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## ASYMPTOTICS OF EIGENSECTIONS ON TORIC VARIETIES

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With an appendix by D. BARLET

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ABSTRACT. — Using exhaustion properties of invariant plurisubharmonic functions along with basic combinatorial information on toric varieties, we prove convergence results for sequences of densities  $|\varphi_n|^2 = |s_N|^2 / \|s_N\|_{L^2}^2$  for eigensections  $s_N \in \Gamma(X, L^N)$  approaching a semiclassical ray. Here  $X$  is a normal compact toric variety and  $L$  is an ample line bundle equipped with an arbitrary positive bundle metric which is invariant with respect to the compact form of the torus. Our work was motivated by and extends that of Shiffman, Tate and Zelditch.

RÉSUMÉ. — En utilisant les propriétés d'exhaustion des fonctions plurisousharmoniques invariantes en combinaison avec les données combinatoires basiques des variétés toriques, nous montrons des résultats de convergence pour des suites de densités  $|\varphi_n| = |s_N|^2 / \|s_N\|_{L^2}^2$  des sections propres  $s_N \in \Gamma(X, L^N)$  approchant un rayon semi-classique. Ici  $X$  est une variété torique normale et  $L$  désigne un fibré en droites ample muni d'une métrique positive quelconque invariante par rapport à l'action de la forme compacte du tore. Notre travail était motivé par ceux de Shiffman, Tate et Zelditch et généralise ceux-ci.

### 1. Notation and statement of results

Let us begin by describing the basic setting of this paper. For this let  $X$  be an  $m$ -dimensional connected normal compact complex space equipped with an effective holomorphic action of a complex torus  $T \cong (\mathbb{C}^*)^m$ . It follows that  $T$  has a (unique, Zariski dense) open orbit  $T.x_0$  where the base point  $x_0$  is fixed for the discussion. We consider a very ample line bundle  $\pi : L \rightarrow X$  to which the elements of  $T$  can be lifted in the sense that for every  $t \in T$  there is a holomorphic bundle mapping  $\hat{t} : L \rightarrow L$

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with  $t\pi = \pi\hat{t}$ . If  $\hat{T}$  denotes the group of bundle transformations which arise in this way, then  $\hat{T} \cong (\mathbb{C}^*)^{m+1}$  and  $\pi$  induces an exact sequence  $1 \rightarrow \mathbb{C}^* \rightarrow \hat{T} \rightarrow T \rightarrow 1$ .

The group  $\hat{T}$  is naturally represented on the space  $\Gamma(X, L)$  of sections by  $\hat{t}(s) := \hat{t}st^{-1}$ . Since  $\hat{T} \cong (\mathbb{C}^*)^{m+1}$ , this representation is completely reducible. Note that if  $s_1, s_2 \in \Gamma(X, L)$  are eigensections which transform by the same character, then  $s_1 s_2^{-1}$  is a  $T$ -invariant meromorphic function on  $X$  which, since  $T$ -has an open orbit, is constant. Hence the representation is multiplicity-free.

In order to lift the  $T$ -action to  $L$  we fix a base eigensection  $s_0 \in \Gamma(X, L)$  and define the base point  $1_{x_0} := s(x_0)$  in the fiber  $L_{x_0}$  over the base point  $x_0$  in the open  $T$ -orbit. Now  $\hat{t}(s_0) = \chi_0(\hat{t})s_0$  for some character  $\chi_0 \in \mathfrak{X}(\hat{T})$ . Thus  $\text{Ker}(\chi_0)$  is identified with  $T$  by its orbit of  $1_{x_0}$  which is mapped bijectively onto the open orbit in  $X$ . We choose this lifting of  $T$  as a group of bundle transformations, i.e.,  $T \cong \text{Ker}(\chi_0) \hookrightarrow \hat{T}$ . Since  $L$  is assumed to be very ample and the associated embedding  $\varphi_L : X \rightarrow \mathbb{P}(\Gamma(X, L)^*)$  is equivariant, it follows from the fact that every holomorphic representation of  $T$  is algebraic that the  $T$ -action on  $X$  is algebraic. Consequently  $X$  is a toric variety (see, e.g., [4] definitions and basic results).

Now let  $L$  be equipped with a smooth Hermitian bundle metric  $h$  which is positive in the sense that for every local section  $s$  the function  $-\log |s|_h^2$  is strictly plurisubharmonic. A function on a complex space is said to be smooth if it can be locally extended to a smooth function in a local embedding space of  $X$  in a complex manifold. It is said to be strictly plurisubharmonic if the extended function is strictly plurisubharmonic. We also assume that  $h$  is invariant with respect to the maximal compact subgroup  $T_{\mathbb{R}}$  of  $T$ . This can be achieved by averaging. Of fundamental importance here is the  $L_2$ -norm  $\|s\|_h^2 := \int_X |s|_h^2 d\lambda$ . The measure  $d\lambda$ , which is normalized so that  $X$  has unit volume, is chosen to be associated to the volume form  $\omega^m$  of a Kähler metric. The latter is defined on a covering  $\{U_\alpha\}$  by strictly plurisubharmonic potential functions  $h_\alpha$  where the differences  $h_\beta - h_\alpha$  are pluriharmonic on the intersections  $U_{\alpha\beta}$ . Thus  $\omega$  is locally the  $(1, 1)$ -form  $\omega_\alpha = \frac{i}{2} \partial\bar{\partial}h_\alpha$ .

### The weight lattice

Having chosen  $1 = 1_{x_0}$ , in every isotypical component  $V_\chi$  we have a unique section  $s$  with  $s(x_0) = 1$ . The character  $\chi$  defines  $s$  on the open orbit by  $t^{-1}s(t.x_0) = \chi(t)s(x_0) = \chi(t) \cdot 1$ . Since  $s_0$  is  $T$ -invariant, it follows that

$s(tx_0) = \chi(t)s_0(tx_0)$ . Consequently, having fixed  $s_0$  all other eigensections are determined by their characters and as a result we turn to the space of characters on  $T$ .

Characters  $\chi : T \rightarrow \mathbb{C}^*$  restrict to characters  $\chi : T_{\mathbb{R}} \rightarrow S^1$  and conversely such compact characters extend uniquely to characters of  $T$ . Thus it is traditional to write a character as a compact character  $\chi = e^{2\pi i\alpha}$  where the linear function  $\alpha \in \mathfrak{t}_{\mathbb{R}}^*$  is required to take on integral values on the kernel of  $\exp : \mathfrak{t}_{\mathbb{R}} \rightarrow T_{\mathbb{R}}$ . Of course we implicitly also regard such linear functions as being complex linear, i.e., in  $\mathfrak{t}^*$  where they define the complex characters  $\chi$ . We denote the space of such functions by  $\mathfrak{t}_{\mathbb{Z}}^*$  and refer to it as the (full) weight lattice. We define the dual lattice  $\mathfrak{t}_{\mathbb{Z}}$  by

$$\mathfrak{t}_{\mathbb{Z}} = \{v \in \mathfrak{t} : u(v) \in \mathbb{Z} \text{ for all } u \in \mathfrak{t}_{\mathbb{Z}}^*\}. \quad (1)$$

and obtain the pairing

$$\langle \cdot, \cdot \rangle : \mathfrak{t}_{\mathbb{Z}}^* \times \mathfrak{t}_{\mathbb{Z}} \rightarrow \mathbb{Z}, \quad \langle u, v \rangle = u(v). \quad (2)$$

Using the identification explained above, if the bundle  $L$  is equipped with a lifting of the  $T$ -action, then  $\Gamma(X, L)$  is described as a  $T$ -representation space by the set of weights in  $\mathfrak{t}_{\mathbb{Z}}^*$  which lie in a certain polygonal region which is defined by the geometry of  $X$  as a  $T$ -space (see §2).

### Sequences of eigensections

The purpose of our work is to explain certain asymptotic phenomena for sequences  $(s_N)$  of sections where  $s_N \in \Gamma(X, L^N)$ . Having fixed the base point  $s_0$  with  $s_0(x_0) =: 1$ , we have the base point  $s_0^N$  for  $\Gamma(X, L^N)$  with  $s_0^N(x_0) =: 1^N$ . Thus we have the correspondence between eigensections and linear functions in  $\mathfrak{t}_{\mathbb{Z}}^*$  at that level as well. Recall this is given by using the lifting of the  $T$ -action via  $s_0^N$ , noting that a given eigensection  $s$  satisfies form  $s(tx_0) = \chi(t)s_0^N(tx_0)$  and expressing  $\chi$  as  $e^{2\pi i\alpha}$ . The correspondence is then defined by  $s \mapsto \alpha$ .

We wish to understand the asymptotic behavior of a sequence of eigensections  $(s_N)$  where the individual elements  $s_N \in \Gamma(X, L^N)$  are chosen so that the associated weights approximate a ray  $R(\xi) := \mathbb{R}^{\geq 0} \cdot \xi$  defined by  $\xi \in \mathfrak{t}_{\mathbb{Z}}^*$ . If  $\alpha_N$  is the integral weight associated to  $s_N$ , then one says that  $(s_N)$  approximates  $R(\xi)$  at infinity if

$$\alpha_N = N \cdot \xi + O(1).$$

It should be underlined that, while our discussion depends on the choice of the lifting of the  $T$ -action to  $L$ , the results are hardly affected. For example,

if the action is lifted via another base eigensection  $\hat{s}_0$  which is associated to  $s_0$  by the character  $\chi_{\hat{\alpha}}$ , then a sequence  $(s_N)$  approximates the ray  $R(\xi)$  with respect to the base section  $s_0$  if and only if it approximates the ray  $R(\xi - \hat{\alpha})$  with respect to the base section  $\hat{s}_0$ .

Our main result states that for every ray  $R(\xi)$  and every sequence  $(s_N)$  which approximates  $R(\xi)$  at infinity the sequence

$$|\varphi_N|_h^2 := \frac{|s_N|_h^2}{\|s_N\|_{L_2}^2}$$

of probability densities converges with precise estimates of both  $|s_N|_h^2$  and  $\|s_N\|_{L_2}^2$  to the integration current of a certain  $T_{\mathbb{R}}$ -orbit  $M$ . In fact  $M$  is the set where a certain canonically associated strictly plurisubharmonic function  $f : X \rightarrow \mathbb{R}^{\geq 0} \cup \{\infty\}$  takes on its minimum. This function arises as follows.

Given  $(s_N)$  one studies the strictly plurisubharmonic functions  $f_N := -\frac{1}{N} \log |s_N|_h^2$ . It is a simple matter to check that these converge (locally on compact subsets) to a smooth strictly plurisubharmonic function  $f$  on the open  $T$ -orbit. However, simple examples show that they do not necessarily converge on  $X$ , even outside the zero sets of the  $s_N$ . However, there exist tame sequences  $s'_N$  which approach the same ray at infinity so that the polar sets of the associated functions  $f'_N$  stabilize for large  $N$  as one ample divisor  $Y$  with  $f'_N \rightarrow f'$  uniformly on compact subsets on the complement  $X \setminus Y$ . Since  $f' = f$  on the open orbit, we may define  $f$  independent of the tame sequence by continuation to  $X$  simply by continuity. The role of  $f$  is emphasized by the following result.

