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THE BRIANÇON-SKODA NUMBER OF ANALYTIC IRREDUCIBLE PLANAR CURVES

by Jacob SZNAJDMAN

ABSTRACT. — The Briançon-Skoda number of a ring R is defined as the smallest integer k, such that for any ideal $I \subset R$ and $l \geqslant 1$, the integral closure of I^{k+l-1} is contained in I^l. We compute the Briançon-Skoda number of the local ring of any analytic irreducible planar curve in terms of its Puiseux characteristics. It turns out that this number is closely related to the Milnor number.

RÉSUMÉ. — Le nombre de Briançon-Skoda d'un anneau R est défini comme le plus petit entier k, tel que pour tout idéal $I \subset R$ et $l \geqslant 1$, la clôture intégrale de I^{k+l-1} est contenu dans I^l. Nous calculons le nombre de Briançon-Skoda de l'anneau local d'une courbe analytique plane et irréductible en fonction de ses exposants caractéristiques de Puiseux. Il s'avère que ce nombre est étroitement lié au nombre de Milnor.

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1. Introduction

The Briançon-Skoda theorem is a famous theorem in commutative algebra. It was first proven in 1974 by Joël Briançon and Henri Skoda [18]. In the original setting, the theorem dealt with the ring of germs of holomorphic functions at $0 \in \mathbb{C}^n$, but other rings have been studied as well.

Given a ring R, one defines the integral closure of an ideal $I \subset R$ as

$$\overline{I} = \{\phi \in R : \ \exists (N \geqslant 1, b_j \in I^j) \ \phi^N + b_1\phi^{N-1} + \cdots + b_N = 0\}. \tag{1.1}$$

We are interested in integers k such that the inclusion $\overline{I^{k+l-1}} \subset I^l$ holds for any ideal $I \subset R$ and $l \geqslant 1$. We will denote the smallest such integer k

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by bs(R). If no such integer exists, we say that bs(R) $= \infty$. Huneke, [9], proved that bs(R) $< \infty$, given some fairly mild assumptions on R. It is desirable to express bs(R) in terms of invariants of (the singularity of) R.

In the case R is the local ring $\mathcal{O}_{Z,z}$ of an analytic variety Z at some point z, Huneke's result was later proven analytically in [3]. In this setting,

$$\overline{I} = \{\phi \in \mathcal{O}_{Z,z} : \ \exists K > 0 \ \ |\phi| \leqslant K|I| \text{ on } Z\}, \tag{1.2}$$

where $|I| = |a_1| + \cdots + |a_m|$, and a_j generate I. A proof of (1.2) is found in [13]. We shall prefer the alternative definition of the Briançon-Skoda number as the smallest integer k such that, for all $\phi \in \mathcal{O}_{Z,z}$ and all ideals $I \subset \mathcal{O}_{Z,z}$,

$$|\phi| \lesssim |I|^{k+l-1} \text{ on } Z \tag{1.3}$$

implies that $\phi \in I^l$.

In the original setting, that is, $R = \mathcal{O}_{\mathbb{C}^n,0}$, the Briançon-Skoda theorem states that $\mathrm{bs}(\mathcal{O}_{\mathbb{C}^n,0}) = n$. The subject of this paper is the case when R is the local ring at 0 of an irreducible analytic curve C in $\mathbb{C}^2$. Our main result is a formula that expresses $\mathrm{bs}(C) := \mathrm{bs}(\mathcal{O}_{C,0})$ in terms of the Puiseux characteristics of C at 0.

Let m be the multiplicity of the curve C at the origin. Recall that $m = 1$ if and only if the curve is smooth near 0. Note that bs(C) $= 1$ if and only if C is smooth; if bs(C) $= 1$, then every weakly holomorphic on C is strongly holomorphic, so C is normal, and therefore smooth. The same steps applied backwards give that bs(C) $= 1$ whenever C is smooth.

When $\dim Z > 1$, one may ask if Z is regular if and only if $\mathrm{bs}(Z) = \dim Z$. We do not know the answer to this question. However, it is known that there are rings with mild singularities so that $\mathrm{bs}(R) = \dim R$, for example, the class of pseudo-rational rings, see [14].

