Arata KOMYO

On compactifications of character varieties of $n$-punctured projective line


<http://aif.cedram.org/item?id=AIF_2015__65_4_1493_0>

© Association des Annales de l’institut Fourier, 2015,
Certains droits réservés.

[BY-ND] Cet article est mis à disposition selon les termes de la licence
Creative Commons Attribution – pas de modification 3.0 France.
http://creativecommons.org/licenses/by-nd/3.0/fr/

L’accès aux articles de la revue « Annales de l’institut Fourier »
(http://aif.cedram.org/), implique l’accord avec les conditions générales
d’utilisation (http://aif.cedram.org/legal/).
ON COMPACTIFICATIONS OF CHARACTER VARIETIES OF $n$-PUncUTURED PROJECTIVE LINE

by Arata KOMYO (*)

Abstract. — In this paper, we construct compactifications of $SL_2(\mathbb{C})$-character varieties of $n$-punctured projective line and study the boundary divisors of the compactifications. This study is motivated by a conjecture for the configurations of the boundary divisors, due to C. Simpson. We verify the conjecture for a few examples.

RÉSUMÉ. — Dans cet article, nous construisons des compactifications de $SL_2(\mathbb{C})$-variétés de caractères d’une droite projective moins $n$ points et étudions les diviseurs au bord des compactifications. Cette étude est motivée par une conjecture, due à C. Simpson, sur les configurations des diviseurs au bord. Nous vérifions quelques cas de la conjecture.

1. Introduction

Let $C$ be a compact Riemann surface of genus $g$, and let $\{t_1, \ldots, t_n\}$ be the set of $n$-distinct points on $C$. For a positive integer $r > 0$, denote by $\mathcal{P}_r$ the set of partitions of $r$, and fix $\mu = (\mu^1, \ldots, \mu^n) \in (\mathcal{P}_r)^n$ where $\mu^i = (\mu^1_i, \ldots, \mu^r_i) \in \mathcal{P}_r$. For each partition $\mu^i \in \mathcal{P}_r$, let us fix semisimple conjugacy classes $C_1, \ldots, C_n \subset SL_r(\mathbb{C})$ which is generic in the sense of [4, Definition 2.1.1] and type $\mu^1, \ldots, \mu^n$, that is, the multiplicities of eigenvalues of matrices in $C_i$ are given by $\mu^i = (\mu^1_i, \mu^2_i, \ldots)$. We consider a monodromy $SL_r(\mathbb{C})$-semisimple representation $\rho : \pi_1(C \setminus \{t_1, \ldots, t_n\}, \ast) \longrightarrow SL_r(\mathbb{C})$

Keywords: character variety, geometric invariant theory.

(*) The author would like to thank Professor Kentaro Mitsui, Professor Masa-Hiko Saito and Professor Carlos Simpson, and for many comments and discussions. He thanks Professor Masa-Hiko Saito for warm encouragement.
of type \((g, \mu)\) which satisfies the condition \(\rho(\gamma_i) \in C_i\) for each \(i\) where \(\gamma_i\) is an ant-clockwise loop around the point \(t_i\). We can define the \(SL_r(\mathbb{C})\)-character variety \(\mathcal{R}_{g,\mu}\) of the \(n\)-punctured compact Riemann surface of genus \(g\) by the following categorical quotient

\[
\mathcal{R}_{g,\mu} := \{ (A_1, B_1, \ldots, A_g, B_g; M_1, \ldots, M_n) \in SL_r(\mathbb{C})^{2g} \times C_1 \times \cdots \times C_n \mid (A_1, B_1) \cdots (A_g, B_g) M_1 \cdots M_n = I_r \} / SL_r(\mathbb{C}).
\]

Here, we set \((A, B) = ABA^{-1}B^{-1}\) and \(I_r\) is the identity matrix. The variety depends on the actual choice of eigenvalues, but for simplicity we drop this choice from the notation. The categorical quotient \(\mathcal{R}_{g,\mu}\) can be considered as a moduli space of monodoromy \(SL_r(\mathbb{C})\)-semisimple representations of type \((g, \mu)\). The variety \(\mathcal{R}_{g,\mu}\), if nonempty, is a nonsingular affine variety of dimension

\[
d_{g,\mu} := r^2(2g - 2 + n) - \sum_{i,j} (\mu_{ij})^2 + 2 - 2g.
\]

(See [4]). In the case where \(g = 0\) and \(d_{g,\mu} = 2\), \(SL_r(\mathbb{C})\)-character varieties can be classified into four cases, which can be listed as follows:

\[
\begin{align*}
\mu &= ((1, 1), (1, 1), (1, 1), (1, 1)) \\
\mu &= ((1, 1, 1), (1, 1, 1), (1, 1, 1)) \\
\mu &= ((2, 2), (1, 1, 1, 1), (1, 1, 1, 1)) \\
\mu &= ((3, 3), (2, 2, 2), (1, 1, 1, 1, 1, 1)).
\end{align*}
\]

In the first and second types, the \(SL_r(\mathbb{C})\)-character varieties are known to be an affine cubic surface. ([3], [10], [9], [12]).

The purpose of this paper is to study the configuration of boundary divisor of compactifications of \(SL_r(\mathbb{C})\)-character varieties. This study is motivated by a conjecture due to Simpson [18], which is explained as follows. We choose a smooth compactification \(\overline{\mathcal{R}_{g,\mu}}\) of \(\mathcal{R}_{g,\mu}\) such that \(D^B_{g,\mu} = \overline{\mathcal{R}_{g,\mu}} \setminus \mathcal{R}_{g,\mu}\) is a divisor with normal crossings. We call the divisor \(D^B_{g,\mu}\) a boundary divisor of the compactification \(\overline{\mathcal{R}_{g,\mu}}\). Let \(N^B_{g,\mu}\) be a small neighborhood of \(D^B_{g,\mu}\) in \(\overline{\mathcal{R}_{g,\mu}}\), and let \(N^B_{g,\mu} = N^B_{g,\mu} \cap \mathcal{R}_{g,\mu} = \overline{N^B_{g,\mu}} \setminus D^B_{g,\mu}\). Let \(\Delta(D^B_{g,\mu})\) be a simplicial complex whose \(n\)-dimensional simplices correspond to the irreducible components of intersections of \(k + 1\) distinct components of \(D^B_{g,\mu}\). This is called the boundary complex or Stepanov complex of a compactification of \(\mathcal{R}_{g,\mu}\) (see [22], [23], and [16]).

**Theorem 1.1** ([22], [23], and [16]). — The homotopy type of boundary complex \(\Delta(D^B_{g,\mu})\) is independent of the choice of compactifications.
We have a continuous map, well-defined up to homotopy,
\[ N^B_{g,\mu} \to \Delta(D^B_{g,\mu}). \] 
On the other hand, let \( \mathcal{M}_{g,\mu} \) be the moduli space of parabolic Higgs bundles, which is diffeomorphic to the character variety \( \mathcal{R}_{g,\mu} \) via the non-abelian Hodge theory [19]. In particular, we have \( \dim \mathcal{M}_{g,\mu} = d_{g,\mu} \). We have the Hitchin fibration \( \mathcal{M}_{g,\mu} \to \mathbb{A}^{d_{g,\mu}} \). The moduli space \( \mathcal{M}_{g,\mu} \) has a canonical orbifold compactification, where the divisor at infinity is the quotient
\[ D^\text{Dol}_{g,\mu} := \mathcal{M}_{g,\mu}^*/\mathbb{C}^*. \]
Here, \( \mathcal{M}_{g,\mu}^* \) is the complement of the nilpotent cone. Let \( \mathcal{N}_{g,\mu}^\text{Dol} \) be a small neighborhood of \( D^\text{Dol}_{g,\mu} \), and let \( N_{g,\mu}^\text{Dol} = \mathcal{N}_{g,\mu}^\text{Dol} \cap \mathcal{R}_{g,\mu} = \mathcal{N}_{g,\mu}^\text{Dol} \setminus D^\text{Dol}_{g,\mu} \). The Hitchin fibration gives us a continuous map to the sphere at infinity in the Hitchin base
\[ N_{g,\mu}^\text{Dol} \to S^{d_{g,\mu}-1}. \]

**Conjecture 1.2 ([18]).**

1. There exists a homotopy-commutative diagram
\[
\begin{array}{ccc}
N_{g,\mu}^\text{Dol} & \xrightarrow{\cong} & N_{g,\mu}^B \\
\downarrow & & \downarrow \\
S^{d_{g,\mu}-1} & \xrightarrow{\cong} & \Delta(D^B_{g,\mu}).
\end{array}
\]

2. In particular, there exists a non-singular compactification of \( \mathcal{R}_{g,\mu} \) such that the boundary complex is a simplicial decomposition of sphere \( S^{d_{g,\mu}-1} \).

**Remark 1.3 (See [18]).** — The assertion (1) of Conjecture 1.2 is true in the first case of the list (1.1).

The main theorem of this paper is the following

**Theorem 1.4 (Theorem 6.2).** — The assertion (2) of Conjecture 1.2 is true in the following cases:

1. \( g = 0, r = 3, n = 3, \mu = ((1, 1, 1), (1, 1, 1), (1, 1, 1)), d_{g,\mu} = 2; \)
2. \( g = 0, r = 2, n = 5, \mu = ((1, 1), (1, 1), (1, 1), (1, 1), (1, 1)), d_{g,\mu} = 4. \)

For the case (1) of Theorem 1.4, the assertion (2) of Conjecture 1.2 can be verified by the classical invariant theory. ([3], [10], [9], [12]). However, it seems that the application of the classical invariant theory is difficult for general cases. Then, we construct compactifications of \( SL_r(\mathbb{C}) \)-character
varieties as follows. Following [13], we can construct a compactification of the representation variety \([13]\)

\[
\text{Rep}_{g,\mu} := \{(A_1, B_1, \ldots, A_g, B_g; M_1, \ldots, M_n) \in SL_r(\mathbb{C})^{2g} \times C_1 \times \cdots \times C_n \mid (A_1, B_1) \cdots (A_g, B_g) M_1 \cdots M_n = I_r\}.
\]

Then, we take the GIT quotient of this compactification of \(\text{Rep}_{g,\mu}\), which gives a compactification \(\overline{\text{R}}_{g,\mu}\) of \(\text{R}_{g,\mu}\). As special cases, we consider the case where \(g = 0, r = 2, n \geq 4, \mu = ((1, 1), \ldots, (1, 1))\). For \(n = 4\), we obtain the same result as the classical invariant theory [3]. For \(n = 5\) (i.e., the case (2) of Theorem 1.4), \(\overline{\text{R}}_{g,\mu}\) has singular points. A suitable blowing up of \(\overline{\text{R}}_{g,\mu}\) shows that the assertion (2) of Conjecture 1.2 holds. It seems that the configuration of the boundary divisor \(D^B_{0,\mu}\) is rather complicated for \(n \geq 6\).

