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# MATHER DISCREPANCY AND THE ARC SPACES

by Shihoko ISHII

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ABSTRACT. — A goal of this paper is a characterization of singularities according to a new invariant, Mather discrepancy. We also show some evidences convincing us that Mather discrepancy is a reasonable invariant in a view point of birational geometry.

RÉSUMÉ. — Le but de ce papier est une caractérisation de certaines singularités en fonction d'un nouvel invariant, la "discrépance de Mather". Nous montrons également quelques résultats nous prouvant que la discrèpance de Mather est un invariant raisonnable d'un point de vue de la géométrie birationnelle.

## 1. Introduction

Let  $Y \rightarrow X$  be a resolution of the singularities of a variety  $X$  over  $\mathbb{C}$  that factors through the Nash blow-up  $\widehat{X} \rightarrow X$ . In [9] we introduce the Mather discrepancy divisor  $\widehat{K}_{Y/X}$  and showed its relation with the geometry of the arc space  $X_\infty$  of  $X$ . Note that  $\widehat{K}_{Y/X}$  coincides with the usual discrepancy divisor  $K_{Y/X}$  when  $X$  is non-singular. For a pair  $(X, \mathfrak{a})$  consisting of a  $\mathbb{Q}$ -Gorenstein variety  $X$  and an ideal  $\mathfrak{a} \subset \mathcal{O}_X$ , the log-canonical threshold  $\text{lct}(X, \mathfrak{a})$  and minimal log-discrepancy  $\text{mld}(X, \mathfrak{a})$  are defined by using the usual discrepancy divisor  $K_{Y/X}$ , where  $Y \rightarrow X$  is a log-resolution of  $(X, \mathfrak{a})$ . In this paper, for a pair  $(X, \mathfrak{a})$  consisting of an arbitrary variety  $X$  and an ideal  $\mathfrak{a} \subset \mathcal{O}_X$ , we define Mather log-canonical threshold  $\widehat{\text{lct}}(X, \mathfrak{a})$  and Mather minimal log-discrepancy  $\widehat{\text{mld}}(X, \mathfrak{a})$  by using the Mather discrepancy divisor  $\widehat{K}_{Y/X}$ , where  $Y \rightarrow X$  is a log-resolution of  $(X, \mathfrak{a})$  factoring through the Nash blow-up  $\widehat{X} \rightarrow X$ . These invariants are defined for an arbitrary variety  $X$  without the condition of  $\mathbb{Q}$ -Gorenstein or even

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normality. On these invariants, we obtain some formulas which are generalizations of known formulas. This shows that the Mather discrepancy is a reasonable invariant and we can expect that it will be useful in birational geometry. The main result of this paper is about  $\widehat{\text{mld}}(x; X, \mathcal{O}_X)$ .

For a closed point  $x \in X$  of a variety of dimension  $n$ , we have the following inequality:

$$\widehat{\text{mld}}(x; X, \mathcal{O}_X) \geq n.$$

We can characterize a singularity  $(X, x)$  with  $\widehat{\text{mld}}(x; X, \mathcal{O}_X) = n$ . Let  $\lambda_m$  be defined by  $\dim \pi_m^{-1}(x) \cap \psi_m(X_\infty) = mn - \lambda_m$  and let  $\lambda_m^0$  be defined by  $\dim \pi_m^{-1}(x) \cap \psi_m(X_\infty \setminus (\text{Sing } X)_\infty) = mn - \lambda_m^0$ , where  $\psi_m: X_\infty \rightarrow X_m$  and  $\pi_m: X_m \rightarrow X$  are the truncation morphisms. Then it turns out that  $\lambda_m$  and  $\lambda_m^0$  are constant for  $m \gg 0$  and they coincide with  $\widehat{\text{mld}}(x; X, \mathcal{O}_X) - n$ . Our characterization of a singularity  $(X, x)$  with  $\widehat{\text{mld}}(x; X, \mathcal{O}_X) = n$  is the following:

**THEOREM 1.1.** — *For a singularity  $(X, x)$  the following are equivalent:*

- (i)  $\widehat{\text{mld}}(x; X, \mathcal{O}_X) = n$ ;
- (ii)  $\lambda_m = 0$  for every  $m \in \mathbb{N}$ ;
- (iii)  $\lambda_m^0 = 0$  for every  $m \in \mathbb{N}$ ;
- (iv)  $\lambda_1^0 = 0$ ;
- (v) *The tangent cone of  $(X, x)$  has a reduced irreducible component.*

This paper is organized as follows: In the second section we put basic facts on Mather discrepancy and the arc spaces, which will be used in this paper. In the third section the invariants  $\widehat{\text{lct}}$  and  $\widehat{\text{mld}}$  are defined by using Mather discrepancy and we prove the formulas on these invariants. These formulas are natural generalization of the known formulas. In the forth section we prove Theorem 1.1 .

We assume that every scheme in this paper is defined over  $\mathbb{C}$ . A variety is an irreducible reduced scheme of finite type over  $\mathbb{C}$ .

The author expresses her hearty thanks to Lawrence Ein and Mircea Mustața for the stimulating discussions. Before the publication of this paper, Tommaso de Fernex and Roi Docampo introduced the same invariant and obtained the results same as our Proposition 3.10 and Corollary 3.15 independently ([8]).

## 2. Preliminaries on Mather discrepancy and arc spaces

Let  $X$  be a  $\mathbb{Q}$ -Gorenstein variety of index  $r$  and  $f: Y \rightarrow X$  a resolution of the singularities of  $X$ . Then the (usual) discrepancy divisor  $K_{Y/X}$  is

the unique  $\mathbb{Q}$ -divisor supported on the exceptional locus of  $f$  such that  $rK_{Y/X}$  is linearly equivalent with  $rK_Y - f^*(rK_X)$ . Note that the usual discrepancy is defined only for a  $\mathbb{Q}$ -Gorenstein variety  $X$ , and the following Mather discrepancy is defined for every variety, even for non-normal variety.

DEFINITION 2.1 ([9]). — *Let  $X$  be a variety of dimension  $n$  and  $f: Y \rightarrow X$  a resolution of the singularities factoring through the Nash blow up. Then, the image of the canonical homomorphism*

$$f^* \wedge^n \Omega_X \rightarrow \wedge^n \Omega_Y$$

*is an invertible sheaf of the form  $J \wedge^n \Omega_Y$ , where  $J$  is the invertible ideal sheaf of  $\mathcal{O}_Y$  that defines an effective divisor supported on the exceptional locus of  $f$ . This divisor is called the Mather discrepancy divisor and denoted by  $\widehat{K}_{Y/X}$ . For every prime divisor  $E$  on  $Y$ , we define*

$$\widehat{k}_E := \text{ord}_E(\widehat{K}_{Y/X}).$$

*More generally, if  $v$  is a divisorial valuation over  $X$ , then we can assume without loss of generality that  $v = q \text{val}_E$  for a prime divisor  $E$  on some  $Y$  and a positive integer  $q$ , and define*

$$\widehat{k}_v := q \cdot \widehat{k}_E.$$

*For a  $\mathbb{Q}$ -Gorenstein variety  $X$ , we define  $k_E := \text{ord}_E(K_{Y/X})$  for a resolution  $f: Y \rightarrow X$  and define also  $k_v$  in the similar way.*

2.2. — Let  $X$  be an  $n$ -dimensional  $\mathbb{Q}$ -Gorenstein variety of index  $r$ . We write the image of the homomorphism

$$(\wedge^n \Omega_X)^{\otimes r} \rightarrow \mathcal{O}_X(rK_X) = \omega_X^{[r]}$$

by  $I_r \otimes \mathcal{O}_X(rK_X)$ , where  $I_r$  is an ideal of  $\mathcal{O}_X$ . Let  $f: Y \rightarrow X$  be a resolution factoring through the Nash blow up. Then, the relation of usual discrepancy and the Mather discrepancy is as follows:

$$f^*(I_r) \otimes \mathcal{O}_Y(r\widehat{K}_{Y/X}) = \mathcal{O}_Y(rK_{Y/X}).$$

In particular  $\widehat{K}_{Y/X} \geq K_{Y/X}$ . Let  $\mathcal{J}_X$  be the Jacobian ideal of  $X$  and let  $J_r = (\mathcal{J}_X^r : I_r)$ , then  $J_r \cdot I_r$  and  $\mathcal{J}_X^r$  have the same integral closure ([6, Corollary 9.4]). If  $X$  is locally a complete intersection, then  $I_1 = \mathcal{J}_X$

DEFINITION 2.3. — *Let  $X$  be a scheme of finite type over  $\mathbb{C}$  and  $K \supset \mathbb{C}$  a field extension. For  $m \in \mathbb{N}$ , a  $\mathbb{C}$ -morphism  $\text{Spec } K[t]/(t^{m+1}) \rightarrow X$  is called an  $m$ -jet of  $X$  and a  $\mathbb{C}$ -morphism  $\text{Spec } K[[t]] \rightarrow X$  is called an arc of  $X$ .*

2.4. — We denote the space of  $m$ -jets of  $X$  by  $X_m$  and the space of arcs by  $X_\infty$ . For terminologies and the basic properties of these spaces, we refer the paper [11].

DEFINITION 2.5. — *Let  $X$  be a variety over  $\mathbb{C}$ . We say an arc  $\alpha: \text{Spec } K[[t]] \rightarrow X$  is thin if  $\alpha$  factors through a proper closed subset of  $X$ . An arc which is not thin is called a fat arc.*

*An irreducible closed subset  $C$  in  $X_\infty$  is called a thin set if the generic point of  $C$  is thin. An irreducible closed subset in  $X_\infty$  which is not thin is called a fat set.*

One typical example of a fat set is a maximal divisorial set which is introduced in [12].