According to Puiseux's theorem, for a suitable choice of coordinates, the curve is locally parametrized by the normalization $\Pi : \Delta \subset \mathbb{C} \to C$, given by $(z, w) = \Pi(t) = (t^m, g(t))$, for some analytic function $g(t) = \sum_{k \geqslant m} c_k t^k$. Puiseux's theorem is proven analytically in [20], Chapter 1, Section 10, and algebraically in [12].

Set $e_0 = m$ and define inductively

$$\beta_j = \min\{k \in \mathbb{N} : c_k \neq 0, e_{j-1} \nmid k\} \tag{1.4}$$

and

$$e_j = \gcd(e_{j-1}, \beta_j) = \gcd(m, \beta_1, \beta_2, \ldots, \beta_j).$$

The construction stops when, for some integer M, one has $e_M = 1$. These numbers are known as the Puiseux characteristics of the curve. Since $c_k = 0$

for $k < m$, one has $\beta_j \geqslant m$. It is not hard to see that $\beta_\bullet$ is strictly increasing and $e_\bullet$ is strictly decreasing. Thus $\beta_j \geqslant \beta_1 > m$.

Recall that $\lceil x \rceil$ is the ceiling function applied to x, that is, the smallest integer n such that $n \geqslant x$.

Theorem 1.1. — *For any germ of an analytic irreducible planar curve C, one has*

$$\mathrm{bs}(C) = \left\lceil \frac{1}{m}\Big(1 + \sum_{i=1}^{M}(e_{i-1} - e_i)\beta_i\Big) \right\rceil.$$

We see from this formula and the comments above that indeed $\mathrm{bs}(C) = 1$ if and only if C is smooth. By Remark 10.10 in [15], this formula can be rewritten as

$$\mathrm{bs}(C) = \left\lceil 1 + \frac{\mu}{m} \right\rceil,$$

where μ is the Milnor number of the germ C.

2. Analytic formulation of the Briançon-Skoda problem

Any ideal $I = (a_1, \dots, a_m) \subset \mathcal{O}_{C,0}$ has a reduction, that is, an ideal $J \subset I$ such that $|J| \simeq |I|$, and $J = (a)$ is generated by $\dim C = 1$ element. This is not hard to see. Indeed, if $\Pi : \mathbb{C} \to C$ is the normalization of C, we see that $|a| \simeq |I|$ holds if and only if

$$|\Pi^* a| \lesssim |\Pi^* a_1| + \cdots + |\Pi^* a_m|.$$

Clearly $|\Pi^* a_1| + \cdots + |\Pi^* a_m| \simeq |\Pi^* a_j|$, where j is an index such that the vanishing order of $\Pi^* a_j$ at 0 is minimal. Thus J is a reduction of I if we take $a = a_j$. For the purposes of finding $\mathrm{bs}(C)$, we can replace I by its reduction J. We henceforth set $I = (a)$.

A (germ of a) meromorphic form on C is defined as a meromorphic form on C_{reg} which is the pull-back of a meromorphic form near $0 \in \mathbb{C}^2$ with respect to the inclusion map $i : C \to \mathbb{C}^2$. A weakly holomorphic function on C is a holomorphic function on C_{reg} that is locally bounded on C. Any weakly holomorphic function is meromorphic.

The following lemma gives an alternative characterization of the number $\mathrm{bs}(C)$.

Lemma 2.1. — *The Briançon-Skoda number* $\mathrm{bs}(C)$ *is the smallest integer $k \geqslant 1$, such that if we are given any $a \in \mathcal{O}_{C,0}$ with $a(0) = 0$, and any weakly holomorphic function ψ, then*

$$|\psi| \lesssim |a|^{k-1} \quad \text{on } C, \tag{2.1}$$

implies that ψ is strongly holomorphic.

Proof. — Assume that k is any integer that satisfies the property given in the lemma. Take any non-trivial ideal $(a) \subset \mathcal{O}_{C,0}$ and a function ϕ so that $|\phi| \lesssim |a|^{k+l-1}$. Then $\psi = \phi/a^l$ is meromorphic and, satisfies (2.1). In particular, ψ is weakly holomorphic. By our assumption on k, ψ is strongly holomorphic, so $\phi \in (a)^l$. We have thus shown that $\mathrm{bs}(C) \leqslant k$.