Conjecture 1.2 is related to the P=W conjecture due to Hausel et al ([1]). First, we consider compact curve cases. The non-abelian Hodge theory for compact curves states that character varieties \(\mathcal{R}\) are diffeomorphic to moduli spaces \(\mathcal{M}\) of semi-stable Higgs bundles. Then, we have the induced isomorphism between the rational cohomology groups of \(\mathcal{R}\) and \(\mathcal{M}\). The P=W conjecture assert that the isomorphism of the rational cohomology groups exchanges the weight filtration on the cohomology groups of \(\mathcal{R}\) with the perverse Leray filtration associated with the Hitchin fibration on the cohomology groups of \(\mathcal{M}\). The P=W conjecture is verified in the case where \(r = 2\) ([1]). We may extend the conjecture to punctured curve cases. On the other hand, there exists a natural isomorphism from the reduced homology of the boundary complex \(\Delta(D^B_{g,\mu})\) to the \(2l\)-th graded piece of the weight filtration on the cohomology of \(\overline{\text{R}}_{g,\mu}\):

\[
\tilde{H}_{i-1}(\Delta(D^B_{g,\mu}), \mathbb{Q}) \cong G_{2l}^W H^{2l-i}(\overline{\text{R}}_{g,\mu}, \mathbb{Q}).
\]

(For example, see [16, Theorem 4.4]). By the isomorphism, the assertion (2) of Conjecture 1.2 implies that there exists only 1-dimensional weight \(2d_{g,\mu}\) part in the middle degree \(d_{g,\mu}\) cohomology of the character variety, which is also a consequence of the P=W conjecture.

**Remark 1.5.** — The structure groups of character varieties studied in [1] are \(GL_n(\mathbb{C})\), \(PGL_n(\mathbb{C})\) and \(SL_n(\mathbb{C})\). However, for \(g = 0\), those character varieties are the same.

The organization of this paper is as follows. In Section 2, we give the definition of a \(SL_r(\mathbb{C})\)-character variety. In Section 3, we consider the case where \(g = 0, r = 2, n = 4\) and \(g = 0, r = 3, n = 3\). In those cases, the character varieties are describe by invariants and a relation of invariants.
We recall that the character varieties are affine cubic surfaces. In Section 4, we consider the construction of compactifications of $SL_2(\mathbb{C})$-character varieties of $g = 0, \mu = ((1,1),\ldots,(1,1))$. In Section 5 and 6, we describe the boundary divisor of the compactifications of the cases where $n = 4$ and $n = 5$.

2. Preliminaries

We fix integers $g, r, n$ with $g \geq 0, r > 0, n > 0$, and let $(C, t) = (C, t_1, \ldots, t_n)$ be an $n$-pointed compact Riemann surface of genus $g$, which consists of a compact Riemann surface $C$ of genus $g$ and a set of $n$-distinct points $t = \{t_i\}_{1 \leq i \leq n}$ on $C$. We put $D(t) = t_1 + \cdots + t_n$ for each $(C, t) = (C, t_1, \ldots, t_n)$. We denote by

\[(2.1) \quad \Gamma_{C,t} := \pi_1(C \setminus D(t), *)\]

the fundamental group of $C \setminus D(t)$ with the base point $* \in C \setminus D(t)$. The group $\Gamma_{C,t}$ is generated by $(2g+n)$-element $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_n$ with one relation

\[(\alpha_1, \beta_1) \cdots (\alpha_g, \beta_g) \gamma_1 \cdots \gamma_n = 1.\]

Here, we set $(\alpha, \beta) = \alpha \beta \alpha^{-1} \beta^{-1}$. The set of generators $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_n$ is called canonical generators of $\Gamma_{C,t}$.

**Definition 2.1.** — An $SL_r(\mathbb{C})$-representation of the fundamental group $\Gamma_{C,t}$ is a group homomorphism

\[(2.2) \quad \rho : \Gamma_{C,t} \longrightarrow SL_r(\mathbb{C}).\]

Let $Hom(\Gamma_{C,t}, SL_r(\mathbb{C}))$ be the set of all $SL_r(\mathbb{C})$-representations of $\Gamma_{C,t}$. If we fix a set of canonical generators of $\Gamma_{C,t}$, we have the identification

\[Hom(\Gamma_{C,t}, SL_r(\mathbb{C})) \cong (\mathbb{C})^{2g+n-1}.\]

**Definition 2.2.** — Two $SL_r(\mathbb{C})$-representations $\rho_1$ and $\rho_2$ are isomorphic to each other, if and only if there exists a matrix $P \in SL_r(\mathbb{C})$ such that

\[\rho_2(\gamma) = P^{-1} \cdot \rho_1(\gamma) \cdot P \quad \text{for all } \gamma \in \Gamma_{C,t}.\]

Let $R_{(g,n-1)}^r$ denote the affine coordinate ring of $SL_r(\mathbb{C})^{2g+n-1}$. We consider the simultaneous action of $SL_r(\mathbb{C})$ on $SL_r(\mathbb{C})^{2g+n-1}$ as

\[P \cdot (A_1, \ldots, A_g, B_1, \ldots, B_g; M_1, \ldots, M_{n-1}) \mapsto (P^{-1}A_1 P, \ldots, P^{-1}A_g P, P^{-1}B_1 P, \ldots, P^{-1}B_g P; P^{-1}M_1 P, \ldots, P^{-1}M_{n-1} P).\]
The invariant ring \((R_{(g,n-1})^{Ad(SL_r(\mathbb{C}))}\) is finitely generated. For any \((C, t)\), there exists the universal categorical quotient map

\[
\Phi^r_{(C, t)} : \text{Hom}(\Gamma_{C, t}, SL_r(\mathbb{C})) \cong SL_r(\mathbb{C})^{2g+n-1} \\
\rightarrow \mathcal{R}^r_{(C, t)} = SL_r(\mathbb{C})^{2g+n-1} // SL_r(\mathbb{C})
\]

where

\[
\mathcal{R}^r_{(C, t)} = \text{Spec}[(R_{r, (g,n-1)})^{Ad(SL_r(\mathbb{C}))}].
\]

The following lemma is due to Simpson.

**Lemma 2.3** ([21, Proposition 6.1]). — The closed points of \(\mathcal{R}^r_{(C, t)}\) represent the Jordan equivalence classes of \(SL_r(\mathbb{C})\)-representations of \(\Gamma_{C, t}\).

Let us set

\[
\mathcal{A}^{(n)}_r := \left\{ a = (a^{(i)}_j)_{1 \leq j \leq r-1 \leq n} \in \mathbb{C}^{nr-n} \right\}.
\]

For \(a = (a^{(i)}_j) \in \mathcal{A}^{(n)}_r\), we set

\[
\chi_i(s) := s^r + a^{(i)}_{r-1}s^{r-1} + \cdots + a^{(i)}_1s + (-1)^r, \ (i = 1, \ldots, n).
\]

Moreover, we define the morphism

\[
\phi^r_{(C, t)} : \mathcal{R}^r_{(C, t)} \rightarrow \mathcal{A}^{(n)}_r
\]

by the relation

\[
\text{det}(sI_r - \rho(\gamma_i)) = \chi_i(s)
\]

where \([\rho] \in \mathcal{R}^r_{(C, t)}\) and \(\gamma_i\) is a anticlockwise loop around the point \(t_i\). The fiber of \(\phi^r_{(C, t)}\) at \(a \in \mathcal{A}^{(n)}_r\) is given by the affine subscheme of \(\mathcal{R}^r_{(C, t)}\):

\[
\mathcal{R}^r_{(C, t), a} := (\phi^r_{(C, t)})^{-1}(a) = \{ [\rho] \in \mathcal{R}^r_{(C, t)} | \text{det}(sI_r - \rho(\gamma_i)) = \chi_i(s), 1 \leq i \leq n \}.
\]

For \(a \in \mathcal{A}^{(n)}_r\), let \(\mu^i = (\mu^i_1, \mu^i_2, \ldots)\) be the partition of \(r\) which implies the multiplicity of the solutions of the equation \(\chi_i(s) = 0\). Put \(\mu = (\mu_1, \ldots, \mu^n)\), called the multiplicity of \(a \in \mathcal{A}^{(n)}_r\). Moreover, we define the subvariety

\[
\mathcal{A}^{(n)}_{r, \mu} := \left\{ a = (a^{(i)}_j)_{1 \leq j \leq r-1 \leq n} \in \mathbb{C}^{nr-n} \mid \text{the multiplicity of } a \text{ is } \mu \right\} \subset \mathcal{A}^{(n)}_r.
\]
**Definition 2.4.** — We fix a $k$-tuple $\mu$ of partitions of $r$. Let $a$ be an element of $A_r^{(n)}$. Then, we define

$$\mathcal{R}^{r,s}_{(C,t),\mu,a} := \{ [\rho] \in \mathcal{R}^{r}_{(C,t)} | \det(sI_r - \rho(\gamma_i)) = \chi_i(s), \rho(\gamma_i) : \text{diagonalizable}, 1 \leq i \leq n \}$$

$$= \{ (A_1, B_1, \ldots, A_g, B_g) \in SL_r(\mathbb{C})^{2g} \times C_1 \times \cdots \times C_n | (A_1, B_1) \cdots (A_g, B_g) M_1 \cdots M_n = I_r \} / SL_r(\mathbb{C})$$

where $C_i = \{ M \in SL_r(\mathbb{C}) | \det(sI_r - M) = \chi_a(s), M : \text{diagonalizable} \}$. In Section 1, we denoted by $\mathcal{R}^{r}_{(C,t),\mu}$ the variety instead of $\mathcal{R}^{r,s}_{(C,t),\mu,a}$, for simplicity. The affine subvariety $\mathcal{R}^{r,s}_{(C,t),\mu,a}$ is called a $SL_r(\mathbb{C})$-character variety of the $n$-punctured compact Riemann surface of genus $g$. In particular, we denote by $\mathcal{R}^{r}_{n,a}$ this variety in the case where $g = 0, \mu = ((1, \ldots, 1), \ldots, (1, \ldots, 1))$.

If we take a generic $a \in A_r^{(n)}$, the affine algebraic variety $\mathcal{R}^{r,s}_{(C,t),\mu,a}$ is a non-singular irreducible variety of dimension

$$d_{g,\mu} := r^2(2g - 2 + n) - \sum_{i,j} (\mu^i_j)^2 + 2 - 2g,$$

and has a holomorphic symplectic structure, if nonempty. (See [4],[6]). In particular, for $g = 0, \mu = ((1, \ldots, 1), \ldots, (1, \ldots, 1))$, the dimension of $\mathcal{R}^{r}_{n,a}$ is

$$d_{0,((1,1),\ldots,(1,1))} = 2n - 6.$$

### 3. Invariant ring

We recall the explicit description of the invariant ring $(\mathcal{R}^{r}_{(g,n-1)})^{Ad(SL_r(\mathbb{C}))}$ for the two cases $g = 0, r = 2, n = 4$ and $g = 0, r = 3, n = 3$. The following proposition follows from the fundamental theorem for matrix invariants. (See [2] or [17]).

