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## BIRATIONAL POSITIVITY IN DIMENSION 4

by Behrouz TAJI

With an appendix by Frédéric CAMPANA

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ABSTRACT. — In this paper we prove that for a nonsingular projective variety of dimension at most 4 and with non-negative Kodaira dimension, the Kodaira dimension of coherent subsheaves of  $\Omega^p$  is bounded from above by the Kodaira dimension of the variety. This implies the finiteness of the fundamental group for such an  $X$  provided that  $X$  has vanishing Kodaira dimension and non-trivial holomorphic Euler characteristic.

RÉSUMÉ. — Dans cet article, nous montrons que pour une variété projective lisse,  $X$ , de dimension au plus 4 et de dimension de Kodaira non négative, la dimension de Kodaira des sous-faisceaux cohérents de  $\Omega^p$  est majorée par la dimension de Kodaira de  $X$ . Cela implique la finitude du groupe fondamental de  $X$  lorsque la dimension de Kodaira de  $X$  est nulle et sa caractéristique holomorphe d'Euler est non nulle.

### 1. Introduction

It is a classical result of Bogomolov known as Bogomolov-De Franchis-Castelnuovo inequality that for projective varieties, Kodaira dimension of rank one subsheaves of  $\Omega^p$  is bounded from above by  $p$ . In [3] Campana has proved that assuming some standard conjectures for non-uniruled varieties, the Kodaira dimension of such subsheaves admits another upper bound, namely the Kodaira dimension of the variety (see Theorem 1.3).

Throughout this paper we will refer to the conjectures of the minimal model program. See [11] for the basic definitions and background. Here we will only state the ones that we need.

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*Keywords:* Kodaira dimension, varieties of Kodaira dimension zero, minimal model theory.

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CONJECTURE 1.1 (The minimal model conjecture for nonsingular varieties). — *Let  $X$  be a nonsingular projective variety over  $Z$ . If  $K_X$  is pseudo-effective/ $Z$ , then  $X/Z$  has a minimal model. Otherwise it has a Mori fiber space/ $Z$ .*

Remember that by  $K_X$  pseudo-effective/ $Z$ , we mean  $K_X$  can be realized as limit of effective divisors in the relative Neron-Severi space  $N^1(X/Z)$ .

CONJECTURE 1.2 (The abundance conjecture for minimal models with terminal singularities). — *Let  $X/Z$  be a normal projective variety with terminal singularities and  $\mathbb{Q}$ -Cartier canonical divisor. If  $K_X$  is nef/ $Z$  then it is semi-ample/ $Z$ , i.e., it is pull back of a divisor that is ample/ $Z$ .*

These two conjectures put together is sometimes referred to as the good minimal model conjecture.

For nonsingular projective varieties, Campana has introduced in [3] a new and a more general notion of Kodaira dimension defined by

$$\kappa^+(X) := \max \{ \kappa(\det \mathcal{F}) \mid \mathcal{F} \text{ is a coherent subsheaf of } \Omega_X^p, \text{ for some } p \}$$

and conjectured that  $\kappa = \kappa^+$  when  $\kappa \geq 0$ . He proves that the conjectured equality holds assuming the good minimal model conjecture:

THEOREM 1.3 (Equality of  $\kappa$  and  $\kappa^+$  when  $\kappa \geq 0$ , cf. [3, Prop. 3.10]). *Let  $X$  be a nonsingular projective variety in dimension  $n$  with non-negative Kodaira dimension. If the good minimal model conjecture holds for nonsingular projective varieties of dimension up to  $n$  and with vanishing Kodaira dimension, then  $\kappa(X) = \kappa^+(X)$ .*

This in particular refines Bogomolov's inequality when the Kodaira dimension is relatively small, for example when  $\kappa(X) = 0$  and  $L \subseteq \Omega_X^p$ , then  $\kappa(L) \leq 0$ .

Note that when  $c_1 = 0$  then we have  $\kappa = \kappa^+$  by Bochner's vanishing coupled with Yau's solution [15] to the Calabi's conjecture.

By Theorem 1.3,  $\kappa(X)$  and  $\kappa^+(X)$  coincide for non-uniruled threefolds as a consequence of the minimal model program or MMP for short (see for example [9]). We prove Campana's conjecture in dimension four and for varieties with positive Kodaira dimension in dimension five:

THEOREM 1.4. — *Let  $X$  be a nonsingular projective variety.*

- (i) *If dimension of  $X$  is at most 4 and  $\kappa(X) \geq 0$ , then  $\kappa = \kappa^+$ .*
- (ii) *If dimension of  $X$  is 5 and  $\kappa(X) \geq 1$ , then  $\kappa = \kappa^+$ .*

Furthermore one can show that the cotangent bundle of such varieties is birationally stable (see the appendix A).