Let K be the smallest integer k with the property given in the lemma. It remains to show that $\mathrm{bs}(C) \geqslant K$. There is a weakly holomorphic but not holomorphic function ψ such that $|\psi| \lesssim |a|^{K-2}$ for some holomorphic non-unit a. Thus $|a\psi| \lesssim |a|^{K-1}$ which implies that $\phi := a\psi$ is holomorphic. Note that ϕ does not belong to (a). Since $|\phi| \lesssim |a|^{K-1}$, we obtain $\mathrm{bs}(C) \geqslant K$. □

We need some preliminaries before stating a criterion for when a weakly holomorphic function is strongly holomorphic. A (p,q)-current T on C is a current acting on $(1-p, 1-q)$-forms in the ambient space, with the additional requirement that $T.\xi = 0$ whenever $i^*\xi = 0$ on C_{reg}. The $\overline{\partial}$-operator is defined as usual by $\overline{\partial}T.\xi = (-1)^{p+q+1}T.\overline{\partial}\xi$, where ξ is a test form. If $i^*\xi = 0$ on C_{reg}, then $i^*\overline{\partial}\xi = \overline{\partial}i^*\xi = 0$ on C_{reg}. Thus $\overline{\partial}T$ is a $(p, q+1)$-current on C. Any meromorphic (p,q)-form η on C can be seen as a (p,q)-current on C which acts by

$$\eta.\xi = \int_C \eta \wedge \xi := \int_{\mathbb{C}} \Pi^*(\eta \wedge \xi),$$

where the right-most side is a principal value integral of a meromorphic form in one variable.

According to Weierstrass preparation theorem, for an appropriate choice of coordinates, we may assume that C is the zero locus of a Weierstrass polynomial $P(z,w) = w^m + b_1(z)w^{m-1} + \cdots + b_m(z)$, where m was defined as the multiplicity of C at 0. Let ω' be any meromorphic form acting on the tangent space of $\mathbb{C}^2$, but defined on C, such that

$$dP \wedge \omega' = dz \wedge dw. \tag{2.2}$$

Then $\omega = i^*\omega'$ is a well-defined meromorphic form on C. We choose the representative

$$\omega' = -\frac{1}{P'_w}dz \tag{2.3}$$

for ω; this shows that (2.2) can be satisfied. Theorem 2.2 below is a reformulation of a result by A. Tsikh, [19]. Tsikh's proof, which is given in Section 3, relies on residue theory in two variables, and implicitly, therefore also Hironaka resolutions. We are not aware of any elementary proof.

Theorem 2.2. — *Let ψ be any meromorphic function on C. Then ψ is strongly holomorphic if and only if $\psi\omega$ is $\overline{\partial}$-closed.*

Remark 2.3. — The form ω generates all $\overline{\partial}$-closed meromorphic $(1,0)$-forms on C. In fact, let ξ be any meromorphic $(1,0)$-form on C. Then there is a meromorphic function α such that $\xi = \alpha\omega$. Hence, if $\overline{\partial}\xi = 0$, we have that $\alpha\omega$ is $\overline{\partial}$-closed, so by Theorem 2.2, $\alpha \in \mathcal{O}_{C,0}$.

Lemma 2.4. — *The Briançon-Skoda number is given by the identity*

$$\mathrm{bs}(C) = \left\lceil (1 + \mathrm{ord}_0(\Pi^* P'_w))/m \right\rceil. \tag{2.4}$$

Proof. — Let k be the right hand side of (2.4). We will first show that $\mathrm{bs}(C) \leqslant k$. Assume that ψ is weakly holomorphic on C and satisfies (2.1). According to Lemma 2.1, it suffices to show that ψ is strongly holomorphic.