**Proposition 3.1.**

$$(\mathcal{R}^{r}_{(0,n-1)})^{Ad(SL_r(\mathbb{C}))} = \mathbb{C}[\text{Tr}(M_{i_1}M_{i_2} \cdots M_{i_k}) | 1 \leq i_1, \ldots, i_k \leq n - 1].$$

In particular, for $r = 2$, the elements $\text{Tr}(M_{i_1}M_{i_2} \cdots M_{i_k})$ of degree $k \leq 3$ generate the invariant ring, that is,

$$(\mathcal{R}^{2}_{(0,n-1)})^{Ad(SL_2(\mathbb{C}))} = \mathbb{C}[\text{Tr}(M_i), \text{Tr}(M_iM_j), \text{Tr}(M_iM_jM_k) | 1 \leq i, j, k \leq n - 1].$$
First, we consider the case where \( g = 0, r = 2, n = 4 \). Let \((i, j, k)\) be a cyclic permutation of \((1, 2, 3)\). Then, the invariant ring \((R_{(0,3)}^2)^{Ad(SL_2(\mathbb{C}))}\) is generated by seven elements \(x_1, x_2, x_3, a_1, a_2, a_3, a_4\) and there exists a relation

\[
f_a(x) := x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1(a)x_1 - \theta_2(a)x_2 - \theta_3(a)x_3 + \theta_4(a) = 0
\]

where

\[
\begin{align*}
\theta_1(a) &= a_ia_4 + a_ja_k \quad (i, j, k), \\
\theta_3(a) &= a_1a_2a_3a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4.
\end{align*}
\]

Therefore, we have an isomorphism

\[
(R_{(0,3)}^2)^{Ad(SL_2(\mathbb{C}))} \cong \mathbb{C}[x_1, x_2, x_3, a_1, a_2, a_3, a_4]/(f_a(x)).
\]

We have the surjective morphism

\[
\phi_{(\mathbb{P}^1,0,1,t,\infty)}^2 : \mathcal{R}_{(\mathbb{P}^1,0,1,t,\infty)}^2 \to \text{Spec}[\mathcal{A}^2_{(4)}] = \text{Spec}[\mathbb{C}[a_1, a_2, a_3, a_4]]
\]

where \( t \) is a point of \( \mathbb{P}^1 \) such that \( t \neq 0, 1, \infty \). The fiber at \( a \in \mathcal{A}^2_{(4)} \), such that the type of the multiplicities of eigenvalues is \((1,1),(1,1),(1,1),(1,1)\), is an affine cubic hypersurface in \( \mathbb{C}^3 \). Hence, the \( SL_2(\mathbb{C}) \)-character variety of the 4-punctured projective line is an affine cubic hypersurface

\[
\mathcal{R}_{4,a} \cong \{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid f_a(x) = 0\}.
\]

The affine cubic hypersurface is called a Fricke-Klein cubic surface.

We consider the natural compactification \( \mathbb{C}^3 \to \mathbb{P}^3 \) as follows. Set \( x_1 = X/W, x_2 = Y/W, x_3 = Z/W \). Then, we obtain the following homogeneous polynomial

\[
XYZ + X^2W + Y^2W + Z^2W - \theta_1(a)XW^2 - \theta_2(a)YW^2 - \theta_3(a)ZW^2 + \theta_4(a)W^3 = 0.
\]

Substitute \( W = 0 \) to this equation. Then, we obtain the equation \( XYZ = 0 \). Hence, the boundary divisor of the natural compactification of \( \mathcal{R}_{4,a} \) consists
of three lines. The boundary complex is shown in Figure 3.1. The boundary complex is a simplicial decomposition of $S^1$.

Next, we consider the case where $g = 0, r = 3, n = 3$. We describe generators and defining relations for the invariant ring $(R^3_{(0,2)})^{Ad(SL_3(\mathbb{C}))}$. The following proposition is due to Lawton [12].

**Proposition 3.3.** — The invariant ring $(R^3_{(0,2)})^{Ad(SL_3(\mathbb{C}))}$ is generated by

$$
\begin{align*}
a_1 := & \text{Tr}(M_1) \\
b_1 := & \text{Tr}(M_2) \\
c_1 := & \text{Tr}(M_1^{-1}M_2^{-1}) = \text{Tr}(M_3) \\
x_1 := & \text{Tr}(M_1M_2^{-1}) \\
x_2 := & \text{Tr}(M_1^{-1}M_2) \\
x_3 := & \text{Tr}(M_1M_2M_1^{-1}M_2^{-1}),
\end{align*}
$$

and there exists a relation

$$x_3^2 - fx_3 + g = 0$$

where $f, g$ are polynomials of $x_1, x_2$ over $\mathbb{C}[a_1, a_2, b_1, b_2, c_1, c_2]$, more precisely,

$$
\begin{align*}
f &= x_1x_2 - a_2b_1x_1 - a_1b_2x_2 + (\text{constant terms in } x_1, x_2) \\
g &= x_1^3 + x_2^3 + (\text{terms that order is at most 2 in } x_1, x_2).
\end{align*}
$$

We consider the subring $A_3^{(3)} = \mathbb{C}[a_1, a_2, b_1, b_2, c_1, c_2]$ of $(R^3_{(0,2)})^{Ad(SL_3(\mathbb{C}))}$. We have a natural morphism

$$
\phi^3_{(\mathbb{P}^1,0,1,\infty)} : \mathcal{R}^3_{(\mathbb{P}^1,0,1,\infty)} = \text{Spec}[(R^3_{(0,2)})^{Ad(SL_3(\mathbb{C}))}] \rightarrow A_3^{(3)} = \text{Spec}[A_3^{(3)}].
$$

The fiber at $a \in A_3^{(3)}$, such that the type of the multiplicities of eigenvalues is $[((1,1,1), (1,1,1), (1,1,1))$, is an affine cubic hypersurface in $\mathbb{C}^3$. Hence,
the $SL_3(\mathbb{C})$-character variety of the 3-punctured projective line is an affine cubic hypersurface

$$\mathcal{R}^3_{3,\alpha} \cong \{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid x_3^2 - fx_3 + g = 0\}.$$  

We consider the compactification $\mathbb{C}^3 \hookrightarrow \mathbb{P}^3$ as follows. Set $x_1 = X/W, x_2 = Y/W, x_3 = Z/W$. Then, we obtain the following homogeneous polynomial

$$X^3 + Y^3 - XYZ + (\text{term containing } W) = 0.$$  

We substitute $W = 0$ to this equation. Then, we obtain the equation $X^3 + Y^3 - XYZ = 0$. This equation defines a plane cubic curve having a node. The boundary complex is shown in Figure 3.2. The boundary complex is a simplicial decomposition of $S^1$.

4. A compactification of the character variety

We construct a compactification of the $SL_2(\mathbb{C})$-character variety $\mathcal{R}_{n,k}$ ($k$ of the $n$-punctured projective line is date of coefficient of characteristic polynomials) by means of the geometric invariant theory for a compactification of the following variety

**Definition 4.1.** — We put

(4.1)

$$\text{Rep}_{n,k} := \{(M_1, \ldots, M_{n-1}) \in C_1 \times \cdots \times C_{n-1} | M_{n-1}^{-1} \cdots M_1^{-1} \in C_n\} = \{(M_1, \ldots, M_{n-1}) \in C_1 \times \cdots \times C_{n-1} | \text{Tr}(M_{n-1}^{-1} \cdots M_1^{-1}) = k_n\}$$

where $C_i = \{M \in SL_2(\mathbb{C}) \mid \text{Tr}(M) = k_i\}$ and $k = (k_1, \ldots, k_n) \in \mathbb{C}^n$. The affine variety $\text{Rep}_{n,k}$ is said to the $SL_2(\mathbb{C})$-representation variety of the $n$-punctured line.
We will introduce a compactification of the representation variety due to Benjamin [13]. First, we consider a construction of a compactification of the algebraic group $SL_2(\mathbb{C})$. We pick an embedding $\alpha: SL_2(\mathbb{C}) \hookrightarrow PGL_3(\mathbb{C})$. Such an embedding always exists: we consider the natural embedding $SL_2(\mathbb{C}) \rightarrow GL_2(\mathbb{C})$ and we take the composition of the embedding and the map $GL_2(\mathbb{C}) \xrightarrow{\xi} GL_3(\mathbb{C}) \rightarrow PGL_3(\mathbb{C})$ where

$$\xi(A) = \begin{pmatrix} A & \vline & 1 \\ \hline & & \end{pmatrix}$$

and the second arrow is the canonical projection. We regard $PGL_3(\mathbb{C})$ as an open subvariety of $\mathbb{P}(M_3(\mathbb{C}))$, and define the compactification $\overline{SL_2(\mathbb{C})}$ of $SL_2(\mathbb{C})$ as the closure of $\alpha(SL_2(\mathbb{C}))$ in $\mathbb{P}(M_3(\mathbb{C}))$, that is,

$$\overline{SL_2(\mathbb{C})} = \left\{ \begin{pmatrix} a & b \\ c & d \\ \hline & e \end{pmatrix} \in \mathbb{P}(M_3(\mathbb{C})) \mid ad - bc = e^2 \right\}.$$

Then, we obtain a compactification of the semisimple conjugacy class $\mathcal{C}_i$, denoted by $\overline{\mathcal{C}_i}$, that is,

$$\overline{\mathcal{C}_i} = \left\{ \begin{pmatrix} a & b \\ c & d \\ \hline & e \end{pmatrix} \in \mathbb{P}(M_3(\mathbb{C})) \mid ad - bc = e^2, \ a + d = k_i e \right\}.$$

We can define a compactification of the representation variety.

**Definition 4.2.** — We put

$$\overline{Rep_{n,k}} := \{(M_1, \ldots, M_{n-1}) \in \overline{\mathcal{C}_1} \times \cdots \times \overline{\mathcal{C}_{n-1}} \mid \text{Tr}(A_1 \cdots A_{n-1}) = k_n e_1 \cdots e_{n-1}\}$$

(4.2)

where

$$M_1 = \begin{pmatrix} A_1 & \vline & e_1 \\ \hline & & \end{pmatrix}, \ldots, M_{n-1} = \begin{pmatrix} A_{n-1} & \vline & e_{n-1} \\ \hline & & \end{pmatrix}.$$

**Remark 4.3.** — In general, for $X \in \overline{SL_2(\mathbb{C})}$, there is no inverse. Since

$$\text{Tr}(A_{n-1}^{-1} \cdots A_1^{-1}) = \text{Tr}(A_1 \cdots A_{n-1})$$

for $\forall A_i \in SL_2(\mathbb{C})$, we use the condition $\text{Tr}(A_1 \cdots A_{n-1}) = k_n$, instead of $\text{Tr}(A_{n-1}^{-1} \cdots A_1^{-1}) = k_n$. 

TOME 65 (2015), FASCICULE 4
We have the following action of $SL_2(\mathbb{C})$ on $\text{Rep}_{n,k}$, which is compatible with the simultaneous action of $SL_2(\mathbb{C})$ on $\text{Rep}_{n,k}$

$$P \curvearrowleft \left( \left( \begin{array}{c|c} A_1 & e_1 \\ \hline & & \end{array} \right), \ldots, \left( \begin{array}{c|c} A_{n-1} & e_{n-1} \\ \hline & & \end{array} \right) \right) \mapsto \left( \left( \begin{array}{c|c} PA_1P^{-1} & e_1 \\ \hline & & \end{array} \right), \ldots, \left( \begin{array}{c|c} PA_{n-1}P^{-1} & e_{n-1} \\ \hline & & \end{array} \right) \right).$$

(4.3)

We regard $\text{Rep}_{n,k} \subset \mathcal{C}_1 \times \cdots \times \mathcal{C}_{n-1}$ as the closed subset in $\mathbb{P}^4 \times \cdots \times \mathbb{P}^4$. Then, we obtain an embedding in the projective space by the Segre embedding. Let $L$ be an ample line bundle associated with this embedding, that is,

$$L = \bigotimes_{i=1}^{n-1} p_i^*(\mathcal{O}_{\mathbb{P}^4}(1))$$

where $p_i: \text{Rep}_{n,k} \to \mathbb{P}^4$ is the $i$-th projection. Then, $L$ admits the $SL_2(\mathbb{C})$-linearization with respect to the action.