Theorem 1.4 is a consequence of a much more general result that we obtain in this paper:

**THEOREM 1.5.** — *Let  $X$  be a nonsingular projective variety of dimension  $n$ . Assume that the good minimal model conjecture holds for terminal projective varieties with zero Kodaira dimension up to dimension  $n - m$ , where  $m > 0$ . If  $\kappa(X) \geq m - 1$  then  $\kappa = \kappa^+$ .*

An important corollary of 1.4 is the finiteness of the fundamental group of 4-dimensional varieties with vanishing Kodaira dimension and non-zero holomorphic Euler characteristic (see 1.7 below) which follows from a remarkable result of Campana:

**THEOREM 1.6** (Finiteness of the fundamental groups, cf. [3, Cor. 5.3]). *Let  $X$  be a nonsingular projective variety. If  $\kappa^+(X) = 0$  and  $\chi(X, \mathcal{O}_X) \neq 0$ , then  $\pi_1(X)$  is finite.*

**THEOREM 1.7.** — *Let  $X$  be a nonsingular projective variety of dimension at most 4. Assume  $\kappa(X) = 0$  and  $\chi(X, \mathcal{O}_X) \neq 0$ , then  $\pi_1(X)$  is finite.*

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## 2. Generic semi-positivity and pseudo-effectivity

Let  $X$  be a non-uniruled nonsingular projective variety. It is a well known result of Miyaoka, cf. [12, 13] that  $\Omega_X$  is generically semi-positive. This means that the determinant line bundle of any torsion free quotient of  $\Omega_X$  has non-negative degree on curves cut out by sufficiently ample divisors. Equivalently we can characterize this important positivity result by saying that  $\Omega_X$  restricted to these general curves is nef unless  $X$  is uniruled. This property is sometimes called *generic nefness*. Since nefness is invariant under taking symmetric powers this result automatically generalizes to  $\Omega_X^p$ . Using the same characteristic  $p$  arguments as Miyaoka and some deep results in differential geometry, Campana and Peternell have shown that in fact such a determinant line bundle is dual to the cone of moving curves, i.e., its restriction to these curves has non-negative degree. By [1] this is the same as saying that it is pseudo-effective.

**THEOREM 2.1** (Pseudo-effectivity of quotients of  $\Omega_X^p$ , cf. [6, Thm. 1.7]). *Let  $X$  be a non-uniruled nonsingular projective variety and let  $\mathcal{F}$  be an  $\mathcal{O}_X$ -module torsion free quotient of  $\Omega_X^p$ . Then  $\det \mathcal{F}$  is a pseudo-effective line bundle.*

### 3. The refined Kodaira dimension

In this section we will use more or less the same ideas as Cascini [7] to show that  $\kappa$  and  $\kappa^+$  coincide for nonsingular projective varieties of dimension four with non-negative Kodaira dimension and also for varieties of dimension five with positive Kodaira dimension. The following proposition is a result of Campana, cf. [3]. We include a proof for completeness.

**PROPOSITION 3.1.** — *Let  $X$  be a nonsingular projective variety with  $\kappa(X) = 0$ . If  $X$  has a good minimal model then  $\kappa = \kappa^+$ .*

*Proof.* — Let  $Y$  be a  $\mathbb{Q}$ -factorial normal variety with at worst terminal singularities serving as a good minimal model for  $X$ . Note that  $K_Y$  is numerically trivial. Let  $\pi : \tilde{Y} \rightarrow Y$  be a resolution. Since  $\kappa(\tilde{Y}) = 0$ ,  $\tilde{Y}$  is not uniruled. Let  $\mathcal{F} \subseteq \Omega_{\tilde{Y}}^p$  be a coherent subsheaf with maximum Kodaira dimension, i.e.,  $\kappa(\det \mathcal{F}) = \kappa^+(\tilde{Y})$ .

