g,1}) \xrightarrow{\cong} \mathfrak{a}_g^-, \quad [x] \mapsto -N\theta(x)$$

is a filter preserving isomorphism of Lie algebras, and induces a filter preserving isomorphism of Lie algebras

$$-\lambda_\theta: L^+(\Sigma_{g,1}, \{ \cdot \}) \xrightarrow{\cong} \mathfrak{l}_g^+.$$

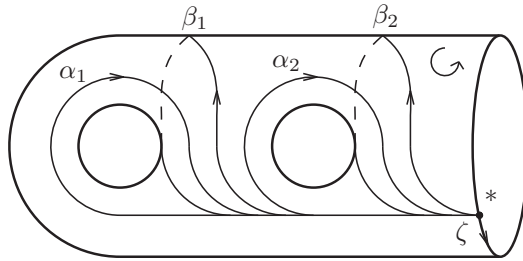


Figure 13. the boundary loop and symplectic generators ($g = 2$)

As was mentioned in [15, §6.3], the composite $-\lambda_\theta \tau$ is exactly Massuyeau’s improvement ρ^θ [20] of the Johnson map introduced by Kawazumi [14]. Its graded quotients with respect to the Johnson filtration on $I_{g,1}$ and the degree filtration on I_g^+ are the Johnson homomorphisms of all degrees introduced by Johnson [13] and improved by Morita [26]. Indeed, it is this context in which the Lie algebra ℓ_g was introduced by Morita [24] [25].

Through the isomorphism $-\lambda_\theta$, the Turaev cobracket δ on $\widehat{\mathbb{Q}\hat{\pi}}(\Sigma_{g,1})$ induces the complete Lie cobracket δ^θ on the Lie algebra \mathfrak{a}_g^- . Namely, δ^θ is defined so that the following diagram commutes:

$$\begin{array}{ccc}
 \widehat{\mathbb{Q}\hat{\pi}}(\Sigma_{g,1}) & \xrightarrow{\delta} & \widehat{\mathbb{Q}\hat{\pi}}(\Sigma_{g,1}) \widehat{\wedge} \widehat{\mathbb{Q}\hat{\pi}}(\Sigma_{g,1}) \\
 -\lambda_\theta \downarrow & & (-\lambda_\theta)^2 \downarrow \\
 \mathfrak{a}_g^- & \xrightarrow{\delta^\theta} & \mathfrak{a}_g^- \widehat{\wedge} \mathfrak{a}_g^- .
 \end{array}$$

The grading on \mathfrak{a}_g^- defines the Laurent expansion of the cobracket δ^θ . Namely, for any $u \in H^m$ we can write

$$\begin{aligned}
 \delta^\theta(N(u)) &= \sum_{p=-} \delta_{(p)}^\theta(N(u)), \quad \text{where} \\
 \delta_{(p)}^\theta(N(u)) &= (\mathfrak{a}_g^- \widehat{\wedge} \mathfrak{a}_g^-)_{(m+p)} := \bigoplus_{k+l=m+p} N(H^{-k}) \wedge N(H^{-l}).
 \end{aligned}$$

In this subsection and §6.5 we prove the following

THEOREM 6.3.2. — *For any symplectic expansion θ we have*

(1) $\delta_{(p)}^\theta = 0$ for $p = 0, -1$, and $p \leq -3$.

(2) $\delta_{(-2)}^\theta$ is the same as Schedler's cobracket [31], i.e.,

$$\begin{aligned} &\delta_{(-2)}^\theta(N(X_1 X_2 \cdots X_m)) \\ &= - \sum_{i < j} (X_i \cdot X_j) \{ N(X_{i+1} \cdots X_{j-1}) \widehat{N}(X_{j+1} \cdots X_m X_1 \cdots X_{i-1}) \\ &\quad - N(X_{j+1} \cdots X_m X_1 \cdots X_{i-1}) \widehat{N}(X_{i+1} \cdots X_{j-1}) \} \end{aligned}$$

for any $X_i \in H$ and $m \geq 1$.

In particular, the (-2) -nd term $\delta_{(-2)}^\theta$, or equivalently Schedler's cobracket, is independent of the choice of a symplectic expansion θ , so that we denote it by δ^{alg} . Since δ^{alg} is the graded quotient of the cobracket δ^θ with respect to the degree filtration, it induces a structure of a complete involutive Lie bialgebra on the Lie algebra \mathfrak{a}_g^- .

Recall that the graded quotients of the geometric Johnson map are the classical Johnson homomorphisms. Hence, as a corollary of Theorem 6.3.2, we obtain

COROLLARY 6.3.3. — For any $k \geq 1$, we have

$$\delta^{\text{alg}} \tau_k = 0: l_{g,1}(k) / l_{g,1}(k+1) \cong (\mathfrak{a}_g^- \widehat{\mathfrak{a}_g^-})_{(k)}.$$

Here $l_{g,1}(k)$ is the k -th term of the Johnson filtration, and τ_k is the k -th Johnson homomorphism.

The proof of Theorem 6.3.2 is based on a theorem of Massuyeau and Turaev [22, Theorem 10.4], which gives a tensorial description of the homotopy intersection form. As is announced in [23, Remark 7.4.3], Massuyeau and Turaev [21] also prove Theorem 6.3.2 in a similar way to ours. In this subsection we prove Theorem 6.3.2 except for the case $p = 0$, which will be proved in §6.5.