DEFINITION 2.6. — *For a divisorial valuation  $v$  over a variety  $X$ , define the maximal divisorial set corresponding to  $v$  as follows:*

$$C_X(v) := \overline{\{\alpha \in X_\infty \mid \text{ord}_\alpha = v\}},$$

where  $\overline{\{\ \}}$  is the Zariski closure in  $X_\infty$ .

PROPOSITION 2.7 ([12]). — *Let  $v = q \text{val}_E$  be a divisorial valuation over a variety  $X$ . Let  $f: Y \rightarrow X$  be a good resolution of the singularities of  $X$  such that the prime divisor  $E$  appears on  $Y$ . Here, a good resolution means a resolution whose exceptional locus is a simple normal crossing divisor. Then,*

$$C_X(v) = \overline{f_\infty(\text{Cont}^q(E))}.$$

*In particular,  $C_X(v)$  is irreducible.*

DEFINITION 2.8 ([4]). — *For an ideal sheaf  $\mathfrak{a}$  on a variety  $X$ , we define*

$$\text{Cont}^e(\mathfrak{a}) = \{\alpha \in X_\infty \mid \text{ord}_\alpha(\mathfrak{a}) = e\}$$

and

$$\text{Cont}^{\geq e}(\mathfrak{a}) = \{\alpha \in X_\infty \mid \text{ord}_\alpha(\mathfrak{a}) \geq e\}.$$

*These subset are called contact loci of an ideal  $\mathfrak{a}$ . The subset  $\text{Cont}^{\geq m}(\mathfrak{a})$  is closed and  $\text{Cont}^m(\mathfrak{a})$  is locally closed. Both are cylinders. We can define in the obvious way also subsets  $\text{Cont}^e(\mathfrak{a})_m$  (if  $e \leq m$ ) and  $\text{Cont}^{\geq e}(\mathfrak{a})_m$  (if  $e \leq m + 1$ ) in  $X_m$ .*

PROPOSITION 2.9 ([9]). — *Let  $X$  be an affine variety, and let  $\mathfrak{a}_i \subset \mathcal{O}_X$  ( $i = 1, \dots, r$ ) be non-zero ideals. Then, for  $e_1, \dots, e_r \in \mathbb{N}$ , every fat irreducible component of the intersection  $\text{Cont}^{\geq e_1}(\mathfrak{a}_1) \cap \dots \cap \text{Cont}^{\geq e_r}(\mathfrak{a}_r)$  is a maximal divisorial set.*

Note that [9, Proposition 2.12] is formulated for the case  $r = 1$ . But its proof works also for  $r > 1$ .

2.10. — As  $X$  is a variety over  $\mathbb{C}$ , the arc space  $X_\infty$  is irreducible ([16]). Therefore  $\psi_m(X_\infty)$  is an irreducible constructible subset in  $X_m$  of dimension  $(m + 1)n$ , where  $n = \dim X$ . Let  $C \subset X_\infty$  be a cylinder  $\psi_p^{-1}(A)$  contained in  $\text{Cont}^e(\mathcal{J}_X)$ . Then, codimension of  $C$  is defined as follows:

$$\text{codim}(C, X_\infty) := (m + 1)n - \dim \psi_m(C)$$

for  $m \geq \max\{p, e\}$ .

For an arbitrary cylinder  $C$ , the codimension is defined as follows:

$$\text{codim}(C, X_\infty) := \min\{\text{codim}(C \cap \text{Cont}^e(\mathcal{J}_X)) \mid e \in \mathbb{N}\}.$$

We sometimes write  $\text{codim}(C)$  for  $\text{codim}(C, X_\infty)$ , when there is no possible confusion. Note that these are well defined by the following lemma. (For details, see [6, Section 5].)

LEMMA 2.11 ([3], [6]). — *Let  $X$  be a variety of dimension  $n$  and  $e$  a nonnegative integer. Let  $\mathcal{J}_X$  be the Jacobian ideal of  $X$ . Fix  $m \geq e$  and let  $\psi_{m+e,m}: X_{m+e} \rightarrow X_m$ ,  $\psi_m: X_\infty \rightarrow X_m$  be the truncation morphisms.*

- (i) *We have  $\psi_m(\text{Cont}^e(\mathcal{J}_X)) = \psi_{m+e,m}(\text{Cont}^e(\mathcal{J}_X)_{m+e})$ .*
- (ii) *The truncation morphism  $\psi_{m+1,m}: X_{m+1} \rightarrow X_m$  induces a piecewise trivial fibration*

$$\psi_{m+1}(\text{Cont}^e(\mathcal{J}_X)) \rightarrow \psi_m(\text{Cont}^e(\mathcal{J}_X))$$

*with fiber  $\mathbb{A}^n$ .*

LEMMA 2.12 ([6]). — *Let  $X$  be a variety of dimension  $n$ . If  $m, p$  and  $e$  are nonnegative integers such that  $2p \geq m \geq p + e$ , then the truncation morphism  $\psi_{m,p}: X_m \rightarrow X_p$  induces a piecewise trivial fibration*

$$\text{Cont}^e(\mathcal{J}_X)_m \rightarrow \text{Cont}^e(\mathcal{J}_X)_p \cap \text{Im}(\psi_{m,p})$$

*with fiber  $\mathbb{A}^{(m-p)n+e}$ .*

PROPOSITION 2.13 ([9]). — *For a divisorial valuation  $q \text{ val}_E$ , we have the codimension of the corresponding maximal divisorial set*

$$q(\widehat{k}_E + 1) = \text{codim}(C_X(q \text{ val}_E)).$$

### 3. Invariants based on Mather discrepancy

First we start this section with the well known invariants.

DEFINITION 3.1. — *Let  $(X, \mathfrak{a})$  be a pair consisting of  $\mathbb{Q}$ -Gorenstein variety  $X$  and a non-zero ideal  $\mathfrak{a}$  of  $\mathcal{O}_X$ . The log-canonical threshold of  $(X, \mathfrak{a})$  is defined as follows:*

$$\text{lct}(X, \mathfrak{a}) = \sup\{c \mid k_E - c \text{ord}_E(\mathfrak{a}) + 1 \geq 0, E \text{ divisor over } X\}.$$

Let  $W$  be a closed subset of  $X$ . The minimal log-discrepancy of  $(X, \mathfrak{a})$  along  $W$  is defined as follows:

If  $\dim X \geq 2$ ,

$$\text{mld}(W; X, \mathfrak{a}) = \inf\{k_E - \text{ord}_E(\mathfrak{a}) + 1 \mid E \text{ divisor over } X \text{ with center in } W\}.$$

When  $\dim X = 1$  we use the same definition of minimal log discrepancy, unless the infimum is negative, in which case we make the convention that  $\text{mld}(W, X, \mathfrak{a}) = -\infty$ .

Remark 3.2.

- (i) The log-canonical threshold is also presented as

$$\text{lct}(X, \mathfrak{a}) = \max\{c \mid k_{E_i} - c \text{ord}_{E_i}(\mathfrak{a}) + 1 \geq 0, \\ E_i : \text{exceptional prime divisor on } Y\}$$

for a fixed log-resolution  $f: Y \rightarrow X$  of  $(X, \mathfrak{a})$ .

- (ii) If  $\text{mld}(W; X, \mathfrak{a}) < 0$ , then  $\text{mld}(W; X, \mathfrak{a}) = -\infty$ . This is known when  $\dim X \geq 2$ , while it follows from the definition when  $\dim X = 1$ .

Now we will define the invariants modified from these invariants.

DEFINITION 3.3. — *Let  $(X, \mathfrak{a})$  be a pair consisting of an arbitrary variety  $X$  and a non-zero ideal  $\mathfrak{a}$  of  $\mathcal{O}_X$ . The Mather log-canonical threshold of  $(X, \mathfrak{a})$  is defined as follows:*

$$\widehat{\text{lct}}(X, \mathfrak{a}) = \sup\{c \mid \widehat{k}_E - c \text{ord}_E(\mathfrak{a}) + 1 \geq 0, E \text{ divisor over } X\}.$$

Let  $W$  be a closed subset of  $X$ . The Mather minimal log-discrepancy of  $(X, \mathfrak{a})$  along  $W$  is defined as follows:

If  $\dim X \geq 2$ ,

$$\widehat{\text{mld}}(W; X, \mathfrak{a}) = \inf\{\widehat{k}_E - \text{ord}_E(\mathfrak{a}) + 1 \mid E \text{ divisor over } X \text{ with center in } W\}.$$

When  $\dim X = 1$  we use the same definition of Mather minimal log discrepancy, unless the infimum is negative, in which case we make the convention that  $\widehat{\text{mld}}(W, X, \mathfrak{a}) = -\infty$ .

Remark 3.4.

(i) The Mather log-canonical threshold is represented as

$$\widehat{\text{lct}}(X, \mathfrak{a}) = \max\{c \mid \widehat{k}_{E_i} - c \text{ord}_{E_i}(\mathfrak{a}) + 1 \geq 0, \\ E_i : \text{exceptional prime divisor on } Y\}$$

for a fixed log-resolution  $f: Y \rightarrow X$  of  $(X, \mathfrak{a})$  factoring through the Nash blow up, because for a sequence  $Y' \xrightarrow{g} Y \xrightarrow{f} X$  of such log resolutions of  $(X, \mathfrak{a})$ , we have  $\widehat{K}_{Y'/X} = K_{Y'/Y} + g^* \widehat{K}_{Y/X}$  with  $K_{Y'/Y} \geq 0$ .

(ii) If  $\widehat{\text{mld}}(W; X, \mathfrak{a}) < 0$ , then  $\widehat{\text{mld}}(W; X, \mathfrak{a}) = -\infty$ . This is proved by using the previous formula of  $\widehat{K}_{Y'/X}$  when  $\dim X \geq 2$ , while it follows from the definition when  $\dim X = 1$ .