**THEOREM 1.1.** — *The smooth strictly plurisubharmonic function  $f$  is an exhaustion of  $X \setminus Y$  which takes on its minimum exactly on the  $T_{\mathbb{R}}$ -orbit  $M$  which is a strong deformation retract of  $X \setminus Y$ .*

The  $T_{\mathbb{R}}$ -orbit  $M$  is contained in the closed  $T$ -orbit  $\mathcal{O}_\tau$  in  $X \setminus Y$ . Both  $Y$  and  $\mathcal{O}_\tau$  are determined by  $\xi$  by the combinatorial polyhedral geometry associated to  $X$  (§3.1). The location of  $M$  in  $\mathcal{O}_\tau$  varies in an explicitly determined way as a function of  $\xi$  and the (positive) metric  $h$  (§3.2).

Normalizing  $f$  so that  $f|_M = 0$  it follows that the sequence of probability densities “localizes” on  $M$ . This statement is made precise in the following theorem. Here  $d\lambda$  is the smooth probability measure introduced above and  $dM$  is the invariant Haar measure on the torus  $M$ . It should be emphasized that for this it is not required that  $(s_N)$  is tame, only that a semiclassical ray is approximated.

THEOREM 1.2. — *The sequence of measures  $|\varphi_N|_h^2 d\lambda$  converges to the Dirac measure  $\delta_M$  on  $M$  in the weak sense. That is, we have*

$$\int_X u |\varphi_N|_h^2 d\lambda \rightarrow \int_M u dM$$

for all continuous functions  $u : X \rightarrow \mathbb{R}$ .

For the proof of Theorem 1.1 it is necessary to estimate the pointwise asymptotic behavior of the probability densities. This comes down to an analysis of the  $L^2$ -norms  $\|s_N\|_{L^2}^2$ . It is only necessary to carry out the estimation locally near  $M$ . This is possible because  $f$  is in fact a Bott-Morse function with its minimum on  $M$ . If  $X$  is smooth, the relevant integrals can be directly computed. In the singular case essentially the same estimate holds, but the proof is more delicate (see the appendix). The final result is of the form

$$|\varphi_N|_h^2 \sim N^\kappa e^{-Nf}. \quad (3)$$

The speed of convergence, determined by the exponent  $\kappa$ , depends on the position of  $M$  in the stratification of  $X$  given by the  $T$ -action. Calculations are straightforward when  $M$  lies in the open and dense orbit  $\mathcal{O}_0$ . In this case  $\kappa$  equals  $\frac{1}{2} \dim X$ . However, if  $M$  lies in some boundary component  $\mathcal{O}_\tau$ , the behavior is more subtle and more technical effort is needed. Two problems arise: The singularities of  $X$  play a role and, as compared to the tame sequence, the sequence  $(s_N)$  may show a irregular behavior as it approaches the semiclassical limit given by  $R(\xi)$ . In fact, as we will illustrate in an example later on, the pointwise asymptotic behavior of  $|\varphi_N|_h^2$  depends on  $s_N$  and is not uniquely determined by the ray  $R(\xi)$ .

In §3 we derive a precise asymptotic formula for tame sequences and discuss their relationship to arbitrary, possibly non-tame sequences. It turns out that the tails of the distribution functions defined by a tame sequence give upper estimates for the original ones. This is the content of the following theorem:

THEOREM 1.3. — *Let  $(s_N)$  be a sequence of sections approximating a ray  $R(\xi)$  at infinity and let*

$$D_N(t) := \text{Vol}\{x \in X; |\varphi_N|_h^2 > t\}$$

be the tail of the associated distribution function. If  $(s'_N)$  is an associated tame sequence, then

$$D_N(t) \leq D'_N(t) \sim \left(\frac{\log N}{N}\right)^\kappa,$$

where  $\kappa = \text{codim } \mathcal{O}_\tau + \frac{1}{2} \dim \mathcal{O}_\tau$ .

It should be mentioned that for any given sequence the difference between  $D_N(t)$  and  $D'_N(t)$  is only of finite order in  $N$ .

### Previous results

In [13] the authors derive formulæ for the pointwise asymptotic behavior of the probability densities  $|\varphi_N|_h^2$  and the distribution functions  $D_N(t)$  in the case of  $X$  being smooth and embedded in projective space with the bundle metric  $h$  being the restriction of the Fubini-Study metric. Their results are valid for arbitrary sequences in the case  $\xi$  is in the interior of  $P_D$  (see below for the definition). In this case every sequence approaching  $R(\xi)$  is tame. If the localization manifold  $M$  is located on the boundary of the open orbit they only consider asymptotic sequences of a special type: The element  $\xi$  defining the ray  $R(\xi)$  is assumed to be integral and the asymptotic sequence of characters  $\alpha_N$  is assumed to be  $\alpha_N = N\xi$ . This is a special case of a tame sequence as it is considered in the present paper.

In the more recent paper [14] the authors also deal with the smooth case. In particular, they derive asymptotic developments of the value of  $|\varphi_N|_N^2$  at the momentum map preimage of  $\frac{\alpha_N}{N}$ . This is carried out without reference to a particular choice of a sequence  $(s_N)$  but instead depends on the location of  $\frac{\alpha_N}{N}$  in the momentum map image  $P$ . This is a delicate matter when  $\frac{\alpha_N}{N}$  approaches a face of the boundary of  $P$  (see §6.3). In our relatively simple considerations, a similar phenomenon arises. In our corresponding situation where  $M \subset \mathcal{O}_\tau$  and  $\tau \neq 0$ , the sequence  $(s_N)$  may or may not be tame. If it is not tame one can not expect a universal scaled probability density. It should be remarked that in the smooth case the limiting function  $f$  can be explicitly computed in the relevant local coordinants. This done in the course of the work in [14] (see also [12]).

In [3] the smooth case is also considered. There the authors take advantage of the Delzant construction which realizes  $X$  as a certain GIT-quotient of the set of stable points of a linear torus action on  $\mathbb{C}^d$ . The flat metric on the trivial bundle on  $\mathbb{C}^d$  defines a positive Hermitian metric on the push-down of the trivial bundle. In the toric setting they derive explicit formulas for the *stability function* which compares the two metrics. In a general situation (see §5) they make a quantitative comparison of the  $L_2$ -norms. This allows them to apply previously obtained results (Lemma 8.1) to derive a formula for the universal scaled probability distribution under special conditions on  $\alpha_N$  (see §8). For more general results in this direction see the work of Ma and Zhang, e.g., Theorem 0.10 of ([10]).

## 2. Preparation on toric varieties

Here we begin by presenting certain background information for dealing with the combinatorial side of the theory of toric varieties. Our main observation is that associated to any ray there is a tame sequence.

### One-parameter subgroups

Recall that by definition there is a open and dense  $T$ -orbit  $\mathcal{O}_0$  in  $X$ . In order to understand the structure of the boundary  $\text{bd}(\mathcal{O}_0)$  one considers algebraic 1-parameter subgroups  $\lambda : \mathbb{C}^* \rightarrow T$ . In a systematic way one determines each *boundary orbit* as the orbit of a limit point

$$x_0^\tau := \lim_{z \rightarrow 0} \lambda(z)x_0 \in X \quad (4)$$

for  $\lambda$  appropriately chosen. Many considerations are in this way reduced to the 1-dimensional case. Since  $T = (\mathbb{C}^*)^m$ , 1-parameter subgroups are in 1-1 correspondence with integral vectors  $v = (n_1, \dots, n_m) \in \mathbb{Z}^m$  by setting

$$\lambda_v : \mathbb{C}^* \rightarrow T, \quad \lambda_v(z) = (z^{n_1}, \dots, z^{n_m}).$$

We therefore may regard the integral lattice  $\mathfrak{t}_{\mathbb{Z}}$  defined in (1) as the space of 1-parameter subgroups of  $T$ . Using the pairing  $\langle \cdot, \cdot \rangle : \mathfrak{t}_{\mathbb{Z}}^* \times \mathfrak{t}_{\mathbb{Z}} \rightarrow \mathbb{Z}$  from (2) we describe one-parameter groups by

$$\chi_\alpha(\lambda_v(z)) = z^{\langle \alpha, v \rangle}$$

for a character  $\chi_\alpha = e^{2\pi i \alpha}$  with  $\alpha \in \mathfrak{t}_{\mathbb{Z}}^*$ .

### 2.1. Orbit structure

In the theory of toric varieties (see [4]) one associates in a one-to-one fashion to each  $T$ -orbit  $\mathcal{O} \subset X$  a set  $\tau \subset \mathfrak{t}_{\mathbb{R}}$  called a *strongly convex rational polyhedral cone*. Such a cone is by definition the set of all convex combinations of a set of integral vectors  $v_1, \dots, v_r \in \mathfrak{t}_{\mathbb{Z}}$ , called the *generators of  $\tau$* , i.e.

$$\tau = \{\lambda_1 v_1 + \dots + \lambda_r v_r : \lambda_j \geq 0\} =: \langle v_1, \dots, v_r \rangle_{\mathbb{R} \geq 0}$$

such that  $\tau \cap (-\tau) = \{0\}$ . The vectors  $v_j$  are chosen in way such that the one-parameter groups defined by  $\text{Int}(\tau) \cap \mathfrak{t}_{\mathbb{Z}}$  close up in the orbit  $\mathcal{O}_\tau$ ; that is, if  $v \in \mathfrak{t}_{\mathbb{Z}}$  is in the relative interior of  $\tau$ , then  $\lim_{z \rightarrow 0} \lambda_v(z)x_0 = x_0^\tau$  is an element of  $\mathcal{O}_\tau$ . The collection of all such cones  $\tau$  constitute a *fan*  $\Sigma(X)$ .



Fans and their exact relation to the orbit structure of  $X$  are studied in detail, e.g., in [4]. What is important for us in the following are two basic facts:

- (1) Each  $T$ -invariant complex irreducible hypersurface  $Y_j$  corresponds to a one-dimensional cone  $\tau_j = \langle v_j \rangle_{\mathbb{R}_{\geq 0}}$  with  $v_j \in \mathfrak{t}_{\mathbb{Z}}$ .
- (2) A cone  $\sigma$  of maximal dimension corresponds to a  $T$ -fixed point  $x_\sigma$ .