Recall the definition of the homotopy intersection form [22]. Taking a path ν and a second base point \bullet as in Figure 7, we identify the fundamental group $\pi_1(\Sigma_{g,1}, \bullet)$ with $\pi = \pi_1(\Sigma_{g,1}, \ast)$ by the isomorphism $\alpha = \bar{\nu}\alpha\nu$. (Note that in [22, §7], ν and $\bar{\nu}$ are denoted by ν_\bullet and $\bar{\nu}_\bullet$, respectively.) Let α be an oriented based immersed loop on $\Sigma_{g,1}$ with base point \bullet , and β an oriented based immersed loop on $\Sigma_{g,1}$ with base point \ast such that their intersections consists of transverse double points. Then the formula

$$(6.3.4) \quad \eta(\alpha, \beta) := \sum_{p \in \alpha \beta} \varepsilon(p; \alpha, \beta) \bar{\nu} \alpha_{\ast p} \beta_p \in \mathbb{Q}\pi$$

gives rise to a well-defined \mathbb{Q} -bilinear map $\eta: \mathbb{Q}\pi \times \mathbb{Q}\pi \rightarrow \mathbb{Q}\pi$ via the identification $\pi_1(\Sigma_{g,1}, \bullet) = \pi$ stated above. The map η is called the *homotopy intersection form* of $\Sigma_{g,1}$ [22], which is essentially the same as what Papakyriakopoulos [28] and Turaev [32] independently introduced.

Massuyeau and Turaev [22] proves that this map η naturally extends to a \mathbb{Q} -bilinear map $\hat{\eta}: \widehat{\mathbb{Q}\pi} \times \widehat{\mathbb{Q}\pi} \rightarrow \widehat{\mathbb{Q}\pi}$, and gives its tensorial description through any symplectic expansion $\theta: \widehat{\mathbb{Q}\pi} \cong \widehat{T}$. Let $\varepsilon: \widehat{T} \rightarrow \widehat{T}/\widehat{T}_1 = \mathbb{Q}$ be the augmentation map. Define a \mathbb{Q} -bilinear map $\rho: \widehat{T}_1 \times \widehat{T}_1 \rightarrow \widehat{T}$ by

$$\left(X_1 \cdots X_m \overset{\circ}{\smile} Y_1 \cdots Y_n \right) := (X_m \cdot Y_1) X_1 \cdots X_{m-1} Y_2 \cdots Y_n \quad H^{m+n-2}$$

for any $n, m \geq 1$, and $X_i, Y_j \in H$. Here $(X_m \cdot Y_1) \in \mathbb{Q}$ is the intersection pairing of X_m and $Y_1 \in H$. A \mathbb{Q} -bilinear map $\rho: \widehat{T} \times \widehat{T} \rightarrow \widehat{T}$ is defined by

$$(6.3.5) \quad \rho(a, b) := (a - \varepsilon(a)) \overset{\circ}{\smile} (b - \varepsilon(b)) + (a - \varepsilon(a))s(\omega)(b - \varepsilon(b))$$

for any a and $b \in \widehat{T}$, where $s(z)$ is the formal power series

$$\begin{aligned} s(z) &= \frac{1}{e^{-z} - 1} + \frac{1}{z} \\ &= -\frac{1}{2} - \sum_{k>1} \frac{B_{2k}}{(2k)!} z^{2k-1} \\ &= -\frac{1}{2} - \frac{z}{12} + \frac{z^3}{720} - \frac{z^5}{30240} + \dots \end{aligned}$$

Massuyeau and Turaev [22] proved the following.

THEOREM 6.3.6 ([22] Theorem 10.4). — *Let $\theta: \pi \rightarrow \widehat{T}$ be a symplectic expansion. Then the following diagram commutes:*

$$\begin{array}{ccc} \widehat{\mathbb{Q}\pi} \times \widehat{\mathbb{Q}\pi} & \xrightarrow{\hat{\eta}} & \widehat{\mathbb{Q}\pi} \\ \theta \times \theta \downarrow & & \theta \downarrow \\ \widehat{T} \times \widehat{T} & \xrightarrow{\rho} & \widehat{T}. \end{array}$$

Recall the \mathbb{Q} -linear map $\kappa: \mathbb{Q}\pi \rightarrow \mathbb{Q}\pi \rightarrow \mathbb{Q}\pi \rightarrow \mathbb{Q}\pi$ introduced in §4.2. Let $\Delta: \mathbb{Q}\pi \rightarrow \mathbb{Q}\pi \rightarrow \mathbb{Q}\pi$ be the coproduct defined by $\alpha \in \pi \rightarrow \alpha \otimes \alpha$, and $\iota: \mathbb{Q}\pi \rightarrow \mathbb{Q}\pi$ the antipode defined by $\alpha \in \pi \rightarrow \alpha^{-1}$. Then, for any α and