PROPOSITION 3.5. — *Let  $X$  be an arbitrary variety and  $\mathfrak{a}$  be a non-zero ideal of  $\mathcal{O}_X$ . Then,*

$$\widehat{\text{lct}}(X, \mathfrak{a}) = \min_{m \in \mathbb{N}} \frac{\text{codim}(\text{Cont}^{\geq m}(\mathfrak{a}))}{m}.$$

Proof. — First we prove that

$$(3.1) \quad \widehat{\text{lct}}(X, \mathfrak{a}) \leq \inf_{m \in \mathbb{N}} \frac{\text{codim}(\text{Cont}^{\geq m}(\mathfrak{a}))}{m}.$$

For  $m \in \mathbb{N}$ , take an irreducible component  $C_X(v) \subset \text{Cont}^{\geq m}(\mathfrak{a})$  which gives the codimension of  $\text{Cont}^{\geq m}(\mathfrak{a})$ . Let  $v = q \text{val}_E$  for a prime divisor  $E$  over  $X$  and let  $r_E := \text{ord}_E(\mathfrak{a})$ . Then,  $C_X(v) \subset \text{Cont}^{\geq m}(\mathfrak{a})$  implies  $qr_E = v(\mathfrak{a}) \geq m$ . Therefore,

$$\text{codim}(\text{Cont}^{\geq m}(\mathfrak{a})) = \text{codim}(C_X(v)) = q(\widehat{k}_E + 1) \geq m \frac{\widehat{k}_E + 1}{r_E}.$$

Here, by the definition of  $\widehat{\text{lct}}$ , the last term is  $\geq m \cdot \widehat{\text{lct}}(X, \mathfrak{a})$ , which gives the inequality (3.1). For the theorem, it is sufficient to prove that there is  $m$  such that

$$(3.2) \quad \frac{\text{codim}(\text{Cont}^{\geq m}(\mathfrak{a}))}{m} \leq \widehat{\text{lct}}(X, \mathfrak{a}).$$

By Remark 3.4 (i), there is a prime divisor  $E$  over  $X$  such that  $\widehat{\text{lct}}(X, \mathfrak{a}) = (\widehat{k}_E + 1)/\text{ord}_E(\mathfrak{a})$ . Let  $m = \text{ord}_E(\mathfrak{a})$ , then we have the inclusion  $C_X(\text{val}_E) \subset \text{Cont}^{\geq m}(\mathfrak{a})$  and this implies the inequality

$$\text{codim}(\text{Cont}^{\geq m}(\mathfrak{a})) \leq \text{codim}(C_X(\text{val}_E)) = \widehat{k}_E + 1 = m \cdot \widehat{\text{lct}}(X, \mathfrak{a}),$$

which gives the required inequality (3.2). □



As a corollary, we obtain the formula of  $\text{lct}$  for non-singular case.

**COROLLARY 3.6** ([4]). — *Let  $(X, \mathfrak{a})$  be a pair consisting of a non-singular variety  $X$  and an ideal  $\mathfrak{a} \subset \mathcal{O}_X$ . Let  $Z$  be the subscheme defined by  $\mathfrak{a}$ . Then the log-canonical threshold is obtained as follows:*

$$\text{lct}(X, \mathfrak{a}) = \min_{m \in \mathbb{N}} \frac{\text{codim}(Z_{m-1}, X_{m-1})}{m}.$$

This follows immediately from the theorem, since the equality  $\text{codim}(\text{Cont}^{\geq m}(\mathfrak{a})) = \text{codim}(Z_{m-1}, X_{m-1})$  holds for non-singular  $X$ .

The next is the formula for the Mather minimal log-discrepancy in terms of the arc space.

**PROPOSITION 3.7.** — *Let  $(X, \mathfrak{a})$  be a pair consisting of an arbitrary variety  $X$  and a non-zero ideal  $\mathfrak{a} \subset \mathcal{O}_X$ . Let  $W$  be a proper closed subset of  $X$  and  $I_W$  be the (reduced) ideal of  $W$ . Then,*

$$(3.3) \quad \widehat{\text{mld}}(W; X, \mathfrak{a}) = \inf_{m \in \mathbb{N}} \{ \text{codim}(\text{Cont}^m(\mathfrak{a}) \cap \text{Cont}^{\geq 1}(I_W)) - m \}.$$

We also have

$$(3.4) \quad \widehat{\text{mld}}(W; X, \mathfrak{a}) = \inf_{m \in \mathbb{N}} \{ \text{codim}(\text{Cont}^{\geq m}(\mathfrak{a}) \cap \text{Cont}^{\geq 1}(I_W)) - m \}.$$

*Proof.* — For the proof of  $\geq$  at (3.3), let  $E$  be any prime divisor over  $X$  with the center in  $W$ . Let  $m = \text{ord}_E(\mathfrak{a})$  and  $v = \text{val}_E$ . Then, there is a non-empty open subset  $C$  of  $C_X(v)$  such that  $C \subset \text{Cont}^m(\mathfrak{a}) \cap \text{Cont}^{\geq 1}(I_W)$ . Hence,

$$\begin{aligned} \widehat{k}_E - \text{ord}_E(\mathfrak{a}) + 1 &= \text{codim}(C_X(v)) - m \\ &\geq \text{codim}(\text{Cont}^m(\mathfrak{a}) \cap \text{Cont}^{\geq 1}(I_W)) - m, \end{aligned}$$

which yields the required inequality unless  $\dim X = 1$  and  $\widehat{\text{mld}}(X, \mathfrak{a}) = -\infty$ .

When  $\dim X = 1$  and  $\widehat{\text{mld}}(X, \mathfrak{a}) = -\infty$ , there is a prime divisor  $E$  over  $X$  with the center in  $W$  such that  $\widehat{k}_E - \text{ord}_E(\mathfrak{a}) + 1 < 0$ . Let  $m = \text{ord}_E(\mathfrak{a})$ , then  $\text{codim } C_X(\text{val}_E) - m < 0$ . Here, for every  $q \in \mathbb{N}$ , by Proposition 2.13,

$$\text{codim } C_X(q \text{val}_E) - qm = q(\text{codim } C_X(\text{val}_E) - m) < 0.$$

As a non-empty open subset of  $C_X(q \text{val}_E)$  is in  $\text{Cont}^{qm} \cap \text{Cont}^{\geq 1}(I_W)$ , we have

$$\begin{aligned} \text{codim}(\text{Cont}^{qm} \cap \text{Cont}^{\geq 1}(I_W)) - qm &\leq \text{codim}(C_X(q \text{val}_E)) - qm \\ &= q(\text{codim}(C_X(\text{val}_E)) - m) < 0. \end{aligned}$$

Here, if  $q \rightarrow \infty$ , then we have

$$\text{codim}(\text{Cont}^{qm} \cap \text{Cont}^{\geq 1}(I_W),) - qm \rightarrow -\infty,$$

which implies the right hand side of (3.3) in the theorem is  $-\infty$ .

For the proof of  $\leq$  at (3.3), we may assume that  $\widehat{k}_E - \text{ord}_E(\mathbf{a}) + 1 \geq 0$  for every prime divisor  $E$  over  $X$  with the center in  $W$ . Indeed if there is a prime divisor  $E$  with the center in  $W$  and  $\widehat{k}_E - \text{ord}_E(\mathbf{a}) + 1 < 0$ , then  $\widehat{\text{mld}}(W; X, \mathbf{a}) = -\infty$  by Remark 3.4, (ii) and therefore the required inequality is trivial.

For  $m \in \mathbb{N}$ , let  $C \subset \text{Cont}^m(\mathbf{a}) \cap \text{Cont}^{\geq 1}(I_W)$  be an irreducible component which gives the codimension of  $\text{Cont}^m(\mathbf{a}) \cap \text{Cont}^{\geq 1}(I_W)$ . Then, the closure  $\overline{C}$  is  $C_X(v)$  for some divisorial valuation  $v$ , since a fat irreducible component of  $\text{Cont}^{\geq m}(\mathbf{a}) \cap \text{Cont}^{\geq 1}(I_W)$  is a maximal divisorial set (Proposition 2.9) and  $\text{Cont}^m(\mathbf{a}) \cap \text{Cont}^{\geq 1}(I_W)$  is an open subset of  $\text{Cont}^{\geq m}(\mathbf{a}) \cap \text{Cont}^{\geq 1}(I_W)$ . Let  $v = q \text{val}_E$  and  $m = v(\mathbf{a})$ , then  $E$  is a prime divisor over  $X$  with the center in  $W$ ,  $m = q \text{ord}_E(\mathbf{a})$  and

$$\begin{aligned} \text{codim}(\text{Cont}^m(\mathbf{a}) \cap \text{Cont}^{\geq 1}(I_W)) - m &= \text{codim}(C_X(v)) - m \\ &= q(\widehat{k}_E + 1) - q \text{ord}_E(\mathbf{a}) \geq \widehat{k}_E + 1 - \text{ord}_E(\mathbf{a}), \end{aligned}$$

which yields the required inequality.