Let  $C$  be an irreducible curve on  $Y$  cut out by sufficiently general hyperplanes and let  $\tilde{C}$  to be the corresponding curve in  $\tilde{Y}$ . Now using the standard isomorphism:  $\Omega_{\tilde{Y}}^p|_{\tilde{C}} \cong K_{\tilde{Y}}|_{\tilde{C}} \otimes \wedge^{n-p} \mathcal{F}_{\tilde{Y}}|_{\tilde{C}}$ , we get  $\mathcal{F}^*|_{\tilde{C}}$  as a quotient of  $K_{\tilde{Y}}^*|_{\tilde{C}} \otimes \Omega_{\tilde{Y}}^{n-p}|_{\tilde{C}}$ . But  $K_{\tilde{Y}}^*$  is numerically trivial on  $\tilde{C}$  and  $\Omega_{\tilde{Y}}^{n-p}|_{\tilde{C}}$  is nef by Miyaoka, so  $\mathcal{F}^*|_{\tilde{C}}$  must also be nef and we have

$$\deg(\det \mathcal{F}|_{\tilde{C}}) \leq 0.$$

But this inequality holds for a covering family of curves and thus  $\kappa(\mathcal{F}) \leq 0$ . ay from these divisors, so they must coincide everywhere and hence they are proportional everywhere. □

As was mentioned in the introduction (Theorem 1.3), assuming the good Minimal Model conjecture for varieties up to dimension  $n$  and with zero Kodaira dimension, we have  $\kappa = \kappa^+$  in the case of  $n$ -dimensional varieties of positive Kodaira dimension as well. See [3, Prop. 3.10] for a proof. The main result of this paper is concerned with replacing this assumption with the abundance conjecture in lower dimensions.

*Remark 3.2.* — Following the recent developments in the minimal model program, we now know that we have a good minimal model when numerical Kodaira dimension is zero. The Proposition 3.1 shows that  $\kappa^+(X)$  also

vanishes in this case. By [3] this implies in particular that nonsingular varieties with vanishing numerical dimension have finite fundamental groups as long as they have non-trivial holomorphic Euler characteristic (see 1.6).

We will need the following lemmas in the course of the proof of our main result.

LEMMA 3.3. — *Let  $f : X \rightarrow Z$  be a surjective morphism with connected fibers between normal projective varieties  $X$  and  $Z$ . Let  $D$  be an effective  $\mathbb{Q}$ -Cartier divisor in  $X$  that is numerically trivial on the general fiber of  $f$ . If  $D$  is  $f$ -nef, then there exist birational morphisms  $\pi : \tilde{Z} \rightarrow Z$ ,  $\mu : \tilde{X} \rightarrow X$ , a  $\mathbb{Q}$ -Cartier divisor  $G$  in  $\tilde{Z}$ , and an equidimensional morphism  $\tilde{f} : \tilde{X} \rightarrow \tilde{Z}$  such that  $\mu^*(D) = \tilde{f}^*(G)$ .*

*Proof.* — The fact that we can modify the base of our fibration to get a morphism whose fibers are of constant dimension is guaranteed by [14]. This is called *flattening* of  $f$ . Let  $\tilde{X}$  be a normal birational model of  $X$  and  $\tilde{Z}$  a smooth birational model for  $Z$  such that  $\tilde{f} : \tilde{X} \rightarrow \tilde{Z}$  is flat.

If general fibers are curves, by assumption the degree of  $\mu^*(D)$  on general fibers of  $\tilde{f}$  is zero. On the other hand  $\mu^*(D)$  is effective and relatively nef, so it must be trivial on all fibers. This implies the existence of the required  $\mathbb{Q}$ -Cartier divisor  $G$  in  $\tilde{Z}$ .

In the case of higher dimensional fibers,  $\mu^*(D)$  must still be numerically trivial on all fibers of  $\tilde{f}$ . To see this, let  $C$  be an irreducible curve contained in a  $d$ -dimensional non-general fiber  $\tilde{F}_0$  of  $\tilde{f}$ . Then for a sufficiently general members  $D_i$  of the linear system of an ample divisor  $H$ , we have

$$D_1 \dots D_{d-1} \tilde{F}_0 = mC + C',$$

where  $C'$  is an effective curve and  $m$  accounts for the multiplicity of the irreducible component of  $F_0$  containing  $C$ . Now since  $\mu^*(D)$  is numerically trivial on the general fiber of  $\tilde{f}$ , we have

$$\mu^*D \cdot (mC + C') = 0.$$

But  $\mu^*(D)$  is  $\tilde{f}$ -nef, so that  $\mu^*(D) \cdot C = 0$ .

We know that  $\mu^*(D)$  is effective, so  $\mu^*(D)$  must be trivial on all fibers. Again this ensures the existence of a  $\mathbb{Q}$ -Cartier divisor  $G$  in  $\tilde{Z}$  such that  $\mu^*(D) = \tilde{f}^*(G)$ . □

In the course of the proof of Lemma 3.3 we repeatedly used the standard fact that given a surjective morphism  $f : X \rightarrow Z$  between normal varieties  $X$  and  $Z$ , where  $Z$  is  $\mathbb{Q}$ -factorial, and an effective  $\mathbb{Q}$ -Cartier divisor  $D$  that is trivial on all fibers, we can always find a  $\mathbb{Q}$ -Cartier divisor  $G$  in

$Z$  such that  $D = f^*(G)$ . One can verify this by reducing it to the case where  $X$  is a surface and  $Z$  is a curve. Here the negative semi-definiteness of the intersection matrix of the irreducible components of singular fibers establishes the claim.