Using $\Pi(t) = (t^m, g(t))$ and (2.1), we see that

$$-\psi\omega.\overline{\partial}\eta = \int_C \frac{\psi}{P'_w} dz \wedge \overline{\partial}\eta = \int_{\mathbb{C}} \frac{h(t) t^{(k-1)\,\mathrm{ord}_0(\Pi^* a)}}{t^{\mathrm{ord}_0(\Pi^* P'_w)}} d(t^m) \wedge \overline{\partial}(\Pi^*\eta), \tag{2.5}$$

where h is holomorphic in t and $\eta \in C_0^\infty(\mathbb{C}^2, 0)$ is an arbitrary test function. In view of Theorem 2.2, we would like to show that $\psi\omega.\overline{\partial}\eta = 0$, because then ψ is holomorphic. The last integral in equation (2.5) vanishes if

$$(k-1)\,\mathrm{ord}_0(\Pi^* a) \geqslant \mathrm{ord}_0(\Pi^* P'_w) - (m-1), \tag{2.6}$$

since then the integrand is the product of a holomorphic function and a $\overline{\partial}$-exact form with compact support. Clearly the worst case in (2.6) occurs when $\mathrm{ord}_0(\Pi^* a)$ is minimal. Since $\mathrm{ord}_0(g(t)) \geqslant m$, the minimal value is $\mathrm{ord}_0(\Pi^* a) = m$, which is attained for example when $a = z$. By the definition of k, we see that (2.6) holds with equality in the worst case.

We will now show that $\mathrm{bs}(C) \geqslant k$. Let $Q = (1 + \mathrm{ord}_0(\Pi^* P'_w))/m$, so that $k = \lceil Q \rceil$. Then

$$|\Pi^* P'_w| \simeq |t|^{\mathrm{ord}_0(\Pi^* P'_w)} = |\Pi^* z|^{\mathrm{ord}_0(\Pi^* P'_w)/m} = |\Pi^* z|^{Q-1/m},$$

and thus $|P'_w| \simeq |z|^{Q-1/m}$. Let us check that $P'_w \notin (z)$, that is, that $P'_w/z \notin \mathcal{O}_{C,0}$. If $m = 1$, then $\mathrm{bs}(C) = 1$ and we have nothing to prove, so assume that $m > 1$. Recall that $e_0 > e_1$, so we have

$$\mathrm{ord}_0(\Pi^* P'_w) \;\geqslant\; \beta_1 \;>\; m \;=\; \mathrm{ord}_0(\Pi^* z).$$

Thus $\psi := P'_w/z$ is weakly holomorphic on C. If ξ is a test function on C such that $\xi(0) \neq 0$, then

$$\int_C \psi\omega \wedge \overline{\partial}\xi = -\int_C \frac{dz}{z} \wedge \overline{\partial}\xi = -m \int_{\mathbb{C}} \frac{dt}{t} \wedge \overline{\partial}\Pi^*\xi = -2m\pi i \xi(0) \neq 0.$$

Theorem 2.2 now gives that ψ is not strongly holomorphic on C. Thus $\mathrm{bs}(C) > Q - 1/m$. Since $Q \in 1/m \cdot \mathbb{Z}$, $\mathrm{bs}(C) = \lceil Q \rceil$ as claimed. ☐

Remark 2.5. — One may wish to remove the restriction on the Briançon-Skoda number that it has to be an integer. It is then natural to consider

$$\kappa := \inf\{k \in \mathbb{R} : |\psi| \lesssim |a|^{k-1} \implies \psi \in \mathcal{O}_{C,0}\}, \tag{2.7}$$

where the implication is assumed to hold for all weakly holomorphic functions ψ on C and all $a \in \mathcal{O}_{C,0}$ that are not invertible. The argument below yields that the set in the right hand side of (2.7) is open. We claim that $\kappa = Q - 1/m$. This can be seen as follows. Note that the same infimum is attained in (2.7) if we assume that $a = z$ (or more generally that $\mathrm{ord}_0(\Pi^* a) = m$). The example $\psi = P'_w/z$ which we considered before, shows that κ cannot be smaller than $Q - 1/m$. If $k = Q - 1/m + \varepsilon$ and $|\psi| \lesssim |a|^{k-1}$, then $\mathrm{ord}_0(\Pi^*\psi) > m(k-1)$. Hence

$$\mathrm{ord}_0(\Pi^*\psi) \geqslant m(k-1) + 1 = m(Q-1) = \mathrm{ord}_0(\Pi^* P'_w) - (m-1),$$

but then the first half of the proof of Lemma 2.4 shows that $\psi \in \mathcal{O}_{C,0}$. We conclude that $k = Q - 1/m + \varepsilon$ is a candidate for the infimum in (2.7), so $\kappa \leqslant Q - 1/m$ since ε is arbitrary.