For the proof of (3.4) of the theorem, let

$$\begin{aligned} a_m &= \text{codim}(\text{Cont}^m(\mathbf{a}) \cap \text{Cont}^{\geq 1}(I_W)) - m, \\ b_m &= \text{codim}(\text{Cont}^{\geq m}(\mathbf{a}) \cap \text{Cont}^{\geq 1}(I_W)) - m. \end{aligned}$$

As  $\text{Cont}^m(\mathbf{a}) \subset \text{Cont}^{\geq m}(\mathbf{a})$ , we have  $a_m \geq b_m$ . Therefore, it follows

$$\inf_m \{a_m\} \geq \inf_m \{b_m\}.$$

Next we prove the converse inequality. For every  $m \in \mathbb{N}$ , let  $C_X(v)$  be the irreducible component of  $\text{Cont}^{\geq m}(\mathbf{a}) \cap \text{Cont}^{\geq 1}(I_W)$  that gives the codimension. Then, for  $m' := v(\mathbf{a}) \geq m$  we have

$$\text{codim}(\text{Cont}^{\geq m}(\mathbf{a}) \cap \text{Cont}^{\geq 1}(I_W)) = \text{codim}(\text{Cont}^{m'}(\mathbf{a}) \cap \text{Cont}^{\geq 1}(I_W)).$$

Hence,  $b_m \geq a_{m'}$ , which yields  $\inf_m \{b_m\} \geq \inf_m \{a_m\}$ . □

*Remark 3.8.* — Our formula can be easily extended for the combination of ideals  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_r$  instead of one ideal  $\mathbf{a}$ . I.e., we have

$$\begin{aligned} \widehat{\text{mld}}(W; X, \mathbf{a}_1^{e_1} \mathbf{a}_2^{e_2} \dots \mathbf{a}_r^{e_r}) &= \inf_{m_i \in \mathbb{N}} \left\{ \text{codim}(\text{Cont}^{m_1}(\mathbf{a}_1) \cap \dots \right. \\ &\quad \left. \dots \cap \text{Cont}^{m_r}(\mathbf{a}_r) \cap \text{Cont}^{\geq 1}(I_W)) - \sum_i m_i e_i \right\}, \end{aligned}$$

where  $e_i$ 's are positive real numbers. Here, any of  $\text{Cont}^{m_i}(\mathbf{a}_i)$ 's can be replaced by  $\text{Cont}^{\geq m_i}(\mathbf{a}_i)$ . For simplicity of the notation and the proofs, we keep formulating the forthcoming formulas for one ideal only. But note that the formulas in this section are also valid under this combination form.

The following lemma is a modified version of [6, Theorem 7.9] for our discussion.

LEMMA 3.9. — *Let  $X$  be a variety of dimension  $n$ , then*

$$\widehat{\text{mld}}(W; X, \mathbf{a}\mathcal{J}_X) = \inf \left\{ (s+1)n - m - \dim \left( \text{Cont}^{\geq m}(\mathbf{a})_s \cap \text{Cont}^e(\mathcal{J}_X)_s \cap \text{Cont}^{\geq 1}(I_W)_s \right) \right\},$$

where the infimum is over those  $m, e \in \mathbb{N}$  and  $s \in \mathbb{N}$  such that  $s \geq 2e, e+m$ .

*Proof.* — Let  $S := \text{Cont}^{\geq m}(\mathbf{a})_s \cap \text{Cont}^e(\mathcal{J}_X)_s \cap \text{Cont}^{\geq 1}(I_W)_s \subset X_s$ . Then, by Lemma 2.12, the truncation morphism  $\psi_{s,s-e}$  induces a piecewise trivial fibration  $S \rightarrow \psi_{s,s-e}(S)$  with fiber  $\mathbb{A}^{e(n+1)}$ . By Lemma 2.11 (i), it follows

$$\psi_{s,s-e}(S) = \psi_{s-e}(\text{Cont}^{\geq m}(\mathbf{a}) \cap \text{Cont}^e(\mathcal{J}_X) \cap \text{Cont}^{\geq 1}(I_W)).$$

On the other hand, by Lemma 2.11 (ii),

$$\begin{aligned} \text{codim}(\text{Cont}^{\geq m}(\mathbf{a}) \cap \text{Cont}^e(\mathcal{J}_X) \cap \text{Cont}^{\geq 1}(I_W)) \\ = (s-e+1)n - \dim \left( \psi_{s-e}(\text{Cont}^{\geq m}(\mathbf{a}) \cap \text{Cont}^e(\mathcal{J}_X) \cap \text{Cont}^{\geq 1}(I_W)) \right), \end{aligned}$$

as  $s-e \geq m, e, 1$ . The last term is equal to  $(s-e+1)n - (\dim S - e(n+1))$ , by the previous discussion. Now we obtain

$$\begin{aligned} \widehat{\text{mld}}(W; X, \mathbf{a}\mathcal{J}_X) &= \inf_{m,e} \left\{ \text{codim}(\text{Cont}^{\geq m}(\mathbf{a}) \right. \\ &\quad \left. \cap \text{Cont}^e(\mathcal{J}_X) \cap \text{Cont}^{\geq 1}(I_W)) - m - e \right\} \\ &= \inf \left\{ (s+1)n - m - \dim S \right\}, \end{aligned}$$

where the infimum is over those  $m, e \in \mathbb{N}$  and  $s \in \mathbb{N}$  such that  $s \geq 2e, e+m$ .  $\square$

The following is a generalization of [6, Theorem 8.1] and [15], whose formulation is in Corollary 3.11. But our proof is just an imitation of the proof of [6, Theorem 8.1] and is even easier.

PROPOSITION 3.10 (Inversion of Adjunction). — *Let  $X$  be an arbitrary variety,  $A$  a non-singular variety containing  $X$  as a closed subvariety of codimension  $c$  and  $W$  a proper closed subset of  $X$ . Let  $\tilde{\mathbf{a}} \subset \mathcal{O}_A$  be an ideal*

such that its image  $\mathfrak{a} := \widetilde{\mathfrak{a}}\mathcal{O}_X \subset \mathcal{O}_X$  is non-zero. Denote the ideal of  $X$  in  $A$  by  $I_X$ . Then,

$$\widehat{\text{mld}}(W; X, \mathfrak{a}\mathcal{J}_X) = \widehat{\text{mld}}(W; A, \widetilde{\mathfrak{a}}I_X^c).$$

*Proof.* — Our proof is based on the proof of [6, Theorem 8.1] which is a formula of usual  $\text{mld}$  for a  $\mathbb{Q}$ -Gorenstein variety  $X$ . For the reader's convenience, here we write down the proof with a care on the difference between their statement and ours.

For the proof of  $\geq$  in the equality of the proposition, let  $\sigma = \widehat{\text{mld}}(W; A, \widetilde{\mathfrak{a}}I_X^c)$  and let  $n = \dim X$ . We assume contrary,  $\widehat{\text{mld}}(W; X, \mathfrak{a}\mathcal{J}_X) < \sigma$ , and will induce a contradiction. By the assumption and Lemma 3.9, there exist  $e, m \in \mathbb{N}$ ,  $s \geq 2e, m + e$  and an irreducible component  $C$  of  $\text{Cont}^{\geq m}(\mathfrak{a})_s \cap \text{Cont}^e(\mathcal{J}_X)_s \cap \text{Cont}^{\geq 1}(I_W)_s$  such that

$$(3.5) \quad (s + 1)n - m - \dim C < \sigma,$$

where  $I_W$  is the defining ideal of  $W$  in  $X$ . As  $C \subset \text{Cont}^{\geq m}(\mathfrak{a})_s \cap \text{Cont}^{\geq 1}(I_W)_s = \text{Cont}^{\geq m}(\widetilde{\mathfrak{a}})_s \cap \text{Cont}^{\geq 1}(\widetilde{I}_W)_s \cap X_s$ , where  $\widetilde{I}_W$  is the defining ideal of  $W$  in  $\mathcal{O}_A$ , we have

$$(\psi_s^A)^{-1}(C) \subset \text{Cont}^{\geq m}(\widetilde{\mathfrak{a}}) \cap \text{Cont}^{\geq 1}(\widetilde{I}_W) \cap \text{Cont}^{\geq s+1}(I_X) =: S,$$

where  $\psi_s^A: A_\infty \rightarrow A_s$  is the truncation morphism. Here we note that  $\text{Cont}^{\geq s+1}(I_X) = \text{Cont}^{\geq c(s+1)}(I_X^c)$ . Now we obtain

$$\begin{aligned} (n + c)(s + 1) - \dim C &= \text{codim}(C, A_s) = \text{codim}((\psi_s^A)^{-1}(C), A_\infty) \\ &\geq \text{codim}(S, A_\infty) \geq \sigma + m + c(s + 1), \end{aligned}$$

which is a contradiction to (3.5).

For the proof of  $\leq$  in the equality of the proposition, let  $\tau = \widehat{\text{mld}}(W; X, \mathfrak{a}\mathcal{J}_X)$ . We assume contrary,  $\tau > \widehat{\text{mld}}(W; A, \widetilde{\mathfrak{a}}I_X^c)$ , and will induce a contradiction. By the assumption and Proposition 3.7, there exists an irreducible component  $C$  of  $\text{Cont}^{\geq m}(\widetilde{\mathfrak{a}}) \cap \text{Cont}^{\geq d}(I_X) \cap \text{Cont}^{\geq 1}(\widetilde{I}_W)$  such that

$$(3.6) \quad \text{codim}(C, A_\infty) < cd + m + \tau.$$

Just as in the proof of [6, Theorem 8.1], there is an open subcylinder  $C_0 \subset C$  such that  $\text{codim}(C_0 \cap X_\infty, X_\infty) \leq \text{codim}(C_0, A_\infty) + e - cd$ , where  $e = \min_{\gamma \in C \cap X_\infty} \{\text{ord}_\gamma(\mathcal{J}_X)\}$ . Therefore, we have

$$\text{codim}(C_0 \cap X_\infty, X_\infty) < m + \tau + e.$$

On the other hand, as  $C_0 \cap X_\infty \subset \text{Cont}^{\geq m}(\mathfrak{a}) \cap \text{Cont}^{\geq e}(\mathcal{J}_X) \cap \text{Cont}^{\geq 1}(I_W)$ , we obtain

$$\begin{aligned} \tau &\leq \text{codim}(\text{Cont}^{\geq m}(\mathfrak{a}) \cap \text{Cont}^{\geq e}(\mathcal{J}_X) \cap \text{Cont}^{\geq 1}(I_W)) - m - e \\ &\leq \text{codim}(C_0 \cap X_\infty, X_\infty) - m - e < \tau, \end{aligned}$$

which is a contradiction. □

As  $\text{mld}(W; X, \mathfrak{a}J_r^{1/r}) = \widehat{\text{mld}}(W; X, \mathfrak{a}\mathcal{J}_X)$  for  $\mathbb{Q}$ -Gorenstein  $X$  we obtain the following:

**COROLLARY 3.11** ([7], [6], [15]). — *Let  $X$  be a normal closed subvariety in a non-singular variety  $A$  of codimension  $c$  and let  $W$  be a proper closed subset of  $X$ . Assume that  $X$  is  $\mathbb{Q}$ -Gorenstein variety of index  $r$ . Let  $\tilde{\mathfrak{a}} \subset \mathcal{O}_A$  be an ideal such that its image  $\mathfrak{a} := \tilde{\mathfrak{a}}\mathcal{O}_X \subset \mathcal{O}_X$  is non-zero. Then,*

$$\text{mld}(W; X, \mathfrak{a}J_r^{1/r}) = \text{mld}(W; A, \tilde{\mathfrak{a}}I_X^c),$$

where  $I_X$  is the defining ideal of  $X$  in  $A$  and  $J_r$  is as in 2.2.