For application a natural setting for Lemma 3.3 is the relative minimal model program. The following is a reformulation of this lemma in this context.

LEMMA 3.4. — *Let  $f : X \rightarrow Z$  be a surjective morphism with connected fibers between nonsingular projective varieties  $X$  and  $Z$  with dimension  $n$  and  $m$  respectively. Assume  $\kappa(X) \geq 0$  and that  $X/Z$  has a minimal model  $Y/Z$ . Denote the morphism between  $Y$  and  $Z$  by  $\psi$ . Also assume that the abundance conjecture for varieties of vanishing Kodaira dimension holds in dimension  $n - m$ . If the Kodaira dimension of the general fiber of  $f$  is zero, then there exist birational morphisms  $\pi : \tilde{Z} \rightarrow Z$ ,  $\mu : \tilde{Y} \rightarrow Y$ , a  $\mathbb{Q}$ -Cartier divisor  $G$  in  $\tilde{Z}$ , and an equidimensional morphism  $\tilde{\psi} : \tilde{Y} \rightarrow \tilde{Z}$  such that  $\mu^*(K_Y) = \tilde{\psi}^*(G)$ .*

$$\begin{array}{ccccc}
 X & \dashrightarrow & Y & \xleftarrow{\mu} & \tilde{Y} \\
 & \searrow & \downarrow \psi & & \downarrow \tilde{\psi} \\
 & & Z & \xleftarrow{\pi} & \tilde{Z}
 \end{array}$$

*Proof.* — Since  $K_Y$  is  $\psi$ -nef and that the dimension of the general fibers is  $n - m$ , we find that the canonical of the general fiber is torsion by the abundance assumption. Now apply Lemma 3.3 to  $\psi : Y \rightarrow Z$  and take  $K_Y$  to be  $D$ . □

We now turn to another crucial ingredient that we shall use in the Proof of 3.7.

LEMMA 3.5. — *Let  $f : X \rightarrow Z$  be a surjective morphism with connected fibers between normal projective varieties  $X$  and  $Z$  of dimension  $n$  and  $k$  respectively. Let  $D$  be a  $\mathbb{Q}$ -Cartier divisor in  $Z$ . If  $f^*D$  is dual to the cone of moving curves then so is  $D$ .*

*Proof.* — First assume that  $f$  is birational. Let  $C$  be a moving curve in  $Z$  and let  $\mu : \tilde{Z} \rightarrow Z$ , be a birational morphism such that  $\mu_*(\tilde{C}) = C$ , where  $\tilde{C}$  is a complete intersection curve cut out by hyperplanes. Let  $\pi : \tilde{X} \rightarrow X$  be a suitable modification such that  $\tilde{f} : \tilde{X} \rightarrow \tilde{Z}$  is a morphism and we have

the following commutative diagram.

$$\begin{array}{ccc}
 \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Z} \\
 \pi \downarrow & & \downarrow \mu \\
 X & \xrightarrow{f} & Z
 \end{array}$$

Now let  $\tilde{C} = H_1 \dots H_{k-1}$ , where  $H_1, \dots, H_{k-1}$  are ample divisors in  $\tilde{Z}$ . We have

$$\begin{aligned}
 \mu^* D \cdot \tilde{C} &= \mu^* D \cdot H_1 \dots H_{k-1} \\
 &= \tilde{f}^*(\mu^* D) \cdot \tilde{f}^* H_1 \dots \tilde{f}^* H_{k-1} \\
 &= \pi^*(f^* D) \cdot \tilde{f}^* H_1 \dots \tilde{f}^* H_{k-1} \quad \text{by commutativity of the diagram.}
 \end{aligned}$$

Clearly  $\pi^*(f^* D)$  is pseudo-effective. Now since nef divisors are numerically realized as limit of ample ones we have

$$\pi^*(f^* D) \cdot \tilde{f}^* H_1 \dots \tilde{f}^* H_{k-1} \geq 0,$$

which implies  $\mu^*(D) \cdot \tilde{C} \geq 0$ . So that  $D \cdot C \geq 0$  as required.

