ANNALES

DE

L'INSTITUT FOURIER

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Tome 62, nº 6 (2012), p. 2257-2314.

<http://aif.cedram.org/item?id=AIF_2012__62_6_2257_0>

Ann. Inst. Fourier, Grenoble
62, 6 (2012) 2257-2314

THE INVISCID LIMIT AND STABILITY OF CHARACTERISTIC BOUNDARY LAYERS FOR THE COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH NAVIER-FRICTION BOUNDARY CONDITIONS

by Ya-Guang WANG & Mark WILLIAMS

Abstract. — We study boundary layer solutions of the isentropic, compressible Navier-Stokes equations with Navier-friction boundary conditions when the viscosity constants appearing in the momentum equation are proportional to a small parameter ϵ. These boundary conditions are *characteristic* for the underlying inviscid problem, the compressible Euler equations.

The boundary condition implies that the velocity on the boundary is proportional to the tangential component of the stress. The normal component of velocity is zero on the boundary. We first construct a high-order approximate solution that exhibits a boundary layer. The main contribution to the layer appears in the tangential velocity and is of width $\sqrt{\epsilon}$ and amplitude $O(\sqrt{\epsilon})$. Next we prove that the approximate solution stays close to the exact Navier-Stokes solution on a fixed time interval independent of ϵ. As an immediate corollary we show that the Navier-Stokes solution converges in L^∞ in the small viscosity limit to the solution of the compressible Euler equations with normal velocity equal to zero on the boundary.

Résumé. — Nous étudions des solutions avec couches limites des équations de Navier-Stokes compressibles isentropiques avec des conditions de frottement de Navier au bord, lorsque la constante de viscosité figurant dans l'équation sur la quantité de mouvement est proportionnelle à un petit paramètre ϵ. Ces conditions aux limites sont caractéristiques pour le problème non visqueux sous-jacent, le système d' équations d'Euler compressibles.

Les conditions aux limites impliquent que la vitesse au bord est proportionnelle à la composante tangentielle des contraintes. La composante normale de la vitesse est nulle au bord. Nous construisons tout d'abord une solution approchée à un ordre élevé de la solution, décrivant la présence d'une couche limite. La contribution principale de la couche limite apparait dans la composante tangentielle de la vitesse, est de taille $\sqrt{\epsilon}$ et d'amplitude $O(\sqrt{\epsilon})$. Nous prouvons ensuite que cette solution approchée est effectivement asymptotique à la solution exacte, sur un intervalle de temps indépendant de ϵ. Un corollaire immédiat est que la solution des équations de Navier-Stokes converge dans L^∞, lorsque la viscosité tend vers 0, vers la solution du système d'Euler compressible avec composante normale de la vitesse nulle au bord.

Keywords: characteristic boundary layers, compressible Navier-Stokes equations, Navier boundary conditions, inviscid limit.
Math. classification: 76N20, 76N17.

1. Introduction

We consider the vanishing viscosity limit and stability of boundary layers for the compressible Navier-Stokes equations with the Navier boundary condition on the half-space $\{(t,x,y) \in \mathbb{R}^{1+d} : y \geqslant 0\}$, $d = 2, 3$. When $d = 2$ we write these simply as

(1.1)
$$\begin{aligned}
&\partial_t \rho^\epsilon + \nabla \cdot (\rho^\epsilon u^\epsilon) = 0 \\
&\partial_t (\rho^\epsilon u^\epsilon) + \nabla \cdot (\rho^\epsilon u^\epsilon \otimes u^\epsilon) + \nabla p(\rho^\epsilon) - \epsilon \left(\lambda \Delta u^\epsilon + \mu \begin{pmatrix} \mathrm{div} \partial_x u^\epsilon \\ \mathrm{div} \partial_y u^\epsilon \end{pmatrix} \right) = 0 \\
&u_1^\epsilon - \alpha \frac{\partial u_1^\epsilon}{\partial y} = 0, \;\; u_2^\epsilon = 0 \quad \text{on} \quad y = 0.
\end{aligned}$$

Here ρ^ϵ, $u^\epsilon = (u_1^\epsilon, u_2^\epsilon)^T$ and $p(\rho^\epsilon)$ for $\epsilon > 0$ denote, respectively, the density, velocity and pressure of a compressible fluid, λ, μ are constant viscosity coefficients with $\lambda > 0$ and $\lambda + \mu > 0$, and the slip length $\alpha > 0$. We assume that $p(\rho)$ is C^k for k large and satisfies

$$p(\rho) > 0, \; p'(\rho) > 0 \text{ for } \rho > 0. \tag{1.2}$$

The boundary condition was introduced by Navier in [16] and expresses the condition that the velocity on the boundary is proportional to the tangential component of the stress. An elementary derivation of the Navier boundary condition for general regions Ω is given in the introduction of [9]. Observe that in the singular limit when $\alpha \to 0$ the boundary conditions in (1.1) formally tend to the no-slip case, $u^\epsilon|_{y=0} = 0$, while when $\alpha \to +\infty$ the boundary conditions in (1.1) tend to the complete slip case, $u_2^\epsilon|_{y=0} = 0$ and $\partial_y u_1^\epsilon|_{y=0} = 0$.

The boundary layer problem for incompressible flow with the no-slip boundary condition was studied formally by Prandtl [15], who showed that the leading profile of the boundary layer could be described by the solution of an initial boundary value problem for a nonlinear degenerate parabolic-elliptic coupled system now called the Prandtl equations. Under a certain monotonicity restriction on the initial velocity, Oleinik established the short time existence of smooth solutions of the Prandtl equations in the 1960s [17]. The long-standing questions of the well-posedness of the Prandtl equations and of the closeness (for ϵ small) of approximate Navier-Stokes solutions constructed from Prandtl solutions to exact solutions of the Navier-Stokes equations with no-slip boundary conditions are still not thoroughly understood, although there are some important negative results bearing on this question. In [4] Grenier exhibited Prandtl layers that failed to describe

Navier-Stokes solutions by taking initial data constructed from a smooth Euler shear flow $u_E(y)$ with an inflection point that is linearly unstable for the Euler equations (a Raleigh instability). More recently, in [3] Gerard-Varet and Dormy proved linearized instability in a class of Sobolev-type spaces for the Prandtl equations linearized about an exact solution of the form $(u_1(t,\eta),0)$, where $u_1(t,\eta)$ is a solution of the heat equation such that $u_1(0,\eta)$ has a nondegenerate critical point. The only work we know of that answers both of the above questions positively is the work of Sammartino and Caflisch [19], who restrict their analysis to a space of functions analytic in x and η.(1)

In two space dimensions the inviscid limit for the incompressible Navier-Stokes equations with Navier boundary conditions, assuming slip length is independent of viscosity, has been studied by several authors. Assuming bounded vorticity, Clopeau, Mikelić and Robert [1] prove convergence to the Euler solutions. This result was extended to L^p vorticities, $p > 2$, by Lopes Filho et. al. [13]. In [11] Kelliher proves convergence in $L^\infty([0,T],L^2(\Omega))$ of Navier-Stokes solutions with Navier boundary conditions to Navier-Stokes solutions with no-slip boundary conditions as $\alpha \to 0$ uniformly on the boundary (assuming H^3 initial velocity and C^3 boundary). These works do not attempt to construct or give a precise description of the boundary layer.

Again for the incompressible Navier-Stokes equations, Iftimie and Sueur [10] give a careful construction of the boundary layer as well as a rigorous error analysis and discussion of the small viscosity limit for Navier boundary conditions (assuming α independent of ϵ) valid in two and three space dimensions. The error analysis of [10], which uses the tools associated with the divergence free condition (e.g., Leray projectors), is naturally quite different from the arguments given in this paper.

The Prandtl boundary layer is of amplitude $O(1)$ and width $O(\sqrt{\epsilon})$ when the viscosity is proportional to ϵ. The width $O(\sqrt{\epsilon})$ is typical of the boundary layers that arise in problems where the boundary conditions are *characteristic* for the underlying inviscid problem. Since the velocity normal to the boundary vanishes under Navier boundary conditions, the Navier boundary layer is also a characteristic boundary layer. The layers constructed in [10] and the layers constructed here in the compressible case are of width $O(\sqrt{\epsilon})$ like the Prandtl boundary layer, but in both cases are of amplitude $O(\sqrt{\epsilon})$ when α is independent of viscosity.

(1) See also [12], where the convergence as $\epsilon \to 0$ of incompressible Navier-Stokes solutions with no-slip boundary conditions to Euler solutions is proved for circularly symmetric 2D flows by a method that does not involve constructing a Prandtl layer.

In contrast noncharacteristic boundary layers such as those associated with inflow or outflow through the boundary are typically of width $O(\epsilon)$. The stability of noncharacteristic layers of amplitude $O(1)$ and the associated small viscosity limits were studied for incompressible Navier-Stokes in [20] and for compressible Navier-Stokes (and viscous MHD) in [6, 7].

Our main result deals with the system (1.1) when the slip length $\alpha > 0$ is independent of ϵ. We first construct a high-order approximate solution of (1.1) that exhibits a boundary layer. Next we prove that the boundary layer is stable; that is, the approximate solution stays close to the exact Navier-Stokes solution on a fixed time interval independent of ϵ (Theorem 1.3). As an immediate corollary of the stability result, we show that the Navier-Stokes solution converges in L^∞ in the small viscosity limit to the solution of the compressible Euler equations with normal velocity equal to zero on the boundary (Corollary 1.4).

1.1. Symmetric forms of the equations

The error estimates will take advantage of the fact that the equations can be put into a symmetric form where pressure and velocity are the unknowns instead of density and velocity. Since we are now assuming that α is a positive constant independent of ϵ, it has no further effect on the analysis to take $\alpha = 1$.

The interior and boundary equations in (1.1) can be written, where $w = (\rho, u)$ and we suppress some epsilons, as

(1.3)
$$\begin{aligned}&D_0(w)w_t + D_1(w)w_x + D_2(w)w_y - \epsilon[B_{11}w_{xx} + B_{12}w_{xy} + B_{22}w_{yy}] = 0\\ &u_1 - u_{1y} = 0, u_2 = 0 \text{ on } y = 0,\end{aligned}$$

where

(1.4)
$$D_0(w) = \begin{pmatrix} 1 & 0 & 0 \\ u_1 & \rho & 0 \\ u_2 & 0 & \rho \end{pmatrix}, \quad D_1(w) = \begin{pmatrix} u_1 & \rho & 0 \\ u_1^2 + p' & 2\rho u_1 & 0 \\ u_1u_2 & \rho u_2 & \rho u_1 \end{pmatrix},$$
$$D_2(w) = \begin{pmatrix} u_2 & 0 & \rho \\ u_1u_2 & \rho u_2 & \rho u_1 \\ u_2^2 + p' & 0 & 2\rho u_2 \end{pmatrix}$$
$$B_{11} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda + \mu & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad B_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu \\ 0 & \mu & 0 \end{pmatrix}, \quad B_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda + \mu \end{pmatrix}.$$

Applying the symmetrizer

$$S(w) = \begin{pmatrix} \frac{p'}{\rho} & 0 & 0 \\ -u_1 & 1 & 0 \\ -u_2 & 0 & 1 \end{pmatrix} \tag{1.5}$$

to the interior equation we obtain

$$C_0(w)w_t + C_1(w)w_x + C_2(w)w_y - \epsilon[B_{11}w_{xx} + B_{12}w_{xy} + B_{22}w_{yy}] = 0, \tag{1.6}$$

where the B_{ij} are unchanged and

$$C_0(w) = \begin{pmatrix} \frac{p'}{\rho} & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{pmatrix}, \; C_1(w) = \begin{pmatrix} \frac{p'u_1}{\rho} & p' & 0 \\ p' & \rho u_1 & 0 \\ 0 & 0 & \rho u_1 \end{pmatrix}, \tag{1.7}$$
$$C_2(w) = \begin{pmatrix} \frac{p'u_2}{\rho} & 0 & p' \\ 0 & \rho u_2 & 0 \\ p' & 0 & \rho u_2 \end{pmatrix}.$$

Finally, changing the dependent variable in (1.6) to $v = (p, u)$ we obtain the problem

(1.8)

$$(a)\; A_0(v)v_t + A_1(v)v_x + A_2(v)v_y - \epsilon[B_{11}v_{xx} + B_{12}v_{xy} + B_{22}v_{yy}] = 0,$$
$$(b)\; u_1 - u_{1y} = 0, u_2 = 0 \text{ on } y = 0,$$

where again the B_{ij} are unchanged and, with $\rho = \rho(p)$ and $\rho' = \frac{d\rho}{dp}$ now,

$$A_0(v) = \begin{pmatrix} \frac{\rho'}{\rho} & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{pmatrix}, \; A_1(v) = \begin{pmatrix} \frac{\rho'u_1}{\rho} & 1 & 0 \\ 1 & \rho u_1 & 0 \\ 0 & 0 & \rho u_1 \end{pmatrix}, \tag{1.9}$$
$$A_2(v) = \begin{pmatrix} \frac{\rho'u_2}{\rho} & 0 & 1 \\ 0 & \rho u_2 & 0 \\ 1 & 0 & \rho u_2 \end{pmatrix}.$$

We shall work with the form of the equations given by (1.8) in the error analysis of section 3. The fact that the off-diagonal elements are constant in the matrix A_2 makes some unwanted commutator terms vanish in the higher derivative estimates below. For later reference we record here the explicit components of (1.8)(a):

(1.10)
$$\begin{aligned}&\frac{\rho'}{\rho}p_t+\frac{\rho'}{\rho}u_1p_x+u_{1x}+\frac{\rho'}{\rho}u_2p_y+u_{2y}=0\quad(\rho'=\frac{d\rho}{dp})\\&\rho u_{1t}+p_x+\rho u_1u_{1x}+\rho u_2u_{1y}-\epsilon\left((\lambda+\mu)u_{1xx}+\mu u_{2xy}+\lambda u_{1yy}\right)=0\\&\rho u_{2t}+\rho u_1u_{2x}+p_y+\rho u_2u_{2y}-\epsilon\left(\lambda u_{2xx}+\mu u_{1xy}+(\lambda+\mu)u_{2yy}\right)=0.\end{aligned}$$

1.2. Error equation, iteration scheme, and overview of the error analysis

Let $w^a=(\rho^a,u^a)$ be the approximate solution constructed in Proposition 2.3. This solution has the expansion (2.52) involving "slow" and "fast" profiles, the $(\rho^{I,j},u^{I,j})$ and $(\rho^{B,j},u^{B,j})$ respectively, where the $(\rho^{B,j},u^{B,j})$ are functions of (t,x,η) that decay rapidly to 0 as $\eta\to+\infty$. Observe from (2.52) that the boundary layer makes it first appearance in the terms of amplitude $O(\epsilon^{\frac12})$ of $u_1^{\epsilon,M}$. Denoting the left side of (1.6) by $\mathbb{E}(w)$, we may write the problem satisfied by w^a as

$$\begin{aligned}&\mathbb{E}(w^a)=\sqrt{\epsilon}^MR^M\text{ on }[-\delta,T_0]\times\{(x,y):y\geqslant0\}\\&u_1^a-u_{1y}^a=0,\ u_2^a=0\text{ on }y=0,\end{aligned}\tag{1.11}$$

where $\rho^a\geqslant C>0$ for some fixed C throughout its domain, and $\delta>0$. We may, for example, take $\delta=\frac{T_0}{3}$ as in Proposition 2.3, where $[-T_0,T_0]$ is the interval of existence of the leading slow profiles $(\rho^{I,0},u^{I,0})$. Denote the same solution in the (p,u) variables by $v^a=(p^a,u^a)$, where

$$p^a(t,x,y):=p(\rho^a(t,x,y)).\tag{1.12}$$

Writing the left side of (1.8)(a) as $\mathcal{E}(v)$, we have now in place of (1.11)

$$\begin{aligned}&\mathcal{E}(v^a)=\sqrt{\epsilon}^MR^M\text{ on }[-\delta,T_0]\times\{(x,y):y\geqslant0\}\\&u_1^a-u_{1y}^a=0,\ u_2^a=0\text{ on }y=0.\end{aligned}\tag{1.13}$$

We seek an exact solution v to (1.8) on $[0,T_0]\times\{(x,y):y\geqslant0\}$ that is close to v^a. To guarantee high-order corner compatibility conditions for v, we introduce a C^∞ cutoff function $\theta(t)$ that is $\equiv1$ in $t\geqslant-\frac{\delta}{3}$ and $\equiv0$ in $t\leqslant-\frac{\delta}{2}$, and look for v of the form

$$v=v^a+\sqrt{\epsilon}^Lz\tag{1.14}$$

where $2 \leqslant L < M - 2k$ (for k specified later) and $z = (p, u)$ satisfies the forward *error problem*:

$$(1.15)\quad \begin{aligned} &(\sqrt{\epsilon})^{-L}\left(\mathcal{E}(v^a + \sqrt{\epsilon}^L z) - \mathcal{E}(v^a)\right) = -\sqrt{\epsilon}^{M-L}\theta(t)R^M \text{ on } (-\infty, T_0] \\ &u_1 - u_{1y} = 0,\ u_2 = 0 \text{ on } y = 0 \\ &z = 0 \text{ in } t \leqslant -\frac{\delta}{2}. \end{aligned}$$

With a small risk of confusion it is convenient to use (p, u) to denote the components of z now. We are as usual suppressing the ϵ-dependence of $z = z^\epsilon$ (and of other functions) in the notation. Our goal is to solve (1.15) for z^ϵ on $t \leqslant T_0$ for $\epsilon \in (0, \epsilon_0]$ for some sufficiently small ϵ_0.

Remark 1.1. — To make sense of (1.15) on $(-\infty, T_0]$ we take a smooth extension of v^a, which is initially defined on $[-T_0, T_0]$, to $t \leqslant -T_0$ such that the extension remains close in L^∞ to $v^a|_{t=-T_0}$. In particular the extended ρ^a satisfies $\rho^a \geqslant C > 0$ on its domain of definition. Different choices of extension satisfying this condition lead to the same solution z.

The problem (1.15) will be solved by the following iteration scheme:

$$(1.16)\quad \begin{aligned} &A_0(v^a + \sqrt{\epsilon}^L z^n)\partial_t z^{n+1} + A_1(v^a + \sqrt{\epsilon}^L z^n)\partial_x z^{n+1} + A_2(v^a + \sqrt{\epsilon}^L z^n)\partial_y z^{n+1} + \\ &\left(z^{n+1} \cdot \int_0^1 \partial_v A_0(v^a + s\sqrt{\epsilon}^L z^n) ds\right)\partial_t v^a + \\ &\left(z^{n+1} \cdot \int_0^1 \partial_v A_1(v^a + s\sqrt{\epsilon}^L z^n) ds\right)\partial_x v^a + \\ &\left(z^{n+1} \cdot \int_0^1 \partial_v A_2(v^a + s\sqrt{\epsilon}^L z^n) ds\right)\partial_y v^a - \\ &\epsilon[B_{11}\partial_{xx} z^{n+1} + B_{12}\partial_{xy} z^{n+1} + B_{22}\partial_{yy} z^{n+1}] = -\sqrt{\epsilon}^{M-L}\theta(t)R^M \end{aligned}$$

with boundary and initial conditions

$$(1.17)\quad \begin{aligned} &u_1^{n+1} - \partial_y u_1^{n+1} = 0, u_2^{n+1} = 0 \text{ on } y = 0, \\ &z^{n+1} = 0 \text{ in } t \leqslant -\frac{\delta}{2}, \end{aligned}$$

where we take the first iterate $z^0 = 0$.

Next we give an overview of the proof of the main estimates used in showing convergence of the iteration scheme. The numbers in boldface refer to the numbered paragraphs in the proof of Proposition 3.14.

One of the main difficulties in the problem arises from the vanishing at $y=0$ of the coefficient of p_y in the first component equation of (1.8)(a), the pressure equation. This is connected to the fact that the boundary is *characteristic* for the hyperbolic problem (Euler equations); that is, the matrix $A_2(v)$ is singular on $y=0$, since $u_2=0$ on $y=0$. Of course, the same applies to the error equation (1.15). An important consequence of this for the iteration scheme is that we are unable to control $\sqrt{\epsilon}^L|\partial_y p^n|_{L^\infty}$ (or equivalently, $\sqrt{\epsilon}^L|\partial_y \rho^n|_{L^\infty}$). Fortunately, we are able to control $\sqrt{\epsilon}^L|y\partial_y p^n|_{L^\infty}$, and this turns out to be enough to eventually close the estimates.