$\beta \in \pi$ in general position as stated above, we have

$$\begin{aligned}
 & - (1 \ \beta)((1 \ \iota)\Delta\eta(\alpha, \beta))(1 \ \alpha) \\
 &= - \sum_{p \ \alpha \ \beta} \varepsilon(p; \alpha, \beta)(1 \ \beta)(\bar{\nu}\alpha \cdot_p \beta_p \ (\beta_p)^{-1}(\alpha \cdot_p)^{-1}\nu)(1 \ \bar{\nu}\alpha\nu) \\
 &= - \sum_{p \ \alpha \ \beta} \varepsilon(p; \alpha, \beta)(\bar{\nu}\alpha \cdot_p \beta_p \ \beta_p \alpha_p \cdot \nu) \\
 &= \kappa(\alpha, \beta).
 \end{aligned}$$

Hence we obtain

$$(6.3.7) \quad \kappa(u, v) = - \sum (1 \ v)((1 \ \iota)(\Delta\eta(u, v)))(1 \ u)$$

for any u and $v \in \mathbb{Q}\pi$. Here we denote $\Delta u = \sum u \ u$ and $\Delta v = \sum v \ v$.

Let $\theta: \pi \xrightarrow{\widehat{T}}$ be a symplectic expansion. Then we have a unique \mathbb{Q} -linear map $\kappa^\theta: \widehat{T} \xrightarrow{\widehat{T}} \widehat{T} \xrightarrow{\widehat{T}}$ such that the diagram

$$\begin{array}{ccc}
 \widehat{\mathbb{Q}\pi} \xrightarrow{\widehat{\mathbb{Q}\pi}} & \xrightarrow{\kappa} & \widehat{\mathbb{Q}\pi} \xrightarrow{\widehat{\mathbb{Q}\pi}} \\
 \theta \downarrow & & \theta \downarrow \\
 \widehat{T} \xrightarrow{\widehat{T}} & \xrightarrow{\kappa^\theta} & \widehat{T} \xrightarrow{\widehat{T}}
 \end{array}$$

commutes. By Theorem 6.3.6 and (6.3.7), the map κ^θ does not depend on the expansion θ . From (6.3.7) we have

$$\begin{aligned}
 (6.3.8) \quad \kappa^\theta(X, Y) &= -(1 \ \widehat{1})((1 \ \iota)\Delta\rho(X, Y))(1 \ \widehat{1}) \\
 &= -(X \cdot Y)(1 \ \widehat{1}) - (1 \ \iota)\Delta(Xs(\omega)Y)
 \end{aligned}$$

for any X and $Y \in H$.

Now we define a \mathbb{Q} -linear map $\mu^\theta: \widehat{T} \xrightarrow{\widehat{T}} \widehat{T} \xrightarrow{\mathfrak{a}_g^-}$ by the commutative diagram

$$\begin{array}{ccc}
 \widehat{\mathbb{Q}\pi} & \xrightarrow{\mu} & \widehat{\mathbb{Q}\pi} \xrightarrow{\widehat{\mathbb{Q}\pi}} \\
 \theta \downarrow & & -\theta \downarrow \lambda_\theta \\
 \widehat{T} & \xrightarrow{\mu^\theta} & \widehat{T} \xrightarrow{\mathfrak{a}_g^-}.
 \end{array}$$

From Corollary 4.3.4 we have

$$\begin{aligned}
 & \mu^\theta(X_1 \cdots X_m) \\
 &= (1 \widehat{-} N) \sum_{1 \leq i < j \leq m} (X_1 \cdots X_{i-1} \widehat{1}) \kappa^\theta(X_i, X_j) \\
 & \qquad \qquad \qquad \times (X_{j+1} \cdots X_m \widehat{X_{i+1} \cdots X_{j-1}}) \\
 & \quad + \sum_{i=1}^m (X_1 \cdots X_{i-1} \widehat{1}) \mu^\theta(X_i) (X_{i+1} \cdots X_m \widehat{1}) \\
 (6.3.9) \quad &= \sum_{1 \leq i < j \leq m} (X_i \cdot X_j) X_1 \cdots X_{i-1} X_{j+1} \cdots X_m \widehat{N(X_{i+1} \cdots X_{j-1})} \\
 & \quad + (1 \widehat{-} N) \sum_{1 \leq i < j \leq m} (X_1 \cdots X_{i-1} \widehat{1}) ((1 \widehat{-} \iota) \Delta(X_i s(\omega) X_j)) \\
 & \qquad \qquad \qquad \times (X_{j+1} \cdots X_m \widehat{X_{i+1} \cdots X_{j-1}}) \\
 & \quad + \sum_{i=1}^m (X_1 \cdots X_{i-1} \widehat{1}) \mu^\theta(X_i) (X_{i+1} \cdots X_m \widehat{1})
 \end{aligned}$$

for any $m \geq 0$ and $X_i \in H$. We remark that the first term in (6.3.9) is of degree $m - 2$, and the second and the third terms are of degree $\geq m$. In fact, $\mu^\theta(X_i) \in \widehat{T} \widehat{-} \mathfrak{a}_g^-$ and $\widehat{T} \widehat{-} \mathfrak{a}_g^-$ starts from degree 1. On the other hand, by Lemma 4.4.1 the maps δ^θ and μ^θ are related by the formula

$$(6.3.10) \quad \delta^\theta \widehat{-} N = (1 - T)(N \widehat{-} 1_{\mathfrak{a}_g^-}) \mu^\theta,$$

where $T: \mathfrak{a}_g^- \widehat{-} \mathfrak{a}_g^- \rightarrow \mathfrak{a}_g^- \widehat{-} \mathfrak{a}_g^-$ is the switch map: $T(u \widehat{-} v) = v \widehat{-} u$. Theorem 6.3.2 except for the case $p = 0$ follows from the above observation and (6.3.10). □