In Section 4 we express $\mathrm{ord}_0(\Pi^* P'_w)$ in terms of Puiseux's invariants in Lemma 4.1. Together with Lemma 2.4, this yields Theorem 1.1.

Remark 2.6 (Outline of alternative proof of Theorem 1.1). — Note that (2.1) is equivalent to

$$\mathrm{ord}_0(\Pi^*\psi) \geqslant (k-1)\,\mathrm{ord}_0(\Pi^* a).$$

Furthermore, one can assume that $\mathrm{ord}_0(\Pi^* a) = m$ since this is the worst case. The integer in the lemma is therefore

$$\begin{aligned} &\min\{k : \forall \psi \in \tilde{\mathcal{O}}_{C,0} \quad \mathrm{ord}_0(\Pi^*\psi) \geqslant (k-1)m \implies \psi \in \mathcal{O}_{C,0}\} = \\ &\min\{k : \forall \psi \in \tilde{\mathcal{O}}_{C,0} \setminus \mathcal{O}_{C,0} \quad \mathrm{ord}_0(\Pi^*\psi) < (k-1)m\}, \end{aligned} \tag{2.8}$$

where $\tilde{\mathcal{O}}_{C,0}$ is the set of weakly holomorphic functions on C. The maximal value of $\mathrm{ord}_0(\Pi^*\psi)$ given that $\psi \in \tilde{\mathcal{O}}_{C,0} \setminus \mathcal{O}_{C,0}$ is $c - 1$, where c is the *conductor number* of the curve. We conclude by Lemma 2.1 and (2.8) that $\mathrm{bs}(C)$ is the smallest integer that is strictly larger than $c/m + 1 - 1/m$, which is the same as $\lceil 1 + c/m \rceil$. Gorenstein [7] and Samuel [17] (see Kodaira [11, 10] for corresponding result in the analytic setting) prove that $c = 2\delta$, where δ is a well-known invariant in singularity theory. Finally, by Theorem 10.5 in [15], $\mu = 2\delta$.

3. Proof of Theorem 2.2

One can define the principal value current $1/P$ as follows: Let χ be a smooth cut-off function such that $\chi \equiv 0$ on some interval $[0, \delta]$ and $\chi \equiv 1$ on $[1, \infty)$. For any full-degree test form ξ, one defines

$$\int_{\mathbb{C}^n} \frac{1}{P}\xi = \lim_{\varepsilon \to 0} \int_{\mathbb{C}^n} \chi(|P|/\varepsilon)\frac{\xi}{P}. \tag{3.1}$$

The existence of such principal values was proved in [8], although with a slightly different definition; however (3.1) is just an avarage of principal values in the sense of Herrera-Lieberman. We now apply the $\overline{\partial}$-operator in the sense of currents to obtain $\overline{\partial}(1/P)$.

Let ψ be a meromorphic function on C and let $\Psi = \Psi_1/\Psi_2$ be a representative in the ambient space such that Ψ_1 and Ψ_2 are relatively prime. Then $\psi\overline{\partial}(1/P)$ can be defined as

$$\lim_{\varepsilon \to 0} \Psi_1 \frac{\chi(|\Psi_2|/\varepsilon)}{\Psi_2}\, \overline{\partial}\, \frac{1}{P}. \tag{3.2}$$

It is not obvious that (3.2) is a valid definition, that is, that the limits exist and does not depend on the choice of χ, nor on the representative Ψ of ψ; see *e.g.* Theorem 1 in [5] and the comments that follow it.

We now formulate (for the special case of hypersurfaces) a criterion which is due to Tsikh, [19]. A generalized version of this criterion can be found in [1], *cf.* Remark 3.2.