Now assume that  $f$  is not birational and let  $C = H_1 \dots H_{k-1}$  be an irreducible curve cut out by ample divisors in  $Z$ . In particular  $C$  is of constant dimension along the image of fibers. After cutting down by general hyperplanes  $H'_1, \dots, H'_{n-k}$ , we can find an irreducible curve

$$C' = H'_1 \dots H'_{n-k} \cdot f^*(H_1) \dots f^*(H_{k-1})$$

that maps surjectively onto  $C$ . Thus we have  $(\deg f|_{C'}) \cdot D \cdot C = f^* D \cdot C' \geq 0$ .

For a moving curve that is not given by intersections of hyperplanes, we repeat the same argument as above after going to a suitable modification. □

*Remark 3.6.* — We know by [1] that for nonsingular projective varieties, pseudo-effective divisors are dual to the cone of moving curves. Using the lemma above, we can easily extend this to normal varieties by going to a resolution. This fact is of course already well known. For convenience we rephrase the Lemma 3.5 as follows:

LEMMA 3.5'. — *Let  $f : X \rightarrow Z$  be a surjective morphism with connected fibers between normal projective varieties. Let  $D$  be a  $\mathbb{Q}$ -Cartier divisor in  $Z$ . If  $f^* D$  is pseudo-effective then so is  $D$ .*

We shall prove Theorem 1.5 as a consequence of the following proposition:



PROPOSITION 3.7. — *Let  $X$  be a nonsingular projective variety of dimension  $n$  with  $\kappa(X) \geq 0$ . Assume that the good minimal model conjecture holds for terminal projective varieties with zero Kodaira dimension up to dimension  $n - m$ , where  $m > 0$ . Let  $\mathcal{F} \subseteq \Omega_X^p$  be a coherent subsheaf and define the line bundle  $L = \det \mathcal{F}$ . If  $\kappa(K_X + L) \geq m$ , then  $\kappa(L) \leq \kappa(X)$ .*

*Proof.* — First a few observations. The isomorphism  $K_X^* \otimes \Omega_X^p \cong \wedge^{n-p} \mathcal{T}_X$  implies that  $K_X^* \otimes \mathcal{F}$  is a subsheaf of  $\wedge^{n-p} \mathcal{T}_X$ . But  $X$  is not uniruled and so by 2.1,  $rK_X - L$  is pseudo-effective as a Cartier divisor, where  $r$  is the rank of  $\mathcal{F}$ .

We can of course assume that  $X$  is not general type. Now if we assume that  $K_X + L$  is big then by using the equality  $(r + 1)K_X = (rK_X - L) + (K_X + L)$  and pseudo-effectivity of  $rK_X - L$ , we conclude that  $K_X$  must be big as well. So we may also assume that  $K_X + L$  is not big and that  $\kappa(L) > 0$ .

Without loss of generality we can also assume that the rational map  $X \dashrightarrow Z$  corresponding to  $K_X + L$  is a morphism, since we can always go to a suitable modification, pull back  $L$  and prove the theorem at this level. Denote this map by  $i_{K_X+L}$  and note that by definition we have  $\kappa((K_X + L)|_F) = 0$ , where  $F$  is the general fiber of  $i_{K_X+L}$ . Finally, we observe that  $\kappa(F) \leq \kappa((K_X + L)|_F) = 0$  and as we are assuming that  $X$  has non-negative Kodaira dimension, we have  $\kappa(F) = 0$ .

Claim 3.8. — Without loss of generality, we can assume  $L$  is the pull back of a  $\mathbb{Q}$ -Cartier divisor  $L_1$  in  $Z$ .

Assuming this claim for the moment, our aim is now to show that after a modification  $\pi : \tilde{Z} \rightarrow Z$ , we can find a big divisor in  $\tilde{Z}$  whose Kodaira dimension matches that of  $X$ . This will imply that  $\kappa(L) = \kappa(L_1) \leq \kappa(X)$ , as required.

To this end take  $Y$  to be a relative minimal model for  $X$  over  $Z$  and denote the birational map between  $X$  and  $Y$  by  $\phi$  and the induced morphism  $Y \rightarrow Z$  by  $\psi$  (see the diagram below). Observe that we can assume that  $\phi$  is a morphism without losing generality. Denote  $\psi^*(L_1)$  by  $L_Y$ . Fix  $K_Y$  to be the cycle theoretic push forward of  $K_X$ .