6.4. The Morita traces

In this subsection we prove that Schedler’s cobracket δ^{alg} restricted to \mathfrak{l}_g^+ recovers the *Morita traces* of all degrees $\text{Tr}_k: (\mathfrak{l}_g^+)_{(k+1)} \rightarrow \text{Sym}^{k-1} H$, $k \geq 4$ [26]. Here $(\mathfrak{l}_g^+)_{(n)}$ is the degree n part of $\mathfrak{l}_g^+ \widehat{-} \mathfrak{a}_g^-$, and $\text{Sym}^n H$ is the n -th symmetric power of the homology group $H = H_1(\Sigma_{g,1}; \mathbb{Q})$.

To state our result precisely, we need some notations. Let

- $p_1: \mathfrak{a}_g^- = \prod_{m=1} N(H \widehat{-}^m) \rightarrow N(H \widehat{-}^1) = H$ be the first projection,
- $i: \mathfrak{a}_g^- = \prod_{m=1} N(H \widehat{-}^m) \hookrightarrow \prod_{m=1} H \widehat{-}^m = \widehat{T}_1$ the inclusion map,
- $\varpi: \widehat{T} \widehat{-} \widehat{\text{Sym}}(H) := \prod_{m=0} \text{Sym}^m(H)$ the natural projection.

We define

$$\mathfrak{s} := \varpi \quad (p_1 \widehat{i}) : \mathfrak{a}_g^- \widehat{\mathfrak{a}}_g^- \quad H \quad \widehat{T}_1 = \widehat{T}_2 \quad \widehat{\text{Sym}}(H).$$

Then we have

THEOREM 6.4.1.

$$\mathfrak{s} \quad \delta^{\text{alg}} /_{(\mathfrak{t}_g^+)_{(m+2)}} = (-1)^m m \times \text{Tr}_{m+1} : (\mathfrak{t}_g^+)_{(m+2)} \quad \text{Sym}^m H$$

for any $m \geq 3$.

From Corollary 6.3.3 $\mathfrak{s} \quad \delta^{\text{lag}} \quad \tau_k = 0 : l_{g,1}(k) / l_{g,1}(k + 1) \quad \text{Sym}^k H$. Hence Theorem 6.4.1 means that all the Morita traces are derived from the fundamental geometric fact that any diffeomorphism preserves the self-intersection of any curve on the surface.

The rest of this subsection is devoted to the proof of Theorem 6.4.1. Let $\widehat{L} \quad \widehat{T}_1$ be the completed free Lie algebra over the vector space H . It is known that (a multiple of) the Dynkin idempotent $\Phi : \widehat{T}_1 \rightarrow \widehat{L}$ defined by $\Phi(X_1 X_2 \cdots X_m) := [X_1, [X_2, [\cdots [X_{m-1}, X_m]]]]$ for $X_i \in H$ and $m \geq 1$ satisfies $\Phi /_{\widehat{L}} \quad H \quad m = m 1_{\widehat{L}} \quad H \quad m$. On the other hand, as was shown in [16, §2.7], we have $\mathfrak{t}_g^+ \rtimes \mathfrak{sp}(H) = N(H \quad \widehat{L}) = \text{Ker}([\cdot, \cdot]) \quad H \quad \widehat{L}$. Here $[\cdot, \cdot] : H \rightarrow \widehat{L} \quad \widehat{L}, X \quad u \quad [X, u]$, is the bracket map. Since $N(Y[X_1, \Phi(X_2 \cdots X_m)]) = N([Y, X_1]\Phi(X_2 \cdots X_m))$ for any $Y \in H$, we have

$$(6.4.2) \quad N([H, H] \quad \widehat{L}) = \mathfrak{t}_g^+.$$

Hence it suffices to prove Theorem 6.4.1 at $N([Y, Z]\Phi(X_1 \cdots X_m))$ for any $Y, Z, X_i \in H$.

Originally the Morita traces were defined as the trace of certain matrix representation of $\text{Hom}(H_{\mathbb{Z}}, \Gamma_k / \Gamma_{k+1})$ using the Fox free derivative. Here $H_{\mathbb{Z}} = \pi^{\text{ab}} = H_1(\Sigma_{g,1}; \mathbb{Z})$ and $\{\Gamma_k\}_k$ is the the lower central series of π (with $\Gamma_1 = \pi$). Actually it is known that the k -th Morita trace $\text{Tr}_k : H \rightarrow L_k \quad \text{Sym}^{k-1}(H)$ coincides with $(-1)^k$ times the map $H \rightarrow L_k \quad H \rightarrow H \quad H \quad k \quad C_{12} \quad H \quad (k-1) \quad \text{Sym}^{k-1}(H)$, where $C_{12} : H \rightarrow H \quad k \quad H \quad (k-1)$ is defined by $X_0 \rightarrow X_1 X_2 \cdots X_k \rightarrow (X_0 \cdot X_1) X_2 \cdots X_k$ for any $X_i \in H$, cf. [27, Remark 22] (note that the convention about indices in [26] [27] is different from ours). We use this description of Tr_k . To compute $\text{Tr}_{m+1}(N([Y, Z]\Phi(X_1 \cdots X_m))) \quad \text{Sym}^m(H)$, note that any tensor including $[Y, Z]$ or $\Phi(X_1 \cdots X_m)$ must vanish in the symmetric power $\text{Sym}^m(H)$. Hence, if we fix Y and Z and define a map $\beta = \beta_{Y,Z} : H \rightarrow H \quad m \quad \text{Sym}^m(H)$