The first step is to get an L^2 estimate for (1.16), (1.17) and to estimate tangential $(\partial_t,\ \partial_x,\ y\partial_y)$ derivatives. The L^2 estimate, Proposition 3.2, is unusual in that it does not require boundedness of the Lipschitz norm of $\sqrt{\epsilon}^L z^n$, even though the matrix coefficients in (1.16) are functions of $\sqrt{\epsilon}^L z^n$. Instead, the estimate assumes only

$$|z^n, \sqrt{\epsilon}^L(\partial_t z^n, \partial_x z^n, y\partial_y p^n, \partial_y u^n)|_{L^\infty} \leqslant 1. \tag{1.18}$$

An argument that recurs often in section 3 is illustrated by the proof of L^∞ boundedness of

$$\partial_y A_2 = (\partial_p A_2)\partial_y(p^a + \sqrt{\epsilon}^L p^n) + (\partial_{u_2} A_2)\partial_y(u_2^a + \sqrt{\epsilon}^L u_2^n) \tag{1.19}$$

in the proof of the L^2 estimate. For the second term on the right boundedness is clear from (1.18). For the first term we use the fact that $\partial_p A_2$ is a *diagonal* matrix with entries that vanish when $y=0$ as a consequence of the Navier boundary conditions. Thus we can extract a factor of y from $\partial_p A_2$ to multiply $\partial_y p^n$ and thereby make use of (1.18). The advantage of using (p,u) coordinates is clearly seen here; the use of $w=(\rho,u)$ coordinates would have led to $\partial_\rho C_2$ (see (1.7)), which has off-diagonal elements that do not vanish on $y=0$, instead of $\partial_p A_2$.

The estimate of higher tangential derivatives in Proposition 3.10 works in any number of space dimensions, and relies on Moser estimates, Lemma 3.6, to estimate the L^2 norm of products like

$$(M^{\alpha_1} z^n_{i_1}) \dots (M^{\alpha_r} z^n_{i_r})(M^{\alpha_{r+1}} z^{n+1}_{i_{r+1}}) \tag{1.20}$$

that arise from commuting tangential derivatives M^α through the equations. Here we have set $M=(M_0,M_1,M_2) := (\partial_t, \partial_x, y\partial_y)$.

To control the L^∞ norm of z^{n+1} it suffices to control the L^2 norm of sufficiently many tangential derivatives (roughly $\frac{d}{2}$) and of at least one normal derivative $\partial_y z^{n+1}$ as well. The presence of viscosity in the second and third component equations of (1.16) gives better control over u^{n+1} than over p^{n+1}; for example, $\sqrt{\epsilon}|\nabla u^{n+1}|_{L^2}$ appears on the left in the L^2

estimate (3.7). If we could solve for p_y^{n+1} in the first equation, we could use the better control over u^{n+1} to estimate $|p_y^{n+1}|_{L^2}$ by a sum of appropriate norms of the other terms appearing in the pressure equation. This strategy works well in noncharacteristic boundary problems for the Navier-Stokes equations (see e.g., [6] (6.87)-(6.88)).

That strategy does not work here, and in order to estimate $|p_y^{n+1}|_{L^2}$ we must do a separate L^2 estimate for $\partial_y z^{n+1}$. If one simply differentiates (1.16) with respect to y and takes an inner product with $\partial_y z^{n+1}$, the estimate fails because differentiation with respect to y destroys the Navier boundary conditions, and one obtains boundary terms from integration by parts that cannot be controlled. Instead, we shall first add a viscosity term $\eta\Delta p^{n+1}$ to the pressure equation as in (3.1) and impose an extra boundary condition, namely $p_y^{n+1} = 0$ on $y = 0$(2). This has several helpful effects, including that of making some undesirable boundary terms arising from integration by parts vanish.

Denoting the unknown in the modified problem (3.1) by $z^{n+1,\eta}$, in Proposition 3.14 we obtain estimates on $|M^j\partial_y z^{n+1,\eta}|_{L^2}$ that are uniform in *both* ϵ and η. We do this by differentiating the equation (3.1) with $M^j\partial_y$, and then taking the L^2 pairing with

$$(\eta M^j\partial_y p^{n+1,\eta}, M^j\partial_y u^{n+1,\eta}). \tag{1.21}$$

One of the challenges here is that we do not have any useful version of the Moser estimates that applies to products (again arising from commutators) of the form

$$(M^{\alpha_1} z_{i_1}^n)\dots(M^{\alpha_r}\partial_y z_{i_r}^n)(M^{\alpha_{r+1}} z_{i_{r+1}}^{n+1}), \tag{1.22}$$

for example, where a normal derivative appears on one of the factors. It turns out that by restricting the space dimension to $d = 2$ or $d = 3$ we are able to use the Sobolev estimates of Corollary 3.8 to control the L^∞ norms of all but one of the factors occurring in such products. This is carried out in steps **4, 5, 6** of the proof of Proposition 3.14. The reason for introducing the factor η in (1.21) is seen, for example, in the estimate (3.55), where the presence of η allows us bypass a problem with estimating $\partial_y^2 p^{n+1,\eta}$ by taking advantage of the equation to estimate $|\eta\partial_y^2 p^{n+1,\eta}|_{L^\infty}$! Similar use of the extra factor of η in (1.21) is made in the estimate of (3.61).

(2) This is a familiar maneuver in the study of compressible flow, used for example in [2].

As a consequence of the extra boundary condition $\partial_y p^{n+1,\eta}|_{y=0}$ in the modified problem (3.1), the boundary terms arising from the integrations by parts involving the $A_2\partial_y$ and $-\eta\Delta$ terms vanish (see the discussion below (3.45) in **2**). The boundary terms arising from the integration involving $\epsilon B_{22}\partial_y^2$ depend only on $u^{n+1,\eta}$ and are estimated in **10** using the Navier boundary conditions and trace estimates like (3.77), in some cases after using the interior equations to rewrite $\epsilon\partial_y^2 u|_{y=0}$.

A helpful device in this argument is to first settle for an intermediate estimate, namely (3.86), which gives a control on $\sqrt{\eta}|M^j\partial_y p^{n+1,\eta}|_{L^2}$ that clearly degenerates as $\eta \to 0$, but which gives good control on $|M^j\partial_y u^{n+1,\eta}|_{L^2}$. The unwanted factor $\sqrt{\eta}$ appears here because we chose to pair the differentiated equation with (1.21). In the final step of the proof of Proposition 3.14 we use the *third* scalar component equation of (1.16), together with the good control on $u^{n+1,\eta}$ from the intermediate estimate, to estimate $|M^j\partial_y p^{n+1,\eta}|_{L^2}$ without the unwanted factor of $\sqrt{\eta}$.

The estimates (3.42) of Proposition 3.14 allow us to construct the $(n+1)$-st iterate z^{n+1} satisfying (1.16) by taking a limit of $z^{n+1,\eta}$ as $\eta \downarrow 0$. That limit is shown in Proposition 3.15 to inherit the same estimates uniform in ϵ that are satisfied by $z^{n+1,\eta}$. The final paragraphs of section 3.3 show that these estimates are strong enough to deduce convergence as $n \to \infty$ of the iteration scheme (1.16)-(1.17) for $t \leqslant T_0$, where $T_0 > 0$ is independent of ϵ. In fact, T_0 can be taken to be the same constant as the one in (1.11), which specified the time interval of existence of the approximate solution.

1.3. Main results

We introduce some notation before stating the main result.[3]

For $T > 0$ let $\Omega_T = \{(t,x,y) : y \geqslant 0, -\infty < t \leqslant T\}$ and set $b\Omega_T = \Omega_T \cap \{y = 0\}$.

Notation 1.2. — 1.) Denote the tangential operators ∂_t, ∂_x, and $y\partial_y$ by M_j, $j = 0,1,2$ respectively, and for $k \in \mathbb{N} = \{0,1,2,\dots,\}$ let M^k denote the collection of operators $M_0^{\alpha_0}M_1^{\alpha_1}M_2^{\alpha_2}$ such that $\alpha_0+\alpha_1+\alpha_2 = k$. Sometimes M^k is used to denote a particular member of this collection of operators. Set $\nabla f = (\partial_x f, \partial_y f)$.

(3) At this point some readers may wish to skip Notation 1.2 and focus on part 2 of Theorem 1.3 and Corollary 1.4.

2.) Let $\mu = \frac{\gamma}{\epsilon}$, where we always suppose $0 < \epsilon \leqslant 1 \leqslant \gamma$. On Ω_T with $z = (p, u)$ set

$$|z|_{k,\mu,\gamma} = \sum_{j=0}^{k} \mu^{k-j} |e^{-\gamma t} M^j z|_{L^2(\Omega_T)},$$
$$\langle z \rangle_{k,\mu,\gamma} = \sum_{j=0}^{k} \mu^{k-j} |e^{-\gamma t} M^j z|_{L^2(b\Omega_T)}. \tag{1.23}$$

3.) Set $\|z\|'_{k,\mu,\gamma} = |\sqrt{\gamma} z, \sqrt{\epsilon} \nabla u|_{k,\mu,\gamma} + \langle \sqrt{\epsilon} u_1 \rangle_{k,\mu,\gamma} + \langle \sqrt{\epsilon} u_{1y} \rangle_{k,\mu,\gamma}$.

4.) Set $\|z\|_{k,\mu,\gamma} = \|z\|'_{k,\mu,\gamma} + |\sqrt{\gamma} \partial_y z|_{k-2,\mu,\gamma} + |\partial_y \sqrt{\epsilon} \nabla u|_{k-2,\mu,\gamma}$.

5.) When there are d space dimensions, define Ω_T and the norms $\|z\|'_{k,\mu,\gamma}$ and $\|z\|_{k,\mu,\gamma}$ with the obvious changes: let $x \in \mathbb{R}^{d-1}$, $\alpha_1 \in \mathbb{N}^{d-1}$, and take $k = \alpha_0 + |\alpha_1| + \alpha_2$.

6.) Set $\|z\|^{**} := |z, Mz, M^2 z, \sqrt{\epsilon} \partial_y u, \sqrt{\epsilon} M \partial_y u|_{L^\infty(\Omega_T)}$.

The main result of the paper is the following theorem.

THEOREM 1.3. — *Let v^a be the approximate boundary layer solution of order M described in Proposition 2.3, but written in (p, u) coordinates, which satisfies* (1.13) *on* $[-\delta, T_0] \times \{(x, y) : y \geqslant 0\}$. *Assume the space dimension $d = 2$ or 3 and assume $L \geqslant 2$, $k \geqslant 5$, and $M - L - 2k > 0$. Suppose that s_0, the index measuring regularity of the leading slow profile $(\rho^{I,0}, u^{I,0})$, satisfies*

$$s_0 > k + 4 + 5M + \frac{d+1}{2}. \tag{1.24}$$

Then there exists a positive constant γ_0 and a positive decreasing function $\epsilon_0(\gamma)$ such that for $\gamma \geqslant \gamma_0$ and $0 < \epsilon \leqslant \epsilon_0(\gamma)$, the nonlinear forward error problem (1.15) *has a unique solution z on Ω_{T_0} satisfying the estimates*

$$(a) \|z\|^{**} \leqslant 1,$$
$$(b) \|z\|_{k,\mu,\gamma} \leqslant 2C(\gamma) \sqrt{\epsilon}^{M-L-2k} \leqslant 1, \tag{1.25}$$

with norms as defined in Notation 3.4 and $C(\gamma)$ as in (3.90).

2. If we take

$$v^\epsilon = v^a + \sqrt{\epsilon}^L z, \tag{1.26}$$

then v is an exact boundary layer solution of the nonlinear problem (1.8) *on Ω_{T_0}, the latter problem being equivalent to the original problem* (1.1).

As an immediate corollary of this theorem and the expansions (2.52), we can describe the small viscosity limit.

COROLLARY 1.4 (Inviscid limit). — *On* $\mathcal{O}_{T_0} := [0, T_0] \times \{(x, y) : y \geqslant 0\}$ *the exact solution* v^ϵ *given by* (1.26) *is related to the approximate boundary layer solution* v^a (2.52) *and to the inviscid solution* $v^0 := (\rho^{I,0}, u^{I,0})$ *(satisfying* (2.3),(2.11)*) as follows:*

$$(1.27) \qquad \begin{aligned} &(a)\ |v^\epsilon - v^a|_{L^\infty(\mathcal{O}_{T_0})} \leqslant C\epsilon^{\frac{L}{2}} \\ &(b)\ |v^\epsilon - v^0|_{L^\infty(\mathcal{O}_{T_0})} \leqslant C\epsilon^{\frac{1}{2}}, \end{aligned}$$

for some $C > 0$ *independent of* ϵ*. The estimate (a) also holds with the* L^∞ *norm replaced by the* $\|\cdot\|^{**}$ *norm (recall Notation 3.13).*

The body of the paper is organized as follows. In section 2.1 we give a formal description of the profile equations in a few representative cases where the slip length is taken to be a power of the viscosity, $\alpha^\epsilon = \epsilon^\delta$, $\delta > 0$. The results of [18] indicate that in many physical situations the slip length should depend on viscosity. This formal analysis indicates that, depending on the size of δ, the main contribution to the boundary layer will either appear in the leading term of the expansion (of amplitude $O(1)$) and satisfy nonlinear Prandtl-type equations ($\delta \geqslant \frac{1}{2}$), or will appear in a subsequent term (of amplitude $O(\epsilon^{\frac{1}{2}-\delta})$) and satisfy linearized Prandtl equations. The formal analysis helps to clarify the sense in which the problems obtained by taking slip length $\alpha^\epsilon = \epsilon^\delta$ for different choices of $\delta \geqslant 0$ "interpolate" between our problem and the classical no slip problem ($\alpha = 0$).

In section 2.2 we return to our main focus, the case where α is a constant independent of viscosity, and give a rigorous solution of the profile equations thereby producing the high-order approximate solutions of (1.1) described in Proposition 2.3. In section 3 we prove the stability of the boundary layer (Theorem 1.3) and show that the approximate solution is $O(\epsilon)$ close in L^∞ to an exact solution of (1.1) on the fixed time interval $[0, T_0] \times \{(x, y) : y \geqslant 0\}$ for $0 < \epsilon \leqslant \epsilon_0$, where ϵ_0 is sufficiently small. The small viscosity limit can then simply be read off from the expansion that gives the approximate solution (Corollary 1.4). Section 3.4 shows how a boundary layer can form as time evolves in an exact solution that does not initially possess a layer.

In section 4.1 of the Appendix we explain how a proof of the short-time existence of smooth solutions of the compressible Navier-Stokes equations with Navier boundary conditions and fixed viscosity ($\epsilon = 1$) can easily be extracted from the estimates of section 3. Here, as in the proof of Proposition 3.14, we are unable to estimate L^2 norms of higher ($k \geqslant 2$) normal derivatives of the pressure in the iteration scheme. The difficulties arise from unmanageable boundary terms that appear after integration by parts, and an associated loss of derivatives in the estimates (see Remark 4.3). Such a loss prevents closure of the iteration scheme for higher normal derivatives, but one can close the scheme to obtain a solution with C^0 bounds on p and C^1 bounds on u. Starting with a solution (p,u) having this regularity, one can use a bootstrapping argument to show that for sufficiently smooth and corner-compatible initial data, $(\rho,u)\in C^{m-1}\times C^m$ for any given m. This argument (see (4.29)) uses integration along characteristics of the velocity field, and takes advantage of the fact that the combination

$$p'(\rho)\partial_y\rho-(\lambda+\mu)\partial_y^2u_2 \tag{1.28}$$

is more regular than the individual terms. We note that in several of his papers (e.g., [8, 9]), David Hoff has used the better regularity of a similar combination, the "effective viscous flux" $p(\rho)-(\lambda+\mu)\text{div}u$, to solve other types of compressible flow problems.

Section 4.2 proves a result on linear, scalar Prandtl-type equations with Neumann boundary conditions needed for the construction of the fast profiles $u_1^{B,j}$ in section 2.2. Finally, in section 4.3 we formulate a few of the open questions that arise when attempting to solve the profile equations in the cases $\alpha^\epsilon=\epsilon$ and $\alpha^\epsilon=\epsilon^{\frac{1}{2}}$.

Remark 1.5. — The results of section 2.2 on approximate solutions hold for space dimensions $d\geqslant 2$, while the main stability results, Theorem 1.3 and Corollary 1.4, hold only for $d=2$ and $d=3$. Throughout the paper we consider $d=2$ for convenience, but identical proofs with only the obvious notational changes (e.g., replace x by $(x_1,\dots,x_{d-1})$, replace the scalar u_1 in (1.1) by $(u_{1j})|_{j=1,\dots,d-1}$, etc.) work in section 2.2 and in Proposition 3.10 for $d\geqslant 2$ and in Proposition 3.14 for $d=2$ and $d=3$.

2. Approximate boundary layer solutions

The main results of this paper are for the case when the slip length is a positive constant $\alpha>0$. We take that constant to be one, since this choice has no effect on the analysis. It will be clear from the proofs that with only minor changes in the analysis, we could just as well take α to be a smooth function $\alpha(t,x)$ satisfying $0<m\leqslant\alpha(t,x)\leqslant M$ for all (t,x).

First, we shall write down the profile equations that arise in the cases

$$\alpha^\epsilon=\epsilon^\delta,\ \text{where}\ \delta=1,\frac{1}{2},\ \text{or}\ 0. \tag{2.1}$$

This requires little extra work and will also help to clarify how the profile equations are affected by the slip length. A rigorous solution of the profile equations in the case $\delta = 0$ is given in section 2.2. Further discussion of the cases $\delta = 1, \frac{1}{2}$ appears in section 4.3.

2.1. Profile equations.

We take the following ansatz:

$$(2.2)\qquad \begin{aligned} \rho^\epsilon(t,x,y) &= \textstyle\sum_{j\geqslant 0} \epsilon^{\frac{j}{2}} (\rho^{I,j}(t,x,y) + \rho^{B,j}(t,x,\frac{y}{\sqrt{\epsilon}})) \\ u^\epsilon(t,x,y) &= \textstyle\sum_{j\geqslant 0} \epsilon^{\frac{j}{2}} (u^{I,j}(t,x,y) + u^{B,j}(t,x,\frac{y}{\sqrt{\epsilon}})) \end{aligned}$$

for the solutions of (1.1) with α^ϵ as in (2.1), where $\rho^{B,j}(t,x,\eta)$ and $u^{B,j}(t,x,\eta)$ are assumed to decay rapidly to 0 as $\eta \to +\infty$. Plugging (2.2) into the equations in (1.1), setting terms of order ϵ^0 equal to zero, and letting $\eta \to +\infty$, one finds that the leading profiles $(\rho^{I,0}, u^{I,0})$ of outer flow should satisfy the compressible Euler equations in $y \geqslant 0$

$$(2.3)\qquad \begin{aligned} &\partial_t \rho^{I,0} + \nabla\cdot(\rho^{I,0}u^{I,0}) = 0 \\ &\partial_t(\rho^{I,0}u^{I,0}) + \nabla\cdot(\rho^{I,0}u^{I,0}\otimes u^{I,0}) + \nabla p(\rho^{I,0}) = 0. \end{aligned}$$

Similarly, for $j \geqslant 1$ one obtains that $(\rho^{I,j}, u^{I,j})$ should satisfy the following linearized Euler equations:

(2.4)
$$\begin{aligned} &\partial_t \rho^{I,j} + \nabla\cdot(\rho^{I,j}u^{I,0} + \rho^{I,0}u^{I,j}) = -\sum_{1\leqslant k\leqslant j-1} \nabla\cdot(\rho^{I,k}u^{I,j-k}) \\ &\partial_t u^{I,j} + (u^{I,0}\cdot\nabla)u^{I,j} + (u^{I,j}\cdot\nabla)u^{I,0} + \tfrac{1}{\rho^{I,0}}\nabla(p'(\rho^{I,0})\rho^{I,j}) - \tfrac{\rho^{I,j}}{(\rho^{I,0})^2}\nabla p(\rho^{I,0}) \\ &\quad = -\sum_{1\leqslant k\leqslant j-1}(u^{I,k}\cdot\nabla)u^{I,j-k} + f_j(\{\rho^{I,k}, \nabla\rho^{I,k}, \nabla^2 u^{I,k-2}\}_{k\leqslant j-1}) + \\ &\qquad \tfrac{1}{\rho^{I,0}}\left(\lambda\Delta u^{I,j-2} + \mu\mathrm{div}(\nabla u^{I,j-2})\right), \end{aligned}$$

where terms with negative superscipts are zero. For the moment we ignore initial conditions and the issue of corner compatibility. We treat those matters carefully in section (2.2) where we solve the profile equations in the case $\alpha = 1$.