For $x = (M_1, \ldots, M_{n-1}) \in \text{Rep}_{n,k}$, we put

$$I^{nil} := \{ i \in \{1, \ldots, n-1\} \mid M_i \text{ is nilpotent i.e. } e_i = 0 \}.$$

If $I^{nil}$ is not empty, we decompose

(4.4) $$I^{nil} = I_1^{nil} \cup \cdots \cup I_k^{nil}$$

where the index set $I_l^{nil} \subset I^{nil}$ ($1 \leq l \leq k$) consists of indexes of same matrices, that is, matrices indexed by elements of $I_l^{nil}$ are same each other and two matrices which respectively have indexes in $I_l^{nil}$ and $I_{l'}^{nil}$ where $l \neq l'$ are not equal. Let $\sharp I_l^{nil}$ be the cardinality of $I_l^{nil}$, and let $m_1$ be a maximum value in $\sharp I_1^{nil}, \ldots, \sharp I_k^{nil}$. We put

$$J_l := \{ j \in \{1, \ldots, n-1\} \mid M_j \text{ is not nilpotent, } M_j \ast M_i = M_i \ast M_j = M_i, i \in I_l^{nil} \}.$$

Here, we define the product $\ast$ as

$$M \ast M' := \left( \begin{array}{c|c} AA' & e \\ \hline e & \end{array} \right) \in \mathbb{P}M_3(\mathbb{C})$$

for $M := \left( \begin{array}{c|c} A & e \\ \hline e & \end{array} \right)$ and $M' := \left( \begin{array}{c|c} A' & e' \\ \hline e' & \end{array} \right)$.

Note that the product $\ast$ is well-defined in the case where $M$ (resp. $M'$) is nilpotent and $M'$ (resp. $M$) is not nilpotent where $M \in \mathcal{C}$ and $M' \in \mathcal{C}'$. Let $m_2$ be a maximum value in $\{ \sharp J_l \mid l \text{ is satisfied } \sharp I_l^{nil} = m_1, 1 \leq l \leq k \}$. If $I^{nil}$ is empty, then we put $m_1 = m_2 = 0$. 

---

**ANNALES DE L’INSTITUT FOURIER**
Remark 4.4. — Let \((M_1, \ldots, M_{n-1}) \in \text{Rep}_{n,k}\). Suppose that \(i \in I^{nil}\). We normalize the nilpotent matrix \(M_i\):

\[(4.5) \quad M_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

For a matrix \(M_j (j \neq i)\), the condition which, by this transformation, the matrix \(M_j\) is transformed to the following form

\[
\begin{pmatrix} a_j & b_j \\ 0 & d_i \end{pmatrix}
\]

is equivalent to the condition \(M_j * M_i = M_i * M_j = M_i\).

Proposition 4.5. — The point \(x = (M_1, \ldots, M_{n-1})\) is semi-stable (resp. stable) point if and only if \(x\) is satisfied the following condition,

\[(4.6) \quad n - 1 \geq 2m_1 + m_2 \quad (\text{resp. } >).
\]

Proof. — For any integer \(r > 0\), let \(\lambda_r\) be the 1-parameter subgroup (1-PS) of \(SL_2(\mathbb{C})\) given by

\[(4.7) \quad \lambda_r : t \mapsto \begin{pmatrix} t^r & 0 \\ 0 & t^{-r} \end{pmatrix}, t \in \mathbb{C}^\times.
\]

The matrix \(\lambda_r(t)\) acts on \(\text{Rep}_{n,k}\) as follows.

\[
\begin{pmatrix} t^r & 0 \\ 0 & t^{-r} \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} a_{n-1} & b_{n-1} \\ c_{n-1} & d_{n-1} \end{pmatrix}.
\]

We put \(n' := 5^{n-1}\). Let \(\mathbb{A}^{n'}\) be the affine cone over the projective space \(\mathbb{P}^{n'-1}\) which is the target space of the Segre embedding. We take a base change of the affine cone \(\mathbb{A}^{n'}\) via \(\text{Rep}_{n,k} \hookrightarrow \mathbb{P}^{n'-1}\), denoted by the same notation \(\mathbb{A}^{n'}\). Let \(x^* = (M_1^*, \ldots, M_{n-1}^*)\) be the closed point of \(\mathbb{A}^{n'}\) lying over \(x \in \text{Rep}_{n,k}\), that is, \(x^* \neq 0\) and \(x^*\) projects to \(x\). The action (4.3) and the linearization \(L\) define a linear action of \(SL_2(\mathbb{C})\) on \(\mathbb{A}^{n'}\). In particular, the matrix \(\lambda_r(t)\) acts on \(\mathbb{A}^{n'}\) as follows. For each \(i = 1, \ldots, n - 1\), let
$e_1^{(i)}, \ldots, e_5^{(i)}$ be a basis of $\mathbb{A}^5$ such that the matrix

$$M_i^* = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$$

is described by

$$M_i^* = a_i e_1^{(i)} + b_i e_2^{(i)} + c_i e_3^{(i)} + d_i e_4^{(i)} + e_i e_5^{(i)}.$$ 

Let $e_{i_1, \ldots, i_{n-1}}$ be the base $e_1^{(i_1)} \otimes \cdots \otimes e_{i_{n-1}}^{(n-1)}$ of $\mathbb{A}^{n'}$ where $i_1, \ldots, i_{n-1} \in \{1, \ldots, 5\}$. Then, the action of $\lambda_r(t)$ of $\mathbb{A}^5$ is given by

$$\lambda_r(t) \cdot e_{i_1, \ldots, i_{n-1}} = t^{2r(r_1^{+}, \ldots, i_{n-1}^{+}) - r_1^{-}, \ldots, i_{n-1}^{-}} e_{i_1, \ldots, i_{n-1}},$$

where $i_1, \ldots, i_{n-1} \in \{1, \ldots, 5\}$ and $r_1^{+}, \ldots, i_{n-1}^{+}$ (resp. $r_1^{-}, \ldots, i_{n-1}^{-}$) is the number of 2 (resp. 3) in the index set $\{i_1, \ldots, i_{n-1}\}$. For $x^* \in \mathbb{A}^{n'}$ lying over $x \in \text{Rep}_{n,k}$, we write $x^* = \sum x^*_{i_1, \ldots, i_{n-1}} e_{i_1, \ldots, i_{n-1}}$, so that

$$\lambda_r(t) \cdot x^* = \sum t^{2r(r_1^{+}, \ldots, i_{n-1}^{+}) - r_1^{-}, \ldots, i_{n-1}^{-}} x^*_{i_1, \ldots, i_{n-1}},$$

where $r_1^{+}, \ldots, i_{n-1}^{+} = r_1^{-}, \ldots, i_{n-1}^{-}$, and we put

$$\mu^L(x, \lambda_r) := \max \{-r_1^{-}, \ldots, i_{n-1}^{-} | i_1, \ldots, i_{n-1} \text{ such that } x^*_{i_1, \ldots, i_{n-1}} \neq 0\}$$

$$= \# \left\{ i \left| M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, c_i \neq 0 \right. \right\} - \# \left\{ i \left| M_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_i = 0 \right. \right\}.$$ (4.8)

On the other hand, we have

$$\# \left\{ i \left| M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, c_i \neq 0 \right. \right\} = (n - 1) - \# \left\{ i \left| M_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_i = 0 \right. \right\}$$

$$- \# \left\{ i \left| M_i = \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix}, e_i \neq 0 \right. \right\}.$$ 

Then, we have

$$\mu^L(x, \lambda_r)$$

(4.9)

$$= (n - 1) - 2 \# \left\{ i \left| M_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_i = 0 \right. \right\}$$

$$- \# \left\{ i \left| M_i = \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix}, e_i \neq 0 \right. \right\}.$$
By the Hilbert-Mumford criterion (see [14, Theorem 2.1] or [15, Proposition 4.11]), the point \(x\) is stable (resp. semi-stable) for this action if and only if \(\mu_L(g \cdot x, \lambda_r) > 0\) (resp. \(\geq 0\)) for every \(g \in SL_2(\mathbb{C})\) and every 1-PS \(\lambda_r\) of the form (4.7). If the point \(x\) satisfies the condition \(2m_1 < \sharp I^{\text{nil}}\), then we have \(\mu_L(g \cdot x, \lambda_r) > 0\) for any \(g \in SL_2(\mathbb{C})\). On the other hand, we consider the case where the point \(x\) satisfies the condition \(2m_1 \geq \sharp I^{\text{nil}}\). There are at most two components of the decomposition (4.4) of \(I^{\text{nil}}\) such that the cardinalities are \(m_1\). We denote by \(I^{\text{nil}}_{\text{max}}\) the union of the components. If the index set \(I^{\text{nil}} \setminus I^{\text{nil}}_{\text{max}}\) is nonempty, then we have \(\mu_L(g \cdot x, \lambda_r) > 0\) for \(g \in SL_2(\mathbb{C})\) such that \(gM_i g^{-1}\) is the matrix (4.5) where \(i \in I^{\text{nil}} \setminus I^{\text{nil}}_{\text{max}}\). For \(g \in SL_2(\mathbb{C})\) such that \(gM_i g^{-1}\) is the matrix (4.5) where \(i \in I^{\text{nil}}_{\text{max}}\), we have (4.10) \(\mu_L(g \cdot x, \lambda_r) \geq (n - 1) - (2m_1 + m_2)\).

If the index \(i \in I^{\text{nil}}_{\text{max}}\) of the normalized matrix is a element of \(I^{\text{nil}}_l\) such that \(\sharp I^{\text{nil}}_l = m_1\) and \(\sharp J_l = m_2\), then the equality of (4.10) holds. For the other matrix \(g \in SL_2(\mathbb{C})\), we have \(\mu_L(g \cdot x, \lambda_r) > 0\). We have thus proved the proposition. \(\Box\)

We obtain a compactification of the character variety \(\mathcal{R}_{n,k}\).

DEFINITION 4.6. —

\[
\overline{\mathcal{R}_{n,k}} := \text{Proj } H^0(\text{Rep}_{n,k}, L^{\otimes r})^{Ad(SL_2(\mathbb{C}))}.
\]

The variety \(\overline{\mathcal{R}_{n,k}}\) is a projective algebraic variety. This variety may have singular points on the boundary. Then, we should take a resolution of singular points of \(\overline{\mathcal{R}_{n,k}}\). In general, it is not easy to give a systematic resolution of singularities for any \(n\). On the following sections, we treat the cases for \(n = 4, 5\). We will show that \(\overline{\mathcal{R}_{n,k}}\) is non-singular and the boundary divisor is a triangle of \(\mathbb{P}^1\). On Section 6, we will treat the case for \(n = 5\).