THEOREM 3.1. — *Let ψ be any meromorphic function on C. Then ψ is strongly holomorphic if and only if $\psi\overline{\partial}(1/P)$ is $\overline{\partial}$-closed.*

Proof. — Assume that $\psi\overline{\partial}(1/P)$ is $\overline{\partial}$-closed and write $\psi = g/h$ for some functions $g, h \in \mathcal{O}_{\mathbb{C}^2,0}$. By basic rules for Coleff-Herrera products, which can be deduced for example from Theorem 1 in [5], we have

$$g\overline{\partial}\frac{1}{h} \wedge \overline{\partial}\frac{1}{P} = \overline{\partial}(\psi\overline{\partial}\frac{1}{P}) = 0.$$

By the duality theorem, [16], [6], it follows that $g \in (h, P)$, so $g - \alpha h \in (P)$ for some $\alpha \in \mathcal{O}_{\mathbb{C}^2,0}$. Thus $\psi = \alpha$ on C. The converse is immediate. □

Remark 3.2. — More generally, if $f = (f_1, \ldots, f_p)$ defines a complete intersection and ψ is a meromorphic function on $Z = Z(f)$, then Tsikh's result states that ψ is strongly holomorphic if and only if $\psi\big[\overline{\partial}(1/f_1) \wedge \ldots \wedge \overline{\partial}(1/f_p)\big]$ is $\overline{\partial}$-closed, where the current in square brackets in the Coleff-Herrera product. The generalization by Andersson [1] to arbitrary Z is an analogous criterion on ψR^Z, where R^Z is the Andersson-Wulcan current,

[4], associated to Z. If Z is a complete intersection, then R^Z coincides with the Coleff-Herrera product.

The form ω is by definition Leray's residue form of $dz\wedge dw/P$, and Leray's residue formula states that

$$\int_C \omega \wedge \xi = \frac{1}{2\pi i} \int \bar\partial \frac{1}{P} \wedge dz \wedge dw \wedge \xi$$

for any $(0,1)$-test form ξ. It follows from this formula that $\psi\omega$ is $\bar\partial$-closed if and only if $\psi\bar\partial(1/P)$ is $\bar\partial$-closed. Therefore, Theorem 2.2 follows from Theorem 3.1.

Remark 3.3. — Andersson and Samuelsson [2] construct an intrinsic form ω on Z – the *the structure form*. The structure form of the plane curve C is (up to a non-vanishing holomorphic factor) precisely $\omega = dz/P'_w$ as above. Theorem 2.2 also has a generalized counterpart; in the complete intersection case it is given simply by replacing C by Z and interpreting ω as the structure form on Z. For the general case see [2], equation (i)'.

4. The singularity of Leray's residue form

In this section, we will prove Lemma 4.1 below, and thereby finish the proof of Theorem 1.1. Previously we have chosen the coordinates (z, w) in $\mathbb{C}^2$, so that C is the zero locus of an irreducible Weierstrass polynomial

$$P(z, w) = w^m + a_1(z)w^{m-1} + \cdots + a_m(z).$$

Lemma 4.1. — *The vanishing order of* $\Pi^* P'_w$ *at* 0 *is*

$$\sum_{l=1}^{M} (e_{l-1} - e_l)\beta_l. \tag{4.1}$$

Proof. — For any (small) $z \neq 0$, let $\alpha_j(z)$, $1 \leqslant j \leqslant m$, be the roots of $w \mapsto P(z, w)$; then $P = \prod_{j=1}^m (w - \alpha_j(z))$. It is well-known that the $\alpha_j(z)$ are holomorphic on some sufficiently small neighbourhood V of z. Recall that $\Pi(t) = (t^m, g(t))$, where $g(t) = \sum_{k\geqslant m} c_k t^k$. Choose t such that $t^m = z$, and let ρ be a primitive m:th root of unity. Then Π maps $\{t, \rho t, \ldots, \rho^{m-1}t\}$ bijectively onto the fibre $(\{z\} \times \mathbb{C}_w) \cap C$. Thus $\{g(\rho^j t) : 1 \leqslant j \leqslant m\}$ are the roots of $w \mapsto P(z, w)$. After possibly possibly renumbering the m roots, we get $g(\rho^j t) = \alpha_j(t^m)$.