In the discussion below we shall denote by $\overline{u}(t,x)$ the trace of a function $u(t,x,y)$ on the boundary $\{y=0\}$. Setting terms of order $O(\epsilon^{-\frac{1}{2}})$ equal to zero, we get

$$(2.5)\qquad \begin{pmatrix} \overline{u_2^{I,0}} + u_2^{B,0} & \overline{\rho^{I,0}} + \rho^{B,0} \\ \frac{p'(\overline{\rho^{I,0}}+\rho^{B,0})}{\overline{\rho^{I,0}}+\rho^{B,0}} & \overline{u_2^{I,0}} + u_2^{B,0} \end{pmatrix} \begin{pmatrix} \partial_\eta \rho^{B,0} \\ \partial_\eta u_2^{B,0} \end{pmatrix} = 0$$

and

$$(\overline{u_2^{I,0}} + u_2^{B,0})\partial_\eta u_1^{B,0} = 0. \tag{2.6}$$

Assume for now the existence of smooth functions $\rho^{I,0}$, $u_2^{I,0}$, $\rho^{B,0}$, $u_2^{B,0}$ satisfying (2.5), (2.6), and

$$\overline{\rho^{I,0}} + \rho^{B,0} > 0. \tag{2.7}$$

The boundary condition given in (1.1) implies

$$\overline{u_2^{I,0}} + u_2^{B,0} = 0 \qquad \text{on } \eta = 0, \tag{2.8}$$

so on $\{\eta = 0\}$ the determinant of the coefficient matrix in (2.5) equals $-p'(\overline{\rho^{I,0}} + \rho^{B,0}) \neq 0$. By continuity there is an $\eta_0 > 0$ such that for $0 \leqslant \eta \leqslant \eta_0$ the coefficient matrix is nonsingular. Thus, from (2.5) we have

$$\partial_\eta \rho^{B,0} = 0, \quad \partial_\eta u_2^{B,0} = 0$$

for $0 \leqslant \eta \leqslant \eta_0$, which implies

$$(\rho^{B,0}, u_2^{B,0})(t,x,\eta) \equiv (\rho^{B,0}, u_2^{B,0})(t,x,0) \tag{2.9}$$

for $0 \leqslant \eta \leqslant \eta_0$. Thus, the coefficient matrix in (2.5) at $\eta = \eta_0$ is the same as at $\eta = 0$. By continuous induction one deduces that the identity (2.9) holds for all $\eta \geqslant 0$. Thus, we get

$$(\rho^{B,0}, u_2^{B,0})(t,x,\eta) \equiv 0. \tag{2.10}$$

From (2.8) we observe that

$$u_2^{I,0}(t,x,0) = 0, \tag{2.11}$$

so the leading slow profiles $(\rho^{I,0}, u^{I,0})$ satisfy the compressible Euler equations (2.3) with the impermeability boundary condition (2.11) and initial conditions to be described later. The validity of the identity (2.6) for any $u_1^{B,0}$ follows immediately from (2.10) and (2.11).

Setting equal to zero the terms of order $O(\epsilon^0)$ in the equations (1.1), we find

$$\overline{\rho^{I,0}}\partial_\eta u_2^{B,1} + \partial_x(\overline{\rho^{I,0}} u_1^{B,0}) = 0 \tag{2.12}$$

$$\begin{aligned}\partial_t u_1^{B,0} &+ (\overline{u_1^{I,0}} + u_1^{B,0})\partial_x u_1^{B,0} \\ &+ (\overline{u_2^{I,1}} + u_2^{B,1} + \eta\overline{\partial_y u_2^{I,0}})\partial_\eta u_1^{B,0} + \overline{\partial_x u_1^{I,0}} u_1^{B,0} = \frac{\lambda}{\overline{\rho^{I,0}}}\partial_\eta^2 u_1^{B,0}\end{aligned} \tag{2.13}$$

and

$$u_1^{B,0}\overline{\partial_x u_2^{I,0}} + \overline{u_2^{I,0}}\partial_\eta u_2^{B,1} + \frac{p'(\overline{\rho^{I,0}})}{\overline{\rho^{I,0}}}\partial_\eta \rho^{B,1} = 0 \tag{2.14}$$

From (2.14) and (2.11) we immediately get

$$\rho^{B,1}(t,x,\eta) \equiv 0. \tag{2.15}$$

The system (2.12), (2.13) can be viewed as a nonlinear system of Prandtl-type equations for the unknowns $(u_1^{B,0}, \widetilde{u_2^{B,1}})$, where

$$\widetilde{u_2^{B,1}}(t,x,\eta) = u_2^{I,1}(t,x,0) + u_2^{B,1}(t,x,\eta). \tag{2.16}$$

Provided these equations can be solved (after adding appropriate initial and boundary conditions), one can obtain $\overline{u_2^{I,1}}$ as a limit

$$\overline{u_2^{I,1}} := \lim_{\eta\to\infty} \widetilde{u_2^{B,1}}(t,x,\eta) \tag{2.17}$$

and then

$$u_2^{B,1}(t,x,\eta) = \widetilde{u_2^{B,1}}(t,x,\eta) - \overline{u_2^{I,1}}(t,x). \tag{2.18}$$

This gives the boundary condition

$$u_2^{I,1}|_{y=0} = \overline{u_2^{I,1}} \tag{2.19}$$

for the linearized Euler system (2.4) for $(\rho^{I,1}, u^{I,1})$.

Similarly, vanishing of the terms of order $O(\epsilon^{\frac{1}{2}})$ in the equations (1.1) implies that $(u_1^{B,1}, \widetilde{u_2^{B,2}})(t,x,\eta)$ satisfy the following linearized Prandtl equations:

(2.20)
$$\begin{cases} \partial_t u_1^{B,1} + [(\overline{u_1^{I,0}} + u_1^{B,0})\partial_x + (\overline{u_2^{I,1}} + u_2^{B,1} + \eta\overline{\partial_y u_2^{I,0}})\partial_\eta \\ \qquad\qquad + \overline{\partial_x u_1^{I,0}}]u_1^{B,1} + \widetilde{u_2^{B,2}}\partial_\eta u_1^{B,0} - \frac{\lambda}{\overline{\rho^{I,0}}}\partial_\eta^2 u_1^{B,1} \\ \qquad = -(\eta\overline{\partial_y u_2^{I,1}} + \frac{\eta^2}{2}\overline{\partial_y^2 u_2^{I,0}})\partial_\eta u_1^{B,0} - \overline{\partial_y u_1^{I,0}}u_2^{B,1} - \frac{\lambda\overline{\rho^{I,1}}}{(\overline{\rho^{I,0}})^2}\partial_\eta^2 u_1^{B,0} \\ \partial_x(\overline{\rho^{I,0}}u_1^{B,1}) + \partial_\eta(\overline{\rho^{I,0}}\widetilde{u_2^{B,2}}) = -\overline{\rho^{I,1}}\partial_\eta u_2^{B,1} - \overline{\partial_y\rho^{I,0}}u_2^{B,1} - \partial_x(\overline{\rho^{I,1}}u_1^{B,0}), \end{cases}$$

where $\widetilde{u_2^{B,2}}(t,x,\eta) = u_2^{I,2}(t,x,0) + u_2^{B,2}(t,x,\eta)$, and

(2.21)
$$\frac{\overline{p'(\rho^{I,0})}}{\overline{\rho^{I,0}}}\partial_\eta\rho^{B,2} = -[\partial_t + (\overline{u_1^{I,0}} + u_1^{B,0})\partial_x + (\overline{u_2^{I,1}} + u_2^{B,1} + \eta\overline{\partial_y u_2^{I,0}})\partial_\eta + \overline{\partial_y u_2^{I,0}} - \frac{\lambda+\mu}{\overline{\rho^{I,0}}}\partial_\eta^2]u_2^{B,1} - u_1^{B,0}\overline{\partial_x u_2^{I,1}}$$

Once $(u_1^{B,0}, u_2^{B,1}, u_2^{I,1})$ is determined, the equation (2.21) together with the fast decay condition on $\rho^{B,2}(t,x,\eta)$ determines $\rho^{B,2}$ uniquely.

This concludes the part of the discussion that is common to the cases $\gamma = 1$, $\frac{1}{2}$, and 0. To a large extent it parallels the description of profile equations in the incompressible case given in Wang-Wang-Xin [21]. To proceed further we must look more closely at the boundary conditions that must be imposed on the systems (2.12)-(2.13) and (2.20). From the boundary conditions in (1.1) we have

$$u_2^{I,j} + u_2^{B,j} = 0 \qquad \text{on} \quad \{y = \eta = 0\} \tag{2.22}$$

for all $j \geqslant 1$, and

$$\begin{aligned}\sum_{j\geqslant 0} \epsilon^{\frac{j}{2}} (u_1^{I,j} + u_1^{B,j}) \\ = \alpha^{\epsilon}\{\epsilon^{-\frac{1}{2}}\partial_\eta u_1^{B,0} + \sum_{j\geqslant 0} \epsilon^{\frac{j}{2}} (\partial_y u_1^{I,j} + \partial_\eta u_1^{B,j+1})\} \text{ on } \{y = \eta = 0\}.\end{aligned} \tag{2.23}$$

2.2. Construction of an approximate solution when $\alpha = 1$.

1. Determination of $(\rho^{B,0}$**,** $u_2^{B,0}$**,** $\overline{u_2^{I,0}})$**.** From (2.10) and (2.11)we have

$$\rho^{B,0} = 0,\ \rho^{B,1} = 0,\ u_2^{B,0} = 0,\ \overline{u_2^{I,0}} = 0. \tag{2.24}$$

2. Determination of $(\rho^{I,0}$**,** $u^{I,0})$**.** The leading slow profiles $\rho^{I,0}$, $u^{I,0}$ are determined by solving the following mixed problem for the Euler equations (2.3) on $[-T_0, T_0] \times \{(x, y) : y \geqslant 0\}$ for some $T_0 > 0$:

$$\begin{aligned}&\mathbb{E}(\rho^{I,0}, u^{I,0}) = 0 \\ &u_2^{I,0}|_{y=0} = 0 \\ &(\rho^{I,0}, u^{I,0})|_{t=-T_0} = (\rho_0^{I,0}, u_0^{I,0}),\end{aligned} \tag{2.25}$$

where $\mathbb{E}$ is defined by (2.3) and the initial data $(\rho_0^{I,0}, u_0^{I,0})$ are chosen to satisfy compatibility conditions at the corner $\{t = -T_0, y = 0\}$ of order $s_0 + 1$ for s_0 sufficiently large to be specified later. In addition we require that there exist positive constants $\rho_1 \leqslant \rho_2$ such that

$$\begin{aligned}&\rho_0^{I,0}(x, y) \geqslant \rho_1 \text{ on } y \geqslant 0 \\ &(\rho_0^{I,0} - \rho_2, u_0^{I,0}) \in H^{s_0+2}(y \geqslant 0).\end{aligned} \tag{2.26}$$

Corner compatibility conditions of order s_0+1 can be simply characterized by the property that if $v = (\rho^{I,0}, u^{I,0})$ is the solution on some finite time interval to the pure initial value problem

$$\begin{aligned}&\mathbb{E}(v)=0\\&v|_{t=-T_0}=(\rho_0^{I,0},u_0^{I,0})_e,\end{aligned}\tag{2.27}$$

where $(\rho_0^{I,0},u_0^{I,0})_e$ denotes an H^{s_0+2} extension of $(\rho_0^{I,0}-\rho_2,u_0^{I,0})$ to $y\leqslant 0$, then the function defined by

$$g(t,x)=\begin{cases}u_2^{I,0}|_{y=0},\ t\geqslant -T_0\\0,\ t\leqslant -T_0\end{cases}\tag{2.28}$$

belongs to $H^{s_0+1}(y=0)$.

In order to state an existence result for the system (2.25), we first introduce the following conormal spaces with respect to $y=0$:

Notation 2.1. — 1. For $T>0$ let $\mathcal{O}_T=\{(t,x,y):-T\leqslant t\leqslant T,y\geqslant 0\}$.

2. For $m\in\{0,1,2,...\}$ and multi-indices $\beta=(\beta_0,\beta_1,\beta_2)$ define $H^{0,m}=\{u\in L^2(\mathcal{O}_T):\partial_t^{\beta_0}\partial_x^{\beta_1}(y\partial_y)^{\beta_2}u\in L^2(\mathcal{O}_T)\text{ for }|\beta|\leqslant m\}$.

3. For $m\in\{2,3,4,...\}$ set $E^m(\mathcal{O}_T)=\{u\in H^{0,m}(\mathcal{O}_T):\partial_y^k u\in H^{0,m-2k}(\mathcal{O}_T)$ for $0\leqslant 2k\leqslant m\}$.

Assuming $s_0>\frac{d}{2}+5$, s_0 is even, and that the initial data satisfies compatibility conditions of order s_0+1, by a result of Guès ([5], Theorem 2) the problem (2.25) has a unique solution

$$(\rho^{I,0}-\rho_2,u^{I,0})\in E^{s_0}(\mathcal{O}_{T_0}),\ \text{for some } T_0>0.\tag{2.29}$$

Here we have used the readily verifiable fact that the initial-boundary value problem for the Euler system (2.25) satisfies the symmetrizability, constant multiplicity, maximal dissipativity, and involutivity hypotheses $\mathcal{H}_1,\dots,\mathcal{H}_4$, of [5], Theorem 2.

3. Determination of $\rho^{B,1}$. From (2.15) we have $\rho^{B,1}=0$.

4. Determination of $(u_1^{B,0}, u_2^{B,1}, \overline{u_2^{I,1}})$. From the boundary condition (2.23) with $\alpha^\epsilon=1$, we obtain

$$\begin{aligned}&\partial_\eta u_1^{B,0}|_{\eta=0}=0\\&\partial_\eta u_1^{B,j}=(u_1^{I,j-1}+u_1^{B,j-1})-\partial_y u_1^{I,j-1}\quad\text{on }\eta=y=0,\ \forall j\geqslant 1.\end{aligned}\tag{2.30}$$

Set $\widetilde{u_2^{B,1}}(t,x,\eta)=u_2^{I,1}(t,x,0)+u_2^{B,1}(t,x,\eta)$. Then from (2.12), (2.13), (2.22) and (2.30) we know that $(u_1^{B,0},\widetilde{u_2^{B,1}})$ satisfy the following problem on $\{(t,x,\eta) : t \in [-T_0, T_0], \eta \geqslant 0\}$.

(2.31)
$$\begin{aligned}
&\partial_t u_1^{B,0} + (\overline{u_1^{I,0}} + u_1^{B,0})\partial_x u_1^{B,0} + (\widetilde{u_2^{B,1}} + \eta\overline{\partial_y u_2^{I,0}})\partial_\eta u_1^{B,0} \\
&\qquad\qquad\qquad\qquad + \overline{\partial_x u_1^{I,0}} u_1^{B,0} = \tfrac{\lambda}{\overline{\rho^{I,0}}}\partial_\eta^2 u_1^{B,0} \\
&\partial_x(\overline{\rho^{I,0}} u_1^{B,0}) + \partial_\eta(\overline{\rho^{I,0}}\widetilde{u_2^{B,1}}) = 0 \\
&\partial_\eta u_1^{B,0} = 0, \quad \widetilde{u_2^{B,1}} = 0 \quad \text{on } \eta = 0.
\end{aligned}$$

Adding the initial condition that

$$u_1^{B,0} = 0 \text{ on } t = -T_0, \tag{2.32}$$

we obtain a solution by inspection, namely, by taking

$$u_1^{B,0} = 0, u_2^{B,1} = 0, \text{ and } \overline{u_2^{I,1}} = 0. \tag{2.33}$$

5. Determination of $(\rho^{I,1}, u^{I,1})$**.** These functions are chosen to satisfy a forward mixed problem for the linearized Euler equations on $\mathcal{O}_{T_0}$:

$$\begin{aligned}
&\mathbb{E}_L(\rho^{I,1}, u^{I,1}) = \chi(t)F \\
&u_2^{I,1}|_{y=0} = 0 \\
&(\rho^{I,1}, u^{I,1}) = 0 \text{ in } t \leqslant -\frac{T_0}{2},
\end{aligned} \tag{2.34}$$

where $\mathbb{E}_L$ is the linearized Euler operator defined by the left sides of (2.4) when $j = 1$, $F \in E^{s_0-2}(\mathcal{O}_T)$ is the known forcing term given by the right side of (2.4) when $j = 1$, and $\chi(t)$ is a smooth cutoff function such that

$$\chi(t) = \begin{cases} 1, & t \in [-\frac{T_0}{3}, T_0] \\ 0, & t \leqslant -\frac{T_0}{2} \end{cases} . \tag{2.35}$$

We can now apply another theorem of Guès ([5], Theorem III.2.1) to conclude that (2.34) has a unique solution

$$(\rho^{I,1}, u^{I,1}) \in E^{s_0-2}(\mathcal{O}_{T_0}). \tag{2.36}$$

Observe that $(\rho^{I,1}, u^{I,1})$ is an exact solution of

$$\begin{aligned}
&\mathbb{E}_L(\rho^{I,1}, u^{I,1}) = F \\
&u_2^{I,1}|_{y=0} = 0
\end{aligned} \tag{2.37}$$

on $[-\frac{T_0}{3}, T_0] \times \{(x,y) : y \geqslant 0\}$.

6. Determination of $\rho^{B,2}$**.** From (2.33) and (2.21) we deduce

$$\partial_\eta \rho^{B,2}(t,x,\eta) = 0, \tag{2.38}$$

so the fast decay requirement implies

$$\rho^{B,2}(t,x,\eta)=0. \tag{2.39}$$

7. Determination of $(u_1^{B,1},u_2^{B,2},\overline{u_2^{I,2}})$**.** From (2.20) we see that $(u_1^{B,1},u_2^{B,2})$ satisfy the linearized Prandtl equations:

(2.40)

$$(a)\ \partial_t u_1^{B,1}+\overline{u_1^{I,0}}\partial_x u_1^{B,1}+(\eta\overline{\partial_y u_2^{I,0}})\partial_\eta u_1^{B,1}+\overline{\partial_x u_1^{I,0}}u_1^{B,1}-\frac{\lambda}{\overline{\rho^{I,0}}}\partial_\eta^2 u_1^{B,1}=0$$

$$(b)\ \partial_x(\overline{\rho^{I,0}}u_1^{B,1})+\partial_\eta(\overline{\rho^{I,0}}u_2^{B,2})=0.$$

The boundary conditions for (2.40) coming from (2.30) are:

$$\partial_\eta u_1^{B,1}|_{\eta=0}=(u_1^{I,0}-\partial_y u_1^{I,0})(t,x,0) \tag{2.41}$$

Observe that the problem for $u_1^{B,I}$ is, as a consequence of our taking $\alpha=1$, *decoupled* from the problem for $u_2^{B,2}$. Since we seek solutions rapidly decaying to 0, once $u_1^{B,I}$ is determined we can simply take

$$u_2^{B,2}(t,x,\eta)=\frac{1}{\overline{\rho^{I,0}}}\int_\eta^{+\infty}\partial_x(\overline{\rho^{I,0}}u_1^{B,1})(t,x,s)ds \tag{2.42}$$

and, as required by (2.22),

$$\overline{u_2^{I,2}}:=-u_2^{B,2}(t,x,0). \tag{2.43}$$

In the initial boundary value problem for $u_1^{B,1}$ we arrange compatibility conditions by again using the cutoff χ (2.35). Letting $\mathbb{P}_L$ denote the operator defined by the left side of (2.40)(a), we obtain $u_1^{B,1}$ as the solution of

$$\begin{aligned}&\mathbb{P}_L(u_1^{B,1})=0\\&\partial_\eta u_1^{B,1}|_{\eta=0}=\chi(t)(u_1^{I,0}-\partial_y u_1^{I,0})(t,x,0)\\&u^{B,1}=0\text{ in }t\leqslant-\frac{T_0}{2}\end{aligned} \tag{2.44}$$

on $[-T_0,T_0]\times\{(x,\eta):\eta\geqslant0\}$. For this we employ Proposition 4.7 of section 4.2, which is a slightly modified version of a result of Xin-Yanagisawa [22]. To describe the regularity of solutions we need some notation:

Notation 2.2. — 1. Let $\mathbb{O}_T:=\{(t,x,\eta):t\in[-T,T],\eta\geqslant0\}$.

2. For $m\in\{0,1,2,\ldots\}$ we say that $u(t,x,\eta)\in P^m(\mathbb{O}_T)$ provided

(2.45)

$$\langle\eta\rangle^l u\in C^k([-T,T];H^{m-k}(\mathbb{R}^2_+))\text{ for }k=0,\ldots,m\text{ and for all }l\in\mathbb{N},$$

where $\langle\eta\rangle=(1+\eta^2)^{\frac12}$.

Since $u_1^{I,0} \in E^{s_0}(\mathcal{O}_{T_0})$ we have

$$\chi(t)(u_1^{I,0} - \partial_y u_1^{I,0})(t,x,0) \in H^{s_0-4}(y=0). \tag{2.46}$$

Hence, it follows from Proposition 4.7 that the solution $u_1^{B,1}$ of (2.44) satisfies

$$u_1^{B,1} \in P^{s_0-7}(\mathbb{O}_{T_0}), \tag{2.47}$$

provided s_0 is large enough (as specified later). From (2.42) we obtain $u_2^{B,2} \in P^{s_0-8}(\mathbb{O}_{T_0})$. We note finally that $(u_1^{B,1}, u_2^{B,2})$ satisfies the equations (2.40) and the boundary condition (2.41) exactly on $[-\frac{T_0}{3}, T_0] \times \{(x,\eta) : \eta \geqslant 0\}$.