### Parameterization of eigensections

Let  $Y_1, \dots, Y_\ell$  be the irreducible  $T$ -invariant hypersurfaces in  $X$ . The base section  $s_0$  chosen in §1 defines a  $T$ -invariant divisor

$$D = \sum_j a_j Y_j. \quad (5)$$

Of course  $L = L(D)$ . Each hypersurface  $Y_j$  is given by a one-dimensional cone  $\tau_j = \langle v_j \rangle_{\mathbb{R}_{\geq 0}}$  in the fan of  $X$ . We define the set

$$P_D = \bigcap_{j=1}^{\ell} \{u \in \mathfrak{t}^* : \langle u, v_j \rangle \geq -a_j\}. \quad (6)$$

The following Proposition characterizes holomorphic eigensections in terms of this polyhedron; it is standard in the theory of toric varieties.

PROPOSITION 2.1. — *The spaces of sections  $\Gamma(X, L^N)$  are given by*

$$\Gamma(X, L^N) = \bigoplus_{\alpha} \mathbb{C} s_{\alpha}, \text{ where } \alpha \in NP_D \cap \mathfrak{t}_{\mathbb{Z}}^*$$

and the  $s_{\alpha}$  are  $T$ -eigensections with  $t(s_{\alpha}) = \chi_{\alpha}(t)s_{\alpha}$ . Furthermore, if  $D$  is ample, then  $P_D$  is a strictly convex polytope. It is bounded precisely when  $X$  is compact.

*Remark.* — Since in our case  $L$  is very ample, one can explicitly construct the Kodaira embedding of  $X$  by using the polytope  $P_D$ . This is initially given by  $x \mapsto [s_{\alpha}(x)]$  where the  $s_{\alpha}$  are the distinguished  $T$ -eigensections in  $\Gamma(X, L)$ . Recalling that  $s_{\alpha}(tx_0) = \chi_{\alpha}(t)s_0(tx_0)$  on the open orbit, the embedding can be written as  $t(x_0) \mapsto [\chi_{\alpha}(t)]$ . In other words  $t(x_0)$  is mapped to  $t[1 : \dots : 1]$  and  $X$  is identified with the closure of the  $T$ -orbit  $T \cdot [1 : \dots : 1]$ . In [13] toric varieties are actually defined this way. To ensure smoothness when taking the closure they require the polytope  $P_D$  to be Delzant.  $\square$

## 2.2. Existence of asymptotic sequences

As an application of Proposition 2.1 we show that, given a ray  $R(\xi)$  there exists a sequence of holomorphic eigensections  $s_N$  approximating  $R(\xi)$  at infinity if and only if  $\xi$  is an element of the polytope  $P_D$ . The necessity can be proved as follows.

**PROPOSITION 2.2.** — *If  $(s_N)$  approximates  $\mathbb{R}^{\geq 0}\xi$  at infinity, then  $\xi \in P_D$ .*

*Proof.* — From the definition of a sequence of characters approximating a ray  $R(\xi)$  we have  $\alpha_N/N = \xi + O(N^{-1})$ . Since each  $s_N$  is a holomorphic eigensection in the bundle  $L^N$ , the corresponding character  $\alpha_N$  must lie in the polytope  $NP_D$ , whose defining equations are given by (6). For the vector  $\xi$  we now have

$$\langle \xi, v_j \rangle = \left\langle \frac{\alpha_N}{N}, v_j \right\rangle - \left\langle \frac{\alpha_N}{N} - \xi, v_j \right\rangle \geq -a_j + O(N^{-1}).$$

The last inequality is true for all  $N$ , thus  $\langle \xi, v_j \rangle \geq -a_j$  for all  $j$ , and hence  $\xi \in P_D$ .  $\square$

The converse is also true but requires more effort.

**PROPOSITION 2.3.** — *If  $\xi \in P_D$ , then there exists a sequence  $\alpha_N \in NP_D \cap \mathfrak{t}_{\mathbb{Z}}^*$  such that  $\alpha_N = N\xi + O(1)$  for all  $N \in \mathbb{N}$ .*

Since for every  $\xi \in \mathbb{R}^m$  there exists a sequence  $(\alpha_N)$  of integral points with  $\|N\xi - \alpha_N\| < 1$ , the proposition is a consequence of the following projection argument.

**LEMMA 2.4.** — *Let  $F$  be a face of the polytope  $P_D$  and  $\xi$  be in the relative interior of  $F$ . If  $\alpha_N \in \mathfrak{t}_{\mathbb{Z}}^*$  is any integral sequence such that  $\alpha_N = N\xi + O(1)$ , then there exists a sequence  $\alpha'_N$  having that same property but in addition  $\alpha'_N/N \in F \cap \mathfrak{t}_{\mathbb{Z}}^*$  for almost all  $N$ .*

*Proof.* — Suppose  $\text{codim } F = 0$ . Then  $\xi$  lies in the interior of  $P_D$  which is open. Since the sequence  $\alpha_N/N$  converges to  $\xi$ , the point  $\alpha_N/N$  will lie in the interior of  $P_D$  for big  $N$  and we can simply set  $\alpha'_N = \alpha_N$ . The other extreme case is  $\text{codim } F = \dim P_D$ , which means that  $F$  is a vertex of  $P_D$ . In this case  $\xi$  is integral and we can set  $\alpha'_N = N\xi$ . The remaining case is characterized by  $0 < \text{codim } F < \dim P_D$ . Here we replace  $\alpha_N/N$  by its projection onto the face  $F$ . For this choose integral vectors  $w_1, \dots, w_k$  and a vertex  $\alpha_\sigma$  of  $F$  such that  $F \subset \alpha_\sigma + \text{span}_{\mathbb{R}}\{w_1, \dots, w_k\}$  and define

the projection

$$\alpha'_N = N(\alpha_\sigma + \sum_{j=1}^k \langle \frac{\alpha_N}{N} - \alpha_\sigma, w_j \rangle w_j).$$

By writing  $\xi = \alpha_\sigma + \sum_j \langle \xi - \alpha_\sigma, w_j \rangle w_j$  one immediately sees

$$\| \frac{\alpha'_N}{N} - \xi \| \leq \frac{C}{N} \sum_{j=1}^k \|w_j\| = O(\frac{1}{N}).$$

Hence, the sequences  $\alpha_N$  and  $\alpha'_N$  approximate the same ray  $\mathbb{R}^{\geq 0}\xi$ . Furthermore, since  $\xi$  lies in the relative interior of  $F$ , the elements  $\alpha'_N/N$  will also be in  $F$  for almost all  $N$ . □

### 3. Strictly plurisubharmonic limit functions

The goal of this section is to prove the existence of a certain strictly plurisubharmonic limit function  $f : X \rightarrow \mathbb{R} \cup \{\infty\}$  which is canonically associated to a semiclassical ray  $R(\xi)$ . This function provides the main tool for studying the asymptotic behavior of the probability densities  $|\varphi_N|_h^2$  and the tails of the distribution functions  $D_N(t)$ .

On the open orbit we may simply define  $f = \lim_{N \rightarrow \infty} -\frac{1}{N} \log |s_N|_h^2$ . However, in order to extend  $f$  to  $X$  we need to know more about the behavior of the  $s_N$  at the boundary. The following example shows that this might be quite irregular.

*Example.* — Let  $X = \mathbb{P}_1$  and  $L = H$  be the hyperplane section bundle equipped with its standard metric  $h$ . If  $[z_0 : z_1]$  are standard homogeneous coordinates and sections  $s \in \Gamma(X, L)$  are regarded as linear functions  $\ell(z_0, z_1)$ , then

$$|s|_h^2 = \frac{|\ell(z_0, z_1)|^2}{|z_0|^2 + |z_1|^2}.$$

More generally if we equip  $L^N$  with the associated tensor power metric and a section  $s \in \Gamma(X, L^N)$  is represented by a homogeneous polynomial  $P$  of degree  $N$ , then

$$|s|^2 = \frac{|P(z_0, z_1)|^2}{(|z_0|^2 + |z_1|^2)^N}.$$

Let us begin with the sequence defined by  $s_N = z_0^N$ . It corresponds to the sequence of characters  $\alpha_N = 0$  for all  $N$ . For the probability density we use the normalized Fubini-Study volume form and compute the integral

$$\int_{\mathbb{P}_1} |s_N|^2 = \int_{\mathbb{C}} \frac{1}{(1 + |z|^2)^N} = \int_0^\infty \frac{1}{(1 + r^2)^N} r dr \sim \frac{1}{N - 1}.$$

If we replace  $s_N$  by the sequence defined by the homogeneous polynomials  $z_0^{N-1}z_1$ , belonging to the sequence of characters  $\alpha_N = 1$  for all  $N$ , then

$$\int_{\mathbb{P}^1} |s_N|^2 = \int_{\mathbb{C}} \frac{|z|^2}{(1 + |z|^2)^N} \sim \frac{1}{(N - 1)(N - 2)}.$$

Thus the integrals are asymptotically different. The situation is even worse if we allow the sections to jump around like in the following example:

$$s_N = \begin{cases} z_0^N & \text{for } N \text{ odd,} \\ z_1 z_0^{N-1} & \text{otherwise.} \end{cases}$$

Nevertheless, in all of the above cases the probability density  $|s_N|^2 / \|s_N\|_{L_2}^2$  converges in measure to the Dirac measure of the point  $[1 : 0]$ . □

### 3.1. Tame sequences

We can avoid the kind of problems illustrated above by replacing the sequence  $(s_N)$  by a new sequence  $(s'_N)$  that approximates the same ray, but whose vanishing orders at the boundary can be better controlled. For the construction of  $(s'_N)$  we first note that the asymptotic vanishing order  $\text{ord}_{Y_j}(s_N)$  along a boundary hypersurface  $Y_j$  is well-defined and completely determined by  $R(\xi)$ .