Now by Lemma 3.4 and the abundance assumption after modifying the base by  $\pi : \tilde{Z} \rightarrow Z$ , we can find a morphism  $\tilde{\psi} : \tilde{Y} \rightarrow \tilde{Z}$  such that the dimension of the fibers of this new fibration are all the same and  $\mu^*(K_Y) =$

$\tilde{\psi}^*(G)$  for some  $\mathbb{Q}$ -Cartier divisor  $G$  in  $\tilde{Z}$ .

$$\begin{array}{ccccc}
 X & \xrightarrow{\phi} & Y & \xleftarrow{\mu} & \tilde{Y} \\
 & \searrow^{i_{K_X+L}} & \downarrow \psi & & \downarrow \tilde{\psi} \\
 & & Z & \xleftarrow{\pi} & \tilde{Z}
 \end{array}$$

Noting that  $Y$  is at worst terminal, i.e.,  $K_X + L = \phi^*(K_Y + L_Y) + E$  for an effective exceptional divisor  $E$ , we have  $\kappa(K_X + L) = \kappa(K_Y + L_Y)$ . We also observe that  $rK_Y - L_Y$  must be pseudo-effective.

Define  $\tilde{L}_1 := \pi^*(L_1)$ , so that  $\mu^*(L_Y) = \tilde{\psi}^*(\tilde{L}_1)$  and  $\tilde{\psi}^*(G + \tilde{L}_1) = \mu^*(K_Y + L_Y)$ . This implies that  $G + \tilde{L}_1$  is big in  $\tilde{Z}$ . We also know that  $\mu^*(rK_Y - L_Y)$  is pseudo-effective and  $\mu^*(rK_Y - L_Y) = \tilde{\psi}^*(rG - \tilde{L}_1)$ . Thus by Lemma 3.4,  $rG - \tilde{L}_1$  is pseudo-effective too. Additionally we have

$$(r + 1)G = (rG - \tilde{L}_1) + (G + \tilde{L}_1),$$

where the right hand side is a sum of pseudo-effective and big divisors. This implies that  $G$  is big and we have

$$\kappa(L) = \kappa(\tilde{L}_1) \leq \kappa(G) = \kappa(\mu^*(K_Y)) = \kappa(K_X).$$

Now it remains to prove 3.8.

*Proof of 3.8.* — Let  $X \dashrightarrow Z'$  be the map given by the global sections of large enough multiple of  $L$ , and let  $i_L : X' \rightarrow Z'$  be the Iitaka fibration corresponding to  $L$ , where  $\mu : X' \rightarrow X$  is a suitable modification of  $X$ . As  $\kappa(K_X) \geq 0$ , we have  $\kappa(L) \leq \kappa(K_X + L)$ , where the right hand side of this inequality is zero on the general fiber of  $i_{K_X+L}$ . On the other hand since we have assumed  $\kappa(L)$  to be positive, we find that  $\kappa(L|_F) = 0$ . Hence  $i_{K_X+L}$  factors through  $i_L$  via a rational map  $g$  and we have the following commutative diagram:

$$\begin{array}{ccc}
 X' & \xrightarrow{i_L} & Z' \\
 \mu \downarrow & \nearrow & \uparrow g \\
 X & \xrightarrow{i_{K_X+L}} & Z
 \end{array}$$

Now by considering suitable modifications of  $X$ ,  $Z$  and  $X'$ , we can assume that  $g$  is a morphism. Define the line bundle  $L' := \mu^*(L) - A = i_L^*(H)$ , where  $A$  is an effective divisor and  $H$  is an ample  $\mathbb{Q}$ -Cartier divisor in  $Z'$ . Let  $L''$  be the pull back of  $H$  in  $X$  via  $g$  and  $i_{K_X+L}$ , so that  $\mu^*(L'') = L'$  and that  $\mu^*(L'') + A = \mu^*(L)$ . We claim that we don't lose generality if we

replace  $L$  by  $L''$ . To see this we need to check the following two properties: (i)  $rK_X - L''$  is pseudo-effective and (ii)  $\kappa(L'') = \kappa(L)$ .

To see that (i) holds, note that we have  $\mu^*(rK_X - L'') = \mu^*(rK_X) - (\mu^*(L) - A) = \mu^*(rK_X - L) + A$ . Now since  $rK_X - L$  is pseudo-effective and  $A$  is effective,  $rK_X - L''$  must also be pseudo-effective.