8. Determination of $(\rho^{I,2}, u^{I,2})$. From (2.47), (2.42), and (2.43) we determine that

$$\overline{u_2^{I,2}} \in H^{s_0-8}(y=0). \tag{2.48}$$

The slow profiles $(\rho^{I,2}, u^{I,2})$ are chosen to satisfy a problem like (2.34) arising from (2.4) with $j=2$, but now with boundary data

$$u_2^{I,2}|_{y=0} = \chi(t)\overline{u_2^{I,2}} \in H^{s_0-8}(y=0). \tag{2.49}$$

Applying Theorem III.2.1 of Guès [5] again, we obtain

$$(\rho^{I,2}, u^{I,2}) \in E^{s_0-10}(\mathcal{O}_{T_0}). \tag{2.50}$$

Here some regularity is lost in reducing to the case of zero boundary data to which Guès's theorem applies.

9. Regularity of subsequent profiles. Using (2.21) and the results of paragraphs **7,8** we find $\rho^{B,3} \in P^{s_0-9}(\mathbb{O}_{T_0})$. Subsequent slow and fast profiles satisfy linearized Euler and Prandtl systems just like (2.34) and (2.44) with interior and boundary data depending on previously determined profiles. Continuing according to the above pattern we obtain successive profiles whose regularity we now summarize:

(2.51)

$$(\rho^{I,j}, u^{I,j}) \in E^{s_0-10(\frac{j}{2})}(\mathcal{O}_{T_0}), j \in \{0,2,\dots\}$$

$$(\rho^{I,j}, u^{I,j}) \in E^{s_0-2-10(\frac{j-1}{2})}(\mathcal{O}_{T_0}), j \in \{1,3,5,\dots\}$$

$$u_1^{B,j} \in P^{s_0-7-10(\frac{j-1}{2})}(\mathbb{O}_{T_0}), j \in \{1,3,\dots\};\ u_1^{B,j} \in P^{s_0-9-10(\frac{j-2}{2})}(\mathbb{O}_{T_0}), j \in \{2,4,\dots\}$$

$$u_2^{B,j} \in P^{s_0-8-10(\frac{j-2}{2})}(\mathbb{O}_{T_0}), j \in \{2,4,\dots\};\ u_2^{B,j} \in P^{s_0-10-10(\frac{j-3}{2})}(\mathbb{O}_{T_0}), j \in \{3,5,\dots\}$$

$$\rho^{B,j} \in P^{s_0-9-10(\frac{j-3}{2})}(\mathbb{O}_{T_0}), j \in \{3,5,\dots\};\ \rho^{B,j} \in P^{s_0-11-10(\frac{j-4}{2})}(\mathbb{O}_{T_0}), j \in \{4,6,\dots\}.$$

10. The approximate solution. For a fixed integer $M \geqslant 1$, let $w^a = (\rho^{\epsilon,M}, u^{\epsilon,M})$, where

$$(2.52)\qquad \begin{aligned} \rho^{\epsilon,M}(t,x,y) &= \textstyle\sum_{k=0}^{M} \epsilon^{\frac{k}{2}} \rho^{I,k}(t,x,y) + \sum_{k=3}^{M} \epsilon^{\frac{k}{2}} \rho^{B,k}(t,x,\frac{y}{\sqrt{\epsilon}}) \\ u_1^{\epsilon,M}(t,x,y) &= \textstyle\sum_{k=0}^{M} \epsilon^{\frac{k}{2}} u_1^{I,k}(t,x,y) + \sum_{k=1}^{M} \epsilon^{\frac{k}{2}} u_1^{B,k}(t,x,\frac{y}{\sqrt{\epsilon}}) \\ u_2^{\epsilon,M}(t,x,y) &= \textstyle\sum_{k=0}^{M} \epsilon^{\frac{k}{2}} u_2^{I,k}(t,x,y) + \sum_{k=2}^{M} \epsilon^{\frac{k}{2}} u_2^{B,k}(t,x,\frac{y}{\sqrt{\epsilon}}). \end{aligned}$$

Then by the above construction $(\rho^{\epsilon,M}, u^{\epsilon,M})$ satisfy
(2.53)

$$\begin{aligned} &\partial_t \rho^{\epsilon,M} + \nabla\cdot(\rho^{\epsilon,M} u^{\epsilon,M}) = f_\rho^{\epsilon,M} \\ &\partial_t u^{\epsilon,M} + (u^{\epsilon,M}\cdot\nabla)u^{\epsilon,M} + \tfrac{1}{\rho^{\epsilon,M}}\nabla p(\rho^{\epsilon,M}) \\ &\qquad\qquad - \tfrac{\epsilon}{\rho^{\epsilon,M}}\left(\lambda\Delta u^{\epsilon,M} + \mu \mathrm{div}(\nabla u^{\epsilon,M})\right) = f_u^{\epsilon,M} \\ &u_2^{\epsilon,M} = 0, \quad u_1^{\epsilon,M} - \tfrac{\partial u_1^{\epsilon,M}}{\partial y} = g^{\epsilon,M}, \quad \text{on} \quad y=0 \end{aligned}$$

on $[-\frac{T_0}{3}, T_0] \times \{(x,y) : y \geqslant 0\}$ with errors

$$(2.54)\qquad (f_\rho^{\epsilon,M}, f_u^{\epsilon,M}) = \epsilon^{\frac{M}{2}}(R_\rho^M, R_u^M) = \epsilon^{\frac{M}{2}} R^M, \quad g^{\epsilon,M} = \epsilon^{\frac{M}{2}} h^{\epsilon,M}$$

described below. It is convenient to remove the boundary forcing term $g^{\epsilon,M}$ in (2.53). This can be done with minimal effect on the approximate solution by adding to $u_1^{\epsilon,M}(t,x,y)$ a term of the form

$$(2.55)\qquad u_{1,c}^{\epsilon}(t,x,y) := \epsilon^{\frac{M}{2}} \phi(y) h^{\epsilon,M}(t,x),$$

where $\phi \in C^\infty(y \geqslant 0)$ with compact support, $\phi(0) = 0$, $\phi'(0) = 1$. So henceforth we take our approximate solution to satisfy the boundary condition

$$(2.56)\qquad u_2^{\epsilon,M} = 0, \quad u_1^{\epsilon,M} - \frac{\partial u_1^{\epsilon,M}}{\partial y} = 0 \text{ on } y = 0.$$

With this change we relabel $w^a = (\rho^a, u^a)$ and summarize the properties of w^a as follows:

Proposition 2.3 (Approximate solutions). — *Let $d \geqslant 2$ be the space dimension. For given integers $k \geqslant 2$, $M \geqslant 1$ let*

$$(2.57)\qquad s_0 > k + 4 + 5M + \frac{d+1}{2},$$

where s_0, taken even for convenience, determines the regularity of the initial data at $t = -T_0$ for $(\rho^{I,0}, u^{I,0})$:

(2.58)

$$(\rho_0^{I,0} - \rho_2, u_0^{I,0}) \in H^{s_0+2}(\{(x,y) : y \geqslant 0)\}) \text{ and satisfies compatibility conditions of order } s_0 + 1;$$

$$\rho_0^{I,0}(x,y) \geqslant \rho_1 \text{ on } y \geqslant 0 \text{ for some } 0 < \rho_1 \leqslant \rho_2.$$

Then the approximate solution w^a defined by the expansion (2.52) *satisfies* (2.53) *on $[-\frac{T_0}{3}, T_0] \times \{(x,y) : y \geqslant 0\}$ with $g^{\epsilon,M} = 0$, where $[-T_0, T_0]$ is the interval of existence of $(\rho^{I,0}, u^{I,0})$. The errors $\epsilon^{\frac{M}{2}} R^M$* (2.54) *satisfy*

$$\begin{aligned} &|\partial_t^{\alpha_0}\partial_x^{\alpha_1}(y\partial_y)^{\alpha_2} R^M|_{L^2(\mathcal{O}_{T_0})} < \infty \text{ for } |\alpha| \leqslant k \\ &|\partial_t^{\alpha_0}\partial_x^{\alpha_1}(y\partial_y)^{\alpha_2}\partial_y R^M|_{L^2(\mathcal{O}_{T_0})} < \infty \text{ for } |\alpha| \leqslant k-2. \end{aligned} \tag{2.59}$$

The profiles appearing in the expansion have the regularity summarized in (2.51)*. Moreover we have*

$$\begin{aligned} &|\partial_t^{\alpha_0}\partial_x^{\alpha_1}(y\partial_y)^{\alpha_2} w^a|_{L^\infty(\mathcal{O}_{T_0})} < \infty \text{ for } |\alpha| \leqslant k \\ &|\partial_t^{\alpha_0}\partial_x^{\alpha_1}(y\partial_y)^{\alpha_2}\partial_y w^a|_{L^\infty(\mathcal{O}_{T_0})} < \infty \text{ for } |\alpha| \leqslant k-2. \end{aligned} \tag{2.60}$$

The estimates (2.59), (2.60) are a direct consequence of (2.51) and Sobolev embedding.

3. Exact boundary layer solutions when the slip length $\alpha = 1$

This section is devoted to the proof of the main stability result, Theorem 1.3. Let us denote the left side of equation (1.16) by $\mathcal{L}(z^n)z^{n+1}$. For later use in estimating $\partial_y z^{n+1}$ and its tangential derivatives, we introduce the modified problem for $z^{n+1,\eta} = (p^{n+1,\eta}, u_1^{n+1,\eta}, u_2^{n+1,\eta})$:

(3.1)

$$(a)\ \mathcal{L}(z^n)z^{n+1,\eta} - \begin{pmatrix} \eta\Delta p^{n+1,\eta} \\ 0 \\ 0 \end{pmatrix} = -\sqrt{\epsilon}^{M-L} R^M \text{ in } y \geqslant 0, \text{ where } \eta \in (0,1],$$

$$(b)\ \partial_y p^{n+1,\eta} = 0,\ u_1^{n+1,\eta} - \partial_y u_1^{n+1,\eta} = 0,\ u_2^{n+1,\eta} = 0 \text{ on } y = 0,$$

$$(c)\ z^{n+1,\eta} = 0 \text{ in } t \leqslant -\frac{\delta}{2}.$$

Here the z^n appearing in the coefficients satisfies the scheme (1.16) and Navier boundary conditions (1.17).

3.1. L^2 and tangential derivative estimates.

We begin by proving L^2 a priori estimates for smooth solutions $z = (p, u)$ of

$$\begin{aligned} &\mathcal{L}(z^n)z = f \\ &u_1 - u_{1y} = g,\ u_2 = 0 \text{ on } y = 0 \\ &z = 0,\ g = 0, \text{ and } f = 0 \text{ in } t \leqslant -\frac{\delta}{2}. \end{aligned} \tag{3.2}$$

and for smooth solutions $z^\eta = (p^\eta, u^\eta)$ of

$$\begin{aligned} &\mathcal{L}(z^n)z^\eta - \begin{pmatrix} \eta\Delta p^\eta \\ 0 \\ 0 \end{pmatrix} = f \\ &p^\eta_y = 0,\ u^\eta_1 - u^\eta_{1y} = g,\ u^\eta_2 = 0 \text{ on } y = 0 \\ &z^\eta = 0,\ g = 0, \text{ and } f = 0 \text{ in } t \leqslant -\frac{\delta}{2}. \end{aligned} \tag{3.3}$$

The estimates are somewhat unusual in that they do not require boundedness of the Lipschitz norm of $\sqrt{\epsilon}^L z^n$, even though the matrix coefficients in $\mathcal{L}(z^n)$ are functions of $v^a + \sqrt{\epsilon}^L z^n$. The estimate makes use of the vanishing of u_2^a and u_2^n at $y = 0$.

Notation 3.1. — 1. For $T > 0$ let $\Omega_T = \{(t, x, y) : y \geqslant 0, -\infty < t \leqslant T\}$ and set $b\Omega_T = \Omega_T \cap \{y = 0\}$.

2. Let (f, g) denote the L^2 pairing on Ω_T and $\langle f, g\rangle$ the L^2 pairing on $b\Omega_T$.

3. Set $|f| = |f|_{L^2(\Omega_T)}$, $\langle g\rangle = |g|_{L^2(b\Omega_T)}$, $|f|_* = |f|_{L^\infty(\Omega_T)}$.

4. Let $\nabla f = (\partial_x f, \partial_y f)$.

5. For $\gamma > 0$ let $\mathcal{L}^\gamma(z^n) = e^{-\gamma t}\mathcal{L}(z^n)e^{\gamma t}$ and $z^\gamma = e^{-\gamma t}z$.

Observe that since

$$\mathcal{L}^\gamma(z^n)z^\gamma = e^{-\gamma t}\mathcal{L}(z^n)z = \mathcal{L}(z^n)z^\gamma + \gamma A_0(v^a + \sqrt{\epsilon}^L z^n)z^\gamma, \tag{3.4}$$

z satisfies (3.2) if and only if z^γ satisfies

$$\begin{aligned} &(a)\ \mathcal{L}^\gamma(z^n)z^\gamma = f^\gamma \\ &(b)\ u^\gamma_1 - u^\gamma_{1y} = g^\gamma,\ u^\gamma_2 = 0 \text{ on } y = 0 \\ &(c)\ z^\gamma = 0,\ g^\gamma = 0, \text{ and } f^\gamma = 0 \text{ in } t \leqslant -\frac{\delta}{2}. \end{aligned} \tag{3.5}$$

PROPOSITION 3.2 (L^2 estimate). — *(a) Suppose there exists $\epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0]$:*

$$|z^n, \sqrt{\epsilon}^L(\partial_t z^n, \partial_x z^n, y\partial_y p^n, \partial_y u^n)|_* \leqslant 1. \tag{3.6}$$

Then there exist positive constants C, γ_0 and ϵ_1 such that for $\gamma \geqslant \gamma_0$ and $\epsilon \in (0, \epsilon_1]$ smooth solutions of (3.2) satisfy

$$\begin{aligned}\sqrt{\gamma}|e^{-\gamma t} z| + |e^{-\gamma t}\sqrt{\epsilon}\nabla u| + \sqrt{\epsilon}\langle e^{-\gamma t} u_1\rangle + \sqrt{\epsilon}\langle e^{-\gamma t} u_{1y}\rangle \\ \leqslant C|(e^{-\gamma t} f, e^{-\gamma t} z)|^{\frac{1}{2}} + C\sqrt{\epsilon}\langle e^{-\gamma t} g\rangle \\ \leqslant \delta\sqrt{\gamma}|e^{-\gamma t} z| + C_\delta\left(\frac{|e^{-\gamma t} f|}{\sqrt{\gamma}} + \sqrt{\epsilon}\langle e^{-\gamma t} g\rangle\right).\end{aligned} \tag{3.7}$$

(b) Smooth solutions z^η of (3.3) satisfy the same estimate with the extra term $|e^{-\gamma t}\sqrt{\eta}\nabla p^\eta|$ on the left.

Proof. — **1.** Take the L^2 inner product of (3.5) with z^γ and integrate by parts. The first term on the left in (3.7) appears due to (3.4) and the positive definiteness of A_0. The second term arises in an obvious way from the viscosity terms, while the third and fourth terms (which are equal to each other mod $\sqrt{\epsilon}\langle e^{-\gamma t} g\rangle$) arise from the boundary terms after integrating $-\epsilon(B_{22} z^\gamma_{yy}, z^\gamma)$ by parts in y and using the boundary conditions.

2. Since

$$|\partial_t v^a, \partial_x v^a, \partial_y v^a|_* \leqslant C, \tag{3.8}$$

for C independent of ϵ, the zero order terms in $\mathcal{L}(z^n)$ contribute errors that can be absorbed by the first term of (3.7). Using (3.6) and the symmetry of A_0, A_1, we similarly absorb the contributions from $(A_0\partial_t z^\gamma, z^\gamma)$ and $(A_1\partial_x z^\gamma, z^\gamma)$.

3. The term $(A_2 z^\gamma_y, z^\gamma)$ requires more care. Since the boundary term vanishes, we have

$$(A_2 z^\gamma_y, z^\gamma) = -\frac{1}{2}(z^\gamma, (\partial_y A_2) z^\gamma), \tag{3.9}$$

where

$$\partial_y A_2 = (\partial_p A_2)\partial_y(p^a + \sqrt{\epsilon}^L p^n) + (\partial_{u_2} A_2)\partial_y(u_2^a + \sqrt{\epsilon}^L u_2^n). \tag{3.10}$$

The second term is bounded uniformly with respect to ϵ by (3.6). To treat the first term we note that $\partial_p A_2$ is a diagonal matrix with factors of $u_2^a + \sqrt{\epsilon}^L u_2^n$ appearing on the diagonal. The Navier boundary condition implies

$$u_2^a + \sqrt{\epsilon}^L u_2^n(t,x,y) = y\int_0^1 \partial_y(u_2^a + \sqrt{\epsilon}^L u_2^n)(t,x,sy)ds := yg^n(t,x,y). \tag{3.11}$$

Since by (3.6) $|\sqrt{\epsilon}^L(y\partial_y p^n, \partial_y u_2^n)|_* \leqslant 1$, we see that the first term in (3.10) is uniformly bounded with respect to ϵ.

□

Remark 3.3. — We do not know how to show that iterates satisfy the bound $|\sqrt{\epsilon}^L\partial_y p^n|_* \leqslant 1$ instead of $|\sqrt{\epsilon}^L y\partial_y p^n|_* \leqslant 1$. However, see Remark 4.3 and the proof of Theorem 4.5 for a discussion (in the case $\epsilon = 1$) of how one can estimate the C^1 and higher C^k norms of the pressure for sufficiently smooth and corner compatible initial data *after* one has taken the limit $n \to \infty$ in the iteration scheme. The latter estimates involve a loss of derivatives that cannot be tolerated in the iteration scheme.

In the higher derivative estimates we will use weighted norms whose definition we recall for easy reference.

Notation 3.4. — 1.) Denote the tangential operators ∂_t, ∂_x, and $y\partial_y$ by M_j, $j = 0, 1, 2$ respectively, and for $k \in \mathbb{N} = \{0, 1, 2, \dots,\}$ let M^k denote the collection of operators $M_0^{\alpha_0}M_1^{\alpha_1}M_2^{\alpha_2}$ such that $\alpha_0 + \alpha_1 + \alpha_2 = k$. Sometimes M^k is used to denote a particular member of this collection of operators.

2.) Let $\mu = \frac{\gamma}{\epsilon}$, where we always suppose $0 < \epsilon \leqslant 1 \leqslant \gamma$. On Ω_T with $z = (p, u)$ set

$$|z|_{k,\mu,\gamma} = \sum_{j=0}^{k} \mu^{k-j}|e^{-\gamma t}M^j z|_{L^2(\Omega_T)},$$

(3.12)

$$\langle z\rangle_{k,\mu,\gamma} = \sum_{j=0}^{k} \mu^{k-j}|e^{-\gamma t}M^j z|_{L^2(b\Omega_T)}.$$

3.) Set $\|z\|'_{k,\mu,\gamma} = |\sqrt{\gamma}z, \sqrt{\epsilon}\nabla u|_{k,\mu,\gamma} + \langle\sqrt{\epsilon}u_1\rangle_{k,\mu,\gamma} + \langle\sqrt{\epsilon}u_{1y}\rangle_{k,\mu,\gamma}$.

4.) Set $\|z\|_{k,\mu,\gamma} = \|z\|'_{k,\mu,\gamma} + |\sqrt{\gamma}\partial_y z|_{k-2,\mu,\gamma} + |\partial_y\sqrt{\epsilon}\nabla u|_{k-2,\mu,\gamma}$.

5.) When there are d space dimensions, define Ω_T and the norms $\|z\|'_{k,\mu,\gamma}$ and $\|z\|_{k,\mu,\gamma}$ with the obvious changes: let $x \in \mathbb{R}^{d-1}$, $\alpha_1 \in \mathbb{N}^{d-1}$, and take $k = \alpha_0 + |\alpha_1| + \alpha_2$.

Remark 3.5. — Observe that

$$|z|_{k-1,\mu,\gamma} \leqslant \frac{1}{\mu}|z|_{k,\mu,\gamma} = \frac{\epsilon}{\gamma}|z|_{k,\mu,\gamma}, \tag{3.13}$$

and a similar estimate holds for boundary norms. This property, which we use repeatedly in the estimates to follow (e.g., in (3.33)), is one advantage of defining the weight μ with ϵ in the denominator. This definition of μ also introduces an occasionally helpful ϵ-dependence in the small factor $\mu^{-(2-\frac{d}{2})}$ ($d = 2, 3$) appearing in the Sobolev estimates of Corollary 3.8.

In preparation for the higher derivative estimates we next state weighted versions of the classical Moser and Sobolev estimates.

Lemma 3.6 (Moser estimate, [5] Lemma 2.1.2).
For $k \in \mathbb{N} = \{0, 1, 2, \dots\}$ *let* $\alpha_1 + \dots + \alpha_r \leqslant j \leqslant k$, $\alpha_i \in \mathbb{N}$. *Then*

$$\mu^{k-j} |(M^{\alpha_1} w_1) \dots (M^{\alpha_r} w_r)|_{0,\mu,\gamma} \leqslant C \sum_{i=1}^{r} |w_i|_{k,\mu,\gamma} \left(\prod_{j \neq i} |w_j|_* \right). \tag{3.14}$$

We will use the following weighted version of the standard Sobolev estimate.