LEMMA 3.1. — *There exist non-negative real numbers  $k_1, \dots, k_\ell$  such that for each boundary component  $Y_j$  we have*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \text{ord}_{Y_j}(s_N) = k_j. \tag{7}$$

*Proof.* — A  $T$ -invariant hypersurface  $Y_j$  is determined by a 1-dimensional cone in the fan of  $X$ . Such a cone is generated by a vector  $v_j \in \mathfrak{t}_{\mathbb{Z}}$ . Thus,  $Y_j$  determines a 1-codimensional face  $H_j$  of the polytope  $P_D$  in the following way (see also (6))

$$H_j = P_D \cap \{u \in \mathfrak{t}^* : \langle u, v_j \rangle = -a_j\}. \tag{8}$$

Choosing a vertex  $\alpha_\sigma \in H_j$  it follows that the corresponding eigensection  $s_\sigma$  does not identically vanish on  $Y_j$ . Here we write  $s_N = s_{N,\sigma} s_\sigma^N$  where  $s_{N,\sigma}$  is a  $T$ -equivariant meromorphic function transforming by the character  $\chi_N \chi_\sigma^{-N}$ . Its order along  $Y_j$  is given by  $\langle \alpha_N - N\alpha_\sigma, v_j \rangle$ . Since the sequence  $\alpha_N/N$  converges to  $\xi$  we have

$$\begin{aligned} \frac{1}{N} \text{ord}_{Y_j}(s_N) &= \left\langle \frac{\alpha_N}{N} - \alpha_\sigma, v_j \right\rangle \rightarrow \langle \xi - \alpha_\sigma, v_j \rangle \\ &= \langle \xi, v_j \rangle + a_j =: k_j. \end{aligned} \tag{9}$$

This proves the existence of the numbers  $k_j$  for any sequence  $s_N$  approximating the ray  $R(\xi)$  at infinity.  $\square$

PROPOSITION 3.2. — *There exists a sequence  $(s'_N)$  such that if  $k_j = 0$ , then  $\text{ord}_{Y_j}(s'_N) = 0$  for almost all  $N \in \mathbb{N}$ .*

*Proof.* — From equation (9) we have  $k_j = \langle \xi, v_j \rangle + a_j$ , where the  $a_j$  are the coefficients defining  $P_D$ , see (6). Thus,  $k_j = 0$  if and only if  $\xi$  lies in the hyperplane  $H_j$  defined by equation (8). A collection  $\{H_j\}$  of such hyperplanes determines a face of the polytope

$$\xi \in F = P_D \cap H_1 \cap \dots \cap H_r. \quad (10)$$

Hence, the condition that  $\text{ord}_{Y_j}(s_N) = 0$  whenever  $k_j = 0$  is equivalent to saying that  $\alpha_N$  is an element of the same face as  $\xi$ . By Lemma 2.4 there exists a sequence  $\alpha'_N$  approximating the same ray  $R(\xi)$  but having in addition the property  $\alpha'_N \in F \cap \mathfrak{t}_{\mathbb{Z}}^*$  for almost all  $N$ . This proves the assertion.  $\square$

DEFINITION. — *A sequence  $(s_N)$  which approximates a ray at infinity said to be tame if it fulfills the properties of Proposition 3.2. The union  $Y$  of the hypersurfaces  $Y_j$  with  $k_j > 0$  is called the limiting support of the sequence.*

*Remarks.* — 1. If  $\xi$  lies in the interior of the polytope  $P_D$ , then every asymptotic sequence approximating the ray  $R(\xi)$  is tame. 2. If  $\alpha'_N$  is the tame sequence constructed from a given one  $\alpha_N$ , then the difference  $\alpha'_N - \alpha_N$  is uniformly bounded in  $N$ .

By inspecting the proofs of Lemma 3.1 and Proposition 3.2 we can give a description of the limiting support  $Y$  of a tame sequence in terms of the polytope  $P_D$ .

PROPOSITION 3.3. — *If  $\{H_j\}$  is the set of all 1-codimensional faces of  $P_D$  which do not contain  $\xi$  and  $\{Y_j\}$  is the set of associated  $T$ -invariant hypersurfaces, then the limiting support of a tame sequence is given by  $Y = \cup_j Y_j$ .*

### On the complex geometry of $X \setminus Y$

If  $(s_N)$  is a tame sequence with asymptotic support  $Y$ , then, since  $L$  is ample, the  $T$ -invariant open set  $X \setminus Y$  is an affine variety. Since  $T$  has an open orbit in  $X \setminus Y$ , it follows it possesses only the constant invariant holomorphic functions and as a result it possesses a unique closed orbit  $\mathcal{O}_\tau$ .

Let us explain how  $\xi$  determines  $\mathcal{O}_\tau$  in the combinatorial language of toric varieties. For this recall that the face  $F$  is defined by the condition that  $\xi$  is in its relative interior. In the following way  $F$  is associated to a cone  $\tau$  in the fan  $\Sigma(X)$ . By definition,  $F$  is the intersection of  $P_D$  with a set of supporting hyperplanes. That is, if  $I \subset \{1, \dots, \ell\}$  is an index set, then  $F$  is defined by

$$F = P_D \cap \bigcap_{i \in I} H_i \quad \text{where} \quad H_i = \{u \in \mathfrak{t}_{\mathbb{R}}^* : \langle u, v_i \rangle = -a_i\}. \quad (11)$$

Since  $D$  is very ample, the vectors  $v_k$  with  $k \in I$  define a cone  $\tau$  in the fan of  $X$ . This cone, regarded as a fan, defines the affine toric variety  $X \setminus Y$ . In particular, the relative interior of  $\tau$  corresponds to the closed (dimension-theoretically minimal) orbit in  $X \setminus Y$ .  $\square$

Again in the setting of a tame sequence we observe that for  $N$  sufficiently large the functions  $f_N = -\frac{1}{N} \log |s_N|^2$  are  $T_{\mathbb{R}}$ -invariant strictly plurisubharmonic exhaustions of  $X \setminus Y$ . In the next section it is shown that they converge to a function  $f$  which is likewise a smooth strictly plurisubharmonic exhaustion. Thus the following is relevant for our considerations.

**THEOREM 3.4.** — *Let  $Z$  be a Stein space equipped with a holomorphic action of a reductive group  $G$  which is the complexification of a maximal compact subgroup  $K$  and let  $\rho : X \rightarrow \mathbb{R}^{\geq 0}$  be a smooth proper  $K$ -invariant strictly plurisubharmonic exhaustion. Assuming that  $\mathcal{O}(Z)^G \cong \mathbb{C}$ , it follows that the minimum set  $M := \{\zeta \in Z; \rho(\zeta) = \min\{\rho(z); z \in Z\}\}$  consists of a single  $K$ -orbit which is contained in the closed  $G$ -orbit in  $Z$ .*

This result is a special case of Corollary 1 in §5.4 of [5]. It is one of the basic first steps for the construction of the analytic Hilbert quotient by the method of Kählerian reduction (see [7]). It should also be mentioned that using the gradient of the norm of the associated moment map one shows that  $M$  is a strong deformation retract of  $Z$  ([6]). Although we apply these results in the case of an affine variety, the plurisubharmonic functions at hand are only smooth and are not of the type where the algebraic theory can be applied (See [11] for basic results in the situation where  $Z$  is equivariantly embedded in a representation space and  $\rho$  is the restriction of a  $K$ -invariant norm-function.).

### Product structure of $\mathbf{X} \setminus \mathbf{Y}$

Continuing in our special setting where  $\mathcal{O}_\tau$  is as above, we let  $T_\tau$  be the connected component at the identity of the isotropy group  $T_{x_0^\tau}$  of any base point in  $\mathcal{O}_\tau$ . This can be alternatively described as the connected

component at the identity of the ineffectivity of the  $T$ -action on  $\mathcal{O}_\tau$ . Let  $T'$  be a complementary complex torus in  $T$ , i.e.,  $T = T_\tau \times T'$ .

Let us choose  $x_0^\tau$  so that there is a 1-parameter subgroup  $\lambda$  with  $\lim_{t \rightarrow 0} \lambda(t)(x_0) = x_0^\tau$ . Clearly  $\lambda(\mathbb{C}^*) \in T_\tau$  and thus  $x_0^\tau$  is in the closure  $\text{cl}(T_\tau.x_0) =: \Sigma_\tau$ . In fact  $\{x_0^\tau\}$  is the unique closed orbit in this closure and therefore the restriction to  $\Sigma_\tau$  of any  $T_\mathbb{R}$ -invariant strictly plurisubharmonic exhaustion has exactly  $\{x_0^\tau\}$  as its minimizing set.

PROPOSITION 3.5. — *The map  $\alpha : T' \times \Sigma_\tau \rightarrow X \setminus Y, (t', x) \mapsto t'(x)$ , establishes an  $T$ -equivariant isomorphism  $\mathcal{O}_\tau \times \Sigma_\tau \rightarrow X \setminus Y$ .*

*Proof.* — Linearizing the action of the maximal compact torus of  $T_\tau$  shows that the only  $T_\tau$ -orbits which have  $x_0^\tau$  in its closure are those in  $\Sigma_\tau$ . Thus if  $t' \in T'$  fixes  $x_0^\tau$ , then  $t'(\Sigma_\tau) = \Sigma_\tau$ . Since  $t'$  centralizes  $T_\tau$ , it stabilizes the open  $T_\tau$ -orbit in  $\Sigma_\tau$  and, since this is contained in the open  $T$ -orbit in  $X$  where  $T$  acts freely, it follows that  $t' = \text{Id}$ . Hence,  $T'$  acts freely on  $\mathcal{O}_\tau$  and  $T_\tau$  is the full isotropy group at  $x_0^\tau$ .

The above shows in particular that we must only prove that  $\alpha$  is an isomorphism. For the surjectivity of  $\alpha$  we implement the Hilbert Lemma which, given  $x \in X \setminus Y$ , provides a 1-parameter subgroup  $\lambda$  with  $x_1 \in \mathcal{O}_\tau \cap \text{cl}(\lambda(\mathbb{C}^*).x)$ . Choosing  $t' \in T'$  with  $t'(x_1) = x_0^\tau$ , since  $x_0^\tau$  is in the closure of  $\lambda(\mathbb{C}^*)t'(x)$ , it follows that  $t'(x) \in \Sigma_\tau$  and the surjectivity is proved.

For the injectivity, if  $x_1, x_2 \in \Sigma_\tau$  and  $t'_1, t'_2 \in T'$  are such that  $t'_1(x_1) = t'_2(x_2)$ , then, defining  $t = (t'_1)^{-1}t'_2$ , it follows that  $t'(x_1) = x_2$ . As we have seen above, this implies that  $t'$  stabilizes  $\Sigma_\tau$  and is consequently in  $T_\tau$ , i.e.,  $t' = \text{Id}$  □

It should be mentioned that this product decomposition can be proved by purely combinatorial means (see [4]).

### 3.2. Existence of the limit function

Recall that our goal is to understand the limiting properties of the probability density function  $|\varphi_N|_h^2$ . The following is the first main step in this direction.

PROPOSITION 3.6. — *If  $(s_N)$  is a tame sequence which approximates a ray  $R(\xi)$  at infinity with limiting support  $Y$ , then the associated sequence*

$$f_N = -\frac{1}{N} \log |s_N|_h^2$$

converges uniformly on compact subsets of  $X \setminus Y$  to a smooth strictly plurisubharmonic function  $f$ .