by

$$\beta(X_1 \cdots X_m) := -(Y \cdot X_1)Z X_2 \cdots X_m - (Y \cdot X_m)Z X_1 \cdots X_{m-1} + (Z \cdot X_1)Y X_2 \cdots X_m + (Z \cdot X_m)Y X_1 \cdots X_{m-1},$$

then we have

$$\text{Tr}_{m+1}(N([Y, Z]\Phi(X_1 \cdots X_m))) = (-1)^{m+1}\beta(\Phi(X_1 \cdots X_m))$$

for any $X_i \in H$.

LEMMA 6.4.3.

- (1) If m is even, then $\beta(\Phi(X_1 \cdots X_m)) = 0$.
- (2) If $m \geq 3$, then $\beta(\Phi(X_1 X_2 X_3 \cdots X_m)) = X_1 X_2 \beta(\Phi(X_3 \cdots X_m))$.

The first assertion was already known by Morita [26, Theorem 6.1(ii)].

Proof.

- (1) Let $\iota: \widehat{T} \rightarrow \widehat{T}$ denote the antipode, i.e., we have $\iota(X_1 \cdots X_m) = (-1)^m X_m \cdots X_1$. The map β is ‘symmetric’. In other words, we have $\beta \iota(X_1 \cdots X_m) = (-1)^m \beta(X_m \cdots X_1) = (-1)^m \beta(X_1 \cdots X_m)$ $\text{Sym}^m(H)$, while we have $\iota|_{\widehat{L}} = -1$. Hence $-\beta \Phi(X_1 \cdots X_m) = \beta \iota \Phi(X_1 \cdots X_m) = (-1)^m \beta \Phi(X_1 \cdots X_m)$, which implies $\beta \Phi(X_1 \cdots X_m) = 0$ if m is even.
- (2) We remark

$$(6.4.4) \quad \begin{aligned} \Phi(X_1 X_2 X_3 \cdots X_m) &= X_1 X_2 \Phi(X_3 \cdots X_m) + \Phi(X_3 \cdots X_m) X_2 X_1 \\ &\quad - X_1 \Phi(X_3 \cdots X_m) X_2 - X_2 \Phi(X_3 \cdots X_m) X_1. \end{aligned}$$

Then $\beta(X_1 \Phi(X_3 \cdots X_m) X_2) = \beta(X_1 \Phi(X_3 \cdots X_m) X_2) = 0$, since $\Phi(X_3 \cdots X_m)$ remains in $\text{Sym}^m(H)$. When we compute the β -image of the first and the second terms in the right hand side, the terms coming from the contraction of X_1 and Y or Z must vanish since they include $\Phi(X_3 \cdots X_m)$. The rest terms equal $X_1 X_2 \beta \Phi(X_3 \cdots X_m)$. This proves the lemma. □

From the definition, we have

$$\begin{aligned} \mathfrak{s}\delta^{\text{alg}}(N(X_1 X_2 \cdots X_m)) \\ = \sum_{i=1}^m (X_{i-1} \cdot X_{i+1}) X_i X_{i+2} \cdots X_m X_1 \cdots X_{i-2} \quad \text{Sym}^{m-2}(H), \end{aligned}$$

where we denote $X_{m+1} = X_1$, $X_0 = X_m$ and so on. Similarly $[Y, Z]$ and $\Phi(X_1 \cdots X_m)$ vanish in the symmetric power $\text{Sym}^m(H)$. Hence, if

we fix Y and Z and define maps $\alpha = \alpha_{Y,Z}: H^m \rightarrow \text{Sym}^m(H)$ and $\gamma = \gamma_{Y,Z}: H^m \rightarrow \text{Sym}^m(H)$ by

$$\begin{aligned} \alpha(X_1 \cdots X_m) &:= -\beta(X_1 \cdots X_m) + \gamma(X_1 \cdots X_m), \quad \text{and} \\ \gamma(X_1 \cdots X_m) &:= -(Y \cdot X_2)Z X_1 X_3 \cdots X_m - (Y \cdot X_{m-1})Z X_1 \cdots X_{m-2} X_m \\ &\quad + (Z \cdot X_2)Y X_1 X_3 \cdots X_m + (Z \cdot X_{m-1})Y X_1 \cdots X_{m-2} X_m, \end{aligned}$$

respectively, then we have

$$\mathfrak{S}\delta^{\text{alg}}(N([Y, Z]\Phi(X_1 \cdots X_m))) = \alpha(\Phi(X_1 \cdots X_m))$$

for any $X_i \in H$.