5. \(n = 4\)

Let

\[
\begin{pmatrix}
\begin{array}{c|c}
 a_1 & b_1 \\
\hline
c_1 & d_1
\end{array}
\end{pmatrix},
\begin{pmatrix}
\begin{array}{c|c}
 a_2 & b_2 \\
\hline
c_2 & d_2
\end{array}
\end{pmatrix},
\begin{pmatrix}
\begin{array}{c|c}
 a_3 & b_3 \\
\hline
c_3 & d_3
\end{array}
\end{pmatrix}
\in \text{Rep}_{4,k}.
\]

The compactification \(\overline{\text{Rep}_{4,k}}\) is defined by the following equations in \(\mathbb{P}^4 \times \mathbb{P}^4 \times \mathbb{P}^4\)

\[
a_i + d_i = k_i e_i, \quad (i = 1, 2, 3),
\]
\[ a_i d_i - b_i c_i = e_i^2, \quad (i = 1, 2, 3), \]

We analyze the stability. If \( e_i = 0 \) and \( e_j e_k \neq 0 \) (\( j, k \in \{1, 2, 3\} \setminus \{i\} \)), then \( x \) is an unstable point if and only if \( x \) is a point of the orbit of \((M_1, M_2, M_3)\) where

\[
M_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}, \quad M_k = \begin{pmatrix} a_k & b_k \\ 0 & d_k \end{pmatrix}.
\]

If \( e_i = 0, e_j = 0 \), then \( x \) is an unstable point if and only if \( x \) is a point of the orbit of \((M_1, M_2, M_3)\) where two matrices in \( M_1, M_2, M_3 \) are

\[
\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

**Lemma 5.1.** — The point \( x \in \overline{\text{Rep}_{4,k}} \) is stable if and only if \( x \) is semistable.

**Proof.** — The point \( x = (M_1, M_2, M_3) \) is not stable if only \( x \) is normalized as follows.

\[
M_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}, \quad M_k = \begin{pmatrix} a_k & b_k \\ 0 & d_k \end{pmatrix},
\]

where \( c_j \neq 0 \),

or

\[
M_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_j = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_k = \begin{pmatrix} a_k & b_k \\ 0 & d_k \end{pmatrix}.
\]

However, the matrices are not satisfied the equation (5.4). Then, there are no strictly semistable points.

The following theorem shows that our compactification \( \overline{\mathcal{R}_{4,k}} \) of \( \mathcal{R}_{4,k} \) has the same configuration of the boundary divisor as the natural compactification of the Fricke-Klein cubic surface.

**Theorem 5.2.** — The boundary divisor of the compactification \( \overline{\mathcal{R}_{4,k}} \) is a triangle of three projective lines.
Proof. — We describe the boundary divisor explicitly. Let $E_i$ be the image of the divisor $[e_i = 0]$ on $\text{Rep}_{4,k}^4$ by the quotient $\text{Rep}_{4,k}^4 \to \mathcal{R}_{4,k}$ $(i = 1, 2, 3)$. First, we describe $[e_1 = 0]$. We normalize $M_1$ by the $SL_2(\mathbb{C})$-conjugate action as the matrix (4.5). The stabilizer subgroup of the matrix is $\left\{ \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \right\}$.

By the stability, we obtain $c_2 \neq 0$ and $c_3 \neq 0$. Since $c_2 \neq 0$, the matrices of the component $[e_1 = 0]$ are normalized by the action of this stabilizer subgroup:

$\left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 \end{pmatrix} \right), \left( \begin{pmatrix} 0 & -e_2^2 \\ c_2^2 & k_2e_2 \\ c_2e_2 \end{pmatrix} \right), \left( \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \\ e_3 \end{pmatrix} \right)$.

The stabilizer subgroup of the normalized matrices is the torus group $\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}$.

Before we consider the quotient by the torus group, we consider the normalized matrices (5.5). The normalized matrices are defined by the following equations

$$\begin{align*}
\begin{cases}
a_3 + d_3 &= k_3e_3, \\
a_3d_3 - b_3c_3 &= e_3^2, \\
c_2a_3 + k_2e_2c_3 &= 0
\end{cases}
\end{align*}$$

(5.6)
in the Zariski open set $c_2c_3 \neq 0$ of $\mathbb{P}^1 \times \mathbb{P}^1$. By the equations $a_3 + d_3 = k_3e_3$ and $a_3d_3 - b_3c_3 = e_3^2$, we obtain the equation

$$(-a_3^2 + k_3a_3e_3 - e_3^2) - b_3c_3 = 0.$$ 

Note that the equation define a hypersurface of degree 2 in $\mathbb{P}^3$, which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. We put the coordinate $([S_3 : T_3], [U_3 : V_3]) \in \mathbb{P}^1 \times \mathbb{P}^1$ such that

$$(S_3U_3)(T_3V_3) = -a_3^2 + k_3a_3e_3 - e_3^2 = -(a_3 - \alpha^+_3)e_3)(a_3 - \alpha^-_3e_3)$$ 

$S_3V_3 = b_3$ 
$T_3U_3 = c_3$
where \( \alpha_i^+, \alpha_i^- \) are eigenvalues of a matrix of the semisimple conjugacy class \( C_i \). Then, we obtain the following transformation from \( \mathbb{P}^1 \times \mathbb{P}^1 \) to the hypersurface of degree 2 on \( \mathbb{P}^3 \):

\[
a_3 = \frac{\alpha_3^- S_3 U_3 + \alpha_3^+ T_3 V_3}{\alpha_3^+ - \alpha_3^-}, \quad b_3 = S_3 V_3,
\]

\[
c_3 = T_3 U_3, \quad d_3 = \frac{\alpha_3^- S_3 U_3 + \alpha_3^+ T_3 V_3}{\alpha_3^+ - \alpha_3^-}, \quad e_3 = \frac{S_3 U_3 + T_3 V_3}{\alpha_3^+ - \alpha_3^-}.
\]

Therefore, the normalized matrices are defined by

\[
c_2(\alpha_3^- S_3 U_3 + \alpha_3^+ T_3 V_3) + k_2(\alpha_3^+ - \alpha_3^-)c_2(T_3 U_3) = 0
\]

in the Zariski open set \( c_2 T_3 U_3 \neq 0 \) of \( \mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1) \).

We consider the quotient by the torus group. The torus action on \( \mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1) \) is

\[
\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mapsto ([c_2 : e_2], [S_3 : T_3], [U_3 : V_3]) \mapsto ([a^{-1}c_2 : ae_2], [aS_3 : a^{-1}T_3], [a^{-1}U_3 : aV_3]).
\]

We consider the \( SL_2(\mathbb{C}) \)-linearization \( L = \bigotimes_{i=1}^3 p_i^*(\mathcal{O}_{\mathbb{P}^1}(1)) \) on \( \text{Rep}_{4,k} \). We take a pull-back of \( L \) via the embedding

\[
p_{e_1} : \mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1) \hookrightarrow \text{Rep}_{4,k}
\]

defined by the matrices (5.5) and the transform (5.7). Let \( L_{e_1} \) be the pullback of \( L \) on \( \mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1) \). We obtain the \( T \)-linearization on \( L_{e_1} \) induced by the \( SL_2(\mathbb{C}) \)-linearization \( L \) on \( \text{Rep}_{4,k} \). We consider the dual action on \( H^0(\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1), L_{e_1}) \). We have the following basis of the subspace consisting of invariant sections:

\[
s_1 = b_1 \otimes c_2^2 \otimes S_3 U_3, \quad s_2 = b_1 \otimes c_2^2 \otimes T_3 V_3, \quad s_3 = b_1 \otimes c_2 e_2 \otimes T_3 U_3
\]

where \( b_1 \in H^0(\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1), (p_{e_1} \circ p_1)^*(\mathcal{O}_{\mathbb{P}^1}(1))) \) corresponding to the \((1,2)\)-entry of the matrix \( M_1 \). The sections have the relation

\[
\alpha_3^- s_1 + \alpha_3^+ s_2 + k_2(\alpha_3^+ - \alpha_3^-)s_3 = 0
\]

by the equation (5.8). Therefore, we obtain \( E_1 \cong \mathbb{P}^1 \). In the same way, we also obtain \( E_i \cong \mathbb{P}^1 \) (\( i = 2, 3 \)).
We show that $E_1$ and $E_2$ intersect at one point. We substitute $e_2 = 0$ for (5.6). Then, we have the following equations

$$\begin{cases}
    a_3 + d_3 = k_3 e_3, \\
    a_3 d_3 - b_3 c_3 = e_3^2, \\
    a_3 = 0.
\end{cases}$$

The locus defined by the equations above is a quadric curve in $\mathbb{P}^2$, which is isomorphic to $\mathbb{P}^1$. There are two unstable points in the locus, $[b_3 : c_3 : e_3] = [0 : 1 : 0]$ and $[b_3 : c_3 : e_3] = [1 : 0 : 0]$. The intersection is the quotient of $\mathbb{P}^1$ minus the two points by the torus action. Then, the intersection is a point. In the same way, the intersection of $E_2$ and $E_3$ (resp. $E_3$ and $E_1$) is a point. □

### 6. $n = 5$

Let

$$\begin{pmatrix}
    a_1 & b_1 \\
    c_1 & d_1 \\
    e_1
\end{pmatrix}, \quad \begin{pmatrix}
    a_2 & b_2 \\
    c_2 & d_2 \\
    e_2
\end{pmatrix}, \quad \begin{pmatrix}
    a_3 & b_3 \\
    c_3 & d_3 \\
    e_3
\end{pmatrix}, \quad \begin{pmatrix}
    a_4 & b_4 \\
    c_4 & d_4 \\
    e_4
\end{pmatrix} \in \text{Rep}_{5,k}.$$

The compactification $\text{Rep}_{5,k}$ is defined by the following equations in $(\mathbb{P}^4)^4$

(6.1) \hspace{1cm} a_i + d_i = k_i e_i, \quad (i = 1, 2, 3, 4),

(6.2) \hspace{1cm} a_i d_i - b_i c_i = e_i^2, \quad (i = 1, 2, 3, 4),

(6.3) \hspace{1cm} \text{Tr} \left( \begin{pmatrix}
    a_1 & b_1 \\
    c_1 & d_1 \\
    e_1
\end{pmatrix}, \begin{pmatrix}
    a_2 & b_2 \\
    c_2 & d_2 \\
    e_2
\end{pmatrix}, \begin{pmatrix}
    a_3 & b_3 \\
    c_3 & d_3 \\
    e_3
\end{pmatrix}, \begin{pmatrix}
    a_4 & b_4 \\
    c_4 & d_4 \\
    e_4
\end{pmatrix} \right) = k_5 e_1 e_2 e_3 e_4.$$

We consider the stability condition.

**Lemma 6.1.** — The closures of orbits of properly semistable points contain the point

(6.4) \hspace{1cm} s_1 = \begin{pmatrix}
    0 & 1 \\
    0 & 0 \\
    0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
    0 & 1 \\
    0 & 0 \\
    0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
    0 & 0 \\
    0 & 1 \\
    0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
    0 & 0 \\
    0 & 1 \\
    0 & 0
\end{pmatrix}.$$
or

\[
(6.5) \quad s_2 = \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right).
\]

Expect for the points of the orbits of \( s_1 \) and \( s_2 \), the stabilizer groups of every points are finite. Each stabilizer group of the orbits of \( s_1 \) and \( s_2 \) is conjugate to the torus group \( T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\} \).