Lemma 3.7 (Sobolev estimate). — *Suppose* $k > \frac{d}{2} + 1$. *Then*

$$|w|_* \leqslant C \mu^{-(k - \frac{d}{2} - 1)} e^{\gamma T} (|w|_{k,\mu,\gamma} + |w_y|_{k-2,\mu,\gamma}) \leqslant C \mu^{-(k - \frac{d}{2} - 1)} e^{\gamma T} \|w\|_{k,\mu,\gamma}. \tag{3.15}$$

Proof. — By using Seeley extensions in the t and $y = x_d$ directions and the observation

$$1 = e^{\gamma t} e^{-\gamma t} \leqslant e^{\gamma (T - t)} \text{ for } T \geqslant t,$$

we reduce to proving the following estimate without exponential weights on $\mathbb{R}^{d+1}$

$$|w|_* \leqslant C \mu^{-(k - \frac{d}{2} - 1)} (|w|_{k,\mu} + |w_y|_{k-2,\mu}), \tag{3.16}$$

where now we define

$$|w|_{k,\mu} = \sum_{j=0}^{k} \mu^{k-j} |M^j z|_{L^2(\mathbb{R}^{d+1})}. \tag{3.17}$$

Letting $\xi = (\xi_0, \xi_1, \dots, \xi_{d-1}, \xi_d) = (\xi', \xi_d) \in \mathbb{R}^{d+1}$ denote the dual variable to $(t, x_1, \dots, x_{d-1}, y)$, we have by taking Fourier transforms and using the Cauchy-Schwartz inequality:

$$\begin{aligned} |w|_* &\leqslant \int |\hat{w}(\xi)| d\xi \leqslant C (|w|_{k,\mu} + |w_y|_{k-2,\mu}) \cdot \sqrt{\iint [(\mu + |\xi'|)^{2k} + \xi_d^2 (\mu + |\xi'|)^{2(k-2)}]^{-1} d\xi' d\xi_d} \\ &= C \sqrt{\pi} \, (|w|_{k,\mu} + |w_y|_{k-2,\mu}) \sqrt{\int (\mu + |\xi'|)^{-2(k-1)} d\xi'} = C' (|w|_{k,\mu} + |w_y|_{k-2,\mu}) \mu^{-(k - \frac{d}{2} - 1)}. \end{aligned} \tag{3.18}$$

Here we have done the ξ_d integral in the first line by setting $s = \xi_d (\mu + |\xi'|)^{k-2}$ and the ξ' integral in the second line using $\eta' = \xi' / \mu$. □

The following corollary is immediate.

Corollary 3.8. — *Suppose $d=2$ or $d=3$. Then for $z=(p,u)$, $\mu=\frac{\gamma}{\epsilon}$, and $k\geqslant 3$ we have*

$$
\begin{aligned}
&|z|_* \leqslant C\mu^{-(2-\frac{d}{2})}e^{\gamma T}\|z\|_{3,\mu,\gamma}\\
&|M^q z|_* \leqslant C\mu^{-(2-\frac{d}{2})}e^{\gamma T}\|z\|_{k,\mu,\gamma} \text{ for } 0\leqslant q\leqslant k-3\\
&|M^q(\sqrt{\epsilon}\nabla u)|_* \leqslant C\mu^{-(2-\frac{d}{2})}e^{\gamma T}\|z\|_{k,\mu,\gamma} \text{ for } 0\leqslant q\leqslant k-3.
\end{aligned}
\tag{3.19}
$$

The following lemma on commutators, which is proved by direct computation, is used in the proof of the next proposition.

Lemma 3.9. — *Let $M^j=M_0^{\alpha_0}M_1^{\alpha_1}M_2^{\alpha_2}$, where $\alpha_0+\alpha_1+\alpha_2=j$ and $\alpha_2>0$. Then*

$$
\begin{aligned}
&[y,M^j]w = M_0^{\alpha_0}M_1^{\alpha_1}(b_0yw+b_1yM_2w+\cdots+b_{\alpha_2-1}yM_2^{\alpha_2-1}w)\\
&[\partial_y,M^j]w = M_0^{\alpha_0}M_1^{\alpha_1}(c_0w_y+c_1M_2w_y+\cdots+c_{\alpha_2-1}M_2^{\alpha_2-1}w_y)\\
&[\partial_y^2,M^j]w = M_0^{\alpha_0}M_1^{\alpha_1}(d_0w_{yy}+d_1M_2w_{yy}+\cdots+d_{\alpha_2-1}M_2^{\alpha_2-1}w_{yy})
\end{aligned}
\tag{3.20}
$$

for some $b_i\in\mathbb{N}$, $c_i\in\mathbb{N}$, $d_i\in\mathbb{N}$. If $\alpha_2=0$, then $[y,M^j]=[\partial_y,M^j]=[\partial_y^2,M^j]=0$.

Proposition 3.10 (Tangential higher derivative estimate). — *Assume $L\geqslant 1$, $k\geqslant 0$, $M-L-2k>0$, and suppose that*

$$
|z^n, Mz^n, \sqrt{\epsilon}\partial_y u^n, \sqrt{\epsilon}M\partial_y u^n|_* \leqslant 1. \tag{3.21}
$$

For γ fixed large enough and $\epsilon\in(0,1]$, $\eta\in(0,1]$, the solution $z^{n+1,\eta}$ of (3.1) *satisfies*

$$
\begin{aligned}
&\|z^{n+1,\eta}\|'_{k,\mu,\gamma}+\sqrt{\eta}|\nabla p^{n+1,\eta}|_{k,\mu,\gamma}\leqslant\\
&\qquad C\left(\frac{1}{\sqrt{\gamma}}+|z^{n+1,\eta},Mz^{n+1,\eta}|_*\right)\|z^n\|'_{k,\mu,\gamma}++C(\gamma)\sqrt{\epsilon}^{M-L-2k}.
\end{aligned}
\tag{3.22}
$$

Proof. —

1. In the proof we will set $z=(p,u)=z^{n+1,\eta}$. To estimate $\|z\|'_{k,\mu,\gamma}$ we apply the L^2 estimate (3.7) to the problem satisfied by $\mu^{k-j}M^jz$. Note that for $\alpha=(\alpha_0,\alpha_1,\alpha_2)$, $|\alpha|=j$, the boundary conditions satisfied by $M^\alpha z$ are

$$
\begin{aligned}
&\partial_y M^\alpha p=0,\ (I-\partial_y)M^\alpha u_1\\
&\qquad=\begin{cases}-M_0^{\alpha_0}M_1^{\alpha_1}u_1, & \alpha_2>0\\ 0, & \alpha_2=0\end{cases},\ M^\alpha u_2=0 \text{ on } y=0.
\end{aligned}
\tag{3.23}
$$

Commuting $\mu^{k-j}M^jz$ through (3.1)(a), we obtain forcing that is a sum of $-\sqrt{\epsilon}^{M-L}\mu^{k-j}M^jR^M$ and commutator terms. Thus, the L^2 estimate gives

$$\begin{aligned}(3.24)\quad &\mu^{k-j}|\sqrt{\gamma}M^jz,\sqrt{\epsilon}\nabla M^ju|_{0,\mu,\gamma}\\ &\quad+\sqrt{\epsilon}\mu^{k-j}\langle M^ju_1,\partial_y(M^ju_1)\rangle+\sqrt{\eta}\mu^{k-j}|\nabla M^jp|_{0,\mu,\gamma}\\ &\leqslant\frac{C}{\sqrt{\gamma}}\left(\sqrt{\epsilon}^{M-L}\mu^{k-j}|M^jR^M|_{0,\mu,\gamma}+\mu^{k-j}|\text{interior commutators}|_{0,\mu,\gamma}\right)\\ &\qquad\qquad+\sqrt{\epsilon}\sum_{l=0}^{k-1}\mu^{k-1-l}\langle M^lu_1\rangle_{0,\mu,\gamma}.\end{aligned}$$

where the boundary term on the right is explained by (3.23). We treat the interior commutators below. The R^M term on the right is $\leqslant C(\gamma)\sqrt{\epsilon}^{M-L-2k}$ and the boundary term is (by an estimate like (3.13)) $\leqslant\frac{\epsilon}{\gamma}\|z\|'_{k,\mu,\gamma}$.

Notation 3.11. — 1. For $s\in\{1,2,3,\dots\}$ and a function w with components w_i, denote by $M^{\langle s\rangle}w$ any set of products of the form $(M^{s_1}w_{i_1})\dots(M^{s_r}w_{i_r})$, where $s_1+\cdots+s_r=s$, $s_i\geqslant 1$. If $s=0$, set $M^{\langle 0\rangle}w=1$.

2. In the estimates below the symbol for a matrix like A_2 (or vector like z) will sometimes represent a single entry (or component) of that matrix (or vector). The correct interpretation should be clear from the context.

2. Consider the interior commutator $\mu^{k-j}|[A_2(v^a+\sqrt{\epsilon}^Lz^n)\partial_y,M^j]z|_{0,\mu,\gamma}$. Observe that

$$\begin{aligned}(3.25)\quad [A_2\partial_y,M^j]z&=A_2\partial_yM^jz-M^j(A_2\partial_yz)=\\ &\quad(\text{terms where } A_2 \text{ is differentiated})+A_2[\partial_y,M^j]z.\end{aligned}$$

Each of the terms where A_2 is differentiated has components which are sums of terms of the form

$$\mathcal{A}(v^a+\sqrt{\epsilon}^Lz^n)\,M^{\langle s\rangle}(v^a+\sqrt{\epsilon}^Lz^n)\,M^t\partial_yz,\tag{3.26}$$

where $\mathcal{A}$ is (an entry in) some derivative $d_v^qA_2$, $q\geqslant 1$, and $s+t=j$, $t\leqslant j-1$. If the factor $M^{s'}(u_2^a+\sqrt{\epsilon}^Lu_2^n)$ appears in (3.26), we use vanishing of the factor at $y=0$ and (3.11) to place a factor of y on $M^t\partial_yz$. For g^n as in (3.11) we have in view of (3.20)

$$M^{s'}(yg^n)=yM^{s'}g^n+(\text{sum of terms of the form } yM^{s''}g^n),\tag{3.27}$$

where $s''\leqslant s'-1$.

We must then estimate terms like

$$C\mu^{k-j}|M^{\langle s-s'\rangle}(v^a+\sqrt{\epsilon}^L z^n)\,(M^l M(g^n(t,x,y))\,(yM^t\partial_y z)|_{0,\mu,\gamma}, \tag{3.28}$$

where $l \leqslant s'-1$. In turn (3.28) is $\leqslant$ a sum of terms like

$$C\mu^{k-j}|(M^{\langle\alpha\rangle}v^a)(M^{\langle\beta\rangle}\sqrt{\epsilon}^L z^n)\,(M^l M(g^n(t,x,y))\,(M^{t'}(y\partial_y z))|_{0,\mu,\gamma}, \tag{3.29}$$

where $\alpha+\beta=s-s'$ and $t'\leqslant t$. Since tangential derivatives of $(v^a,\nabla v^a)$ are bounded, the Moser estimates imply that terms of type (3.29) in which $\sqrt{\epsilon}^L M^l M\int_0^1 \partial_y u_2^n(t,x,sy)ds := G^n$ appears are dominated by

$$\begin{aligned}
&C|\sqrt{\epsilon}^L z^n)|_*^r\,|M(\sqrt{\epsilon}^L\partial_y u_2^n))|_*\,|y\partial_y z|_{k-1,\mu,\gamma}+\\
&\quad C|\sqrt{\epsilon}^L z^n)|_*^r\,|y\partial_y z|_*\,|M(\sqrt{\epsilon}^L\partial_y u_2^n))|_{k-1,\mu,\gamma}+\\
&\qquad C|\sqrt{\epsilon}^L z^n)|_*^{r-1}\,|y\partial_y z|_*\,|M(\sqrt{\epsilon}^L\partial_y u_2^n))|_*\,|\sqrt{\epsilon}^L z^n)|_{k-1,\mu,\gamma},
\end{aligned} \tag{3.30}$$

where r is the number of factors appearing in $M^{\langle\beta\rangle}\sqrt{\epsilon}^L z^n$. From (3.21) it follows that $\frac{1}{\sqrt{\gamma}}$(3.30) is dominated by the right side of (3.22)$+\frac{C}{\sqrt{\gamma}}\|z\|'_{k,\mu,\gamma}$. Terms of type (3.29) in which G^n is not present are handled similarly.

If the factor $M^{s'}(u_2^a+\sqrt{\epsilon}^L u_2^n)$ does not appear in (3.26), then $\mathcal{A}=d_v^q A_2$ must be $d_p^q A_2(v^a+\sqrt{\epsilon}z^n)$ (recall A_2 is independent of u_1). The latter matrix is a diagonal matrix with factors of $u^a+\sqrt{\epsilon}^L u_2^n$ appearing on the diagonal, so again we can extract a factor of y to place on $M^t\partial_y z$ and proceed as above.

3. Consider now $\mu^{k-j}A_2[\partial_y,M^j]z$, where $M^j=M_0^{\alpha_0}M_1^{\alpha_1}M_2^{\alpha_2}$ with $\alpha_2>0$. Using (3.15)(b) and recalling the form (1.9) of A_2, we see that the term requiring the most care is $\mu^{k-j}M_0^{\alpha_0}M_1^{\alpha_1}M_2^{\alpha_2-1}p_y$, which appears in the third component of $\mu^{k-j}A_2[\partial_y,M^j]z$. To handle this term we must use the stronger form of the L^2 estimate (3.7) in which the pairing $|(e^{-\gamma t}f,e^{-\gamma t}z)|^{\frac12}$ appears on the right. By (3.20) we have

$$|\left(e^{-\gamma t}\mu^{k-j}M_0^{\alpha_0}M_1^{\alpha_1}M_2^{\alpha_2-1}p_y,e^{-\gamma t}\mu^{k-j}M^j u_2\right)|^{\frac12} \tag{3.31}$$

is $\leqslant$ a sum of terms of the form

$$C|\left(e^{-\gamma t}\mu^{k-j}M_0^{\alpha_0}M_1^{\alpha_1}M_2^{\alpha_2'-1}yp_y,e^{-\gamma t}\mu^{k-j}M^{j-1}\partial_y u_2\right)|^{\frac12},\ \alpha_2'\leqslant\alpha_2. \tag{3.32}$$

Here we have extracted a factor of y from one of the M_2 derivatives on the right side of (3.31). Since

$$|\mu^{k-j}M^{j-1}\partial_y u_2|_{0,\mu,\gamma} \leqslant \frac{1}{\sqrt{\epsilon}}\|z\|'_{k-1,\mu,\gamma} \leqslant \frac{1}{\sqrt{\epsilon}}\frac{\epsilon}{\gamma}\|z\|'_{k,\mu,\gamma}, \tag{3.33}$$

we see that terms like (3.32) are dominated by $C\frac{1}{\sqrt{\gamma}}\|z\|'_{k,\mu,\gamma}$.

4. The interior commutators involving $A_0\partial_t$ and $A_1\partial_x$ are handled similarly but more easily, since there is no need to extract factors of y as above, and $[\partial_t, M^j] = [\partial_x, M^j] = 0$.

5. The interior commutators involving $\int_0^1 \partial_v A_i ds$ are still easier to treat, since the corresponding terms in $\mathcal{L}(z^n)$ (1.16) are of order zero and $M^j(\partial_t, \nabla)v^a$ is uniformly bounded with respect to $\epsilon \in (0,1]$.

6. Now we examine $[\eta\Delta, M^j]p$, where $M^j = M_0^{\alpha_0}M_1^{\alpha_1}M_2^{\alpha_2}$. This commutator is zero unless $\alpha_2 > 0$, in which case by Lemma 3.9

$$\begin{aligned}[\eta\Delta, M^j]p = [\eta\partial_y^2, M^j]p = \eta M_0^{\alpha_0}M_1^{\alpha_1}[\partial_y^2, M_2^{\alpha_2}]p = \\ \eta M_0^{\alpha_0}M_1^{\alpha_1}\left(d_0 p_{yy} + d_1 M_2 p_{yy} + \cdots + d_{\alpha_2-1}M_2^{\alpha_2-1}p_{yy}\right),\end{aligned} \tag{3.34}$$

for some constants d_i. Next use the equation (3.1)(a) to write

$$\eta p_{yy} = -\eta p_{xx} + (\mathcal{L}(z^n)z + \sqrt{\epsilon}^{M-L}R^M)_1, \tag{3.35}$$

where the subscript 1 denotes the first component, and substitute (3.35) into (3.34). We have, for example,

$$\frac{1}{\sqrt{\gamma}}\mu^{k-j}|M_0^{\alpha_0}M_1^{\alpha_1}M_2^{\alpha_2-1}\eta p_{xx}|_{0,\mu,\gamma} \leqslant \frac{1}{\sqrt{\gamma}}\sqrt{\eta}\mu^{k-j}|\nabla M^j p|_{0,\mu,\gamma} \quad (\eta \in (0,1]), \tag{3.36}$$

which can be absorbed by the left side of (3.24). The term u_{2y} appears on the right in (3.35) (recall (1.10)) and

$$\begin{aligned}\frac{1}{\sqrt{\gamma}}\mu^{k-j}|M_0^{\alpha_0}M_1^{\alpha_1}M_2^{\alpha_2-1}u_{2y}|_{0,\mu,\gamma} &\leqslant \frac{1}{\sqrt{\epsilon}\sqrt{\gamma}}\|u_2\|'_{k-1,\mu,\gamma} \\ &\leqslant \frac{1}{\sqrt{\epsilon}\sqrt{\gamma}}\frac{\epsilon}{\gamma}\|u_2\|'_{k,\mu,\gamma}.\end{aligned} \tag{3.37}$$

Corresponding to the term $\frac{\rho'}{\rho}(p^a + \sqrt{\epsilon}^L p_n)\ (u_2^a + \sqrt{\epsilon}^L u_2^n)p_y$ in (3.35) we must estimate, for example,

$$\frac{1}{\sqrt{\gamma}}\mu^{k-j}|M_0^{\alpha_0}M_1^{\alpha_1}M_2^{\alpha_2-1}\left(\frac{\rho'}{\rho}(p^a + \sqrt{\epsilon}^L p_n)(u_2^a + \sqrt{\epsilon}^L u_2^n)p_y\right)|_{0,\mu,\gamma}, \tag{3.38}$$

which is a sum of terms of the form

$$\frac{1}{\sqrt{\gamma}}\mu^{k-j}|\mathcal{A}\;(M^{\langle r\rangle}v^a)\;(M^{\langle s\rangle}(\sqrt{\epsilon}^L z^n))\;M^t\partial_y p|_{0,\mu,\gamma}, \tag{3.39}$$

where $\mathcal{A}$ has the same meaning as before and $r+s+t=j-1$. These terms can now be estimated by the same arguments used for (3.26). The remaining terms on the right in (3.35) are similar, but simpler to handle.

7. Finally we examine the commutator terms involving $\epsilon[B_{12}\partial_{xy}, M^j]$ and $\epsilon[B_{22}\partial_{yy}, M^j]$. Consider for example $\mu^{k-j}[\epsilon\partial_y^2, M_0^{\alpha_0}M_1^{\alpha_1}M_2^{\alpha_2}]u_2$, where $\alpha_2>0$. Rewriting this term using (3.20), and substituting for $\epsilon\partial_y^2 u_2$ its expression coming from the third component of equation (3.1)(a), we obtain several terms including, for example, $\mu^{k-j}M_0^{\alpha_0}M_1^{\alpha_1}M_2^{\alpha_2-1}p_y$. If we use the strong form of the L^2 estimate in which the pairing $|(e^{-\gamma t}f, e^{-\gamma t}z)|^{\frac{1}{2}}$ appears on the right, this term is paired with $\mu^{k-j}M^j u_2$, and so we again obtain the pairing (3.31). The remaining terms are also handled by arguments used above.

□

Remark 3.12. — If we had used the symmetric form (1.7) of the equations, where $w=(\rho,u)$ is the dependent variable, the argument in step **2** of the above proof involving extraction of the factor y would fail due to the presence of off-diagonal terms in $\partial_\rho^q A_2$ that do not vanish when $y=0$. Instead of assuming $|p^n, Mp^n|_*\leqslant 1$ we would need to assume $|\rho^n, M\rho^n, \partial_y\rho^n|_*\leqslant 1$; yet we are not able to prove a uniform bound for $|\partial_y\rho^n|_*$, or even $\sqrt{\epsilon}^L|\partial_y\rho^n|_*$.

3.2. Normal derivative estimates.

In the estimates of normal derivatives one of the main challenges is that we do not have any useful version of the Moser estimates that applies to products of the form

$$(M^{\alpha_1}w_1)\cdots(M^{\alpha_{r-1}}w_{r-1})(M^{\alpha_r}\partial_y w_r) \tag{3.40}$$

rather than (3.14). Such products arise in the estimates of commutators below. This forces us to control the L^∞ norms of all but one of the factors appearing in products like (3.40) and leads to the restriction to dimensions $d=2,3$ in the next Proposition.