For (ii) it suffices to show  $\kappa(L) = \kappa(L')$  which is a consequence of the following inequality:

$$\kappa(L) = \kappa(\mu^*L) \leq \dim Z' = \kappa(L').$$

This finishes off the proof of Claim 3.8 after a possible base change corresponding to  $K_X + L''$ .  $\square$

Now our main result immediately follows:

*Proof of Theorem 1.5.* — Let  $\mathcal{F} \subseteq \Omega_X^p$  be a coherent subsheaf with maximum Kodaira dimension, i.e.,  $\kappa(L) = \kappa^+(X)$ , where  $L = \det(\mathcal{F})$ . Assume that  $\kappa(L) > \kappa(X)$ . Then  $\kappa(L) \geq m$  and in particular we have  $\kappa(K_X + L) \geq m$ . Now the proposition above implies that  $\kappa(L) \leq \kappa(X)$ , which is a contradiction.  $\square$

As we discussed in the introduction, this greatly improves the Bogomolov's inequality for projective varieties of dimension at most five and with relatively small Kodaira dimension.

*Remark 3.9 (Birational stability in dimension 4).* — Theorem 1.4 can be further strengthened by replacing  $\kappa^+$  by a stronger birational invariant  $\omega(X)$  (see A for the definition) which measures the maximal positivity of coherent rank one subsheaves of  $\Omega_X^1 \otimes^{\otimes m}$ , for any  $m > 0$ , i.e.,  $\kappa(X)$  and  $\omega(X)$  coincide for fourfolds with non-negative Kodaira dimension. The proof is identical to that of Theorem 1.4 by observing that pseudo-effectivity of  $rK_X - L$  in 3.7 can be replaced by that of  $mK_X - L$ , where  $m$  denotes the tensorial power of cotangent bundle containing the line bundle  $L$ .

*Remark 3.10.* — We would like to point out that when  $\kappa(X) \geq \dim X - 3$ , we have  $\kappa = \kappa^+$  by [3, Prop. 10.9], where 3 in this inequality comes from the abundance result for varieties of dimension at most 3. So the real improvement provided by 1.4 is when  $\kappa = 0$  in dimension 4 and  $\kappa = 1$  in dimension 5.

## Appendix A. Birational stability of the cotangent bundle (by Frédéric Campana)

We present here a new birational invariant  $\omega(X)$  similar to the Kodaira dimension  $\kappa(X)$ , at least equal to  $\kappa(X)$  and to our previous  $\kappa^+(X)$  and

$\kappa_+(X)$ , and conjecturally equal to all these when  $X$  is not uniruled. The preceding arguments of Behrouz Tajiri directly apply to show this conjecture in dimension 4 as well, and also under the situations considered in his Theorem 1.5 above. This invariant can be introduced in the orbifold compact Kähler case as well, with similar expected properties.

A major aim of algebraic geometry consists in deriving the qualitative geometry of a complex connected projective manifold  $X$  from the positivity/negativity properties of its canonical bundle  $K_X$  (e.g: it has been shown that  $X$  is rationally connected, hence simply connected when its canonical bundle is anti-ample). An intermediate step consists in relating the positivity/negativity of its cotangent bundle  $\Omega_X^1$  from the one of  $K_X$ .

The positivity of  $K_X$  is suitably measured by the canonical (or Kodaira) dimension of its canonical algebra  $K(X) := \bigoplus_{m \geq 0} H^0(X, K_X^{\otimes m})$ , defined as:  $\kappa(X) := \max_{\{m > 0\}}(\dim \Phi_m(X)) \in \{-\infty, 0, 1, \dots, n\}$ , where  $\Phi_m : X \dashrightarrow \mathbb{P}(H^0(X, K_X^{\otimes m})^*)$  is the rational map defined by the linear system  $H^0(X, K_X^{\otimes m})$  if this is nonzero, and is  $-\infty$  otherwise.

In a similar way, we define:  $\Omega(X) := \bigoplus_{m \geq 0} H^0(X, \Omega_X^1{}^{\otimes m})$ , to be the cotangent algebra of  $X$ , and its dimension to be:  $\omega(X) := \max_{\{m > 0, L\}}(\dim \Phi_L(X)) \in \{-\infty, 0, 1, \dots, n\}$ , where  $L \subset \Omega_X^1{}^{\otimes m}$  ranges over all of its coherent rank one subsheaves, with  $m > 0$  arbitrary. Here,  $\Phi_L X \dashrightarrow \mathbb{P}(H^0(X, L)^*)$  is the rational map associated with the linear system defined by the sections of  $L$ .

**Basic properties.** — 1.  $\omega(X) \geq \kappa(X)$  is a birational invariant of  $X$ . We say that  $\Omega_X^1$  is birationally stable if  $\omega(X) = \kappa(X)$ , which means that  $\kappa(X, L) \leq \kappa(X)$ , for any  $L \subset \Omega_X^1{}^{\otimes m}$ , coherent of rank 1, if  $m > 0$  is arbitrary. We shall see below that this happens, conjecturally, if and only if  $X$  is not uniruled (or if and only if  $\kappa(X) \geq 0$ ).