**COROLLARY 3.12** (Adjunction formula). — *Let  $X$  be a closed subvariety of a variety  $X'$  of codimension  $c$  and let  $W$  be a proper closed subset of  $X$ . Let  $\mathfrak{a}' \subset \mathcal{O}_{X'}$  be an ideal such that its image  $\mathfrak{a} := \mathfrak{a}'\mathcal{O}_X \subset \mathcal{O}_X$  is non-zero. Let  $I_{X/X'}$  be the defining ideal of  $X$  in  $X'$ . Then,*

$$\widehat{\text{mld}}(W; X, \mathfrak{a}\mathcal{J}_X) \geq \widehat{\text{mld}}(W; X', \mathfrak{a}'\mathcal{J}_{X'}I_{X/X'}^c).$$

*Proof.* — We may assume that  $X'$  is affine. Let  $A$  be a non-singular variety containing  $X'$  and  $I_{X/A}$  and  $I_{X'/A}$  be the ideals of  $X$  and  $X'$  in  $A$ , respectively. Let  $\tilde{\mathfrak{a}} \subset \mathcal{O}_A$  is an ideal whose image in  $\mathcal{O}_{X'}$  is  $\mathfrak{a}'$ . Let  $c'$  be the codimension of  $X'$  in  $A$ . By Proposition 3.10, we have

$$\widehat{\text{mld}}(W; X, \mathfrak{a}\mathcal{J}_X) = \widehat{\text{mld}}(W; A, \tilde{\mathfrak{a}}I_{X/A}^{c+c'}),$$

and

$$\widehat{\text{mld}}(W; X', \mathfrak{a}'\mathcal{J}_{X'}I_{X/X'}^c) = \widehat{\text{mld}}(W; A, \tilde{\mathfrak{a}}I_{X'/A}^c I_{X'/A}^{c'}).$$

As we have an inclusion

$$\tilde{\mathfrak{a}}I_{X/A}^{c+c'} \supset \tilde{\mathfrak{a}}I_{X'/A}^c I_{X'/A}^{c'}$$

of ideals, the inequality of our corollary follows. □

**Example 3.13** (L. Ein, M. Mustață). — The inequality in Corollary 3.12 is not the equality in general. Indeed the following is a counter example. Let  $A = \mathbb{A}_{\mathbb{C}}^4$ ,  $X' \subset A$  be a hypersurface defined by  $xy - zw = 0$  and  $X \subset A$  be defined by  $x = z = 0$ . Let  $0$  be the origin of  $A$ . Then,

$$\widehat{\text{mld}}(0; X, \mathcal{J}_X) = 2 > 1 \geq \widehat{\text{mld}}(0; X', \mathcal{J}_{X'}I_{X/X'}).$$

Actually, as  $X$  is a non-singular surface,  $\widehat{\text{mld}}(0; X, \mathcal{J}_X) = \text{mld}(0; X, \mathcal{O}_X) = 2$ . On the other hand, by Proposition 3.10, it follows

$$\widehat{\text{mld}}(0; X', \mathcal{J}_{X'} I_{X'/X'}) = \text{mld}(0; A, I_{X'/A} I_{X'/A}).$$

Let  $E$  be the exceptional divisor of the blow-up  $\tilde{A} \rightarrow A$  at the origin 0. Then,  $\text{ord}_E I_{X'/A} I_{X'/A} = 3$ , since the ideal is a homogeneous ideal generated by polynomials of degree 3. Hence,

$$\text{ord}_E K_{\tilde{A}/A} - \text{ord}_E I_{X'/A} I_{X'/A} + 1 = 1,$$

which yields  $\text{mld}(0; A, I_{X'/A} I_{X'/A}) \leq 1$ .

**COROLLARY 3.14.** — *Let  $(X, \mathfrak{a})$  be a pair consisting of an arbitrary variety  $X$  and a non-zero ideal  $\mathfrak{a} \subset \mathcal{O}_X$ , then the function  $x \mapsto \widehat{\text{mld}}(x; X, \mathfrak{a}\mathcal{J}_X)$ , ( $x \in X$  closed point) is lower semicontinuous.*

*Proof.* — The following proof is the same as in [5]. Since the statement is local, it is sufficient to prove the corollary for an affine variety  $X$ . So, we may assume that  $X$  is a closed subvariety of an affine space  $A = \mathbb{A}^N$ . By Proposition 3.10, for  $x \in X$ ,

$$\widehat{\text{mld}}(x; X, \mathfrak{a}\mathcal{J}_X) = \widehat{\text{mld}}(x; A, \tilde{\mathfrak{a}}I_X^c).$$

Here, the right hand side is  $\text{mld}(x; A, \tilde{\mathfrak{a}}I_X^c)$  since  $A$  is non-singular and it is known that the function  $x \mapsto \text{mld}(x; A, \tilde{\mathfrak{a}}I_X^c)$  is lower semicontinuous for non-singular  $A$  ([1], [7]). Now we have only to restrict the function on  $X$ . □

In general, the map  $x \mapsto \widehat{\text{mld}}(x; X, \mathfrak{a})$ , ( $x \in X$  closed point) is not necessarily lower semi-continuous (see Example 4.9).

**COROLLARY 3.15.** — *Let  $X$  be a variety of dimension  $n$ . Then, for every closed point  $x \in X$ , the following inequality holds:*

$$\widehat{\text{mld}}(x; X, \mathcal{J}_X) \leq n,$$

where the equality holds if and only if  $(X, x)$  is non-singular.

*Proof.* — The first statement is an immediate consequence of the lower semicontinuity. But for the second statement we give a different and direct proof of the first statement. We may assume that  $X$  is affine, therefore it is a closed subvariety of an affine space  $A = \mathbb{A}^N$ . Let  $I_X$  and  $M_x$  be the defining ideals of  $X$  and  $x$  in  $A$  and  $c = N - n$ . By Proposition 3.10,

$$\widehat{\text{mld}}(x; X, \mathcal{J}_X) = \widehat{\text{mld}}(x; A, I_X^c).$$

By Proposition 3.7 and Remark 3.8

$$\begin{aligned} \widehat{\text{mld}}(x; A, I_X^c) &= \inf_m \{ \text{codim}(\text{Cont}^{\geq m}(I_X) \cap \text{Cont}^{\geq 1}(M_x), A_\infty) - cm \} \\ &= \inf_m \{ \text{codim}(X_{m-1} \cap \pi_{m-1}^{-1}(x), A_{m-1}) - cm \}, \end{aligned}$$

where  $\pi_{m-1}: A_{m-1} \rightarrow A$  is the canonical projection. Since the restriction  $\overline{(X_{reg})_{m-1}} \rightarrow X$  of the projection has the relative dimension  $(m-1)n$ , it follows

$$(3.7) \quad (m-1)n \leq \dim \overline{(X_{reg})_{m-1}} \cap \pi_{m-1}^{-1}(x) \leq \dim X_{m-1} \cap \pi_{m-1}^{-1}(x).$$

Therefore, for every  $m \in \mathbb{N}$ ,

$$\widehat{\text{mld}}(x; A, I_X^c) \leq \dim A_{m-1} - (m-1)n - cm = n.$$

This shows the first statement.

If  $(X, x)$  is non-singular, then  $\widehat{\text{mld}}(x; X, \mathcal{J}_X) = \text{mld}(x; X, \mathcal{O}_X)$  and it is well known that  $\text{mld}(x; X, \mathcal{O}_X) = n$ . Conversely, if  $\widehat{\text{mld}}(x; X, \mathcal{J}_X) = n$ , then the equalities in (3.7) hold for every  $m \in \mathbb{N}$ . Hence, in particular for  $m = 2$ , we have  $\dim X_1 \cap \pi_1^{-1}(x) = n$ . This yields  $\dim T_{X,x} = n$ , which means that  $(X, x)$  is non-singular.  $\square$

In [17] Shokurov posed the following conjecture:

CONJECTURE 3.16. — *Let  $X$  be a  $\mathbb{Q}$ -Gorenstein variety of dimension  $n$ . Then, for every closed point  $x \in X$ , the following inequality holds:*

$$\text{mld}(x; X, \mathcal{O}_X) \leq n,$$

where the equality holds if and only if  $(X, x)$  is non-singular.

Our Corollary 3.15 is the answer to a modified version of this conjecture. In particular, if  $(X, x)$  is a complete intersection, then the affirmative answer to this conjecture follows from our corollary, because  $\text{mld}(x; X, \mathcal{O}_X) = \widehat{\text{mld}}(x; X, \mathcal{J}_X)$ . This is already observed by Florin Ambro (private communication to the author) who published its special case in [2].