Notation 3.13. — 1. For $z=(p,u)$ set

$$\|z\|^{**} := |z, Mz, M^2 z, \sqrt{\epsilon}\partial_y u, \sqrt{\epsilon}M\partial_y u|_*.$$

2. Let b_{ij} denote the lower right 2×2 block of the viscosity matrix B_{ij}.

Proposition 3.14. — *Suppose $d=2$ or $d=3$. Assume $L\geqslant 2$, $k\geqslant 3$, $M-L-2k>0$, and suppose that*

$$\text{(3.41)}\qquad \begin{aligned}&(a)\ \|z^n\|^{**}\leqslant 1\\ &(b)\ \|z^n\|_{k,\mu,\gamma}\leqslant 1.\end{aligned}$$

For γ fixed large enough there exists $\epsilon_0(\gamma)$ such that for $0<\epsilon\leqslant\epsilon_0(\gamma)$ and $\eta\in(0,1]$, the solution $z^{n+1,\eta}$ of (3.1) satisfies

$$\text{(3.42)}\qquad \begin{aligned}&\|z^{n+1,\eta}\|_{k,\mu,\gamma}+\sqrt{\eta}|\nabla p^{n+1,\eta}|_{k,\mu,\gamma}+\eta|\nabla p_y^{n+1,\eta}|_{k-2,\mu,\gamma}\leqslant\\ &\qquad C\left(\frac{1}{\sqrt{\gamma}}+\|z^{n+1,\eta}\|^{**}\right)\|z^n\|_{k,\mu,\gamma}+C(\gamma)\sqrt{\epsilon}^{M-L-2k}.\end{aligned}$$

The constants on the right depend only on k.

Proof. — **1.** As explained in the Introduction, we cannot use the L^2 estimate (3.7) now. Instead we study the problem satisfied by

$$\text{(3.43)}\qquad Z^j=(P^j,U_1^j,U_2^j):=\mu^{k-2-j}M^j\partial_y z,\ \ j\leqslant k-2,$$

where as before $z=(p,u):=z^{n+1,\eta}$ satisfies (3.1). From (3.1)(a) we obtain

(3.44)

$$\begin{aligned}&(a)\ \mathcal{L}(z^n)Z^j-\begin{pmatrix}\eta\Delta P^j\\0\\0\end{pmatrix}=\\ &\qquad -\mu^{k-2-j}M^j\partial_y(\sqrt{\epsilon}^{M-L}R^M)+(\text{sum of interior commutators}):=\mathcal{F},\\ &(b)\ P^j=0\text{ on }y=0.\end{aligned}$$

The estimate will be proved by taking the L^2 pairing of (3.44)(a) with $(\eta P^j,U_1^j,U_2^j)$. The reason for the factor η on P^j will be seen in steps **4**, **5**, and **6** below.

2. Arguing as in the proof of the L^2 estimate, Proposition 3.2, we obtain for γ large

(3.45)

$$\begin{aligned}&\sqrt{\gamma}\left|\begin{pmatrix}\sqrt{\eta}P^j\\U^j\end{pmatrix}\right|_{0,\mu,\gamma}+\sqrt{\epsilon}|\nabla U^j|_{0,\mu,\gamma}+\eta|\nabla P^j|_{0,\mu,\gamma}\leqslant\\ &\quad C\left|\left(e^{-\gamma t}\mathcal{F},e^{-\gamma t}\begin{pmatrix}\eta P^j\\U^j\end{pmatrix}\right)\right|^{\frac12}+C\sqrt{\epsilon}|\langle e^{-\gamma t}b_{22}\partial_yU^j,e^{-\gamma t}U^j\rangle|^{\frac12}\leqslant\\ &\quad \frac{C_\delta}{\sqrt{\gamma}}|\mathcal{F}|_{0,\mu,\gamma}+\delta\sqrt{\gamma}\left|\begin{pmatrix}\eta P^j\\U^j\end{pmatrix}\right|_{0,\mu,\gamma}+C\sqrt{\epsilon}|\langle e^{-\gamma t}b_{22}\partial_yU^j,e^{-\gamma t}U^j\rangle|^{\frac12},\end{aligned}$$

where the first pairing in the second line is an interior pairing and the second is a boundary pairing. Note that the boundary terms arising from integration by parts in the $A_2\partial_y$ and $-\eta\Delta$ terms both vanish. In deriving (3.45) we have absorbed terms corresponding to the order zero terms of $\mathcal{L}(z^n)$ using the first term on the left. It remains to estimate the interior commutator terms in $\mathcal{F}$ and the boundary terms.

The control on P^j in the first term on the left of (3.45) degenerates as $\eta \to 0$, and so (3.45) is at first sight too weak to imply (3.42). We address this problem in step **11** below, where we estimate the appropriate derivatives of p in terms of u.

3. When $M^j = M_0^{\alpha_0} M_1^{\alpha_1} M_2^{\alpha_2}$ with $\alpha_2 > 0$, for the commutator $[A_2\partial_y, M^j\partial_y]$ we have

(3.46)
$$[A_2\partial_y, M^j\partial_y]z = A_2[\partial_y, M^j]z_y + (\text{terms where } A_2 \text{ is differentiated}).$$

By (3.20) the first term on the right is a linear combination of terms of the form

$$A_2 M_0^{\alpha_0} M_1^{\alpha_1} M_2^{j'} z_{yy},\ 0 \leqslant j' \leqslant \alpha_2 - 1. \tag{3.47}$$

In any term of (3.47) where a diagonal entry of A_2 occurs, we can extract a y from $u_2^a + \sqrt{\epsilon}^L u_2^n$ and estimate

$$\frac{1}{\sqrt{\gamma}}\mu^{k-2-j}|M_0^{\alpha_0} M_1^{\alpha_1} M_2^{j'+1} z_y|_{0,\mu,\gamma} \leqslant \frac{1}{\sqrt{\gamma}}\|z\|_{k,\mu,\gamma}. \tag{3.48}$$

A term of (3.47) in which the 1 from the first row of A_2 occurs is readily estimated using (3.13), since $\partial_y^2 u_2$ is the component of z_{yy} that appears. A term in which the 1 from the third row of A_2 occurs is easily estimated using the interior pairing

$$\left|\left(e^{-\gamma t}\mu^{k-2-j} M_0^{\alpha_0} M_1^{\alpha_1} M_2^{j'} p_{yy}, e^{-\gamma t}\mu^{k-2-j} M^j \partial_y u_2\right)\right|^{\frac{1}{2}} \tag{3.49}$$

by moving a factor of y from one of the M_2 derivatives occurring on the right to the left.

4. The terms of (3.46) in which A_2 is differentiated are of two types:

(3.50)

(a) $\mathcal{A}M^{\langle s\rangle}(v^a + \sqrt{\epsilon}^L z^n)(M^t\partial_y^2 z),$ where $s + t = j \leqslant k-2,\ t \leqslant j-1$

(b) $\mathcal{A}M^{\langle s-s'\rangle}(v^a + \sqrt{\epsilon}^L z^n)\left(M^{s'}\partial_y(v^a + \sqrt{\epsilon}^L z^n)\right)(M^t\partial_y z),$ where $s + t = j,\ t \leqslant j,$

where as before $\mathcal{A}$ represents some derivative of A_2.

In case (a) either the factor $M^r(u_2^a+\sqrt{\epsilon}^L u_2^n)$ appears or it does not. In each case by arguing as in step **2** of the proof of Proposition 3.10, we extract a factor of y to multiply $M^t\partial_y^2 z$. Using (3.20) we have

$$\mu^{k-2-j}|yM^t\partial_y^2 z|_{0,\mu,\gamma}\leqslant C\|z\|_{k,\mu,\gamma}. \tag{3.51}$$

If $t>0$ then $s\leqslant k-3$, and the L^∞ norms of the factors remaining in (3.50)(a) (after extraction of y from one of them) are controlled by Corollary 3.8. So with (3.51) we obtain the estimate

$$\begin{aligned}\mu^{k-2-j}|\mathcal{A}M^{\langle s\rangle}(v^a+\sqrt{\epsilon}^L z^n)(M^t\partial_y^2 z)|_{0,\mu,\gamma}&\\ \leqslant C\left(1+\mu^{-(2-\frac{d}{2})}e^{\gamma T}\|z^n\|_{k,\mu,\gamma}\right)^q\ \|z\|_{k,\mu,\gamma}&\end{aligned} \tag{3.52}$$

where q is the number of factors in $M^{\langle s\rangle}(v^a+\sqrt{\epsilon}^L z^n)$. By assumption (3.41)(b) for $0<\epsilon<\epsilon(\gamma)$ the right side of (3.52) is $\leqslant C\|z\|_{k,\mu,\gamma}$.

If $t=0$ and more than one factor appears in $M^{\langle s\rangle}(v^a+\sqrt{\epsilon}^L z^n)$, we can apply Corollary 3.8 again to control L^∞ norms of individual factors and obtain the estimate (3.52).

Finally, consider the case when $t=0$ and only one factor appears in $M^{\langle s\rangle}(v^a+\sqrt{\epsilon}^L z^n)$, say $M^j(u_2^a+\sqrt{\epsilon}^L u_2^n)$. After extraction of y we obtain for example

$$\mu^{k-2-j}|\mathcal{A}M^j(\sqrt{\epsilon}^L\partial_y u_2^n)\ (y\partial_y^2 u)|_{0,\mu,\gamma}\leqslant C\sqrt{\epsilon}^{L-1}\|z^n\|_{k,\mu,\gamma}\ |\sqrt{\epsilon}(y\partial_y)\partial_y u|_*. \tag{3.53}$$

More delicate is the case $\mathcal{A}M^j(\sqrt{\epsilon}^L u_2^n)\ (\partial_y^2 p)$ (where we have not yet extracted a y). We use the strong form of (3.45) to estimate

$$\left|\left(e^{-\gamma t}\mu^{k-2-j}\mathcal{A}M^j(\sqrt{\epsilon}^L u_2^n)\partial_y^2 p, e^{-\gamma t}\eta\mu^{k-2-j}M^j\partial_y p\right)\right|^{\frac{1}{2}}. \tag{3.54}$$

Here we have used the fact that derivatives of A_2 are diagonal matrices to determine the entry on the right in the pairing in (3.54). After moving the η factor from right to left, we use the first component of equation (3.35) (where a y can be extracted from the coefficient of p_y) to find

$$|\eta p_{yy}|_*\leqslant C|z,Mz,\sqrt{\epsilon}\partial_y u,M(\sqrt{\epsilon}\partial_y u),p_{xx}|_*+C\sqrt{\epsilon}^{M-L}. \tag{3.55}$$

Thus, (3.54) is $\leqslant$

$$C\|z\|^{**}\|z^n\|_{k-2,\mu,\gamma}+\frac{C}{\sqrt{\gamma}}\|z\|_{k,\mu,\gamma}, \tag{3.56}$$

and this in turn is dominated by the right side of (3.42)$+\frac{C}{\sqrt{\gamma}}\|z\|_{k,\mu,\gamma}$. Other subcases when $t=0$ and only one factor appears in $M^{\langle s\rangle}(v^a+\sqrt{\epsilon}^L z^n)$ are handled similarly or more easily.

5. Consider case (b) of (3.50) first when $M^{s'}\partial_y(p^a+\sqrt{\epsilon}^L p^n)$ appears:

$$\mathcal{A}M^{\langle s-s'\rangle}(v^a+\sqrt{\epsilon}^L z^n)\left(M^{s'}\partial_y(p^a+\sqrt{\epsilon}^L p^n)\right)(M^t\partial_y z), \tag{3.57}$$

where $s+t=j,\ t\leqslant j$.
We observe that either a factor $M^r(u_2^a+\sqrt{\epsilon}^L u_2^n)$ appears in $M^{\langle s-s'\rangle}(v^a+\sqrt{\epsilon}^L z^n)$ or it does not. Thus, we can extract a factor of y (either from $M^r(u_2^a+\sqrt{\epsilon}^L u_2^n)$ or from $\mathcal{A}$) to multiply $M^{s'}\partial_y(p^a+\sqrt{\epsilon}^L p^n)$ or $M^t\partial_y z$. If $t=j$ (resp. $t=j-1$), then $s=0$ (resp. $s=1$), and we multiply $M^{s'}\partial_y(p^a+\sqrt{\epsilon}^L p^n)$ by y. We have

$$\mu^{k-2-j}|M^t\partial_y z|_{0,\mu,\gamma}\leqslant\|z\|_{k,\mu,\gamma} \tag{3.58}$$

and the L^∞ norms of the remaining factors in (3.50)(b) are bounded by assumption (3.41)(a). If $2\leqslant t\leqslant j-2$, (3.58) still holds, we multiply $M^{s'}\partial_y(p^a+\sqrt{\epsilon}^L p^n)$ by y, and since $s-s'\leqslant j-2$, $s'\leqslant j-2$, the L^∞ norm of this product and that of the remaining factors are controlled using Corollary 3.8. We obtain

$$\begin{aligned}\mu^{k-2-j}|\mathcal{A}M^{\langle s-s'\rangle}(v^a+\sqrt{\epsilon}^L z^n)\left(M^{s'}\partial_y(p^a+\sqrt{\epsilon}^L p^n)\right)(M^t\partial_y z)|\leqslant\\ C\left(1+\mu^{-(2-\frac{d}{2})}e^{\gamma T}\|z^n\|_{k,\mu,\gamma}\right)^q\|z\|_{k,\mu,\gamma}\end{aligned} \tag{3.59}$$

for some q, and treat (3.59) as we did (3.52). If $t=1$, we have $|yMz_y|_*\leqslant C$ by assumption (3.41)(a). Since $s-s'\leqslant j-1$, $s'\leqslant j-1\leqslant k-3$, we have

$$\mu^{k-2-j}|M^{s'}\partial_y(\sqrt{\epsilon}^L p^n)|_{0,\mu,\gamma}\leqslant C\|z\|_{k-1,\mu,\gamma}, \tag{3.60}$$

and the L^∞ norm of the remaining factors is controlled using Corollary 3.8.

Now suppose $t=0$ and $1\leqslant s'\leqslant j$. We have $|yz_y|_*\leqslant 1$ by assumption (3.41)(a), and (3.60) holds with $C\|z\|_{k,\mu,\gamma}$ on the right. Since $0\leqslant s-s'\leqslant j-1$ the L^∞ norm of the remaining factors is again controlled using Corollary 3.8. Finally, suppose $t=0$ and $s'=0$. In this case (3.50) is

$$\mathcal{A}M^{\langle j\rangle}(v^a+\sqrt{\epsilon}^L z^n)\,\partial_y(p^a+\sqrt{\epsilon}^L p^n)z_y. \tag{3.61}$$

We put the extracted y on $\partial_y(p^a+\sqrt{\epsilon}^L p^n)z_y$, and use $|yp_y^n|_*\leqslant 1$. If u_y appears we borrow $\sqrt{\epsilon}$ from $\sqrt{\epsilon}^L$ and use $|\sqrt{\epsilon}u_y|_*\leqslant\|z\|^{**}$. The remaining factors are estimated using Moser estimates. If p_y appears we write

$$p_y(t,x,y)=y\int_0^1 p_{yy}(t,x,sy)ds. \tag{3.62}$$

As in step **4** (see (3.54)) we use the strong form of (3.45) to borrow a factor of η from the right member of the L^2 pairing. We then control the L^∞ norm of ηp_{yy} using the equation as in (3.55).

6. To finish case (b) of (3.50) we must consider

$$\mathcal{A}M^{\langle s-s'\rangle}(v^a+\sqrt{\epsilon}^L z^n)\left(M^{s'}\partial_y(u_2^a+\sqrt{\epsilon}^L u_2^n)\right)(M^t\partial_y z), \tag{3.63}$$

where $s+t=j,\ t\leqslant j$.
The cases $t=j$, $t=j-1$ and $2\leqslant t\leqslant j-2$ are treated as in step **5**, but note that, because of the better control on u^n, it is not necessary (and in fact may not be possible) to extract a factor of y to multiply $M^{s'}\partial_y(u_2^a+\sqrt{\epsilon}^L u_2^n)$. When $t=1$ we have

$$\mu^{k-2-j}|Mz_y|_{0,\mu,\gamma}\leqslant\|z\|_{k,\mu,\gamma}. \tag{3.64}$$

Since $s'\leqslant j-1$ and $s-s'\leqslant j-1\leqslant k-3$, the L^∞ norm of the remaining factors can be controlled using Corollary 3.8.

Finally, suppose $t=0$ and that both $s'\leqslant j-1$ and $s-s'\leqslant j-1\leqslant k-3$. We have

$$\mu^{k-2-j}|z_y|_{0,\mu,\gamma}\leqslant\|z\|_{k,\mu,\gamma}, \tag{3.65}$$

and the L^∞ norm of the remaining factors can be controlled using Corollary 3.8. Two cases remain:

$$\begin{aligned}&(a)\ \mathcal{A}\ M^{\langle j\rangle}(v^a+\epsilon^L z^n)\ \partial_y(u_2^a+\sqrt{\epsilon}^L u_2^n)\ z_y\\ &(b)\ \mathcal{A}\ (M^j\partial_y(u_2^a+\sqrt{\epsilon}^L u_2^n))\ z_y.\end{aligned} \tag{3.66}$$

In case (a) if u_y appears in a product where $\sqrt{\epsilon}^L$ is a factor, we use $|\sqrt{\epsilon}u_y|_*\leqslant\|z\|^{**}$, $|\sqrt{\epsilon}\partial_y u_2^n|_*\leqslant 1$, and use Moser estimates to handle the remaining factors. (The case where $\sqrt{\epsilon}^L$ does not appear is handled using (3.65).) If p_y appears, the only change is to use (3.62) and argue as at the end of step 5 to control $|\eta p_{yy}|_*$.

In case (b) of (3.66) if u_y appears in a product where $\sqrt{\epsilon}^L$ is a factor, we use $|\sqrt{\epsilon}u_y|_*\leqslant\|z\|^{**}$ and

$$\mu^{k-2-j}|M^j\partial_y u_2^n|_{0,\mu,\gamma}\leqslant\|z^n\|_{k,\mu,\gamma}. \tag{3.67}$$

If p_y appears, the only change is to use (3.62) and argue as at the end of step 5 to control $|\eta p_{yy}|_*$.

This completes the treatment of commutators involving $A_2\partial_y$.

7. The commutators involving $A_0\partial_t$, $A_1\partial_x$, and the order zero terms in $\mathcal{L}(z^n)$ are handled similarly but more easily, since at most one purely normal derivative ∂_y appears in all terms *except* commutator terms like

$$\mathcal{A}M^{\langle r\rangle}(v^a+\sqrt{\epsilon}^L z^n)M^s(\partial_y^2 v_a)M^t z,\ r+s+t=j\leqslant k-2, \tag{3.68}$$

which arise from the third line of (1.16). Although $|M^s(\partial_y^2 v_a)|_* \leqslant \frac{C}{\sqrt{\epsilon}}$, this is not a problem since Moser estimates imply

$$(3.69)\quad \mu^{k-2-j}|\mathcal{A}M^{\langle r\rangle}(v^a+\sqrt{\epsilon}^L z^n)M^s(\partial_y^2 v^a)M^t z|_{0,\mu,\gamma} \leqslant \frac{C}{\sqrt{\epsilon}}\left(|z|_*|z^n|_{k-2,\mu,\gamma}+|z|_{k-2,\mu,\gamma}\right),$$

and $|z|_{k-2,\mu,\gamma} \leqslant (\frac{\epsilon}{\gamma})^2|z|_{k,\mu,\gamma}$.

8. Consider the commutators $\epsilon[B_{12}\partial_{xy}, M^j\partial_y]z$ and $\epsilon[B_{22}\partial_{yy}, M^j\partial_y]z$, $j \leqslant k-2$, which are nonzero when $M^j = M_0^{\alpha_0}M_1^{\alpha_1}M_2^{\alpha_2}$ with $\alpha_2 > 0$. Care is needed for terms like

$$(3.70)\qquad \epsilon[\partial_y^2, M^j\partial_y]u_2 = \epsilon[\partial_y^2, M^j]\partial_y u_2,$$

which by (3.20) is a linear combination of terms of the form $\epsilon M_0^{\alpha_0}M_1^{\alpha_1}M_2^t\partial_y^3 u_2$, where $t \leqslant \alpha_2 - 1$. Rewriting this term by substituting for $\epsilon\partial_y^2 u_2$ its expression coming from the third component of equation (3.1)(a), we obtain several terms including, for example, $M_0^{\alpha_0}M_1^{\alpha_1}M^t p_{yy}$. In the strong form of (3.45) this term gives rise to

$$(3.71)\qquad \left|\left(\mu^{k-2-j}M_0^{\alpha_0}M_1^{\alpha_1}M^t p_{yy}, \mu^{k-2-j}M^j\partial_y u_2\right)\right|^{\frac{1}{2}}.$$

We extract a factor of y from one of the M_2 derivatives on the right side of the pairing, and move it to the left. Using (3.20) we estimate the left side of the pairing

$$(3.72)\qquad |\mu^{k-2-j}yM_0^{\alpha_0}M_1^{\alpha_1}M^t p_{yy}|_{0,\mu,\gamma} \leqslant C\|p\|_{k,\mu,\gamma}.$$

On the right for $s \leqslant \alpha_2 - 1$ we must estimate terms like

$$(3.73)\quad |\mu^{k-2-j}M_0^{\alpha_0}M_1^{\alpha_1}M^s\partial_y^2 u_2|_{0,\mu,\gamma} \leqslant \frac{1}{\sqrt{\epsilon}}\|z\|_{k-1,\mu,\gamma} \leqslant \frac{1}{\sqrt{\epsilon}}\frac{\epsilon}{\gamma}\|z\|_{k,\mu,\gamma}.$$

The other estimates involving B_{22} and B_{12} use earlier arguments.