**Proof.** — Let \( x = (M_1, \ldots, M_4) \) be a property semistable point. By Proposition 4.5, we have \( 2m_1 + m_2 = 4 \). First, we consider the case where \( m_1 = 1, m_2 = 2 \). We put

\[
M_{i_1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad M_{i_2} = \begin{pmatrix} \ast & \ast \\ 0 & \ast \end{pmatrix} ,
\]

\[
M_{i_3} = \begin{pmatrix} \ast & \ast \\ 0 & \ast \end{pmatrix} , \quad M_{i_4} = \begin{pmatrix} \ast & \ast \\ \ast & \ast \end{pmatrix}
\]

(6.6)

where \( \{i_1, \ldots, i_4\} = \{1, \ldots, 4\} \) and \( c_{i_4} \neq 0 \). However, by the condition \( c_{i_4} \neq 0 \), the matrices do not satisfy the equation (6.3).

Second, we consider the case where \( m_1 = 2, m_2 = 0 \). We put

\[
M_{i_1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad M_{i_2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ,
\]

\[
M_{i_3} = \begin{pmatrix} \ast & \ast \\ \ast & \ast \end{pmatrix} , \quad M_{i_4} = \begin{pmatrix} \ast & \ast \\ c_{i_4} & \ast \end{pmatrix}
\]

(6.7)

where \( \{i_1, \ldots, i_4\} = \{1, \ldots, 4\} \), \( c_{i_3} \neq 0 \), and \( c_{i_3} \neq 0 \). If \( (i_1, i_2) = (1, 3) \) or \( (2, 4) \), then the matrices do not satisfy the equation (6.3). Therefore, we consider the case where \( (i_1, i_2) = (1, 2), (2, 3), \) or \( (3, 4) \). The 1-parameter subgroup (4.7) acts on the matrices (6.7). For the matrices \( M_{i_1} \) and \( M_{i_2} \), the action is trivial. The actions of the 1-parameter subgroup \( \lambda_r(t) \) on \( M_{i_3} \)
and $M_{i_4}$ are

\begin{equation}
\lambda_r(t) \cdot M_{i_3} = \begin{pmatrix}
* & t^{2r} \\
-2r c_{i_3} & *
\end{pmatrix}
\begin{pmatrix}
* & t^{2r} \\
-2r c_{i_4} & *
\end{pmatrix}
= \begin{pmatrix}
* & t^{2r} \\
-2r c_{i_3} & *
\end{pmatrix}
\begin{pmatrix}
* & t^{2r} \\
-2r c_{i_4} & *
\end{pmatrix}.
\end{equation}

Then, the limit $\lim_{t \to 0} \lambda_r \cdot M$ is the matrices (6.4) or (6.5).

Since the orbits of the points $s_1$ and $s_2$ are closed, the orbits have the maximum dimension of the stabilizer group, which is one dimension. \qed

We consider a resolution of properly semistable points. We take the blowing up along the orbits of $s_1$ and $s_2$:

\begin{equation}
\widetilde{\text{Rep}}_{5,k} \longrightarrow \overline{\text{Rep}}_{5,k}.
\end{equation}

The simultaneous action of $SL_2(\mathbb{C})$ on $\overline{\text{Rep}}_{5,k}$ induces an action on $\widetilde{\text{Rep}}_{5,k}$. By taking the blowing up (6.9), the condition for stability and unstability is unchanging. On the other hand, the points of the exceptional divisors are stable points. The points of orbits which are not closed are unstable points. Hence, there is no properly semistable point in $\overline{\text{Rep}}_{5,k}$. (See [11, Section 6]). We will show that the quotient of the blowing up is non-singular. First, we describe the blowing up of $\overline{\text{Rep}}_{5,k}$ along the orbit of $s_1$. Let $U_1$ and $U_2$ be the Zariski open sets $U_1 = \{ b_1 \neq 0, b_2 \neq 0, c_3 \neq 0, c_4 \neq 0 \}$ and $U_2 = \{ c_1 \neq 0, c_2 \neq 0, b_3 \neq 0, b_4 \neq 0 \}$ of $\overline{\text{Rep}}_{5,k} \subset \mathbb{C}_1 \times \cdots \times \mathbb{C}_4$. Note that the orbit of $s_1$ is contained in $U_1 \cup U_2$. Since $\mathbb{C}_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ for $i = 1, \ldots, 4$ by the transformation (5.7), we have

\begin{equation}
U_i \subset \overline{\text{Rep}}_{5,k} \subset (\mathbb{P}^1 \times \mathbb{P}^1)^4 \text{ for } i = 1, 2.
\end{equation}

In the open sets $U_1$ and $U_2$, we put the following affine coordinates

\begin{equation}
([1 : x_1], [y_1 : 1]), ([1 : x_2], [y_2 : 1]), ([x_3 : 1], [1 : y_3]), ([x_4 : 1], [1 : y_4]),
\end{equation}

and

\begin{equation}
([z_1 : 1], [1 : w_1]), ([z_2 : 1], [1 : w_2]), ([1 : z_3], [w_3 : 1]), ([1 : z_4], [w_4 : 1]),
\end{equation}
respectively. In the open set $U_1$, the ideal of the orbit of $s_1$ is $(X_1, X_2, X_3, X_4, X_5)$ where

\[
X_0 := e_1 = \frac{y_1 + x_1}{\alpha_1^+ - \alpha_1^-}, \quad X_1 := e_2 = \frac{y_2 + x_2}{\alpha_2^+ - \alpha_2^-},
\]

\[
X_2 := e_3 = \frac{y_3 + x_3}{\alpha_3^+ - \alpha_3^-}, \quad X_3 := e_4 = \frac{y_4 + x_4}{\alpha_4^+ - \alpha_4^-},
\]

\[
X_4 := x_1 - x_2, \quad X_5 := x_3 - x_4.
\]

We can extend the torus action on $\widetilde{\text{Rep}}_{5,k}$ to the torus action on $\text{Rep}_{5,k}$ by

\[
\left(\begin{array}{cc}
a & 0 \\
0 & a^{-1}
\end{array}\right) \cdot [X_0 : X_1 : X_2 : X_3 : X_4 : X_5] \mapsto [a^{-2}X_0 : a^{-2}X_1 : a^2X_2 : a^2X_3 : a^{-2}X_4 : a^2X_5].
\]

On the other hand, in the open set $U_2$, the ideal of the orbit of $s_1$ is $(Y_1, Y_2, Y_3, Y_4, Y_5)$ where

\[
Y_0 := e_1 = \frac{z_1 + w_1}{\alpha_1^+ - \alpha_1^-}, \quad Y_1 := e_2 = \frac{z_2 + w_2}{\alpha_2^+ - \alpha_2^-},
\]

\[
Y_2 := e_3 = \frac{z_3 + w_3}{\alpha_3^+ - \alpha_3^-}, \quad Y_3 := e_4 = \frac{z_4 + w_4}{\alpha_4^+ - \alpha_4^-},
\]

\[
Y_4 := z_1 - z_2, \quad Y_5 := z_3 - z_4.
\]

We can extend the torus action on $\widetilde{\text{Rep}}_{5,k}$ to the torus action on $\text{Rep}_{5,k}$ by

\[
\left(\begin{array}{cc}
a & 0 \\
0 & a^{-1}
\end{array}\right) \cdot [Y_0 : Y_1 : Y_2 : Y_3 : Y_4 : Y_5] \mapsto [a^2Y_0 : a^2Y_1 : a^{-2}Y_2 : a^{-2}Y_3 : a^2Y_4 : a^{-2}Y_5].
\]

Hence, we have

\[
\widetilde{\text{Rep}}_{5,k,s_1} \hookrightarrow (\text{Rep}_{5,k} \setminus U_1 \cup U_2) \cup (U_1 \times \mathbb{P}^5) \cup (U_2 \times \mathbb{P}^5)
\]

where $\widetilde{\text{Rep}}_{5,k,s_1}$ is the blowing up along the orbit of $s_1$. The stabilizer group of any point in the exceptional divisor is

\[
\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.
\]

This action is trivial. In the same way, we can describe the blowing up along the orbit of $s_2$.

**Theorem 6.2.** — In the case of $n = 5$, there exists a non-singular compactification of $\mathcal{R}_{5,k}$ such that the boundary complex is a simplicial decomposition of sphere $S^3$. 

Annales de l'Institut Fourier
Proof. — The outline of the proof is as follows. We put

$$\widetilde{R}_{5,k} := \widetilde{\text{Rep}_{5,k}/SL_2(\mathbb{C})}.$$ 

We have the six components of the boundary divisor of $\widetilde{R}_{5,k}$: the quotients of the proper transformations of the divisors $[e_1 = 0], [e_2 = 0], [e_3 = 0], [e_4 = 0]$ of $\text{Rep}_{5,k}$ and the quotients of the exceptional divisors associated with blowing up along $s_1$ and $s_2$. We denote by $E_1, E_2, E_3, E_4$ and $e_1, e_2, e_3, e_4$ each component. In Step 1, we describe the components $E_1, E_2, E_3$ and $E_4$ explicitly. In Step 2, we describe the intersections $E_i \cap E_j$, $i \neq j$. In particular, the intersections $E_i \cap E_{i+1}$, $i = 1, 2, 3, 4$ (where $E_5$ implies $E_1$) are nonempty and irreducible. On the other hand, the intersections $E_i \cap E_{i+2}$, $i = 1, 2$ are not irreducible. The intersection $E_i \cap E_{i+2}$ consists of two components, denoted by $E_{i,i+2}^+, E_{i,i+2}^-$. Then, we take the blowing up along the components $E_{1,3}^+, E_{1,3}^-, E_{2,4}^+, E_{2,4}^-$. 

(6.11)

$$\tilde{X} \rightarrow X := \widetilde{R}_{5,k}.$$ 

We use the same notation $E_i$ which is the proper transform of $E_i$. We denote by $e_{x_{1,3}}, e_{x_{1,3}}^-, e_{x_{2,4}}, e_{x_{2,4}}^+$ the exceptional divisors associated with the blowing up (6.11). Consequently, the components of the boundary divisor of the compactification $\tilde{X}$ of $R_{5,k}$ are

$$E_1, E_2, E_3, E_4, e_{x_1}, e_{x_2}, e_{x_{1,3}}, e_{x_{1,3}}^-, e_{x_{2,4}}^+, e_{x_{2,4}}^-.$$ 

Next, we see how $e_{x_i}$ and the other components intersect. In Step 3, we describe the 2-dimensional simplices and the 3-dimensional simplices. Finally, we can describe the boundary complex of the boundary divisor of the compactification of the character variety.