#### 4. A characterization of the singularities with the minimal Mather discrepancy

In this section we think of only the pair  $(X, \mathcal{O}_X)$  for a variety  $X$  of dimension  $n \geq 2$ . We write  $\widehat{\text{mld}}(x; X)$  for  $\widehat{\text{mld}}(x; X, \mathcal{O}_X)$ . We denote the canonical projections of jet schemes by  $\psi_m: X_\infty \rightarrow X_m$ ,  $\psi_{m',m}: X_{m'} \rightarrow X_m$  ( $m' > m$ ),  $\pi_m: X_m \rightarrow X$  and  $\pi: X_\infty \rightarrow X$ . When we denote the projections for a variety, say  $Y$ , different from  $X$ , we put the symbol of the variety on the shoulder like  $\psi_m^Y$ , etc.

DEFINITION 4.1. — Let  $x$  be a closed point of a variety  $X$  of dimension  $n$ . For every  $m \in \mathbb{N}$ , define  $\lambda_m$  and  $\lambda_m^0$  as follows:

$$\begin{aligned} \dim \psi_m(\pi^{-1}(x)) &= mn - \lambda_m, \\ \dim \psi_m(\pi^{-1}(x) \setminus (\text{Sing } X)_\infty) &= mn - \lambda_m^0, \end{aligned}$$

where  $\text{Sing } X$  is the singular locus of  $X$ .

The invariant  $\lambda_m$  is introduced by Lawrence Ein. Note that  $\lambda_m = \lambda_m^0$  ( $m \in \mathbb{N}$ ) if  $(X, x)$  is an isolated singularity by [13, Lemma 2.12]. (This is also proved by the fact that  $(\text{Sing } X)_\infty$  is consisting of the trivial arc on  $x$ .) We have some basic properties of these invariants as follows:

LEMMA 4.2.

- (i) For every  $m \in \mathbb{N}$ , it follows  $\lambda_m^0 \geq \lambda_m \geq 0$ .
- (ii) For  $m \geq 0$ , it holds that  $\lambda_m = \lambda_m^0 = \lambda$  (constant).
- (iii)  $\lambda = \widehat{\text{mld}}(x; X) - n$ .

*Proof.* — By the definition, it is clear that  $\lambda_m \leq \lambda_m^0$ . By Denef-Loeser [3, Lemma 4.3], the canonical projection  $\psi_{m,l}$  induces a map  $\psi_m(X_\infty) \rightarrow \psi_l(X_\infty)$  whose fibers have dimension  $\leq (m-l)n$ , where  $m > l$  are non-negative integers, it follows that  $\psi_m(\pi^{-1}(x)) = \psi_m(X_\infty) \cap \pi_m^{-1}(x)$  is of dimension  $\leq mn$  by thinking of  $l = 0$ . This yields (i).

For (ii), remember that  $\pi^{-1}(x)$  is a contact locus and therefore has a finite number of irreducible components  $C_1, \dots, C_r, Z_1, \dots, Z_s$ , where  $C_i$ 's are fat, while  $Z_j$ 's are thin and contained in  $(\text{Sing } X)_\infty$  ([9]). As  $Z_j$  is thin, we have

$$\dim \psi_m(Z_j) \leq m(n-1)$$

for every  $m \in \mathbb{N}$  and  $j$  by the discussion on (i). On the other hand, by Proposition 2.13, for  $m \gg 0$ ,

$$\dim \psi_m(C_i) = (m+1)n - (\widehat{k}_{E_i} + 1),$$

where  $C_i = C_X(\text{val}_{E_i})$ . (Note that the valuation is reduced, since  $C_i$  is a component of  $\pi^{-1}(x)$ .) Let  $\lambda = \min_i \{\widehat{k}_{E_i} + 1\} - n$ , then for  $m \gg 0$  we have  $mn - \lambda_m = \dim \psi_m(\pi^{-1}(x)) = mn - \lambda$ . For  $\lambda_m^0$ , we have the required form from

$$\dim \psi_m(\pi^{-1}(x) \setminus (\text{Sing } X)_\infty) = \dim \psi_m(C_1 \cup \dots \cup C_r)$$

for  $m \gg 0$ .

For (iii), we have only to note that

$$\pi^{-1}(x) = \bigcup_v C_X(v),$$



up to a thin set, where  $v$  are all divisorial valuation centered at  $\{x\}$ . Thus,  $\lambda = \widehat{\text{mld}}(x; X) - n$ .  $\square$

The following is a trivial consequence from the lemma.

**COROLLARY 4.3.** — *For every closed point  $x$  of a variety  $X$  of dimension  $n$ ,*

$$\widehat{\text{mld}}(x; X) \geq n.$$

For the characterization of singularities with  $\widehat{\text{mld}}(x; X) = n$ , we prepare a few lemmas.

**LEMMA 4.4.** — *Let  $b: Y \rightarrow X$  be the blow-up at the point  $x$  and  $g: \tilde{Y} \rightarrow Y$  be a resolution of the singularities of  $Y$ . Let  $h: \tilde{Y} \rightarrow X$  be the composite  $b \circ g$ . Let  $\pi_1^{\tilde{Y}}: \tilde{Y}_1 \rightarrow \tilde{Y}$  be the canonical projection from 1-jet scheme  $\tilde{Y}_1$  of  $\tilde{Y}$  to  $\tilde{Y}$ . Denote the scheme theoretic fibers  $b^{-1}(x)$  and  $h^{-1}(x)$  by  $E$  and  $E'$ , respectively. Then,*

$$\overline{\psi_1(\pi^{-1}(x) \setminus (\text{Sing } X)_\infty)} = h_1\left(\overline{(\pi_1^{\tilde{Y}})^{-1}(E')}\right),$$

and if this subset contains a non-trivial jet, then we also have the following equality:

$$h_1\left(\overline{(\pi_1^{\tilde{Y}})^{-1}(E')}\right) = \overline{h_1\left((\pi_1^{\tilde{Y}})^{-1}(E'_{reg})\right)},$$

where  $E'_{reg}$  is the non-singular locus of the scheme  $E'$ .

*Proof.* — As  $h$  is a proper morphism, by the valuation criteria of properness, every arc in  $\pi^{-1}(x) \setminus (\text{Sing } X)_\infty$  can be lifted to an arc in  $(\pi_1^{\tilde{Y}})^{-1}(E') \setminus (h^{-1}(\text{Sing } X))_\infty$ . (When  $x$  is a non-singular point, just use the surjectivity of  $h_\infty: \tilde{Y}_\infty \rightarrow X_\infty$  on a neighborhood of  $x$ .) Therefore,

$$\psi_1(\pi^{-1}(x) \setminus (\text{Sing } X)_\infty) = h_1\psi_1^{\tilde{Y}}\left(\overline{(\pi_1^{\tilde{Y}})^{-1}(E') \setminus (h^{-1}(\text{Sing } X))_\infty}\right).$$

Noting that  $(\pi_1^{\tilde{Y}})^{-1}(E') \setminus (h^{-1}(\text{Sing } X))_\infty$  is dense in  $(\pi_1^{\tilde{Y}})^{-1}(E')$ , we have the first equality of the lemma. As for the second equality, the inclusion  $\supset$  is trivial. To show the inclusion  $\subset$ , take any non-trivial jet  $h_1(\alpha) \in h_1(\pi_1^{\tilde{Y}})^{-1}(E')$  for  $\alpha \in (\pi_1^{\tilde{Y}})^{-1}(E')$ . Let  $P = \pi_1^{\tilde{Y}}(\alpha) \in E'$ . Then  $\alpha$  corresponds to a ring homomorphism

$$\alpha^*: \mathcal{O}_{\tilde{Y}, P} \rightarrow \mathbb{C}[t]/(t^2)$$

which sends the maximal ideal  $\mathfrak{m}_{\tilde{Y}, P}$  to  $(t)$ . If the homomorphism  $h^*: \mathcal{O}_{X, x} \rightarrow \mathcal{O}_{\tilde{Y}, P}$  corresponding to  $h$  sends  $\mathfrak{m}_{X, x}$  to  $\mathfrak{m}_{\tilde{Y}, P}^2$ , then  $\alpha^*h^*(\mathfrak{m}_{X, x}) = 0$

which means that  $h_1(\alpha)$  is the trivial 1-jet, a contradiction to the definition of  $h_1(\alpha)$ . Therefore the principal ideal  $\mathfrak{m}_{X,x}\mathcal{O}_{\tilde{Y},P} \not\subset \mathfrak{m}_{\tilde{Y},P}^2$ , which yields that  $E'$  is non-singular at  $P$ . Now we have

$$\alpha \in (\pi_1^{\tilde{Y}})^{-1}(E'_{reg}).$$

On the other hand, the trivial jet is contained in the subsets

$$\overline{h_1\left((\pi_1^{\tilde{Y}})^{-1}(E')\right) \setminus \{\text{the trivial jet}\}},$$

by the previous argument, the trivial jet is contained in  $\overline{h_1\left((\pi_1^{\tilde{Y}})^{-1}(E'_{reg})\right)}$ , which completes the proof of the second equality  $\square$

4.5. — Let  $T_{X,x}$  be the tangent space of  $X$  at  $x$  and  $C_{X,x}$  the tangent cone of  $X$  at  $x$ . Let  $b: Y \rightarrow X$  be the blow-up of  $X$  at  $x$  and  $E = b^{-1}(x)$  the scheme theoretic fiber. Then, it is well known that there is an isomorphism

$$E \simeq (C_{X,x} \setminus \{0\})/\mathbb{C}^* \subset (T_{X,x} \setminus \{0\})/\mathbb{C}^*.$$

Denote the projection  $T_{X,x} \setminus \{0\} \rightarrow (T_{X,x} \setminus \{0\})/\mathbb{C}^*$  by  $\rho$ .