**Zusatz.** *Any sequence approximating the same ray at infinity converges uniformly on compact subsets of the open orbit to  $f$ . Thus  $f$  could be alternatively defined as the extension by continuity to  $X \setminus Y$  of this limit function.*

*Remarks.* — As the reader will note in the proof, the convergence  $f_N \rightarrow f$  is locally given by the convergence of the sequence  $(s_N^{1/N})$  of holomorphic functions. Thus the convergence in the  $C^\infty$ -topology is also guaranteed. Finally, if  $\widehat{s}_N$  is any sequence approximating  $R(\xi)$ , then on the open orbit

$$f_N - \widehat{f}_N = 2\operatorname{Re}\left(\frac{\beta_N}{N}\right)$$

where  $\beta_N$  is a sequence of linear functions which are contained in a bounded set. Consequently, on the open orbit  $f$  and  $\widehat{f}$  agree. As a result the limiting function defined by a tame sequence is unique: Take any sequence which approximates  $R(\xi)$  at infinity and define  $f$  to be the function on  $X$  which is obtained by extending by continuity the uniquely defined function on the open orbit to all of  $X$ . Below we will also show that that the limiting measure of the probability densities  $|\varphi_N|_h^2$  also only depends on the ray and not on the particular sequence which approximates it at infinity. This result holds for every sequence approximating the ray. It should be reemphasized, however, that a precise asymptotic development only holds for tame sequences.  $\square$

The following fact plays an essential role in our proof of Proposition 3.6.

**LEMMA 3.7.** — *Let  $Z$  be a compact toric variety of a group  $T \cong (\mathbb{C}^*)^k$  and  $z_0 \in Z$  be a  $T$ -fixed point. If  $L$  is a very ample  $T$ -line bundle on  $Z$ , it follows that up to constant multiples there is exactly one eigensection which does not vanish at  $z_0$ .*

*Proof.* — For every point  $z$  of the (Zariski open) saturation  $\mathcal{S}(z_0) := \{z \in X; z_0 \in \operatorname{cl}(T.z)\}$  there is a 1-parameter group  $\lambda : \mathbb{C}^* \rightarrow T$  with  $z_0$  in the (compact) closure  $C = \operatorname{cl}(\lambda(\mathbb{C}^*).z_0)$  in  $Z$ . Let  $\widehat{C}$  be the normalization of such a curve. Note that  $\widehat{C} \cong \mathbb{P}_1$  and that the lifted action of  $\mathbb{C}^*$  has two fixed points, one over  $z_0$  and another  $\widehat{z}_1$  over some other point in  $C$ . The pull-back  $\widehat{L} \rightarrow \widehat{C}$  of the restriction of  $L$  to  $C$  is isomorphic to some positive power of  $H^k$  of the hyperplane section bundle. If  $s_1, s_2 \in \Gamma(Z, L) \setminus \{0\}$  are eigensections, neither of which vanishes at  $z_0$ , then the lifts  $\widehat{s}_1$  and  $\widehat{s}_2$  of their restrictions to  $C$  are  $\mathbb{C}^*$ -eigensections of  $\widehat{L}$  on  $\widehat{C}$  which only vanish at



$\widehat{z}_1$ . Thus  $\widehat{s}_1$  is a constant multiple of  $\widehat{s}_2$  and consequently  $s_1|_C$  is a constant multiple  $s_2|_C$ . Since this holds for every curve  $C$  constructed in this way and the constant is uniquely determined by  $s_1(z_0)$  and  $s_2(z_0)$ , the desired result follows.  $\square$

We apply this Lemma to restrictions of eigensections to the closures of fibers of the  $T_\tau$ -invariant projection map  $q : \Sigma_\tau \times \mathcal{O}_\tau \rightarrow \mathcal{O}_\tau$ .

**COROLLARY 3.8.** — *If the base section  $s_0$  is chosen so that  $s_0(x_0^\tau) \neq 0$  and  $s_N$  is a  $T$ -eigensection in  $\Gamma(X, L^N)$  which also does not vanish at  $x_0^\tau$ , then the restrictions of  $s$  and  $s_0^N$  to any  $q$ -fiber agree. In particular, the group  $T_\tau$  is in the kernel of the character  $\chi_{\alpha_N}$  associated to  $s_N$  by the base eigensection  $s_0^N$ . Furthermore, the relation  $s(tx_0^\tau) = \chi_{\alpha_N}(t)s_0(tx_0^\tau)$  holds on  $\mathcal{O}_\tau$*

*Proof.* — This follows immediately from the above proposition by applying it in the case of the  $T_\tau$ -action on the closure of a  $q$ -fibers. It is applicable because the closed  $T_\tau$ -orbit is its fixed point which is the intersection point of the  $q$ -fiber with  $\mathcal{O}_\tau$ . Furthermore,  $s_N(x_0) = s_0(x_0)^N$ ; so  $s_N|_{\Sigma_\tau} = cs_0|_{\Sigma_\tau}$  with  $c = 1$ . Equality on the other fibers follows from the fact that  $T$  acts transitively on the set of these fibers.  $\square$

Of course the original base section  $s_0$  may vanish at  $x_0^\tau$ . If it does, using the fact that  $L$  is very ample, we know that there is a  $T$ -eigensection  $\widehat{s}_0$  which does not vanish there. As we have already noted, changing from  $s_0$  to  $\widehat{s}_0$  only has the effect of translating  $R(\xi)$  to  $R(\xi - \hat{\alpha})$  where the character associated to  $\widehat{s}_0$  for the base point  $s_0$  is  $\chi_{\hat{\alpha}}$ .

*Proof of Proposition 3.6:* It follows from the above corollary and the remark that a base change has no influence on the discussion that we must only prove this for the sequence  $(s_N|_{\mathcal{O}_\tau})$ . In more detail, this is a consequence of the product decomposition of  $X \setminus Y$  and the fact that on  $t'(\Sigma_\tau)$  the sections  $s_N$  and  $s_0^N$  for all  $t' \in T'$ . The convergence on  $\mathcal{O}_\tau$  is just the (possibly lower-dimensional) case of the open orbit! In that case

$$f_N = -\frac{1}{N} \log |\chi_{\alpha_N}|^2 + \log |s_0|_h^2$$

and the desired convergence is guaranteed by the fact that  $\frac{\alpha_N}{N} = \xi_{\alpha_N} + O(N^{-1})$ .

Note that since  $-\frac{1}{N} \log |\chi_{\alpha_N}|^2$  converges, given a point in  $\mathcal{O}_\tau$  we may choose a subsequence so that  $(s_{N_k})^{\frac{1}{N_k}}$  converges locally uniformly. Since the local limit section is holomorphic, it follows that the limit function  $f$  is smooth and strictly plurisubharmonic.  $\square$

## Exhaustion property

As we have seen above the functions  $f_N$  and the limit  $f$  are smooth strictly plurisubharmonic exhaustions of  $X \setminus Y$ . Let  $M := \{z \in X \setminus Y; f(z) = \min\{f(x); x \in X \setminus Y\}\}$  and define  $M_N$  to be the corresponding set for  $f_N$ . We know that  $M_N = T_{\mathbb{R}}.x_N$  and  $M = T_{\mathbb{R}}.x_0^\tau$  for points  $x_N, x_0^\tau \in \mathcal{O}_\tau$ . Here we fix  $x_0^\tau$  for the discussion and normalize  $f$  so that  $f(x_0^\tau) = 0$ . Let us underline that the functions  $f_N|_{\mathcal{O}_\tau}$  and  $f|_{\mathcal{O}_\tau}$  have particularly strong convexity properties.

**PROPOSITION 3.9.** — *Let  $\rho$  is smooth  $(S^1)^k$ -invariant strictly plurisubharmonic function on  $(\mathbb{C}^*)^k$  which takes on a local minimum at a point  $z_0$ , then this is a global minimum, the set  $M := \{z \in X \setminus Y; \rho(z) = \min\{\rho(x); x \in X \setminus Y\}\}$  is the  $(S^1)^k$ -orbit of  $z_0$  and  $\rho$  is an exhaustion of  $(\mathbb{C}^*)^k$ .*

*Proof.* — Using polar coordinates we write  $\rho = e^{r(t)}$  where  $r$  is a strictly convex function on  $\mathbb{R}^k$  which takes on a local minimum at  $\log z_0$ .  $\square$

**COROLLARY 3.10.** — *The points  $x_N$  can be chosen in  $M_N$  so that that they converge to  $x_0^\tau$ .*

*Proof.* — If  $U = U(M)$  is an arbitrary  $T_{\mathbb{R}}$ -invariant relatively compact neighborhood of  $M$  in  $X \setminus Y$ , then the restriction of  $f$  to the boundary of  $U(M) \cap T.x_0^\tau$  is strictly larger than 0. Since  $f_N$  converges uniformly to  $f$  on  $U(M)$ , this implies that for  $N$  sufficiently large  $f_N$  attains a local minimum in the interior of  $U(M)$ . But the strong convexity of  $f_N$  (see the argument above) implies that this local minimum is global, i.e.,  $M_N \subset U(M)$ .  $\square$

## Dependency on the metric and the ray

The minimizing set  $M$  of  $f$  is contained in the closed  $T$ -orbit  $\mathcal{O}_\tau$  in  $X \setminus Y$  where  $Y$  and  $\mathcal{O}_\tau$  are completely determined by the face of  $P_D$  which contains  $\xi$ . The exact location of  $M$  in  $\mathcal{O}_\tau$  depends on the metric  $h$  and  $\xi$ . With respect to metrics we underline that we only consider those which are positive in the sense that  $-\log |s|_h^2$  is strictly plurisubharmonic for any nowhere zero local holomorphic section  $s$ . If  $\hat{h} = e^\phi h$  is such a metric and  $\hat{f}$  is the associated strictly plurisubharmonic limit function associated to a ray, then a direct calculation shows that  $\hat{f} = -\phi + f$  which is still a  $T_{\mathbb{R}}$ -invariant strictly plurisubharmonic function on  $X \setminus Y$ . The translation by  $-\phi$  only changes the location of the minimizing set  $\hat{M}$  in  $\mathcal{O}_\tau$ .

Concerning the dependence on  $\xi$ , recall that the convergence  $f_N \rightarrow f$  might require a change of the base eigensection so that  $\alpha_N = N(\hat{\xi}) + O(1)$  where  $\hat{\xi} = \xi - \hat{\alpha}$ . Then,  $f|_{\mathcal{O}_\tau} = -\log |\hat{s}_0|_h^2 - \delta$  where

$$\delta = \lim_{N \rightarrow \infty} \frac{1}{N} \log |\chi_{\alpha_N}|^2.$$

In order to interpret this correctly, we must identify  $\mathcal{O}_\tau$  with the subgroup  $T' \cong (\mathbb{C}^*)^k$ , use the  $T'_\mathbb{R}$ -invariance of  $|\hat{s}_0|_h^2$  and express these quantities in polar coordinates as in Proposition 3.9. At that level, i.e., in logarithmic coordinates, we see that  $f$  is the translate by the linear function  $2\pi \text{Im}(-\hat{\xi})$  of the strictly convex exhaustion of  $\mathbb{R}^k$  determined by  $-\log |\hat{s}_0|_h^2$ . The influence of such a linear translation on the minimizing set of a strictly convex function is easily understood. It also should be noted that  $M$  is just the preimage of 0 under the moment map defined by the Kählerian potential  $f$  on  $X \setminus Y$  (see, e.g., [7]).