Step 1. — We describe the component $E_i$ (i.e. $[e_i = 0]/SL_2(\mathbb{C})$) explicitly. We consider the case where $e_i = 0$. Let $D_i$ be the divisor $[e_i = 0]$ on $\text{Rep}_{5,k}$ for $i = 1, \ldots, 4$. Let $(M_1, \ldots, M_4)$ be a point on $D_1$. We normalize the matrix $M_1$ by the $SL_2(\mathbb{C})$-conjugate action as the matrix (4.5). The stabilizer subgroup of the matrix is the group of upper triangular matrices. From the stability, we obtain $c_2 \neq 0$, $c_3 \neq 0$ or $c_4 \neq 0$. In the case of $c_2 \neq 0$, the matrices of the divisor $D_1$ are normalized by the action of this stabilizer subgroup:

(6.12)

$$\begin{pmatrix}
0 & 1 \\
0 & 0 \\
0 & 0 
\end{pmatrix},
\begin{pmatrix}
0 & -e_2^2 \\
e_2^2 & k_2c_2e_2 \\
c_2e_2 & 0 
\end{pmatrix},
\begin{pmatrix}
a_3 & b_3 \\
c_3 & d_3 \\
e_3 & 0 
\end{pmatrix},
\begin{pmatrix}
a_4 & b_4 \\
c_4 & d_4 \\
e_4 & 0 
\end{pmatrix}.$$
Then, we have the locus defined by the following equations

\[
\begin{aligned}
  a_3 + d_3 &= k_3 e_3 \\
  a_3 d_3 - b_3 c_3 &= e_3^2 \\
  a_4 + d_4 &= k_4 e_4 \\
  a_4 d_4 - b_4 c_4 &= e_4^2 \\
  c_2 a_3 a_4 + k_2 e_2 c_3 a_4 + c_2 b_3 c_4 + k_2 e_2 d_3 c_4 &= 0
\end{aligned}
\]  

(6.13)

in \((\mathbb{P}^1 \times (\mathbb{P}^4 \times \mathbb{P}^4)) \cap [c_2 \neq 0]\). The locus defined by \(a_i + d_i = k_i e_i\) and \(a_i d_i - b_i c_i = e_i^2\) in \(\mathbb{P}^4\) is isomorphic to \(\mathbb{P}^1 \times \mathbb{P}^1\). We put the coordinates \(S_3, T_3, U_3, V_3\) and \(S_4, T_4, U_4, V_4\) of \((\mathbb{P}^1 \times \mathbb{P}^1)^2\) in the same way as in Section 5. Then, the locus of the normalized matrices is defined by the following equation

\[
\begin{aligned}
  c_2 (\alpha_3^3 S_3 U_3 + \alpha_3^2 T_3 V_3)(\alpha_4^3 S_4 U_4 + \alpha_4^2 T_4 V_4) \\
  + k_2 e_2 (\alpha_3^3 - \alpha_3^2)(T_3 U_3)(\alpha_4^3 S_4 U_4 + \alpha_4^2 T_4 V_4) \\
  + c_2 (\alpha_3^3 - \alpha_3^2)(\alpha_4^3 S_3 V_3)(T_4 U_4) \\
  + k_2 e_2 (\alpha_3^3 - \alpha_4^3)(\alpha_3^2 S_3 U_3 + \alpha_3^2 T_3 V_3)(T_3 U_4) &= 0
\end{aligned}
\]

in \((\mathbb{P}^1)^5 \cap [c_2 \neq 0]\). Let \(D_1^{c_2 \neq 0}\) be the Zariski open set of the hypersurface in \((\mathbb{P}^1)^5\). The torus action on \(D_1^{c_2 \neq 0}\) is the following action:

\[
\begin{pmatrix}
a & 0 \\
0 & a^{-1}
\end{pmatrix} \sim ([c_2 : e_2], [S_3 : T_3], [U_3 : V_3], [S_4 : T_4], [U_4 : V_4])
\]

\[
\mapsto ([a^{-1} c_2 : a e_2], [a S_3 : a^{-1} T_3], [a U_3 : a V_3], [a S_3 : a^{-1} T_3], [a^{-1} U_3 : a V_3]).
\]

In the same way as in the case \(c_2 \neq 0\), we have the Zariski open sets of the hypersurfaces in \((\mathbb{P}^1)^5\) corresponding to \(c_3 \neq 0\) and \(c_4 \neq 0\), denoted by \(D_1^{c_3 \neq 0}\) and \(D_1^{c_4 \neq 0}\). We glue \(D_1^{c_2 \neq 0}\), \(D_1^{c_3 \neq 0}\) and \(D_1^{c_4 \neq 0}\), denoted by \(D_1'\). We take the blowing up (6.9). Let \(\tilde{D}_1'\) be the proper transform of \(D_1'\). Then, the component of the boundary divisor \(E_1\) is the quotient of \(\tilde{D}_1'\) by the torus action. Similarly, we may describe the components \(E_j\) \((j = 2, 3, 4)\).

**Step 2.** We denote by \(D_{i,j}\) the intersection of the divisors \([e_i = 0]\) and \([e_j = 0]\) on \(\text{Rep}_{5,k}\). First, we consider the intersection of \(E_1\) and \(E_2\). We substitute \(e_2 = 0\) for (6.13). Then, we have the locus defined by the
the locus defined by the following equations
\[
\begin{align*}
& a_3 + d_3 = k_3 e_3 \\
& a_3 d_3 - b_3 c_3 = e_3^2 \\
& a_4 + d_4 = k_4 e_4 \\
& a_4 d_4 - b_4 c_4 = e_4^2 \\
& a_3 a_4 + b_3 c_4 = 0
\end{align*}
\]
in \((\mathbb{P}^1 \times (\mathbb{P}^1)^2) \cap \{ c_2 \neq 0 \} \). By the transform (5.7), we have the Zariski open set of the hypersurface in \((\mathbb{P}^1)^5 \), denoted by \( D_{c_2 \neq 0}^{c_3 \neq 0} \). Next, we consider the case where \( c_3 \neq 0 \). In the same way as in the case where \( c_2 \neq 0 \), we have the locus defined by the following equations
\[
\begin{align*}
& a_2 + d_2 = 0 \\
& a_2 d_2 - b_2 c_2 = 0 \\
& a_4 + d_4 = k_4 e_4 \\
& a_4 d_4 - b_4 c_4 = e_4^2 \\
& d_2 e_2^2 a_4 - c_2 e_3^2 c_4 + k_3 d_2 c_3 e_3 c_4 = 0
\end{align*}
\]
in \((\mathbb{P}^1 \times (\mathbb{P}^1)^2) \cap \{ c_3 \neq 0 \} \). Since we may put \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} st & s^2 \\ -t^2 & -st \end{pmatrix} \) where \( a + d = 0, ad - be = 0 \), we have
\[
\begin{align*}
& a_4 + d_4 = k_4 e_4 \\
& a_4 d_4 - b_4 c_4 = e_4^2 \\
& (sc_3^2 a_4 - te_3^2 c_4 + k_3 sc_3 e_3 c_4) = 0.
\end{align*}
\]
By the transform (5.7), we have the Zariski open set of the hypersurface in \((\mathbb{P}^1)^5 \), denoted by \( D_{c_2 \neq 0}^{c_3 \neq 0} \). The locus \( D_{c_2 \neq 0}^{c_3 \neq 0} \) is not irreducible. Now, we take the blowing up along the orbits of \( s_1 \) and \( s_2 \). Let \( \widetilde{D}_{c_2 \neq 0}^{c_3 \neq 0} \) be the proper transform of \( D_{c_2 \neq 0}^{c_3 \neq 0} \). Since an orbit of a point of
(6.14) \[ [t = 0] \setminus ([t = 0] \cap [sc_3^2 a_4 - te_3^2 c_4 + k_3 sc_3 e_3 c_4]) \subset \widetilde{D}_{c_2 \neq 0}^{c_3 \neq 0} \]
are not closed, the points of the inverse image of (6.14) on \( \widetilde{D}_{c_2 \neq 0}^{c_3 \neq 0} \) are unstable (see [11, Lemma 6.6]). Then, the quotient of \( \widetilde{D}_{c_2 \neq 0}^{c_3 \neq 0} \) by the torus action is irreducible. Next, we consider the case where \( c_4 \neq 0 \). In the same way as in the case where \( c_3 \neq 0 \), we have the Zariski open set of the hypersurface in \((\mathbb{P}^1)^5 \), denoted by \( D_{c_2 \neq 0}^{c_4 \neq 0} \). We glue \( D_{c_2 \neq 0}^{c_3 \neq 0} \), \( D_{c_2 \neq 0}^{c_4 \neq 0} \) and \( D_{c_4 \neq 0}^{c_4 \neq 0} \), denoted by \( D'_{c_2} \). We take the proper transform of \( D'_{c_2} \) of the blowing up along the orbits of \( s_1 \) and \( s_2 \), denoted by \( \widetilde{D}'_{c_2} \). Then, the intersection of \( E_1 \) and \( E_2 \) is
the quotient of $\tilde{D}_{1,2}$ by the torus action, denoted by $E_{1,2}$. The intersection $E_{1,2}$ is irreducible.

Second, we consider the intersection of $E_1$ and $E_3$. We substitute $e_3 = 0$ for $(6.13)$. Then, we have the locus defined by the following equations

$$
\begin{align*}
\begin{cases}
  a_3 + d_3 = 0 \\
  a_3 d_3 - b_3 c_3 = 0 \\
  a_4 + d_4 = k_4 e_4 \\
  a_4 d_4 - b_4 c_4 = e_4^2 \\
  c_2 a_3 a_4 + k_2 e_2 c_3 a_4 + c_2 b_3 c_4 + k_2 e_2 d_3 c_4 = 0
\end{cases}
\end{align*}
$$

in $(\mathbb{P}^1 \times (\mathbb{P}^4)^2) \cap [c_2 \neq 0]$. We put $a_3 = st, b_3 = s^2, c_3 = -t^2, d_3 = -st$.

Then, we have the equations

$$
(6.15)
\begin{align*}
\begin{cases}
  a_4 + d_4 = k_4 e_4 \\
  a_4 d_4 - b_4 c_4 = e_4^2 \\
  (ta_4 + sc_4)(c_2 s - k_2 e_2 t) = 0
\end{cases}
\end{align*}
$$

We denote the two components $[ta_4 + sc_4 = 0]$ and $[c_2 s - k_2 e_2 t = 0]$ by $D_{1,3}^{c_2 \neq 0, +}$ and $D_{1,3}^{c_2 \neq 0, -}$.