LEMMA 4.6. — *Let  $\pi_1: X_1 \rightarrow X$  be the canonical projection from the 1-jet scheme. Then, we have the following:*

- (i) *For a closed point  $x \in X$ , there is an isomorphism*

$$\phi: \pi_1^{-1}(x) \xrightarrow{\sim} T_{X,x},$$

*which sends the trivial jet to the origin 0 and sends the subset  $\psi_1(\pi^{-1}(x))$  into  $C_{X,x}$ .*

- (ii) *For a point  $P \in E$ , take a 1-jet  $\alpha \in (\pi_1^Y)^{-1}(P)$  of  $Y$ . If  $b_1(\alpha)$  is a non-trivial jet, then, the point  $\rho \circ \phi(b_1(\alpha)) \in (C_{X,x} \setminus \{0\})/\mathbb{C}^*$  corresponds to  $P$  under the isomorphism  $(C_{X,x} \setminus \{0\})/\mathbb{C}^* \xrightarrow{\sim} E$ .*

*Proof.*

(i) The existence of the isomorphism  $\phi$  is well known by the definitions of  $\pi_1^{-1}(x)$  and Zariski tangent space. Here, we give a concrete description of  $\phi$  for the rest of the statements (i).

Let  $(X, x) \subset (\mathbb{A}^N, 0)$  and let  $I$  be the defining ideal of  $X$  in  $\mathbb{A}^N$ . A jet  $\alpha \in \pi_1^{-1}(x)$  corresponds to a ring homomorphism

$$\alpha^*: \mathbb{C}[x_1, \dots, x_N] \rightarrow \mathbb{C}[t]/(t^2); \quad x_i \mapsto a_i^{(1)}t,$$

with  $\alpha^*(I) = 0$ . This homomorphism corresponds exactly to the point

$$(a_1^{(1)}, \dots, a_N^{(1)}) \in Z(I_1) \subset \mathbb{A}^N,$$

where  $I_1$  is the ideal of the linear terms of  $I$ . Here,  $Z(I_1)$  is the tangent space. This correspondence  $\alpha \mapsto (a_1^{(1)}, \dots, a_N^{(1)})$  is  $\phi$ . It is clear that the trivial jet corresponds to the origin. If a 1-jet  $\alpha$  can be lifted to an arc  $\tilde{\alpha}$ , then there is a commutative diagram of ring homomorphisms:

$$\begin{array}{ccc} & & \mathbb{C}[[t]] \\ & \nearrow \tilde{\alpha}^* & \downarrow \\ \mathbb{C}[x_1, \dots, x_N] & \xrightarrow{\alpha^*} & \mathbb{C}[t]/(t^2) \end{array}$$

with  $\tilde{\alpha}^*(I) = 0$ , which yields  $f_*(a_1^{(1)}, \dots, a_N^{(1)}) = 0$  for the initial term  $f_*$  of every element  $f \in I$ . Therefore, this jet corresponds to a point

$$(a_1^{(1)}, \dots, a_N^{(1)}) \in Z(I_*) \subset \mathbb{A}^N,$$

where  $I_*$  is the ideal of the initial terms of  $I$ . Here,  $Z(I_*)$  is the tangent cone. This completes the proof of (i).

For the proof of (ii), we may assume that  $(X, x) = (\mathbb{A}^N, 0)$ . The blow-up  $Y$  is covered by  $U_{(i)} = \text{Spec } \mathbb{C}[x_i, \frac{x_1}{x_i}, \dots, \frac{x_N}{x_i}]$  ( $i = 1, \dots, N$ ). We may assume that  $P \in U_{(1)}$ . Then,  $\alpha$  corresponds to the ring homomorphism

$$\begin{aligned} \alpha^* : \mathbb{C} \left[ x_1, \frac{x_2}{x_1}, \dots, \frac{x_N}{x_1} \right] &\longrightarrow \mathbb{C}[t]/(t^2), \\ x_1 &\mapsto a_1^{(1)}t, \quad \frac{x_i}{x_1} \mapsto a_i^{(0)} + a_i^{(1)}t, \quad (i \neq 1). \end{aligned}$$

Here, we note that the homogeneous coordinates of  $P \in E \simeq \mathbb{P}^{N-1}$  is

$$(1 : a_2^{(0)} : \dots : a_N^{(0)}).$$

As  $b_1(\alpha)$  corresponds to the ring homomorphism

$$b_1(\alpha)^* = \alpha^* \circ b^* : \mathbb{C}[x_1, \dots, x_N] \longrightarrow \mathbb{C}[t]/(t^2)$$

with  $x_1 \mapsto a_1^{(1)}t$ ,  $x_i \mapsto a_i^{(0)}a_1^{(1)}t$ , ( $i \neq 1$ ), it corresponds to a point

$$(a_1^{(1)}, a_2^{(0)}a_1^{(1)}, \dots, a_N^{(0)}a_1^{(1)})$$

in  $\mathbb{A}^N \simeq C_{X,x}$ . Since  $b_1(\alpha)$  is not a trivial jet, we have  $a_1^{(1)} \neq 0$  and the image of  $(a_1^{(1)}, a_2^{(0)}a_1^{(1)}, \dots, a_N^{(0)}a_1^{(1)})$  by  $\rho$  has the homogeneous coordinates  $(1 : a_2^{(0)} : \dots : a_N^{(0)})$  that represents  $P$ . □

**THEOREM 4.7.** — *For a singularity  $(X, x)$  the following are equivalent:*

- (i)  $\widehat{\text{mld}}(x; X, \mathcal{O}_X) = n$ ;
- (ii)  $\lambda_m = 0$  for every  $m \in \mathbb{N}$ ;
- (iii)  $\lambda_m^0 = 0$  for every  $m \in \mathbb{N}$ ;

- (iv)  $\lambda_1^0 = 0$ ;
- (v) *The tangent cone of  $(X, x)$  has a reduced irreducible component, where a reduced irreducible component means an irreducible component which is reduced at the generic point.*

*Proof.* — The implications (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) are trivial by Lemma 4.2, (iii). The implication (i) $\Rightarrow$ (ii) follows from  $0 = \lambda = \lambda_m$  for  $m \gg 0$ . Indeed, fix  $m \gg 0$ , then by [3, Lemma 4.3] and  $\lambda_m = 0$ , for every  $i < m$ ,

$$in \geq \dim \psi_i(\pi^{-1}(x)) \geq mn - (m - i)n = in,$$

which means  $\lambda_i = 0$ . The implication (i) $\Rightarrow$ (iii) follows similarly. (iii) $\Rightarrow$ (iv) is trivial. (iv) $\Rightarrow$ (v) is proved as follows: By (iv) we have  $\dim \psi_1(\pi^{-1}(x) \setminus (\text{Sing } X)_\infty) = n$ . Then, by Lemma 4.4,

$$n = \dim h_1 \left( (\pi_1^{\tilde{Y}})^{-1}(E'_{reg}) \right) = \dim b_1 \left( g_1(\pi_1^{\tilde{Y}})^{-1}(E'_{reg}) \right).$$

Therefore,  $\dim \rho \circ \phi \left( b_1 \left( g_1(\pi_1^{\tilde{Y}})^{-1}(E'_{reg}) \right) \right) = n - 1$ . By Lemma 4.6, we have

$$\rho \circ \phi \left( b_1 \left( g_1(\pi_1^{\tilde{Y}})^{-1}(E'_{reg}) \right) \setminus \{0\} \right) \simeq \pi_1^Y \left( g_1(\pi_1^{\tilde{Y}})^{-1}(E'_{reg}) \right).$$

On the other hand, by commutativity of the diagram, it follows

$$\pi_1^Y \left( g_1(\pi_1^{\tilde{Y}})^{-1}(E'_{reg}) \right) = g(E'_{reg}).$$

Now we obtain

$$\dim g(E'_{reg}) = n - 1.$$

This shows that  $E = b^{-1}(x)$  is reduced at an irreducible component, which implies (v). The proof of (v) $\Rightarrow$ (iii) is the following: Let  $C$  be the reduced irreducible component of  $E = b^{-1}(x)$ . Then, by restricting  $Y$  and  $C$  by appropriate open subsets, we may assume that a pair  $(Y, C)$  is a non-singular pair of a variety and a divisor by [10, Chapter 0, 17.1.8]. As  $\psi_m((\pi^Y)^{-1}(C)) = \overline{(\pi_m^Y)^{-1}(C)}$  is of dimension  $mn + n - 1$  and in general  $\dim \psi_m(\pi^{-1}(x) \setminus (\text{Sing } X)_\infty) \leq mn$ , it is sufficient to show that the general fiber of the morphism

$$b_m : \psi_m((\pi^Y)^{-1}(C)) \longrightarrow \overline{\psi_m(\pi^{-1}(x) \setminus (\text{Sing } X)_\infty)}$$

onto its image has dimension  $n - 1$ . Hence, we have only to prove that

$$\dim b_m^{-1}b_m(\alpha) = n - 1$$

for a general  $\alpha \in (\pi_m^Y)^{-1}(C)$ .