2. Recall that we defined in [3] and [4] two other invariants:  $\kappa^+(X) := \max_{\{F \subset \Omega_X^p, p > 0\}}(\kappa(\det(F)))$ , and:  $\kappa_+(X) := \max_{\{L \subset \Omega_X^p, p > 0\}}(\kappa(L))$ , where  $F$  and  $L$  are arbitrary coherent subsheaves,  $L$  being of rank one.

We thus have:  $\omega(X) \geq \kappa^+(X) \geq \kappa_+(X) \geq \kappa(X)$  in general.

3. Moreover,  $\Omega(X)$  is (in contrast to  $K(X)$ ) functorial in  $X$ , that is: any dominant rational map  $f : X \dashrightarrow Y$  induces naturally an injective algebra morphism  $f^* : \Omega(Y) \rightarrow \Omega(X)$ , and so  $\omega(X)$  is increasing:  $\omega(X) \geq \omega(Y)$  in this situation. The invariant  $\omega$  thus provides an obstruction to the existence of such an  $f$ .

4. Let  $f : X \rightarrow Y$  be a surjective morphism between projective manifolds  $X, Y$ . Let  $X_y$  be its general fibre. If  $\omega(X_y) = -\infty$  (resp. if  $\omega(X_y) = 0$ ), then  $\omega(X) = \omega(Y)$  (resp.  $\omega(X) \leq \dim(Y)$ ). We omit the easy proof.

This inequality is, in general, optimal, as seen by considering the Moishezon-Iitaka fibration  $f : X \rightarrow Y$  of a manifold  $X$  with  $\kappa(X) \geq 0$ , such that  $c_1(X_y) = 0$  (the examples below indeed show that  $\omega(X_y) = \kappa(X_y) = 0$ , then).

If  $r_X : X \rightarrow R(X)$  is the rational quotient of  $X$  (defined in [2], and called “MRC fibration” in [10]), the first statement implies that  $\omega(X) = \omega(R(X))$ . Recall that  $r$  is the only fibration on  $X$  having rationally connected fibres and non-uniruled base  $R(X)$  (by [8]).

*Examples A.1.* — 1. If  $X$  is rationally connected,  $\omega(X) = -\infty$ . Conjecturally, the opposite implication holds as well, and follows from the Abundance conjecture.

2. If  $c_1(X) = 0$ , then  $\omega(X) = 0$ , as seen from the existence of a Ricci-flat Kähler metric and Bochner principle. Alternatively, the general semi-positivity Theorem of Miyaoka shows that  $\omega(X) = 0$  if  $X$  has a birational (normal) model  $X'$  such that  $K_{X'}$  is numerically trivial over its non-singular locus (i.e., such that its degree is zero on each projective curve not meeting its singular locus). The existence of such a model is implied by the Abundance conjecture, too. The Abundance conjecture (and Miyaoka’s Theorem) thus imply that  $\omega(X) = 0$  if  $\kappa(X) = 0$ .

CONJECTURE A.2. — *For any  $X$ , we have:  $\omega(X) = \kappa(R(X))$  (by convention:  $\kappa(pt) = -\infty$ ).*

*Remark A.3.* — This conjecture follows from the Abundance conjecture. Indeed: we need only check that  $\omega(R(X)) = \kappa(R(X))$ , since  $\omega(X) = \omega(R(X))$ , by the property 4 above. Since  $R(X) := Y$  is not uniruled, we have:  $\kappa(Y) \geq 0$ , by Abundance. Let  $f : Y \rightarrow Z$  be the Moishezon-Iitaka fibration of  $Y$ . Since its general fibres have  $\kappa = 0$ , they have  $\omega = 0$  as well by Abundance (example 2 above). Thus  $\omega(Y) \leq \dim(Z) = \kappa(Y) \leq \omega(Y)$ . Thus  $\dim(Z) = \kappa(R(X)) = \omega(Y) = \omega(R(X))$ .

*Remark A.4.* — The definition of  $\Omega(X)$  and  $\omega(X)$  with entirely similar properties can be extended to the case when  $X$  is compact Kähler, and more importantly, when  $X$  (possibly compact Kähler) is equipped with an orbifold divisor  $\Delta := \sum_j a_j \cdot D_j$ , where the  $D_j$ ’s are irreducible pairwise distinct divisors on  $X$  whose union is of normal crossings, and the  $a_j$ ’s are

in  $\mathbb{Q} \cap [0, 1]$ . See [4] and [5] for the relevant definitions. The details will be written elsewhere.

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