### 3.3. Localization

In order to obtain estimates of integrals involving the functions  $f_N$  we will use information on their restrictions to orbits of 1-parameter subgroups which close up to points on the closed orbit  $T.x_0^\tau$ . For this we use the following fact.

LEMMA 3.11. — *If  $\lambda : \mathbb{C}^* \rightarrow T$  is a 1-parameter subgroup and  $x \in X \setminus Y$  is such that the orbit  $\lambda(\mathbb{C}^*).x$  is not closed, then the closure in  $X \setminus Y$  consists of the orbit plus one additional point  $b$  with  $f_N(b)$  strictly less than  $f_N(x)$  for all  $N$ .*

*Proof.* — Since  $X \setminus Y$  is affine, the closure  $C$  of such an orbit is not compact and therefore contains exactly one additional point. Of course  $C$  may be singular, but it is locally irreducible so that the normalization  $\widehat{C} \rightarrow C$  is injective and the pullback of  $f_N$  to  $\widehat{C} \cong \mathbb{C}$  is an  $S^1$ -invariant plurisubharmonic exhaustion which is strictly plurisubharmonic outside of the preimage  $\widehat{b}$  of  $b$ . If strictly inequality would not hold on some circle  $S^1.z$  for  $z \in \widehat{C} \setminus \{\widehat{b}\}$ , then the maximum principle would be violated. □

After these preparations we are now in a position to prove the desired estimates for tame sequences.

PROPOSITION 3.12 (Localization Lemma). — *If  $U(M)$  is a  $T_\mathbb{R}$ -invariant neighborhood of  $M$  which is relatively compact in  $X \setminus Y$ , then there exists*

$N_0$  and  $\varepsilon > 0$  so that if  $N > N_0$  it follows that  $f_N > \varepsilon$  on the complement of  $U(M)$  in  $X \setminus Y$ .

*Proof.* — First recall that we have normalized the sections so that  $f_N(x_0^\tau) = f(x_0^\tau) = 0$ . Then choose  $N_0$  so that  $N > N_0$  implies that the minimum set  $M_N$  is contained in  $U(M)$ . Since  $f_N$  converges uniformly to the strictly plurisubharmonic function  $f$  on  $U(M)$  we may also assume that  $f_N > \varepsilon > 0$  on the boundary of  $U(M)$ . Given  $x \in X \setminus Y$  the Hilbert Lemma guarantees the existence of a 1-parameter group  $\lambda : \mathbb{C}^* \rightarrow T$  whose orbit  $\lambda(\mathbb{C}^*).x$  closes up to the closed  $T$ -orbit  $T.x_0^\tau$ . By Lemma 3.11, if  $x \notin T.x_0^\tau$ , then  $f_N(x) > f_N(b)$  where  $b$  is the additional point in the closure. If  $b \notin U(M)$ , then by connecting  $b$  to  $M_N$  by a real 1-parameter group and using the strong convexity of  $f_N$  along that orbit, we see that  $f_N(b)$  is larger than the value of  $f_N$  at the intersection of that orbit with the boundary of  $U(M)$ . Thus, unless  $x$  is already in  $U(M)$ , it follows that  $f_N(x) > \varepsilon$ .  $\square$

We refer to the above result as a *Localization Lemma*, because it implies that integrals localize at  $M$ . Here is an example of what we mean by this.

**COROLLARY 3.13.** — For  $U(M)$ ,  $\varepsilon$  and  $N_0$  as above, given  $h \in L_1(X)$  it follows that

$$\int h e^{-N f_N} d\lambda \leq \|h\|_{L_1} e^{-N\varepsilon} + \int_{U(M)} h e^{-N f_N} d\lambda.$$

Below we prove precise estimates which lead to the desired result that the probability function  $|\varphi_N|^2$  converges to the Dirac measure of  $M$ .

### 3.4. Morse property

Here we show that  $f$  is a Bott-Morse function near  $M$ . For this the essential point is to understand the behavior of the extension of  $f$  to a smooth embedding space of  $\Sigma_\tau$ .

Extending from  $\Sigma_\tau$

Recall that  $X \setminus Y$  is naturally identifiable with the product  $\Sigma_\tau \times \mathcal{O}_\tau$  with  $M$  contained in  $\mathcal{O}_\tau$  as the  $T_{\mathbb{R}}$ -orbit  $T_{\mathbb{R}}.x_1$ . We begin by analyzing the local behavior of  $f$  on  $\Sigma_\tau$ . For this we first  $T_\tau$ -equivariantly embed  $\Sigma_\tau$  in a complex vector space  $W$  where  $x_0^\tau$  is mapped to the origin  $0 \in W$ .

Now recall that since  $x_0^r$  is the  $T_\tau$ -fixed point in  $\Sigma_\tau$ , there exists a 1-parameter subgroup  $\lambda$  of  $T_\tau$  with the property that  $\lim_{t \rightarrow 0} \lambda(t).x = x_0^r$  for every  $x \in \Sigma_\tau$ . We refer to  $x_0^r$  as being *attractive* for  $\lambda$ . Any linearization such as  $W$  splits  $W = W_- \oplus W_0 \oplus W_+$  with respect to any such 1-parameter subgroup where  $W_-$  are the points in  $W$  for which 0 is attractive for  $\lambda$ ,  $W_0$  is the set of  $\lambda$ -fixed points and  $W_+$  is the set of points for which 0 is repulsive. In our case every point of  $\Sigma_\tau$  is in  $W_-$ . So we have the following remark.

PROPOSITION 3.14. — *There exists an equivariant embedding  $\Sigma_\tau \hookrightarrow W$  so that 0 is attractive for  $\lambda$  for every point in  $W$ . In particular, if in its linearization  $\lambda(t) := \text{Diag}(\chi_1(t), \dots, \chi_n(t))$ , then the weights defining the  $\chi_j$  are all negative, e.g., it is never the case that  $\chi_i \chi_j = 1$ .*

Since  $f$  is a smooth strictly plurisubharmonic function, it extends to a neighborhood  $U$  of 0 in  $W$  as a smooth strictly plurisubharmonic function which after averaging is invariant with respect to the compact form  $(T_\tau)_\mathbb{R}$ . We may assume that  $U$  is  $(T_\tau)_\mathbb{R}$ -invariant and that  $\lambda(t)(U) \subset U$  as  $t \rightarrow 0$ . For  $x \in U$  we consider the closure  $\text{cl}(\lambda(\mathbb{C}^*).x)$  and let  $C(x)$  be the connected (irreducible) component containing 0 of its intersection with  $U$ .

PROPOSITION 3.15. — *The origin  $0 \in U$  is the absolute minimum point of  $f$  in  $U$ , i.e., if  $0 \neq x \in U$ , then  $0 = f(0) < f(x)$ .*

*Proof.* — The normalization  $\widehat{C}(x)$  of  $C(x)$  may be identified with the unit disk so that the pullback  $\widehat{f}$  of  $f$  is plurisubharmonic and strictly plurisubharmonic outside of the origin. Since  $\widehat{f}$  is  $S^1$ -invariant, the desired result follows immediately from the meanvalue property and the maximum principle.  $\square$

Now let us consider the Taylor development of  $f$  at 0,

$$f(z) = Q(z) + O(3).$$

The second order terms are of the form  $Q(z) = \bar{z}^T H z + R(z)$  where  $H$  is a Hermitian matrix and  $R(z) = \text{Re}(\sum a_{ij} z_i z_j)$ . We have implicitly chosen the coordinates  $z$  where  $\lambda$  is linearized and therefore with the origin being attractive for  $\lambda$ .

PROPOSITION 3.16. — *In the above coordinates  $R(z) \equiv 0$ .*

*Proof.* — The function  $f$  is invariant with respect to the  $S^1$ -action defined by  $\lambda$ . Therefore  $R$  is invariant as well. On the other hand  $R(t(z)) = \text{Re}(\sum a_{ij} \chi_i(t) \chi_j(t) z_i z_j)$  for  $t \in S^1$ . Hence the desired result follows from Proposition 3.14.  $\square$

COROLLARY 3.17. — *The extended strictly plurisubharmonic function  $f$  is a  $(T_\tau)_\mathbb{R}$ -invariant Morse function with absolute minimum at 0.*

Morse property on the full neighborhood of  $M$

Having shown that  $f|_{\Sigma_\tau}$  can be regarded as the restriction of a  $(T_\tau)_\mathbb{R}$ -norm function we now deal with the restriction of  $f$  to the orbit  $\mathcal{O}_\tau$ . Let  $T'$  be a toral subgroup complementary to  $T_\tau$  which acts freely and transitively on  $\mathcal{O}_\tau$ . Using polar coordinates we regard the quotient  $\mathcal{O}_\tau/T'_\mathbb{R}$  as a vector space  $V$  with the image of  $M$  being the origin. The function  $f|_{\mathcal{O}_\tau}$  is the pullback of a strictly convex function on  $V$  which attains its minimum as a nondegenerate critical point at the origin in  $V$ .

Now recall that the product structure  $X \setminus Y = \Sigma_\tau \times \mathcal{O}_\tau$  is defined by the  $T'$ -action as  $X \setminus Y = \Sigma_\tau \times T'$ . In this way we regard its quotient by  $T'_\mathbb{R}$  by as the product  $\Sigma_\tau \times V$  and we embed this in  $W \times V$  where  $\Sigma_\tau \hookrightarrow W$  is embedded as in the previous section. We regard  $f$  as being defined on this quotient. For the following result recall that  $U$  is the neighborhood of 0 to which  $f|_{\Sigma_\tau}$  extends as a Morse function with 0 its absolute minimum as a nondegenerate critical point.

PROPOSITION 3.18. — *The extended function  $f$  on  $U \times V$  is a Morse function with its only critical point being the origin which is its absolute minimum.*

*Proof.* — Locally near the origin it is clear  $df|_U$  and  $df|_V$  only vanish at the origin. Thus the origin is an isolated critical point. The Hessian of  $f$  in the vector space coordinates in  $U$  and  $V$  is positive definite, it follows that the origin is a nondegenerate critical point which is (locally) an absolute minimum for  $f$ .  $\square$

It is in the sense of this proposition that we refer to  $f$  as being a Bott-Morse function with critical set  $M = T_\mathbb{R}.x_0^\tau$ .

#### 4. Proofs of the main results

Here we apply the results of §3 to prove the theorems which are announced in §1. We begin by commenting on the difference between the estimates for an arbitrary sequence approximating a ray and a tame sequence approximating the same ray.

### 4.1. Arbitrary sequences

Recall that we replace a given sequence  $(s_N)$  by a *tame* sequence  $(s'_N)$  with  $f'_N$  converging to a strictly plurisubharmonic function  $f'$  on a certain Zariski open set  $X \setminus Y$  which is canonically defined by the ray  $R(\xi)$ . The essential results for  $(s'_N)$  are proved in the following paragraphs of this section. In particular it is shown that the probability density  $|s'_N|/\|s'_N\|_{L_2}^2$  converges in measure (with precise estimates) to the Dirac measure of a canonically associated  $T_{\mathbb{R}}$ -orbit  $M$ . Here  $M$  is the set where  $f'$  takes on its minimum.