9. Next consider $[\eta\Delta, M^j\partial_y]p$, $j \leqslant k-2$. Parallel to (3.70) $[\eta\partial_y^2, M^j\partial_y]p$ is a linear combination of terms of the form $\eta M_0^{\alpha_0}M_1^{\alpha_1}M_2^t\partial_y^3 p$, where $t \leqslant \alpha_2 - 1$. Rewriting ηp_{yy} using equation (3.35), we obtain, for example,

$$(3.74)\qquad \mu^{k-2-j}\eta\left|\left(M_0^{\alpha_0}M_1^{\alpha_1}M^t p_{xxy}\right)\right|_{0,\mu,\gamma} \leqslant C\eta|p_y|_{k-1,\mu,\gamma},$$

since the total order of tangential derivatives on p_y is $\leqslant j+1 \leqslant k-1$ and $k-2-j = (k-1)-(j+1)$. The right side of (3.74) is dominated by the left side of the tangential estimate (3.22). The other terms arising from this commutator are handled by earlier arguments.

10. Now we estimate the boundary term appearing in the right side of (3.45):

$$\sqrt{\epsilon}|\langle e^{-\gamma t}b_{22}\partial_y U^j, e^{-\gamma t}U^j\rangle|^{\frac{1}{2}}. \tag{3.75}$$

Writing $M^j = M_0^{\alpha_0}M_1^{\alpha_1}M_2^{\alpha_2}$ and using (3.20) and the fact that $M_2 w = 0$ on $y = 0$, we see that it suffices to estimate for $j \leqslant k-2$:

$$\begin{aligned}\sqrt{\epsilon}\left|\left\langle e^{-\gamma t}\mu^{k-2-j}M_0^{\alpha_0}M_1^{\alpha_1}\partial_y^2 u, e^{-\gamma t}\mu^{k-2-j}M^j\partial_y u\right\rangle\right|^{\frac{1}{2}} \leqslant \\ \sqrt{\epsilon}\langle e^{-\gamma t}\mu^{k-2-j}M_0^{\alpha_0}M_1^{\alpha_1}\partial_y^2 u\rangle + \sqrt{\epsilon}\langle e^{-\gamma t}\mu^{k-2-j}M^j\partial_y u\rangle\end{aligned} \tag{3.76}$$

When u in (3.76) is replaced by u_2, we have by a standard trace estimate

$$\begin{aligned}\sqrt{\epsilon}\langle e^{-\gamma t}\mu^{k-2-j}M^j\partial_y u_2\rangle \leqslant \sqrt{|\partial_y u_2|_{k-2,\mu,\gamma}}\sqrt{\epsilon|\partial_y^2 u_2|_{k-2,\mu,\gamma}} \leqslant \\ C_\delta|\partial_y u_2|_{k-2,\mu,\gamma} + \delta\epsilon|\partial_y^2 u_2|_{k-2,\mu,\gamma},\end{aligned} \tag{3.77}$$

and each of these terms can be absorbed by the left side of (3.45), after summing over j.

To estimate the first term on the right in (3.76), we use the third component of equation (3.1)(a) and the Navier boundary conditions satisfied by u to write

$$\epsilon(\lambda+\mu)\partial_y^2 u_2 = -\epsilon\mu\partial_{xy}u_1 + L.O.T. = -\epsilon\mu\partial_x u_1 + L.O.T. \text{ on } y=0, \tag{3.78}$$

where $L.O.T.$ represents the contribution from the zero order terms of $\mathcal{L}(z^n)$ (recall (1.16)). Thus,

$$\begin{aligned}\sqrt{\epsilon}\langle e^{-\gamma t}\mu^{k-2-j}M_0^{\alpha_0}M_1^{\alpha_1}\partial_y^2 u_2\rangle \leqslant \sqrt{\epsilon}\langle\partial_y^2 u_2\rangle_{k-2,\mu,\gamma} \leqslant \\ C\sqrt{\epsilon}\langle\partial_x u_1\rangle_{k-2,\mu,\gamma} + \frac{C}{\sqrt{\epsilon}}\langle L.O.T.\rangle_{k-2,\mu,\lambda}.\end{aligned} \tag{3.79}$$

The first term on the right can be absorbed by the boundary norms in $\|z\|'_{k,\mu,\gamma}$. The second term is dominated by a sum of terms of the form

$$\frac{C}{\sqrt{\epsilon}}\mu^{k-2-j}\langle M^j\left(\mathcal{B}(v^a,\partial_t v^a,\nabla v^a,\sqrt{\epsilon}^L z^n)z\right)\rangle_{0,\mu,\gamma} \tag{3.80}$$

where $\mathcal{B}$ is a smooth function of its arguments. By Moser estimates (3.80) is $\leqslant$

$$\frac{C}{\sqrt{\epsilon}}\left(|z|_*\langle\sqrt{\epsilon}^L z^n\rangle_{k-2,\mu,\gamma} + \langle z\rangle_{k-2,\mu,\gamma}\right). \tag{3.81}$$

Since

$$\langle z\rangle_{k-2,\mu,\gamma} \leqslant \sqrt{|z|_{k-2,\mu,\gamma}}\sqrt{|\partial_y z|_{k-2,\mu,\gamma}} \leqslant \frac{|z|_{k-2,\mu,\gamma}}{\sqrt{\epsilon}} + \sqrt{\epsilon}|\partial_y z|_{k-2,\mu,\gamma}, \tag{3.82}$$

(3.81) is dominated by the right side of (3.42)$+\frac{C}{\sqrt{\gamma}}\|z\|_{k,\mu,\gamma}$.

To estimate (3.76) when u is replaced by u_1, we use the second component of equation (3.1)(a) and the Navier boundary conditions satisfied by u to write

$$\begin{aligned}\epsilon\lambda\partial_y^2 u_1 = -\epsilon(\lambda+\mu)\partial_{xx}u_1 - \epsilon\mu\partial_{xy}u_2 + \rho(p^a + \sqrt{\epsilon}^l p^n)\partial_t u_1 + p_x +\\ \rho(p^a + \sqrt{\epsilon}^l p^n)(u_1^a + \sqrt{\epsilon}^L u_1^n)\partial_x u_1 + L.O.T.\,.\end{aligned} \tag{3.83}$$

Using the Navier boundary condition to write the pairing in (3.76) now as

$$\left|\left\langle e^{-\gamma t}\mu^{k-2-j}M_0^{\alpha_0}M_1^{\alpha_1}\epsilon\partial_y^2 u_1, e^{-\gamma t}\mu^{k-2-j}M^j u_1\right\rangle\right|^{\frac{1}{2}}, \tag{3.84}$$

and substituting for $\epsilon\partial_y^2 u_1$ using (3.83), we estimate for example

$$\begin{aligned}&\left|\left\langle e^{-\gamma t}\mu^{k-2-j}M_0^{\alpha_0}M_1^{\alpha_1}p_x, e^{-\gamma t}\mu^{k-2-j}M^j u_1\right\rangle\right|^{\frac{1}{2}}\\ &\quad= \left|\left\langle e^{-\gamma t}\mu^{k-2-j}M_0^{\alpha_0}M_1^{\alpha_1}p, e^{-\gamma t}\mu^{k-2-j}M^j \partial_x u_1\right\rangle\right|^{\frac{1}{2}}\\ &\quad\leqslant \langle p\rangle_{k-2,\mu,\gamma} + \langle u_1\rangle_{k-1,\mu,\gamma}\\ &\quad\leqslant |p|_{k-2,\mu,\gamma} + |p_y|_{k-2,\mu,\gamma} + \frac{\epsilon}{\gamma}\langle u_1\rangle_{k,\mu,\gamma}.\end{aligned} \tag{3.85}$$

The u_1 term can be absorbed by the boundary norms in $\|z\|'_{k,\mu,\gamma}$ and the p term is dominated by the right side of (3.42)$+\frac{C}{\sqrt{\gamma}}\|z\|_{k,\mu,\gamma}$.

Similarly, the term where $\epsilon\partial_{xy}u_2$ appears in place of p_x in (3.85) is treated by integrating by parts in x and using (3.77). The term where $\rho(p^a + \sqrt{\epsilon}^L p^n)\partial_t u_1$ appears in place of p_x in (3.85) is easily estimated by applying Moser estimates to the left entry of the pairing. The remaining terms arising from (3.84) require no new arguments, so this completes the estimates of boundary terms.

11. Combining the tangential estimate (3.22) with what we have proved in steps **1-10** above, we conclude

$$\begin{aligned}&\|z\|'_{k,\mu,\gamma} + \sqrt{\eta}|\nabla p|_{k,\mu,\gamma} + \eta|\nabla p_y|_{k-2,\mu,\gamma}\\ &\qquad + \sum_{j=0}^{k-2}\sqrt{\gamma}\left|\begin{pmatrix}\sqrt{\eta}P^j\\ U^j\end{pmatrix}\right|_{0,\mu,\gamma} + |\partial_y\sqrt{\epsilon}\nabla u|_{k-2,\mu,\gamma}\\ &\leqslant C\left(\frac{1}{\sqrt{\gamma}} + \|z\|^{**}\right)\|z^n\|_{k,\mu,\gamma} + \frac{C}{\sqrt{\gamma}}\|z\|_{k,\mu,\gamma} + C(\gamma)\sqrt{\epsilon}^{M-L-2k}.\end{aligned} \tag{3.86}$$

for γ, ϵ, and η as described in the statement of Proposition 3.14. This does not immediately imply the desired estimate (3.42) because of the factor $\sqrt{\eta}$ appearing in the sum on the left side of (3.86). To finish we show for

$j \leqslant k-2$ that

$$\sqrt{\gamma}\mu^{k-2-j}|M^j p_y|_{0,\mu,\gamma} \leqslant (\text{right side of (3.86)}). \tag{3.87}$$

Rewriting p_y using the third component of equation (1.16), we must estimate several terms including for example

$$\sqrt{\gamma}\mu^{k-2-j}|M^j(\epsilon\partial_y^2 u_2)|_{0,\mu,\gamma} \leqslant |\partial_y\sqrt{\epsilon}\nabla u|_{k-2,\mu,\gamma} \text{ for } \epsilon < \epsilon(\gamma). \tag{3.88}$$

Another term is

$$\begin{aligned}\sqrt{\gamma}\mu^{k-2-j}\left|M^j\left(\rho(p^a+\sqrt{\epsilon}^L p^n)(u_2^a+\sqrt{\epsilon}^L u_2^n)\partial_y u_2\right)\right|_{0,\mu,\gamma}\\ \leqslant C\sqrt{\gamma}\left(|z|_{k-1,\mu,\gamma}+|z|_*\|z^n\|'_{k-1,\mu,\gamma}\right).\end{aligned} \tag{3.89}$$

Here we have extracted a factor of y from $(u_2^a+\sqrt{\epsilon}^L u_2^n)$ to multiply $\partial_y u_2$ and applied Moser estimates. Now (3.13) yields the required estimate. The remaining terms require no new arguments. This completes the proof of Proposition 3.14.

□

3.3. Convergence of the iteration scheme.

The next step is to construct the $(n+1)$-st iterate z^{n+1} solving (1.16)-(1.17) by taking a suitable limit as $\eta \to 0$ of the functions $z^{n+1,\eta}$ estimated in Proposition 3.14.

PROPOSITION 3.15 (Induction step). — *Consider the iteration scheme* (1.16)-(1.17) *and suppose* $d=2$ *or* $d=3$. *Assume* $L \geqslant 2$, $k \geqslant 5$, *and* $M-L-2k>0$. *For* $C(\gamma)$ *as in* (3.42) *assume*

$$\begin{aligned}&(a)\ \|z^n\|^{**} \leqslant 1\\ &(b)\ \|z^n\|_{k,\mu,\gamma} \leqslant 2C(\gamma)\sqrt{\epsilon}^{M-L-2k} \leqslant 1\end{aligned} \tag{3.90}$$

for $\gamma \geqslant \gamma_1$ *large enough and* $0<\epsilon\leqslant\epsilon_1(\gamma)$ *sufficiently small. There exists* γ_0 *and a positive decreasing function* $\epsilon_0(\gamma)$ *such that for* $\gamma\geqslant\gamma_0$ *and* $0<\epsilon\leqslant\epsilon_0(\gamma)$, *we have:*

(i) $z^{n+1,\eta}$ *as in* (3.1) *satisfies the estimates* (3.90);

(ii) the $(n+1)$*-st iterate* z^{n+1} *as in* (1.16)-(1.17) *exists and satisfies the estimates* (3.90). *The choices of* γ_0 *and* $\epsilon_0(\gamma)$ *can be made independently of* $\eta\in(0,1]$ *and* $n\in\mathbb{N}$.

Proof. —

1. First we show that (i) holds. Choose $\epsilon_0(\gamma) \leqslant \epsilon_1(\gamma)$ so that $0 < \epsilon \leqslant \epsilon_0(\gamma)$ implies $2C(\gamma)\sqrt{\epsilon}^{M-L-2k} \leqslant 1$. Since $k \geqslant 5$, Corollary 3.8 implies

$$\|z^{n+1,\eta}\|^{**} \leqslant C\mu^{-(2-\frac{d}{2})}e^{\gamma T}\|z^{n+1,\eta}\|_{k,\mu,\gamma}. \tag{3.91}$$

The estimate (3.42) implies

$$\|z^{n+1,\eta}\|_{k,\mu,\gamma} \leqslant C_1\left(\frac{1}{\sqrt{\gamma}} + C_2\mu^{-(2-\frac{d}{2})}e^{\gamma T}\|z^{n+1,\eta}\|_{k,\mu,\gamma}\right)2C(\gamma)\sqrt{\epsilon}^{M-L-2k} \tag{3.92}$$

$$+C(\gamma)\sqrt{\epsilon}^{M-L-2k}. \tag{3.93}$$

Choose $\gamma_0 \geqslant \gamma_1$ and decrease $\epsilon_0(\gamma)$ if necessary so that for $\gamma \geqslant \gamma_0$ and $0 < \epsilon \leqslant \epsilon_0(\gamma)$

$$\frac{2C_1}{\sqrt{\gamma}} \leqslant \frac{1}{3} \quad \text{and} \quad 2C_1C(\gamma)\sqrt{\epsilon}^{M-L-2k}C_2\mu^{-(2-\frac{d}{2})}e^{\gamma T} \leqslant \frac{1}{3} \quad (\text{recall } \mu = \frac{\gamma}{\epsilon}). \tag{3.94}$$

For such γ and ϵ (3.92) now implies

$$\|z^{n+1,\eta}\|_{k,\mu,\gamma} \leqslant 2C(\gamma)\sqrt{\epsilon}^{M-L-2k} \text{ for } \eta \in (0,1]. \tag{3.95}$$

2. For ϵ and n fixed we consider a sequence z^{n+1,η_k}, where $\eta_k \to 0$. The uniform estimates in (i) imply that a subsequence has a weak limit which satisfies the same estimates. We will show that there exists Z such that $\|Z\|'_{0,\mu,\gamma}$ is finite and

$$\lim_{\eta_k \to 0} \|z^{n+1,\eta_k} - Z\|'_{0,\mu,\gamma} = 0. \tag{3.96}$$

Thus, the weak limit must equal Z, it satisfies the linear problem (1.16)-(1.17), and $z^{n+1} := Z$ satisfies the estimates (3.90).

3. To prove (3.96) we first need to improve the estimate on $|\nabla p_y^{n+1,\eta}|_{0,\mu,\gamma}$ over what is given by (3.86). We will show

$$|\sqrt{\eta}\nabla p_y^{n+1,\eta}|_{0,\mu,\gamma} \leqslant C \tag{3.97}$$

for C independent of n and of ϵ, η, and γ as described (after enlarging γ_0 and shrinking $\epsilon_0(\gamma)$ if necessary. Setting $z = (p,u) = z^{n+1,\eta}$ as before and differentiating (3.1)(a) with respect to y, we obtain

$$\mathcal{L}(z^n)z_y - \begin{pmatrix} \eta\Delta p_y \\ 0 \\ 0 \end{pmatrix} = -\sqrt{\epsilon}^{M-L}\partial_y R^M + (\text{interior commutators}) := \mathcal{G} \tag{3.98}$$

$$p_y = 0,\ u_1 - \partial_y u_1 = 0,\ u_2 = 0 \text{ on } y = 0.$$

Pairing with z_y and arguing as in the proof of the L^2 estimate, Proposition 3.2, we obtain

$$\begin{aligned}\sqrt{\gamma}|z_y|_{0,\mu,\gamma} + \sqrt{\eta}|\nabla p_y|_{0,\mu,\gamma} + \sqrt{\epsilon}|\nabla u_y|_{0,\mu,\gamma}& \\ \leqslant \frac{C}{\sqrt{\gamma}}|\mathcal{G}|_{0,\mu,\gamma} + C\sqrt{\epsilon}|\langle e^{-\gamma t}u_{yy}, e^{-\gamma t}u_y\rangle|^{\frac{1}{2}}&.\end{aligned} \tag{3.99}$$

Here we have used the fact that the boundary terms associated to $A_2\partial_y$ and $\eta\Delta$ are both zero. The argument in step **10** of the proof of Proposition 3.14 shows that the boundary term on the right side of (3.99) is bounded. Consider the interior commutator

$$|[A_2\partial_y, \partial_y]z|_{0,\mu,\gamma} = |(\partial_y A_2)z_y|_{0,\mu,\gamma}. \tag{3.100}$$

After extracting a factor of y from $\partial_p A_2$ to multiply $\partial_y(p^a + \sqrt{\epsilon}^L p^n)$ in the expression for $\partial_y A_2$ given by (3.10), we see that $|\partial_y A_2|_* \leqslant C$, so the contribution from this commutator can be absorbed by the left side of (3.99). The other interior commutators are treated similarly. This gives (3.97).

Set $\zeta_{k_i,k_j} = (P, U) := z^{n+1,\eta_{k_i}} - z^{n+1,\eta_{k_j}}$. From (3.1) we find

$$\begin{aligned}&\mathcal{L}(z^n)\zeta_{k_i,k_j} = \begin{pmatrix}\eta_{k_i}\Delta P\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}(\eta_{k_i} - \eta_{k_j})\Delta p^{n+1,\eta_{k_j}}\\ 0\\ 0\end{pmatrix}\\ &P_y = 0,\ U_1 - \partial_y U_1 = 0, U_2 = 0 \text{ on } y = 0\\ &\zeta_{k_i,k_j} = 0 \text{ in } t \leqslant -\frac{\delta}{2}.\end{aligned} \tag{3.101}$$

Applying the L^2 estimate of Proposition 3.2(b) to solutions of (3.101) we have, provided $\eta_{k_i} \leqslant \eta_{k_j}$,

$$\begin{aligned}\|\zeta_{k_i,k_j}\|'_{0,\mu,\gamma} &\leqslant \frac{C}{\sqrt{\gamma}}\left|(\eta_{k_i} - \eta_{k_j})\Delta p^{n+1,\eta_{k_j}}\right|_{0,\mu,\gamma}\\ &\leqslant \frac{C}{\sqrt{\gamma}}\sqrt{\eta_{k_j}}\left(\sqrt{\eta_{k_j}}|\Delta p^{n+1,\eta_{k_j}}|_{0,\mu,\gamma}\right).\end{aligned} \tag{3.102}$$

With (3.97) and (i) this implies that the sequence z^{n+1,η_k} (indexed by η_k) is Cauchy in the $\|\cdot\|'_{0,\mu,\gamma}$ norm and hence there exists Z as in (3.96). □

End of the proof of Theorem 1.3. It remains to show that the sequence of iterates z^n just constructed converges to a solution of the nonlinear error problem (1.15).