LEMMA 6.4.5.

(1) If $m \geq 4$, then

$$\gamma(X_1 X_2 \cdots X_{m-1} X_m) = X_1 X_m \beta(X_2 \cdots X_{m-1}).$$

(2) If m is even, then $\alpha(\Phi(X_1 \cdots X_m)) = \gamma(\Phi(X_1 \cdots X_m)) = 0$.

(3) If $m \geq 5$, then

$$\gamma(\Phi(X_1 X_2 X_3 \cdots X_m)) = X_1 X_2 \gamma(\Phi(X_3 \cdots X_m)) - 2X_1 X_2 \beta(\Phi(X_3 \cdots X_m)).$$

Proof.

- (1) is clear from the definition.
- (2) is proved in a similar way to Lemma 6.4.3 (1), since the map γ is also ‘symmetric’ with respect to the antipode ι .
- (3) Recall the equation (6.4.4)

$$\begin{aligned} \Phi(X_1 X_2 X_3 \cdots X_m) &= X_1 X_2 \Phi(X_3 \cdots X_m) + \Phi(X_3 \cdots X_m) X_2 X_1 \\ &\quad - X_1 \Phi(X_3 \cdots X_m) X_2 - X_2 \Phi(X_3 \cdots X_m) X_1. \end{aligned}$$

By (1) the γ -image of each of the third and the fourth terms in the right hand side equals $-X_1 X_2 \beta(\Phi(X_3 \cdots X_m))$. When we compute the γ -image of the first and the second terms, the terms coming from the contraction of X_2 and Y or Z must vanish, since they include $\Phi(X_3 \cdots X_m)$. Hence the γ -image of the sum of the first and the second terms equals $X_1 X_2 \gamma(\Phi(X_3 \cdots X_m))$. This proves the lemma. \square

LEMMA 6.4.6. — If $m \geq 3$, then we have

$$\gamma(\Phi(X_1 X_2 \cdots X_m)) = -(m - 1)\beta(\Phi(X_1 X_2 \cdots X_m)).$$

Proof. — If m is even, $\gamma(\Phi(X_1 X_2 \cdots X_m)) = \beta(\Phi(X_1 X_2 \cdots X_m)) = 0$ from Lemma 6.4.3 (1) and Lemma 6.4.5 (2). Hence it suffices to prove the lemma in the case m is odd ≥ 3 by induction.

For $m = 3$ we compute the both sides in the lemma explicitly. Then we have

$$\begin{aligned} \gamma(\Phi(X_1 X_2 X_3)) &= -4(Y \cdot X_2)Z X_1 X_3 + 4(Z \cdot X_2)Y X_1 X_3 \\ &\quad + 4(Y \cdot X_3)Z X_1 X_2 - 4(Z \cdot X_3)Y X_1 X_2 \\ &= -2\beta(\Phi(X_1 X_2 X_3)), \end{aligned}$$

as was to be shown.

Next suppose $m \geq 5$. From Lemma 6.4.5 (1) we have

$$\gamma(\Phi(X_1 X_2 X_3 \cdots X_m)) = X_1 X_2 \gamma(\Phi(X_2 \cdots X_m)) - 2X_1 X_2 \beta(\Phi(X_3 \cdots X_m)),$$

which equals

$$\begin{aligned} &- (m - 3)X_1 X_2 \beta(\Phi(X_3 \cdots X_m)) - 2X_1 X_2 \beta(\Phi(X_3 \cdots X_m)) \\ &= -(m - 1)X_1 X_2 \beta(\Phi(X_3 \cdots X_m)) \end{aligned}$$

by the inductive assumption. By Lemma 6.4.3 (2) we have

$$\beta(\Phi(X_1 X_2 X_3 \cdots X_m)) = X_1 X_2 \beta(\Phi(X_3 \cdots X_m)).$$

Hence we obtain

$$\gamma(\Phi(X_1 X_2 X_3 \cdots X_m)) = -(m - 1)\beta(\Phi(X_1 X_2 X_3 \cdots X_m)).$$

This completes the induction and the proof of the lemma. □

As a corollary of Lemma 6.4.6, we have

$$\begin{aligned} \alpha(\Phi(X_1 \cdots X_m)) &= -\beta(\Phi(X_1 \cdots X_m)) + \gamma(\Phi(X_1 \cdots X_m)) \\ &= -m\beta(\Phi(X_1 \cdots X_m)). \end{aligned}$$

This completes the proof of Theorem 6.4.1. □

6.5. The 0-th term of the Laurent expansion of the Turaev cobracket

In this subsection we prove Theorem 6.3.2 for $p = 0$.