To complete the project we return to our considerations of the original sequence. Let  $(s_N)$  be a given sequence which approximates the ray  $R(\xi)$  at infinity and let  $(s'_N)$  an associated tame sequence. It follows that there exists a bounded sequence of linear functions  $\beta_N \in \mathfrak{t}^*$  so that

$$\left| \frac{s_N}{s'_N} \right|^2 = e^{2\operatorname{Re}(\beta_N)}.$$

In other words there are characters  $\chi_N$  belonging to a finite set so that

$$|s_N|^2 = |\chi_N|^2 |s'_N|^2$$

on the open orbit. In the example at the beginning of §3, using the coordinate  $z = z_1 z_0^{-1}$  the coefficient  $|\chi_N|^2$  is just  $|z|^2$ .

Thus for  $\|s_N\|^2$  we must compute the integral

$$\int_X |\chi_N|^2 e^{-Nf'}.$$

The key is that the characters  $\chi_N$  which arise here belong to a finite set. Furthermore, except on arbitrarily small neighborhoods of the minimizing orbit  $M$  of  $f'$ , the term  $e^{-Nf'}$  kills the effect of the these coefficients. Finally, the  $\chi_N$  extend to holomorphic functions  $m_N$  on  $X \setminus Y$ . These have a certain vanishing order which contributes to the integral  $\|s_N\|^2$  just as in the case of the example of  $\mathbb{P}_1$ . In order to show that the probability densities converge to the Dirac measure on  $M$  it is enough to show that they do so for a subsequence where the coefficient characters are constant with vanishing order  $d$  along a divisor containing  $M$ . In this case, when computing  $\|s_N\|^2$  we have a correction term of order  $N^{-d}$ . However, this only has an effect on the speed of convergence to  $\delta_M$ . Of course the function  $D_N(t) := \operatorname{Vol}\{|\varphi_N|_h^2 > t\}$  will be affected, but the upper estimate for this will be given by the tame sequence.

## 4.2. Pointwise asymptotics of probability densities

In this paragraph only tame sequences  $(s_N)$  are considered. Using the global product structure of  $X \setminus Y$  we introduce a local basis of  $T_{\mathbb{R}}$ -invariant neighborhoods  $U(M)$  of  $M$  in  $X \setminus Y$  which is appropriate for our purposes. For this let  $V(M)$  be an arbitrarily small  $T_{\mathbb{R}}$ -invariant neighborhood of  $M$  in  $\mathcal{O}_\tau$  and  $\Delta$  an arbitrarily small  $T_\tau$ -invariant neighborhood of the fixed point  $x_0^\tau \in \Sigma_\tau$ . Using the map  $\alpha$  we regard  $U(M) := V(M) \times \Delta$  as an arbitrarily small  $T$ -invariant neighborhood of  $M$  in  $X \setminus Y$ . Recall that we are only dealing with a tame sequence which defines the strictly plurisubharmonic function  $f$  which takes on its minimum exactly on  $M$  and that this minimum value has been normalized to be zero.

Since  $f_N \rightarrow f$  uniformly on  $U(M)$ , it is a simple matter to make pointwise estimates of  $|s_N|^2 = e^{-Nf_N}$ . This is due to the fact that  $f$  is a smooth strictly plurisubharmonic function which takes on its minimum along  $M$  and is a Bott-Morse function there. In the orbit direction of  $V(M)$  we can choose coordinates so that it is realized as the pullback of a Morse function from  $\mathcal{O}_\tau$  and in the slice  $\Delta$  we can write it as a Morse function at the fixed point. Combining estimates using these quadratic forms, we will determine precise estimates for  $D_N(t)$ .

In order to provide the necessary estimates for the probability density function we must approximate the  $L_2$ -norm  $\|s_N\|^2$ . Using localization and the fact that  $f_N \rightarrow f$  uniformly on compact subsets of  $X \setminus Y$  it is sufficient to obtain estimates for

$$I_N = \int_{U(M)} e^{-Nf}$$

where  $U(M) = V(M) \times \Delta$  is an arbitrarily small product neighborhood of  $M$ .

PROPOSITION 4.1. — *There exists a positive constant  $c$  such that*

$$I_N \sim cN^{-\kappa}$$

where  $\kappa$  is the sum of the complex dimension of  $\Sigma_\tau$  and one-half the complex dimension of  $\mathcal{O}_\tau$ .

The notation here means that  $\lim_{N \rightarrow \infty} N^\kappa I_N = c$ . It should be remarked that by using the asymptotic development in §5 and an elementary asymptotic development of the integral along  $V(M)$ , it would be possible to give a more complete asymptotic development of  $I_N$ .



*Proof of Proposition 4.1.* We write  $I_N$  as a double integral

$$I_N = \int_{x \in V(M)} e^{-Nf(x)} \int_{\Delta_x} e^{f-f(x)}$$

where  $\Delta_x$  is the fiber over  $x$  of the production  $V(M) \times \Delta \rightarrow V(M)$  and  $f(x)$  denotes the value of  $f$  at the point in  $\Delta_x$  where it takes on its minimum, i.e., at the intersection of that fiber with  $\mathcal{O}_\tau$ . Of course we view  $x \in V(M) \subset \mathcal{O}_\tau$ . For each fiber  $\Delta_x$  we may apply Corollary 5.5 which implies that there is a positive continuous function  $c = c(x)$  so that

$$\int_{\Delta_x} e^{f-f(x)} \sim c(x)N^{-d}$$

where  $d$  is the complex dimension of  $\Delta_x$ . Thus it remains to compute

$$\int_{x \in V(M)} c(x)e^{-Nf(x)}. \quad (12)$$

As noted above we may assume that  $V(M)$  is the product of the torus  $T'$  and a ball  $B$  so that  $f = f(x)$  is the lift from the ball of a positive definite quadratic form. The orbit  $M$  is the preimage of the origin in the projection  $T \times B \rightarrow B$ . Explicit computation of an integral of the form

$$\int_{y \in B} e^{-N\|y\|^2}$$

shows that the integral in (12) is just a constant times  $N^{-\frac{d}{2}}$  where  $d$  is the complex dimension of  $V(M)$ .  $\square$

**COROLLARY 4.2.** — *For a tame sequence  $\{s_N\}$  the pointwise asymptotic behavior of the associated probability functions  $|\varphi_N|_h^2$  is given by*

$$|\varphi_N|_h^2 \sim N^\kappa e^{-Nf} \quad (13)$$

where  $\kappa = \dim \Sigma_\tau + \frac{1}{2} \dim \mathcal{O}_\tau$ .

### 4.3. Distribution functions

In §3.4 we showed that  $f$  can be regarded as a smooth Morse function in the appropriate embedding space for  $\Sigma_\tau$ . Using this, the estimate for the volume  $D_N(t)$  in the tails of the distribution follows from known estimates for the volume of an analytic set embedded in a ball in  $\mathbb{C}^n$ . It should again be emphasized that this can only be carried out for tame sequences.

## The volume estimate

Recall that we are interested in computing

$$D_N(t) := \text{Vol}\{x \in X; |\varphi_N|_h^2 > t\}.$$

Since we have localized the integral to an arbitrarily small neighborhood of  $M$  and since the free  $T_{\mathbb{R}}'$ -action leaves all relevant quantities invariant, it is enough to compute  $D_N(t)$  in the local  $T'$ -quotient of  $X$  in  $U \times V$  where the quotient of  $X$  is realized as  $(\Sigma_{\tau} \cap U) \times V$ . Now, using the Morse Lemma we choose coordinates for  $U$  and  $V$  so that the extended function  $f$  is a sum of norm functions:  $f = \|\cdot\|_U^2 + \|\cdot\|_V^2 =: \|\cdot\|^2$ . For the computation of  $D_N(t)$  it is then enough to compute  $\text{Vol}\{x \in (\Sigma_{\tau} \cap U) \times V =: A; \|x\|^2 < r\}$ . In other words we consider the ball  $\mathbb{B}(r)$  and compute the volume of the (pure dimensional) analytic subset  $A \cap \mathbb{B}(r)$ . It is known that this is  $c(r)r^d$  where  $c(r)$  is a continuous function bounded from above and below, which is closely related to the degree of  $A$ , and  $d$  is the (real) dimension of  $A$ . In other words this volume is asymptotically the same as the volume of a linear submanifold of the same dimension. This allows us to immediately compute  $D_N(t)$

PROPOSITION 4.3. — *The unscaled volume for a tame sequence  $\{s_N\}$  is given by*

$$D_N(t) \sim \left(\frac{\log N}{N}\right)^{\kappa}.$$

*Proof.* — For  $N \gg 0$  we may replace  $f_N$  by  $f$  and  $|\varphi_N|_h^2$  by  $N^{\kappa}e^{-Nf}$ . We then note that  $N^{\kappa}e^{-Nf} > t$  is the same as

$$f < \frac{\kappa \log N}{N} - \frac{\log t}{N} t \sim \frac{\log N}{N} =: r$$

where we compute  $\text{Vol}(\{f < r\})$  as above.  $\square$

This completes the proof of Theorem 1.3.

*Remarks.* — 1. If  $(s_N)$  is not tame, then (localized near  $M$ ) the norm  $|s_N|_h^2$  may become smaller than that of an associated tame sequence. Thus the above precise asymptotic expression for  $D_N(t)$  would become an estimate from above. 2. In [13] asymptotic expressions for  $D_N(t)$  are obtained in certain situations (see Theorem 1.3 of that paper).

## 4.4. Convergence of measures

Our goal here is to prove Theorem 1.2. As in the the statement of that theorem we do not assume that the given sequence of eigensections is tame.

However, as is pointed out in §4.1 it is sufficient to prove the result, i.e.,  $|\varphi_N|_h^2$  converges weakly to the Dirac measure on  $M$ , for  $(s_N)$  tame. More precisely, for any continuous function  $u$

$$\int_X u |\varphi_N|_h^2 d\lambda \rightarrow \int_M u dM$$

where  $d\lambda$  is a smooth probability measure on  $X$  and  $dM$  is the invariant probability measure on the  $T_{\mathbb{R}}$ -orbit  $M$  which is the critical set of  $f$ . It suffices to show this for a smooth, compactly supported function  $u$  (see, e.g., Chpt. II in [8]).