Let  $(X, x) \subset (\mathbb{A}^N, 0)$  and let  $\tilde{b}: \tilde{\mathbb{A}}^N \rightarrow \mathbb{A}^N$  be the blow-up at the origin 0. As  $\tilde{\mathbb{A}}^N$  is covered by  $U_{(i)} = \text{Spec } \mathbb{C}[x_i, \frac{x_1}{x_i}, \dots, \frac{x_N}{x_i}]$  ( $i = 1, \dots, N$ ), we may assume that

$$(Y, C) \subset (U_{(1)}, Z(x_1)).$$

Take  $\alpha \in (\pi_m^Y)^{-1}(C)$ . Then, the  $m$ -jet  $\alpha: \text{Spec } \mathbb{C}[t]/(t^{m+1}) \rightarrow Y \subset U_{(1)}$  corresponding to a ring homomorphism

$$\begin{aligned} \mathbb{C} \left[ x_1, \frac{x_2}{x_1}, \dots, \frac{x_N}{x_1} \right] &\rightarrow \mathbb{C}[t]/(t^{m+1}), \\ x_1 &\mapsto \sum_{j=1}^m a_1^{(j)} t^j, \quad \frac{x_i}{x_1} \mapsto \sum_{j=0}^m a_i^{(j)} t^j \quad (i \neq 1) \end{aligned}$$

has the coordinates

$$\left( \overbrace{(0, a_2^{(0)}, \dots, a_N^{(0)})}^N, \overbrace{(a_1^{(1)}, \dots, a_N^{(1)})}^N, \dots, \overbrace{(a_1^{(m)}, \dots, a_N^{(m)})}^N \right)$$

in  $(U_{(1)})_m = \mathbb{A}^{(m+1)N}$ . The reason why  $a_1^{(0)} = 0$  is because  $\pi_m^Y(\alpha) \in C \subset Z(x_1)$ .

Here, as  $Y$  is non-singular and  $x_1$  is a member of a regular system of parameters of  $Y$  at all points in  $C$ , it follows  $a_1^{(1)} \neq 0$  for general  $\alpha \in (\pi_m^Y)^{-1}(C)$ . Indeed, we can take a regular system  $x_1, z_2, \dots, z_n$  of parameters of  $\mathcal{O}_{Y,P}$  for a general point  $P \in C$ . Then, we obtain an  $m$ -jet  $\mathbb{C}[[x_1, z_2, \dots, z_n]] \rightarrow \mathbb{C}[t]/(t^{m+1})$  which sends  $x_1$  to  $\sum_{j=0}^m a_1^{(j)} t^j$  with  $a_1^{(0)} = 0, a_1^{(1)} \neq 0$ . This shows that a general  $\alpha \in (\pi_m^Y)^{-1}(C)$  satisfies  $a_1^{(1)} \neq 0$ .

Then,  $b_m(\alpha)$  corresponds to the ring homomorphism

$$\begin{aligned} \mathbb{C}[x_1, \dots, x_N] &\rightarrow \mathbb{C}[t]/(t^{m+1}), \\ x_1 &\mapsto \sum_{j=1}^m a_1^{(j)} t^j, \quad x_i \mapsto \sum_{j=1}^m \sum_{l+k=j} a_1^{(l)} a_i^{(k)} t^j \quad (i \neq 1), \end{aligned}$$

because  $x_i = x_1 \frac{x_i}{x_1}$ . Therefore,  $b_m(\alpha)$  has the coordinates

$$(4.1) \quad \left( \overbrace{(0, \dots, 0)}^N \mid \overbrace{(a_1^{(1)}, a_1^{(1)} a_2^{(0)}, \dots, a_1^{(1)} a_N^{(0)})}^N \mid \dots, \right. \\ \left. \overbrace{\left( a_1^{(m)}, \sum_{l=1}^m a_1^{(l)} a_2^{(m-l)}, \dots, \sum_{l=1}^m a_1^{(l)} a_N^{(m-l)} \right)}^N \right)$$

in  $(\mathbb{A}^N)_m = \mathbb{A}^{(m+1)N}$ . Here, in the coordinates we put slits to clarify each block of  $N$ -coordinates for the convenience.

Let  $\beta \in b_m^{-1}b_m(\alpha)$  and let the coordinates of  $\beta$  in  $(U_{(1)})_m = \mathbb{A}^{(m+1)N}$  be

$$\left( \overbrace{(0, d_2^{(0)}, \dots, d_N^{(0)})}^N, \overbrace{(d_1^{(1)}, \dots, d_N^{(1)})}^N, \dots, \overbrace{(d_1^{(m)}, \dots, d_N^{(m)})}^N \right).$$

Then,  $b_m(\beta)$  has the coordinates

$$(4.2) \quad \left( \overbrace{(0, \dots, 0)}^N \mid \overbrace{(d_1^{(1)}, d_1^{(1)}d_2^{(0)}, \dots, d_1^{(1)}d_N^{(0)})}^N \mid \dots, \right. \\ \left. \overbrace{(d_1^{(m)}, \sum_{l=1}^m d_1^{(l)}d_2^{(m-l)}, \dots, \sum_{l=1}^m d_1^{(l)}d_N^{(m-l)})}^N \right).$$

Assume that  $\alpha \in (\pi_m^Y)^{-1}(C)$  is general, then as  $b_m(\beta) = b_m(\alpha)$ , we compare the coordinates in (4.1) and (4.2). First, by the comparison of the first coordinates of each block in (4.1) and (4.2), we obtain

$$d_1^{(j)} = a_1^{(j)}, \quad (j = 1, \dots, m).$$

Next, by the comparison of the second blocks in (4.1) and (4.2), we have

$$d_i^{(0)} = \frac{1}{d_1^{(1)}} a_1^{(1)} a_i^{(0)} = a_i^{(0)}, \quad (i = 2, \dots, N),$$

since  $a_1^{(1)} = d_1^{(1)} \neq 0$ . Then, by the comparison of the third blocks in (4.1) and (4.2), we have

$$d_i^{(1)} = \frac{1}{d_1^{(1)}} (a_1^{(2)} a_i^{(0)} + a_1^{(1)} a_i^{(1)} - d_1^{(2)} d_i^{(0)}) = \frac{1}{a_1^{(1)}} (a_1^{(1)} a_i^{(1)}) = a_i^{(1)}.$$

In the similar way, we obtain successively

$$d_i^{(j)} = a_i^{(j)}, \quad (j = 1, \dots, m - 1, i = 1, \dots, N)$$

and

$$d_1^{(m)} = a_1^{(m)}.$$

Therefore, for a general  $\alpha$ , a jet  $\beta \in b_m^{-1}b_m(\alpha)$  has the coordinates of the form

$$\left( 0, a_2^{(0)}, \dots, a_N^{(0)} \mid a_1^{(1)}, \dots, a_N^{(1)} \mid \dots \right. \\ \left. \mid a_1^{(m-1)}, \dots, a_N^{(m-1)} \mid a_1^{(m)}, c_2, c_3, \dots, c_N \right)$$

for any  $c_2, \dots, c_N$  so that  $\beta$  is in  $Y_m$ . Therefore,

$$b_m^{-1}b_m(\alpha) = (\psi_{m,m-1}^Y)^{-1}(\psi_{m,m-1}^Y(\alpha)) \cap Z(x_1^{(m)} - a_1^{(m)}).$$

As  $Y$  is non-singular of dimension  $n$ , we have

$$(\psi_{m,m-1}^Y)^{-1}(\psi_{m,m-1}^Y(\alpha)) \simeq \mathbb{A}^n.$$

Again as  $Y$  is non-singular and  $x_1$  is a member of a regular system of parameters at a point in  $C$ , the equation  $x_1^{(m)} - a_1^{(m)} = 0$  is not trivial on this space  $\mathbb{A}^n$ . Indeed, we can take a regular system  $x_1, z_2, \dots, z_n$  of parameters of  $\mathcal{O}_{Y,P}$ , where  $P = \pi_1^Y(\alpha)$ . Let an  $m$ -jet

$$\alpha' : \widehat{\mathcal{O}}_{Y,P} = \mathbb{C}[[x_1, z_2, \dots, z_n]] \longrightarrow \mathbb{C}[t]/(t^{m+1})$$

be defined by

$$\begin{aligned} x_1 &\mapsto \sum_{j=1}^{m-1} a_1^{(j)} t^j + at^m \quad \text{with } a \neq a_1^{(m)}, \\ z_i &\mapsto \sum_{j=0}^m a_1^{(j)} t^j, \quad (i = 2, \dots, N). \end{aligned}$$

Then,  $\alpha'$  gives a point in  $(\psi_{m,m-1}^Y)^{-1}(\psi_{m,m-1}^Y(\alpha)) \simeq \mathbb{A}^n$  which does not vanish by  $x_1^{(m)} - a_1^{(m)}$ .

By this,  $x_1^{(m)} - a_1^{(m)} = 0$  is not trivial equation in  $\mathbb{A}^n$ , this equation gives a hypersurface. Hence

$$\dim b_m^{-1}b_m(\alpha) = n - 1.$$

□

*Example 4.8.* — Let  $X$  be a hypersurface in  $\mathbb{A}^{n+1}$  defined by an equation  $f = 0$ . Then  $\widehat{\text{mld}}(x, X) = n$  if and only if the initial term  $f_*$  of  $f$  has an irreducible factor of multiplicity one.

*Example 4.9.* — In general, the map  $x \mapsto \widehat{\text{mld}}(x; X, \mathfrak{a})$ , ( $x \in X$  closed point) is not necessarily lower semi-continuous. Let  $(X, P)$  be a singularity of a variety  $X$  of dimension  $n$ . Assume  $\widehat{\text{mld}}(P; X, \mathcal{O}_X) > n$  (Such an example actually exists, because we have only to take an hypersurface singularity whose defining polynomial has the initial term whose factors are all with multiple powers. See Example 4.8) Every open neighborhood  $U$  of  $P$  contains a non-singular closed point  $y \in X$ . As

$$\widehat{\text{mld}}(y; X, \mathcal{O}_X) = \text{mld}(y; X, \mathcal{O}_X) = n,$$

the map  $x \mapsto \widehat{\text{mld}}(x; X, \mathfrak{a})$ , ( $x \in X$  closed point) is not lower semi-continuous.

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