Using the grading on \mathfrak{a}_g^- , for any $u \in H^{-m}$ we can write

$$\begin{aligned} \mu^\theta(u) &= \sum_{p=-} \mu_{(p)}^\theta(u), \quad \text{where} \\ \mu_{(p)}^\theta(u) \quad (\widehat{T} \widehat{\mathfrak{a}}_{g^-})_{(m+p)} &:= \bigoplus_{k+l=m+p} H^{-k} \otimes N(H^{-l}). \end{aligned}$$

From (6.3.9) we have $\mu_{(p)}^\theta = 0$ for $p \leq -3$ and $p = -1$, and $\mu_{(-2)}^\theta$ is given by the first term of (6.3.9). Since it does not depend on the choice of θ , we denote it by μ^{alg} . Namely,

$$\begin{aligned} \mu_{(-2)}^\theta(X_1 \cdots X_m) &= \mu^{\text{alg}}(X_1 \cdots X_m) \\ &= \sum_{1 \leq i < j \leq m} (X_i \cdot X_j) X_1 \cdots X_{i-1} X_{j+1} \cdots X_m \\ &\quad N(X_{i+1} \cdots X_{j-1}). \end{aligned}$$

Therefore, by (6.3.10), Theorem 6.3.2 for $p = 0$ is deduced from the following

THEOREM 6.5.1. — *For any symplectic expansion θ and $u \in H^m$, we have*

$$\mu_{(0)}^\theta(u) = -\frac{1}{2}(1 - N(u)).$$

The rest of this subsection is devoted to the proof of Theorem 6.5.1.

First we prove Theorem 6.5.1 for $m = 1$, i.e.,

$$(6.5.2) \quad \mu_{(0)}^\theta(X) = -\frac{1}{2}(1 - X), \quad X \in H.$$

Let $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g \in \pi$ be symplectic generators of $\pi = \pi_1(\Sigma_{g,1})$. See Figure 13. The homology classes $A_i = [\alpha_i], B_i = [\beta_i] \in H$ ($1 \leq i \leq g$) constitute a symplectic basis for H . By (3.3.2) we have $\mu(\alpha_i) = 0$ and $\mu(\beta_i) = 1 - |\beta_i|$. From the definition of μ^θ , we obtain

$$\mu^\theta(\theta(\alpha_i)) = 0 \quad \text{and} \quad \mu^\theta(\theta(\beta_i)) = -(\theta \wedge \lambda_\theta)(1 - |\beta_i|) = -1 - N\theta(\beta_i).$$

We denote by θ_k the degree k part of θ . Looking at the degree 1 part of the above equation, for $1 \leq i \leq g$, we have

$$(6.5.3) \quad \begin{aligned} \mu_{(0)}^\theta(A_i) &= -\mu^{\text{alg}}(\theta_3(\alpha_i)), \\ \mu_{(0)}^\theta(B_i) &= -\mu^{\text{alg}}(\theta_3(\beta_i)) - 1 - B_i. \end{aligned}$$

LEMMA 6.5.4. — *Let $\widehat{L} \subset \widehat{T}$ be the set of primitive elements of \widehat{T} . In other words, \widehat{L} is the completed free Lie algebra generated by H . Then for any $u \in \widehat{L} \cap H^3$, we have $\mu^{\text{alg}}(u) = 0$.*

Proof. — It is sufficient to prove the formula for $u = [X, [Y, Z]]$ where $X, Y, Z \in H$. Since $[X, [Y, Z]] = [X, YZ - ZY] = XYZ - XZY - YZX + ZYX$, we compute

$$\begin{aligned} \mu^{\text{alg}}([X, [Y, Z]]) &= (X \cdot Z)1 - Y - (X \cdot Y)1 - Z - (Y \cdot X)1 - Z + (Z \cdot X)1 - Y \\ &= 0. \end{aligned}$$

□

Now we prove (6.5.2). We denote $\ell^\theta := \log \theta: \pi \widehat{T} \rightarrow \widehat{L}$. Note that the logarithm $\log: 1 + \widehat{T}_1 \rightarrow \widehat{T}_1, u \mapsto \sum_{n=1}^\infty ((-1)^{n-1}/n)(u-1)^n$ gives a bijection from the set of group-like elements of \widehat{T} to \widehat{L} . For $x \in \pi$ let $\ell_k^\theta(x)$ be the degree k part of $\ell^\theta(x) \in \widehat{L}$. We remark that $\ell_1^\theta(x) = [x] \in H$.

Step 1. — Let θ^0 be a symplectic expansion satisfying

$$\ell_2^{\theta^0}(\alpha_i) = \frac{1}{2}[A_i, B_i], \quad \ell_2^{\theta^0}(\beta_i) = -\frac{1}{2}[A_i, B_i].$$

Such a symplectic expansion does exist. For example, see Massuyeau [20] and Kuno [18]. Since $\theta^0 = \exp(\ell^{\theta^0})$,

$$\begin{aligned} \theta_3^0(\alpha_i) &= \ell_3^{\theta^0}(\alpha_i) + \frac{1}{2}(A_i \ell_2^{\theta^0}(\alpha_i) + \ell_2^{\theta^0}(\alpha_i) A_i) + \frac{1}{6} A_i A_i A_i \\ &= \ell_3^{\theta^0}(\alpha_i) + \frac{1}{4}(A_i[A_i, B_i] + [A_i, B_i] A_i) + \frac{1}{6} A_i A_i A_i. \end{aligned}$$