**Remark 6.3.** — Any point $(M_1, M_2, M_3, M_4)$ on $D_{1,3}^{c_2 \neq 0, +}$ is conjugate to the following matrices

$$
(6.16)
\begin{pmatrix}
  0 & 1 & & & \\
  0 & 0 & & & \\
  0 & 0 & & & \\
  0 & 0 & & & \\
\end{pmatrix},
\begin{pmatrix}
  a_2 & b_2 & & & \\
  c_2 & d_2 & & & \\
  e_2 & & & & \\
\end{pmatrix},
\begin{pmatrix}
  0 & 0 & & & \\
  0 & 1 & & & \\
  0 & 0 & & & \\
\end{pmatrix},
\begin{pmatrix}
  0 & b_4 & & & \\
  c_4 & d_4 & & & \\
  e_4 & & & & \\
\end{pmatrix}
\end{pmatrix}
$$

In fact, we normalize the third matrix $M_3$ instead of $M_2$. Then, we have

$$
M_3 = \begin{pmatrix}
  0 & 1 & & & \\
  0 & 0 & & & \\
  0 & 0 & & & \\
  0 & 0 & & & \\
\end{pmatrix}
\text{ or } \begin{pmatrix}
  0 & 0 & & & \\
  0 & 1 & & & \\
  0 & 0 & & & \\
  0 & 0 & & & \\
\end{pmatrix}
$$

In the former case, by the stability, we have $c_4 \neq 0$. However, the matrices do not satisfy the condition $(6.3)$. In the latter case, the equation $ta_4 + sc_4 = 0$ implies that $a_4 = 0$. On the other hand, any point on $D_{1,3}^{c_2 \neq 0, -}$ is conjugate to the following matrices

$$
(6.17)
\begin{pmatrix}
  0 & 1 & & & \\
  0 & 0 & & & \\
  0 & 0 & & & \\
  0 & 0 & & & \\
\end{pmatrix},
\begin{pmatrix}
  a_2 & b_2 & & & \\
  c_2 & 0 & & & \\
  e_2 & & & & \\
\end{pmatrix},
\begin{pmatrix}
  0 & 0 & & & \\
  0 & 1 & & & \\
  0 & 0 & & & \\
\end{pmatrix},
\begin{pmatrix}
  a_4 & b_4 & & & \\
  c_4 & d_4 & & & \\
  e_4 & & & & \\
\end{pmatrix}
$$
We consider the cases where \( c_3 \neq 0 \) and \( c_4 \neq 0 \). In the same way as in the case where \( c_2 \neq 0 \), we have the Zariski open sets
\[
D_{1,3}^{c_3 \neq 0, +}, D_{1,3}^{c_3 \neq 0, -}, D_{1,3}^{c_4 \neq 0, +}, D_{1,3}^{c_4 \neq 0, -}
\]
of the hypersurfaces in \((\mathbb{P}^1)^5\). We glue \( D_{1,3}^{c_2 \neq 0, +}, D_{1,3}^{c_2 \neq 0, +} \) and \( D_{1,3}^{c_2 \neq 0, +} \) (resp. \( D_{1,3}^{c_2 \neq 0, -}, D_{1,3}^{c_2 \neq 0, -} \) and \( D_{1,3}^{c_2 \neq 0, -} \)), denoted by \( \tilde{D}_{1,3}^+ \) (resp. \( \tilde{D}_{1,3}^- \)). We take the blowing up \((6.9)\). Let \( \tilde{D}_{1,3}^+ \) and \( \tilde{D}_{1,3}^- \) be the proper transforms of \( D_{1,3}^+ \) and \( D_{1,3}^- \), respectively. Then, the intersections of \( E_1 \) and \( E_3 \) are the quotients of \( \tilde{D}_{1,3}^+ \) and \( \tilde{D}_{1,3}^- \) by the torus action, denoted by \( E_1^+ \) and \( E_1^- \).

We consider the intersections \( E_2 \cap E_3, E_3 \cap E_4 \) and \( E_1 \cap E_4 \). In the same way as in the case \( E_1 \cap E_2 \), the intersections are irreducible, denoted by \( E_{2,3}, E_{3,4} \) and \( E_{1,4} \).

We consider the intersection of \( E_2 \) and \( E_4 \). In the same way as in the case \( E_1 \cap E_3 \), the intersection \( E_2 \cap E_4 \) is not irreducible. The intersection has two components, denoted by \( E_{2,4}^+ \) and \( E_{2,4}^- \). Here, the components \( E_{2,4}^+ \) and \( E_{2,4}^- \) correspond respectively to the following matrices
\[
\begin{pmatrix}
a_1 & b_1 \\
c_1 & d_1 \\
e_1
\end{pmatrix}, \begin{pmatrix}
a_3 & b_3 \\
c_3 & d_3 \\
e_3
\end{pmatrix}, \begin{pmatrix}
a_3 & 0 \\
0 & 1 \\
0 & 0
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
0 & b_1 \\
c_1 & d_1 \\
e_1
\end{pmatrix}, \begin{pmatrix}
0 & 1 \\
0 & 0 \\
0 & 0
\end{pmatrix}, \begin{pmatrix}
a_3 & b_3 \\
c_3 & d_3 \\
e_3
\end{pmatrix}, \begin{pmatrix}
0 & 0 \\
0 & 1 \\
0 & 0
\end{pmatrix}.
\]

Now, we take the blowing up along the components \( E_{1,3}^+, E_{1,3}^-, E_{2,4}^+, E_{2,4}^- \):
\[
\tilde{X} \longrightarrow X := \mathcal{R}_{5,k}.
\]

We use the same notation \( E_i \) which is the proper transforms of \( E_i \). We denote by \( ex_{1,3}^+, ex_{1,3}^-, ex_{2,4}^+, ex_{2,4}^- \) the quotients of the exceptional divisors associated with this blowing up. Consequently, we have the ten components of the boundary divisor of the compactification \( \tilde{X} \) of \( \mathcal{R}_{5,k} \)
\[
E_1, E_2, E_3, E_4, ex_1, ex_2, ex_{1,3}^+, ex_{1,3}^-, ex_{2,4}^+, ex_{2,4}^-,
\]
and we obtain that the intersections
\[
E_1 \cap E_2, \ E_2 \cap E_3, \ E_3 \cap E_4, \ E_4 \cap E_1
\]
and
\[
E_1 \cap ex_{1,3}^+, \ E_3 \cap ex_{1,3}^-, \ E_2 \cap ex_{2,4}^+, \ E_4 \cap ex_{2,4}^-,
\]
are nonempty and irreducible.
We describe the intersections of the other pairs. We consider the intersection of \(\text{ex}^+_{1,3}\) and \(E_4\). If we substitute \(e_4 = 0\) for the matrix (6.16), then we have \(d_4 = 0\). Moreover, we have \(b_4 = 0\) or \(c_4 = 0\). Then, we obtain that
\[ D_{1,3}^+ \cap [e_4 = 0] = \{s_1, s_2\} \cup \{\text{points whose orbits are not closed}\}. \]
By the blowing up along \(s_1\) and \(s_2\), we obtain that the intersection of \(E_{1,3}^+\) and \(E_4\) is empty (see [11, Lemma 6.6]). Then, the intersection of \(\text{ex}^+_{1,3}\) and \(E_4\) is empty. In the same way as above, the intersections
\[ \text{ex}^-_{1,3} \cap E_2, \quad \text{ex}^+_{2,4} \cap E_3, \quad \text{ex}^-_{2,4} \cap E_1 \]
are empty. On the other hand, the intersections
\[ \text{ex}^+_{1,3} \cap E_2, \quad \text{ex}^-_{1,3} \cap E_4, \quad \text{ex}^+_{2,4} \cap E_1, \quad \text{ex}^-_{2,4} \cap E_3, \quad \text{ex}^+_{1,3} \cap \text{ex}^-_{1,3}, \quad \text{ex}^+_{2,4} \cap \text{ex}^-_{2,4} \]
are nonempty and irreducible. Next, we consider the intersections of the pairs containing \(\text{ex}_1\) or \(\text{ex}_2\). The orbit of the point \(s_1\) (resp. \(s_2\)) is contained in the components \(D_1, \ldots, D_4\) and \(D_{1,3}^+, D_{2,4}^\pm\), respectively. Here, \(D_{1,3}^+\) and \(D_{2,4}^\pm\) are the irreducible components of \(D_{1,3}\) and \(D_{2,4}\). Then, the intersections \(\text{ex}_i \cap E_j\) and \(\text{ex}_i \cap \text{ex}^\pm_{k,k+2}\) are nonempty and irreducible for \(i = 1, 2, j = 1, \ldots, 4\) and \(k = 1, 2\). On the other hand, the orbits of the point \(s_1\) and \(s_2\) are not intersect. Then, the intersection of \(\text{ex}_1\) and \(\text{ex}_2\) is empty.

**Step 3.** — We draw the vertexes and the 1-dimensional simplices except \(\text{ex}_1\) and \(\text{ex}_2\). Then, we obtain the graph of Figure 6.1. We consider the following sphere
\[ \mathbb{R}^4 \cap S^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\}. \]
We arrange the vertexes except \(\text{ex}_1\) and \(\text{ex}_2\) on \(S^2 = S^3 \cap [w = 0]\) and arrange the vertexes \(\text{ex}_1\) and \(\text{ex}_2\) at \((0, 0, 0, 1)\) and \((0, 0, 0, -1)\) respectively.
We glue together the vertex $ex_i$ $(i = 1, 2)$ and each vertex on $S^2 = S^3 \cap \{w = 0\}$.

Next, we describe the 2-dimensional simplices. First, we consider the intersections $E_1 \cap E_2 \cap ex_1^{+}$ and $E_2 \cap E_3 \cap ex_1^{+}$. The intersection $E_1 \cap E_2 \cap E_3 = E_1^{+} \cap E_2$ is nonempty and irreducible in $\mathcal{R}_{5,k}$. We take the blowing up along $E_1^{+}$. Then, the intersections $E_1 \cap E_2 \cap ex_1^{+}$ and $E_2 \cap E_3 \cap ex_1^{+}$ are irreducible. Second, we consider the intersections $E_1 \cap ex_1^{+} \cap ex_1^{-}$ and $E_3 \cap ex_1^{+} \cap ex_1^{-}$. We substitute $d_2 = 0$ for the matrices (6.16). Then, we have that $D_1 \cap [d_2 = 0]$ is irreducible. Therefore, the intersection $E_1^{+} \cap E_3^{-}$ is irreducible. We take the blowing up along $E_1^{+}$. Then, the intersections $E_1 \cap ex_1^{+} \cap ex_1^{-}$ and $E_3 \cap ex_1^{+} \cap ex_1^{-}$ are irreducible. Then, we glue together the triangles

$$(E_1, E_2, ex_1^{+}), (E_2, E_3, ex_1^{+}), (E_1, ex_1^{+}, ex_1^{-}) \text{ and } (E_3, ex_1^{+}, ex_1^{-})$$

in the graph of Figure 6.1. In the same way as above, we glue together each triangle. Then, we obtain that the complex of Figure 3 is a simplicial decomposition of $S^2$. Third, we consider the intersection of 3-tuple of components of the boundary divisor containing $ex_1$ or $ex_2$. The divisors $ex_1$ and $ex_2$ are the exceptional divisors of the blowing up along the orbits of $s_1$ and $s_2$. The orbits of $s_1$ and $s_2$ are contained in $D_1 \cap D_{i+1}$ $(i = 1, \ldots, 4)$, $D_1^{+}$, $D_1^{-}$, $D_2^{+}$, and $D_2^{-}$, respectively. Then, the intersections $E_i \cap E_{i+1} \cap ex_j$, $E_{k}^{+} \cap ex_j$ and $E_{k}^{-} \cap ex_j$ are nonempty and irreducible for $i = 1, \ldots, 4$, $j = 1, 2$, and $k = 1, 2$. We take the blowing up along $E_1^{+}$ and $E_1^{-}$. Then, we can glue together the 3-tuples which have either $ex_i$ or $ex_i$ in the graph.

Lastly, we describe the 3-dimensional simplices. We can glue together the 4-tuples of components of the boundary divisor such that the 4-tuples have either $ex_i$ or $ex_i$ and 3-tuples expect $ex_i$ or $ex_i$ are glued together. On the other hand, the intersections of the 4-tuples which have the vertexes expect $ex_i$ or $ex_i$ are empty. Then, we obtain that the boundary complex of the compactification $\tilde{X}$ of $\mathcal{R}_{5,k}$ is simplicial decomposition of $S^3$. \[\square\]

**BIBLIOGRAPHY**


Manuscrit reçu le 6 juin 2013,
révisé le 24 octobre 2014,
accepté le 27 novembre 2014.

Arata KOMYO
Department of Mathematics,
Graduate School of Science, Kobe University,
1-1 Rokkodai-cho, Nada-ku, Kobe, 657-8501 (Japan)
akomyo@math.kobe-u.ac.jp