We claim that

$$\Pi^* P'_w = \prod_{j=1}^{m-1} (g(t) - g(\rho^j t)). \tag{4.2}$$

Since we are asserting the identity of two holomorphic functions, it is enough to prove equality locally outside of $\{t=0\}$. We have

$$P'_w = \sum_{l=1}^{m} \prod_{1\leqslant j\leqslant m,\ j\neq l} (w-\alpha_j(z)),$$

so

$$\Pi^* P'_w = \sum_{l=1}^{m} \prod_{1\leqslant j\leqslant m,\ j\neq l} (g(t)-\alpha_j(t^m)) = \prod_{1\leqslant j\leqslant m-1} (g(t)-g(\rho^j t)).$$

Since $g(t) = \sum_{k=m}^{\infty} c_k t^k$, we have that

$$g(t)-g(\rho^j t) = \sum_{k=m}^{\infty} c_k(1-\rho^{kj})t^k. \tag{4.3}$$

Let $k_j^* = \mathrm{ord}_0(g(t)-g(\rho^j t))$. Then by (4.3),

$$\begin{aligned} k_j^* &= \min\{k : c_k \neq 0, (1-\rho^{kj}) \neq 0\} \\ &= \min\{k : c_k \neq 0, m \nmid kj\}. \end{aligned} \tag{4.4}$$

We also consider the number

$$r(j) = \min\{l : m \nmid j\beta_l\},$$

for each $1 \leqslant j \leqslant m-1$. The sequence $\beta_1, \beta_2, \dots, \beta_M$ is strictly increasing, so

$$m \mid j\beta_l \quad \text{for all} \quad l < r(j), \tag{4.5}$$

and

$$m \nmid j\beta_{r(j)}. \tag{4.6}$$

Since $je_l = \gcd(jm, j\beta_1, \dots, j\beta_l)$, the statements (4.5) and (4.6) together imply

$$m \mid je_l \quad \text{if and only if} \quad l < r(j). \tag{4.7}$$

Now note that $k_j^* \leqslant \beta_{r(j)}$. We claim that, in fact, $k_j^* = \beta_{r(j)}$. Assume to the contrary that $k_j^* < \beta_{r(j)}$. Then $e_{r(j)-1} \mid k_j^*$ by (1.4), so we have

$$je_{r(j)-1} \mid jk_j^*.$$

Together with (4.7), this gives $m \mid jk_j^*$, contradicting the definition of k_j^*.

By (4.2), we have

$$\operatorname{ord}_0(\Pi^* P'_w) = \sum_{j=1}^{m-1} \operatorname{ord}_0(g(t) - g(\rho^j t)) \tag{4.8}$$
$$= \sum_{j=1}^{m-1} k_j^* = \sum_{l=1}^{M} \#\{j : k_j^* = \beta_l\}\beta_l,$$

where the last equality follows since $k_j^* = \beta_{r(j)}$.

Using that $k_j^* = \beta_{r(j)}$ and the strict monotonicity of the sequence $\beta_1, \beta_2, \ldots$, we see that $k_j^* \geqslant \beta_l$ is equivalent to $r(j) \geqslant l$. Thus (4.7) gives that

$$\#\{j : k_j^* \geqslant \beta_l\} = \#\{j \in [1, m-1] : m \mid je_{l-1}\}$$
$$= \#\{\frac{m}{e_{l-1}}, 2\frac{m}{e_{l-1}}, \ldots, (e_{l-1} - 1)\frac{m}{e_{l-1}}\} = e_{l-1} - 1.$$

Clearly, $\#\{j : k_j^* = \beta_l\} = \#\{j : k_j^* \geqslant \beta_l\} - \#\{j : k_j^* \geqslant \beta_{l+1}\} = e_{l-1} - e_l$. We substitute this into (4.8), and thereby obtain the desired formula (4.1). □

Remark 4.2. — Recall that $\omega = -i^*[(P'_w)^{-1}dz]$, where i is the inclusion of $C \setminus \{0\}$ into $\mathbb{C}^2$. It follows from Lemma 4.1 and Remark 10.10 in Milnor's book [15] that $\Pi^*\omega = u(t)t^{-\mu}dt$, where u is holomorphic and non-vanishing and μ is the Milnor number of C. By Remark 2.3, we then have that the maximal singularity of any $\overline{\partial}$-closed $(1,0)$-form on C is precisely the Milnor number.

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