By Lemma 6.5.4, $\mu^{\text{alg}}(\ell_3^{\theta^0}(\alpha_i)) = 0$ and clearly $\mu^{\text{alg}}(A_i A_i A_i) = 0$. Therefore,

$$\begin{aligned} \mu^{\text{alg}}(\theta_3^0(\alpha_i)) &= \frac{1}{4} \mu^{\text{alg}}(A_i[A_i, B_i] + [A_i, B_i] A_i) \\ &= \frac{1}{4} (1 - A_i + 1 - A_i) = \frac{1}{2} (1 - A_i). \end{aligned}$$

Similarly, we obtain $\mu^{\text{alg}}(\theta_3^0(\beta_i)) = (-1/2)(1 - B_i)$. Substituting these equations into (6.5.3), we have $\mu_{(0)}^{\theta^0}(A_i) = (-1/2)(1 - A_i)$ and $\mu_{(0)}^{\theta^0}(B_i) = (-1/2)(1 - B_i)$. This completes the proof of (6.5.2) for θ^0 .

Step 2. — Let θ be an arbitrarily symplectic expansion. Then there exist $u_1 \in \Lambda^3 H$ and $u_2 \in \text{Hom}(H, \widehat{L} \otimes H^{\otimes 3})$ such that

$$(6.5.5) \quad \theta_3(x) = \theta_3^0(x) + (u_1 \otimes 1 + 1 \otimes u_1) \theta_2^0(x) + u_2([x])$$

for any $x \in \pi$. This is proved by a similar way to the proof of Lemma 6.4.2 in [16]. Here $\Lambda^3 H$ is the third exterior power of H and we regard u_1 as an element of $\text{Hom}(H, H^{\otimes 2})$ by the inclusion $\Lambda^3 H \otimes H^{\otimes 3} = H \otimes H^{\otimes 2} = \text{Hom}(H, H^{\otimes 2})$, $X \otimes Y \otimes Z \mapsto XYZ - XZY - YXZ + YZX + ZXY - ZYX$. Notice that $u_1(H) \subset \Lambda^2 H$.

LEMMA 6.5.6. — *For any $u_1 \in \Lambda^3 H$ and $X \in H$ we have*

$$\mu^{\text{alg}}((u_1 \otimes 1 + 1 \otimes u_1) X X) = 0.$$

Proof. — It suffices to consider the case $X = 0$. There exists a \mathbb{Q} -symplectic basis $\{A_j, B_j\}_j \in H$ such that $X = A_1$. By linearity, we may assume that $u_1 = Y \otimes Z \otimes W$ where $Y, Z, W \in \{A_j, B_j\}_j$. Now the assertion is proved by a direct computation. □

Let $x \in \pi$. Since $u_1(H) \cong \Lambda^2 H$, we have $(u_1^{-1} + 1 - u_1)\ell_2^\theta(x) \in \widehat{H}^3$. By Lemma 6.5.4, $\mu^{\text{alg}}((u_1^{-1} + 1 - u_1)\ell_2^\theta(x)) = 0$. By $\theta_2^\theta(x) = \ell_2^{\theta_0}(x) + (1/2)[x][x]$ and Lemma 6.5.6, we conclude

$$\mu^{\text{alg}}((u_1^{-1} + 1 - u_1)\theta_2^\theta(x)) = 0$$

for any $x \in \pi$. Also, we have $\mu^{\text{alg}}(u_2([x])) = 0$ by Lemma 6.5.4. Therefore by (6.5.5) $\mu^{\text{alg}}(\theta_3(x)) = \mu^{\text{alg}}(\theta_3^0(x))$ for any $x \in \pi$. Now from Step 1 and (6.5.3) we have $\mu_{(0)}^\theta(A_i) = (-1/2)(1 - A_i)$ and $\mu_{(0)}^\theta(B_i) = (-1/2)(1 - B_i)$. This completes the proof of (6.5.2). \square

Next we compute $\mu_{(0)}^\theta(X_1 \cdots X_m)$ for $X_1, \dots, X_m \in H$, $m \geq 2$. Since the constant term of $s(z)$ is $-1/2$, by (6.3.9) we have

$$\begin{aligned} \mu_{(0)}^\theta(X_1 \cdots X_m) &= -\frac{1}{2}(1 - N) \sum_{1 \leq i < j \leq m} (X_1 \cdots X_{i-1} - 1)((1 - \iota)\Delta(X_i X_j)) \\ &\quad \times (X_{j+1} \cdots X_m - X_{i+1} \cdots X_{j-1}) \\ &\quad + \sum_{i=1}^m (X_1 \cdots X_{i-1} - 1)\mu_{(0)}^\theta(X_i)(X_{i+1} \cdots X_m - 1). \end{aligned} \tag{6.5.7}$$

Note that we have

$$\begin{aligned} (1 - \iota)(\Delta(X_i X_j)) &= (1 - \iota)(X_i X_j - 1 + X_i - X_j + X_j - X_i + 1 - X_i X_j) \\ &= X_i X_j - 1 - X_i - X_j - X_j - X_i + 1 - X_j X_i. \end{aligned} \tag{6.5.8}$$

Substituting (6.5.2), (6.5.8) into (6.5.7), and computing directly, we obtain

$$\mu_{(0)}^\theta(X_1 \cdots X_m) = -\frac{1}{2}(1 - N(X_1 \cdots X_m)).$$

This completes the proof of Theorem 6.5.1. \square

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