For the proof recall that  $U(M)$  is a  $T_{\mathbb{R}}$ -invariant neighborhood of  $M$  on which the functions  $-\frac{1}{N} \log |s_N|_h^2$  converge uniformly to the strictly plurisubharmonic limit function  $f$  discussed in §3. Outside this neighborhood the functions  $|\varphi_N|_h^2 = |s_N|_h^2 / \|s_N\|_{L^2}^2$  are rapidly converging to zero (see Corollary 3.13). Combining this with the asymptotic formula for the probability densities (see Corollary 4.2) we obtain

$$\int_X u |\varphi_N|_h^2 d\lambda = N^d \int_{U(M)} u e^{-Nf} d\lambda + O(e^{-\varepsilon N}).$$

Recall that for a tame sequence the exponent  $d$  in the above equation is precisely given by  $\kappa$  in Corollary 4.2. For a non-tame sequence it might be bigger since the vanishing order of the  $s_N$  increase (see remarks in §4.1). However, this only improves the convergence of the measures.

We continue by using the Morse property of  $f$  as explained in §4.3. For this we write  $U(M) = T'_{\mathbb{R}} \times U \times V$ , where  $U \times V$  is the product neighborhood provided by Proposition 3.18. Expressing  $u$  in the respective coordinates we obtain

$$\int_{U(M)} u e^{-Nf} d\lambda = C \int_{T'_{\mathbb{R}}} \int_{U \times V} u(\vartheta, z, \mu) e^{-N(\|z\|^2 + \|\mu\|^2)} d\lambda d\vartheta$$

Finally, using the Taylor expansion of  $u(\vartheta, \cdot)$  we see that the right-hand side of the above equation converges to

$$\int_{T'_{\mathbb{R}}} u(\vartheta, 0, 0) d\vartheta = \delta_M(u).$$

This completes the proof of Theorem 1.2.

### 5. Appendix

The aim of this Appendix is to prove the following theorem :

THEOREM 5.1. — *Let  $X \subset U$  be an analytic set in open neighbourhood  $U$  of the origin in  $\mathbb{R}^m$  which is assumed to be of pure dimension  $n + 1$ . Let  $f : U \rightarrow \mathbb{R}^+$  be a real analytic function which is strictly positive on  $X$  outside  $\{0\}$ , and let  $\omega$  be a smooth compactly support  $(n + 1)$ -form on  $U$ . Define the function  $F : \mathbb{R}^+ \rightarrow \mathbb{R}$  by putting for  $t \in \mathbb{R}^+$*

$$F(t) := \int_X e^{-t \cdot f} \cdot \omega.$$

Then as  $t \rightarrow \infty$  the function  $F$  admits an infinitely differentiable asymptotic expansion of the following form:

$$F(t) \simeq \sum_{i \in [1, p]} \sum_{j \in [1, n]} \sum_{\nu \in \mathbb{N}^*} c_{i, \nu}^j \cdot t^{-(\alpha_i + \nu)} \cdot (\text{Log } t)^j \quad (1)$$

where  $\alpha_1, \dots, \alpha_p$  are rational numbers strictly bigger than  $-1$ .

Moreover, the rational numbers  $\alpha_1, \dots, \alpha_p$  and the exponents of  $\text{Log } t$  have to be present in the asymptotic expansion at  $s \rightarrow 0^+$  for the fiber integral

$$\varphi(s) := \int_{X \cap \{f=s\}} \omega / df.$$

For the proof we begin with the remark that the Fubini theorem gives

$$F(t) = \int_0^{+\infty} e^{-t \cdot s} \varphi(s) \cdot ds \quad (2)$$

where  $\varphi$  is the fiber-integral of the statement. Now it known<sup>(1)</sup> that  $\varphi$  admits an infinitely differentiable asymptotic expansion when  $s \rightarrow 0^+$  of the form

$$\varphi(s) \simeq \sum_{i \in [1, p]} \sum_{j \in [1, n]} \sum_{\nu \in \mathbb{N}} \gamma_{i, \nu}^j \cdot s^{\alpha_i + \nu} \cdot (\text{Log } s)^j \quad (3)$$

so the proof of the theorem is a consequence of the following proposition □

PROPOSITION 5.2. — *Let  $\varphi : ]0, +\infty[ \rightarrow \mathbb{R}$  be a continuous function with support in  $]0, A]$ , which admits, when  $s \rightarrow 0^+$ , an asymptotic expansion of the type (3). Then the function defined in (2) admits when  $t \rightarrow +\infty$  an infinitely differentiable asymptotic expansion of the form (1). Moreover, the rational numbers  $\alpha_i$  which appear in the expansion of  $F$  have to appear in the expansion (3). For each  $i \in [1, p]$ , the maximal power of  $\text{Log}$  which appears in (1) with some  $s^{\alpha_i + \nu}$  in front is bounded by its analog in (3).*

The proof of this proposition is contained in the following elementary results.

---

<sup>(1)</sup> See [9], [1] or [2] for more information on the exponents in the case of an isolated singularity.

LEMMA 5.3. — For any  $\alpha > -1$  and any  $j \in \mathbb{N}$  there is a monic polynomial  $P_j$  of degree  $j$  such that

$$\int_0^A e^{-t.s}.s^\alpha.(\text{Log } s)^j ds \simeq \frac{(-1)^j}{\Gamma(\alpha + 1)}.P_j(\text{Log } t).\frac{1}{t^{\alpha+1}} + 0(e^{-\varepsilon.t})$$

for any given  $\varepsilon > 0$  small enough and for  $t \gg 1$ .

*Proof.* — Let us first show that

$$\int_0^{+\infty} e^{-t.s}.s^\alpha.(\text{Log } s)^j ds = \frac{(-1)^j}{\Gamma(\alpha + 1)}.P_j(\text{Log } t).\frac{1}{t^{\alpha+1}}$$

where  $P_j$  is a monic polynomial of degree  $j$ . This formula is easy for  $j = 0$ . The general case is obtained by  $j$  derivations in  $\alpha$ .

Now, as the function  $e^{-u/2}.u^\alpha.(\text{Log } u)^j$  is decreasing for  $u \gg 1$ , it follows that for  $t \gg 1$  the function  $\int_{t.A}^{+\infty} e^{-u}.u^\alpha.(\text{Log } u)^j$  is  $0(e^{-\varepsilon.t})$  for any given  $\varepsilon > 0$  small enough. □

*Remark.* — The preceding proof gives a more precise relation between the expansions (3) and (1).

Now the following elementary lemma allows one to cut asymptotic expansions.

LEMMA 5.4. — In the same situation as in the previous lemma assume that

$|\varphi(s)| \leq C.s^N$  for  $s \in ]0, A]$ . Then

$$\left| \int_0^A e^{-t.s}.\varphi(s).ds \right| \leq C.\frac{(N + 1)!}{t^{N+1}}.$$

*Remark.* — Assume that in the situation of the theorem the function  $f$  is no longer real analytic, but still continuous and with an isolated zero on  $X$  at the origin. If we have two analytic functions  $g_-$  and  $g_+$  such that they satisfy the hypothesis of the theorem and the inequalities :

$$g_- \leq f \leq g_+$$

near 0 in  $X$ , then we may deduce some estimation for  $F(t)$  when  $t \rightarrow +\infty$ . For instance if  $f$  is a positive Morse function of class  $\mathcal{C}^3$  near 0 in  $\mathbb{R}^m$  such that  $f(0) = 0$  and  $df_0 = 0$ , we may use  $g_-(x) = (1 - \varepsilon).q(x)$  and  $g_+(x) = (1 + \varepsilon).q(x)$  where  $q := d^2 f_0$  is a non degenerate positive quadratic form to obtain, at least, the first term of an expansion of  $F(t)$  when  $t \rightarrow +\infty$ .

In the case  $X$  is a complex analytic subset of dimension pure  $n$  near the origin in  $\mathbb{C}^N$  we obtain the following corollary.

COROLLARY 5.5. — *Let  $X \subset U$  be a complex analytic subset of pure dimension  $n$  in an open neighbourhood  $U$  of the origin in  $\mathbb{C}^N$ . Assume that  $0 \in X$  and that  $f : U \rightarrow \mathbb{R}^+$  is a  $C^3$  Morse function on  $U$  with only one critical point at 0. Let  $\omega$  be any  $C^\infty$  Kähler form on  $U$  and  $\rho : U \rightarrow \mathbb{R}^+$  be a  $C^0$  function with compact support which is equal to 1 in a neighbourhood of the origin. Then the limit*

$$\lim_{t \rightarrow +\infty} t^n \cdot \int_X e^{-t \cdot f} \cdot \rho \cdot \omega^{\wedge n}$$

*exists and is finite and strictly positive.*

*Proof.* — First we remark that the choice of  $\rho$  is irrelevant, because the change of  $\rho$  will add a  $0(e^{-\varepsilon \cdot t})$  for some  $\varepsilon > 0$ . The observation made before this corollary now implies that for any  $\varepsilon > 0$ , the inequalities

$$\int_X e^{-t \cdot g_+} \cdot \rho \cdot \omega^{\wedge n} \leq \int_X e^{-t \cdot f} \cdot \rho \cdot \omega^{\wedge n} \leq \int_X e^{-t \cdot g_-} \cdot \rho \cdot \omega^{\wedge n}$$

and if the result is proved for  $f = q$ , we deduce the result for  $f$ .

Now we may use the local parametrization theorem for the complex analytic set  $X$  near zero and see that if  $X \subset P \times B$  where  $P$  is a polydisc in  $\mathbb{C}^n$  such the projection on  $P$  gives a proper and finite map  $\pi : X \rightarrow P$ , which is also proper and finite on the Zariski tangent cone of  $X$  at the origin, the result is true for  $f(u, x) = \|u\|^2 + \|x\|^2$ ,  $\omega$  a Kähler form on  $P$  and  $\rho = \pi^*(\sigma)$  with  $\sigma$  a  $C^0$  function with compact support in  $P$  which is equal to 1 in a neighbourhood of the origin.

In this case, the inequality  $\|x\| \leq C \cdot \|u\|$  which is true for  $(u, x) \in X$  near enough  $(0, 0)$ , thanks to the transversality of  $\{0\} \times B$  to  $C_{X,0}$ , gives

$$k \cdot \int_P e^{-t \cdot (1+C^2) \cdot \|u\|^2} \cdot \sigma \cdot \omega^{\wedge n} \leq \int_X e^{-t \cdot f} \cdot \rho \cdot \omega^{\wedge n} \leq k \cdot \int_P e^{-t \cdot \|u\|^2} \cdot \sigma \cdot \omega^{\wedge n}$$

where  $k$  is the degree of  $X$  on  $P$ .

Finally, we reach the case where  $\omega$  is a Kähler form on  $U = P \times B$  using the fact that we may find a finite number of projections such that the given Kähler form is bounded by the sum of the pull back of the Kähler form on  $P$  by these projections. Because we know that the asymptotic expansion exists, this again shows that there exists a non-zero limit when we multiply by a suitable factor  $t^\alpha / (\text{Log } t)^j$  and that we have  $\alpha = n$  and  $j = 0$  as in the case where  $X = P$ ,  $f = \|u\|^2$  and  $\omega^{\wedge n} = (-2i)^n \cdot du \wedge d\bar{u}$ .  $\square$

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