Set $\zeta^{n+1} = (P^{n+1}, U^{n+1}) := z^{n+1} - z^n$. From (1.16)-(1.17) we see that ζ^{n+1} satisfies

$$\begin{aligned}&\mathcal{L}(z^n)\zeta^{n+1} = -\left(\mathcal{L}(z^n) - \mathcal{L}(z^{n-1})\right) z^n := \mathcal{H}^n\\ &U_1^{n+1} - \partial_y U_1^{n+1} = 0,\ U_2^{n+1} = 0 \text{ on } y = 0\\ &\zeta^{n+1} = 0 \text{ in } t \leqslant -\frac{\delta}{2}.\end{aligned} \tag{3.103}$$

Applying the L^2 estimate of Proposition 3.2(a) to solutions of (3.103) we have

$$\|\zeta^{n+1}\|'_{0,\mu,\gamma} \leqslant \frac{C}{\sqrt{\gamma}} |\mathcal{H}^n|_{0,\mu,\gamma} \leqslant \frac{C}{\sqrt{\gamma}} \sqrt{\epsilon}^{L-1} \|\zeta^n\|'_{0,\mu,\gamma}. \tag{3.104}$$

In deriving the second inequality we consider, for example,

$$\begin{aligned}&|\left(A_2(v^a + \sqrt{\epsilon}^L z^n) - A_2(v^a + \sqrt{\epsilon}^L z^{n-1})\right) \partial_y z^n|_{0,\mu,\gamma} =\\ &\quad |\sqrt{\epsilon}^L \zeta^n \cdot \left(\int_0^1 d_v A_2(v^a + \sqrt{\epsilon}^L z^{n-1} + s\sqrt{\epsilon}^L (z^n - z^{n-1}))ds\right) \partial_y z^n|_{0,\mu,\gamma}\end{aligned} \tag{3.105}$$

In the case of the term involving $\partial_p A_2$ we extract a factor of y from this term and use the uniform estimate $\|z^n\|^{**} \leqslant 1$, (3.90)(a). In the case of the term involving $\partial_{u_2} A_2$ we extract a factor of y from $\sqrt{\epsilon}^L (U_2^n - U_2^{n-1})$ and use (3.90)(a) and

$$\sqrt{\epsilon}^{L-1} |(\sqrt{\epsilon}\partial_y (U_2^n - U_2^{n-1})|_{0,\mu,\gamma} \leqslant \sqrt{\epsilon}^{L-1} \|\zeta^n\|'_{0,\mu,\gamma}. \tag{3.106}$$

The terms like (3.105) associated to $A_0\partial_t$, $A_1\partial_x$ as well as those coming from the zero order terms of $\mathcal{L}(z^n)$ are handled similarly but more easily.

Increasing γ_0 if necessary, we conclude from (3.104) that for $\gamma \geqslant \gamma_0$ and $0 < \epsilon \leqslant \epsilon_0(\gamma)$ the sequence z^n converges in the $\|\cdot\|'_0$ norm to some z. On the other hand the uniform estimates (3.90)(a) and (b) imply that the sequence z^n has a subsequence converging weakly to some limit z' that also satisfies (3.90)(a) and (b). Necessarily, we have $z = z'$. Moreover, by interpolating between the low norm $\|\cdot\|'_{0,\mu,\gamma}$ and the high norm $\|\cdot\|_{k,\mu,\gamma}$ we conclude that the convergence $z^n \to z$ is strong enough to imply that z is a solution of the nonlinear error problem (1.15). This finishes the proof of Theorem 1.3.

3.4. Layer formation as an exact solution evolves.

In Theorem 1.3 we have proved the existence of an exact solution to (1.1) close to a given approximate solution whose expansion exhibits a boundary layer. We can modify this result slightly to rigorously conclude under certain circumstances that a boundary layer will form as time evolves in an exact solution that does not initially possess a layer. Consider for example the problem

(3.107)
$$\begin{aligned}
&\partial_t\rho^\epsilon+\nabla\cdot(\rho^\epsilon u^\epsilon)=0\\
&\partial_t(\rho^\epsilon u^\epsilon)+\nabla\cdot(\rho^\epsilon u^\epsilon\otimes u^\epsilon)+\nabla p(\rho^\epsilon)-\epsilon\left(\lambda\Delta u^\epsilon+\mu\begin{pmatrix}\mathrm{div}\partial_x u^\epsilon\\ \mathrm{div}\partial_y u^\epsilon\end{pmatrix}\right)=F\\
&u_1^\epsilon-\frac{\partial u_1^\epsilon}{\partial y}=0,\ \ u_2^\epsilon=0\quad\text{on}\quad y=0\\
&(\rho^\epsilon,u^\epsilon)|_{t\leqslant 0}=(\underline{\rho},0),
\end{aligned}$$

where the forcing term $F(t,x,y)$ is (say) a C^∞ compactly supported function on $\{(t,x,y): y\geqslant 0\}$ that is supported in $t\geqslant 0$, and $\underline{\rho}$ is a positive constant. Observe that compatibility conditions hold to all orders at the corner $\{t=0,y=0\}$, and that $(\underline{\rho},0)$ is an exact solution in $t\leqslant 0$. An approximate high-order boundary layer solution w^a can be constructed just as before, except that now the inviscid solution $w^0:=(\rho^{I,0},u^{I,0})$ should be constructed to satisfy

$$\begin{aligned}
&\partial_t\rho^{I,0}+\nabla\cdot(\rho^{I,0}u^{I,0})=0\\
&\partial_t(\rho^{I,0}u^{I,0})+\nabla\cdot(\rho^{I,0}u^{I,0}\otimes u^{I,0})+\nabla p(\rho^{I,0})=F\\
&u_2^{I,0}=0 \text{ on } y=0\\
&(\rho^{I,0},u^{I,0})=(\underline{\rho},0)\text{ in } t\leqslant 0.
\end{aligned}\tag{3.108}$$

Since $u_1^{B,1}$ satisfies (2.40), (2.41) and the boundary data

$$\partial_\eta u_1^{B,1}|_{\eta=0}=(u_1^{I,0}-\partial_y u_1^{I,0})(t,x,0)\tag{3.109}$$

in (2.41) is generally nonzero, a boundary layer of amplitude $O(\sqrt{\epsilon})$ and width $\sqrt{\epsilon}$ will form in u_1 as before. The proof that the exact solution $w^\epsilon=(\rho^\epsilon,u^\epsilon)$ of (3.107) is close in the sense of Corollary 1.4 to the approximate solution w^a and to the inviscid solution w^0 goes through exactly as before. In fact now we obtain an exact solution on $(-\infty,T_0]\times\{(x,y):y\geqslant 0\}$ for a fixed T_0 independent of ϵ small.

4. Appendix

4.1. Existence of smooth solutions for fixed viscosity

In this section we take $\epsilon = 1$ and prove the short-time existence of smooth solutions to (1.1) assuming that initial data is given at $t = 0$ satisfying corner compatibility conditions. The first part of the argument parallels closely, but is simpler than, the arguments given in the proof of Theorem 1.3. This part provides a good estimate of $|(\partial_y^m \rho, \partial_y^n u)|_{L^2}$ for $m \leqslant 1$, $n \leqslant 2$, and similar estimates when ρ and u are replaced by arbitrarily high order tangential derivatives $M^r\rho$, $M^r u$. By Sobolev this is enough to conclude that ρ is continuous and u is C^1 (see Theorem 4.2).

The second part of the argument takes advantage of the fact that the combination

$$p'(\rho)\partial_y \rho - (\lambda + \mu)\partial_y^2 u_2 \tag{4.1}$$

is more regular than the individual terms to show that for sufficiently smooth (and corner-compatible) initial data, $(\rho, u) \in C^{m-1} \times C^m$ for any given m. We note that in several of his papers (e.g., [8, 9]), David Hoff has used the better regularity of a similar combination, the "effective viscous flux" $p(\rho) - (\lambda + \mu)\text{div} u$, in other compressible flow problems.

Estimates without loss of derivatives. We work with the symmetric form of the equations given by (1.8), where $v = (p, u)$:

(4.2)

$$(a)\ \mathcal{E}(v) := A_0(v)v_t + A_1(v)v_x + A_2(v)v_y - (B_{11}v_{xx} + B_{12}v_{xy} + B_{22}v_{yy}) = 0,$$

$$(b)\ u_1 - u_{1y} = 0, u_2 = 0 \text{ on } y = 0,$$

$$(c)\ v|_{t=0} = (p^0, u^0),$$

where $(p^0, u^0) \in H^s$ for s large, satisfies corner compatibility conditions (defined below) to integral order $r \leqslant s$, and $p^0 \geqslant C > 0$ on its domain. We will focus on the changes that are needed in the proof of Theorem 1.3 to obtain estimates like those just described. For the moment we assume s and r are large; precise restrictions are given later.

The first step is to take an H^s extension of the initial data to $y \leqslant 0$ and obtain a smooth solution $\tilde{v}$ (see [14]) to the pure initial value problem on $[0, T_0] \times \mathbb{R}^2$ for some $T_0 > 0$.

$$\begin{aligned} &\mathcal{E}(\tilde{v}) = 0 \\ &\tilde{v}|_{t=0} = (p^0, u^0). \end{aligned} \tag{4.3}$$

Let us write the boundary conditions in (4.2) as $\Gamma v|_{y=0} = 0$. Corner compatibility to order r of the initial data in (4.2) is characterized by the property that $g \in H^r(\{y = 0\})$, where

$$g = (g_1, g_2) := \begin{cases} \Gamma\tilde{v}|_{y=0}, \text{ in } t \geqslant 0 \\ 0 \text{ in } t < 0 \end{cases} . \tag{4.4}$$

After taking an H^s extension of $\tilde{v}$ into $t \leqslant 0$ that remains close in $L^\infty(t \leqslant 0)$ to $\tilde{v}|_{t=0}$, we look for a solution to (4.2) of the form

$$v = \tilde{v} + V \tag{4.5}$$

where for some $T_1 > 0$ V is a solution of the forward problem on $\Omega_{T_1} := (-\infty, T_1] \times \{(x, y) : y \geqslant 0\}$,

$$\begin{aligned} &\mathcal{E}(\tilde{v} + V) - \mathcal{E}(\tilde{v}) = 0 \\ &\Gamma V = -(g_1, g_2) \text{ on } y = 0 \\ &V = 0 \text{ in } t \leqslant 0. \end{aligned} \tag{4.6}$$

The nonvanishing of g_2 would lead to an unmanageable boundary term (associated to $(B_{22}V_{yy}, V)$ in the L^2 estimate. So we lift $g_2(t, x)$ to a function $G_2(t, x, y) \in H^r(\Omega_{T_0})$ satisfying

$$G_2|_{y=0} = g_2, \;\; G_2 = 0 \text{ in } t \leqslant 0, \tag{4.7}$$

set $G := (0, 0, -G_2)$ and $V = G + z$, and reduce to solving the following problem for $z = (p', u_1', u_2')$:

$$\begin{aligned} &\mathcal{E}(\tilde{v} + G + z) - \mathcal{E}(\tilde{v}) = 0 \\ &u_1' - \partial_y u_1' = -g_1, \;\; u_2' = 0 \text{ on } y = 0 \\ &z = 0 \text{ in } t \leqslant 0. \end{aligned} \tag{4.8}$$

Since $\tilde{v} + G$ plays a role below similar to that of v^a in the proof of Theorem 1.3, we set

$$v^a := \tilde{v} + G \tag{4.9}$$

and find after a short computation that (4.8) has the form

(4.10)

$$\begin{aligned} (a)\; & A_0(v^a + z)\partial_t z + A_1(v^a + z)\partial_x z + A_2(v^a + z)\partial_y z \\ &\quad + \mathcal{C}_1(\tilde{v}, \partial_t\tilde{v}, \nabla\tilde{v}, G + z)z - (B_{11}z_{xx} + B_{12}z_{xy} + B_{22}z_{yy}) \\ &\qquad = -\mathcal{C}_1(\tilde{v}, \partial_t\tilde{v}, \nabla\tilde{v}, G + z)G + \mathcal{C}_2(v^a + z)(\partial_t G, \nabla G) + F \\ (b)\; & u_1' - \partial_y u_1' = -g_1, \;\; u_2' = 0 \text{ on } y = 0 \\ (c)\; & z = 0 \text{ in } t \leqslant 0. \end{aligned}$$

Here the zero-order terms $\mathcal{C}_1$, $\mathcal{C}_2$ are smooth functions of their arguments and

$$F = B_{11}G_{xx} + B_{12}G_{xy} + B_{22}G_{yy} \in H^{r-2}(\Omega_{T_0}). \tag{4.11}$$

Henceforth we drop the primes and write $z = (p, u)$.

The problem (4.10) can be solved by an obvious iteration scheme similar to (1.16), (1.17). Again one considers a modified problem like (3.3) with a term $\eta\Delta p^\eta$ in the mass equation and the extra boundary condition $p_y^\eta = 0$, and proves L^2 estimates like those in Proposition 3.2 for the linearized problems (3.2) and (3.3). The L^2 estimate has the same form as (3.7) with ϵ set equal to 1. We are no longer free to take ϵ small, of course, but we are free to take T small. Thus, for example, the hypothesis of Proposition 3.2: "Suppose there exists $\epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0]$:

$$|z^n, \sqrt{\epsilon}^L(\partial_t z^n, \partial_x z^n, y\partial_y p^n, \partial_y u^n)|_* \leqslant 1\text{"} \tag{4.12}$$

should now be replaced by: Suppose there exists $T > 0$ such that

$$|z^n, \partial_t z^n, \partial_x z^n, y\partial_y p^n, \partial_y u^n)|_{L^\infty(\Omega_T)} \leqslant 1. \tag{4.13}$$

The norms $\|z\|'_{k,\mu,\lambda}$, $\|z\|_{k,\mu,\lambda}$, and $\|z\|^{**}$ are defined just as before, but now with $\epsilon = 1$ and hence $\mu = \lambda$. Similarly, the Moser (Lemma 3.6) and Sobolev (Lemma 3.7 and Corollary 3.8) estimates have the same form as before, but with $\mu = \lambda$. The present analogue of (3.1) is

$$\begin{aligned} &(a)\ \mathcal{L}(z^n)z^{n+1,\eta} - \begin{pmatrix} \eta\Delta p^{n+1,\eta} \\ 0 \\ 0 \end{pmatrix} = \mathcal{F} \text{ in } y \geqslant 0, \text{ where } \eta \in (0,1], \\ &(b)\ \partial_y p^{n+1,\eta} = 0,\ u_1^{n+1,\eta} - \partial_y u_1^{n+1,\eta} = -g_1,\ u_2^{n+1,\eta} = 0 \text{ on } y = 0, \\ &(c)\ z^{n+1,\eta} = 0 \text{ in } t \leqslant 0, \end{aligned} \tag{4.14}$$

where $\mathcal{L}(z^n)$ is defined in the obvious way using the left side of (4.10)(a).

Setting $\|z\|_{k,\gamma,\gamma} := \|z\|_{k,\gamma}$, $|z|_{k,\gamma,\gamma} = |z|_{k,\gamma}$, etc., we now have in place of Proposition 3.14

Proposition 4.1. — *Suppose $d = 2$ or $d = 3$ and $k \geqslant 3$ and suppose that for some $T_1 > 0$*

$$\begin{aligned} &(a)\ \|z^n\|^{**} \leqslant 1 \\ &(b)\ \|z^n\|_{k,\gamma} \leqslant 1 \end{aligned} \tag{4.15}$$

on $(-\infty, T_1] \times \{(x,y) : y \geqslant 0\}$. There exist positive constants γ_0 and C_0 such that for $\gamma \geqslant \gamma_0$, $\eta \in (0,1]$, and $T_0(\gamma) := \frac{1}{\gamma}$, the solution $z^{n+1,\eta}$ of

(4.14) *satisfies*

(4.16)
$$\|z^{n+1,\eta}\|_{k,\gamma} + \sqrt{\eta}|\nabla p^{n+1,\eta}|_{k,\gamma} + \eta|\nabla p_y^{n+1,\eta}|_{k-2,\gamma} \leqslant C\left(\frac{1}{\sqrt{\gamma}} + \|z^{n+1,\eta}\|^{**}\right)\|z^n\|_{k,\gamma} + \frac{C_0}{\sqrt{\gamma}}\left(|\mathcal{F}|_{k,\gamma} + |\partial_y\mathcal{F}|_{k-2,\gamma}\right) + C_0\langle g_1\rangle_{k,\gamma}$$

on $(-\infty, T_0(\gamma)] \times \{(x,y) : y \geqslant 0\}$.

This Proposition is proved by repeating the arguments using in proving Proposition 3.14 with the small changes already noted. In particular, note that a factor like

$$1 + \mu^{-(2-\frac{d}{2})}e^{\gamma T}\|z^n\|_{k,\mu,\gamma} \tag{4.17}$$

appearing on the right in (3.52) is now $\leqslant 2$ for $\mu = \gamma$ large and $T = \frac{1}{\gamma}$. In place of Theorem 1.3 we now obtain

Theorem 4.2. — *Suppose $d = 2$ or $d = 3$ and that the initial data (p^0, u^0) in (4.2) belongs to H^s and satisfies corner compatibility conditions to integral order r, where*

$$s \geqslant r \text{ and } r - 2 \geqslant k \geqslant 5. \tag{4.18}$$

There exists $\gamma_0 > 0$ and a positive decreasing function $T_1(\gamma) \leqslant \frac{1}{\gamma}$ such that for $\gamma \geqslant \gamma_0$, the nonlinear forward error problem (4.10) has a unique solution $z = (p, u)$ satisfying the estimates

$$\begin{aligned} &(a)\|z\|^{**} \leqslant 1 \\ &(b)\|z\|_{k,\gamma} \leqslant 2\left(\frac{C_0}{\sqrt{\gamma}}\left(|F|_{k,\gamma} + |\partial_y F|_{k-2,\gamma}\right) + C_0\langle g_1\rangle_{k,\gamma}\right) \leqslant 1 \end{aligned} \tag{4.19}$$

on $(-\infty, T_1(\gamma)] \times \{(x,y) : y \geqslant 0\}$, where F and g_1 are as in (4.10) and C_0 is as in (4.16).

2. In particular if we take $\gamma = \gamma_0$ and $v = v^a + z$ for v^a as in (4.9), then v is an exact solution of the original initial boundary value problem (4.2) on $[0, T_1(\gamma_0)] \times \{(x,y) : y \geqslant 0\}$. The solution $v = (p, u)$ has $\|v\|^{**} \leqslant C\|v\|_{k,\gamma_0} < \infty$, so p is C^0 and u is C^1.

The theorem is proved by the arguments of section 3.3 with the changes indicated above. Observe also that, where before we decreased $\epsilon_0(\gamma)$ if necessary to arrange (3.94) and (3.95), now we use the fact that F and g_1 are 0 in $t \leqslant 0$ to choose $T_1(\gamma)$ so that

$$\frac{C_0}{\sqrt{\gamma}}\left(|F|_{k,\gamma} + |\partial_y F|_{k-2,\gamma}\right) + C_0\langle g_1\rangle_{k,\gamma} \tag{4.20}$$

is as small as necessary on $(-\infty, T_1(\gamma)] \times \{(x,y) : y \geqslant 0\}$ to carry out the induction step.

Remark 4.3. — The estimate (4.19) (b) is an estimate without loss of derivatives for the nonlinear forward error problem (4.10). The norm $\|z\|_{k,\gamma}$ for $k = 5$ is strong enough when the space dimension $d = 2$ or 3 to dominate

$$\|z\|^{**} = |z, Mz, M^2 z, \partial_y u, M\partial_y u|_{L^\infty} \tag{4.21}$$

and to construct a solution of (4.10) by a simple iteration scheme. This implies $p \in C^0$ and $u \in C^1$ along with some higher tangential regularity. Since $k \leqslant r-2$ the estimate (4.19) does involve a loss of derivatives relative to the second component $g_2 \in H^r(y=0)$ of the boundary data (g_1, g_2) of the earlier problem (4.6), and thus also with respect to the initial data of the original problem (4.2). It would be interesting to determine whether and by how much this loss can be reduced.

We do not know how to prove an estimate without loss of derivatives analogous to (4.19) with terms $|\partial_y^k p|_{L^2}$ for $k \geqslant 2$ on the left. Unmanageable boundary terms appear when higher normal derivatives are taken. This difficulty does not arise, of course, for the pure initial value problem [14]. However, in the next section we show that one can still deduce higher regularity of the solution constructed above when the initial data in (4.2) is corner-compatible and regular to high order.

Higher regularity. We introduce the following notation.

Notation 4.4. — Let f be a function defined on $[0,T] \times \{(x,y) : y \geqslant 0\}$ for some $T > 0$. We write $f \in C^j$ whenever f and its partial derivatives up to order j are continuous and uniformly bounded on $[0,T] \times \{(x,y) : y \geqslant 0\}$.

Consider now the solution $v = (p,u) = v^a + z$ of (4.2) constructed in Theorem 4.2 for $k \geqslant 5$. The Sobolev estimates (3.19) with $\epsilon = 1$ imply

$$M^m v \in C^0 \text{ and } M^m \partial_y u \in C^0 \text{ for } m \leqslant k-3. \tag{4.22}$$

The same regularity therefore holds for the solution in the original variables $w = (\rho, u)$. Using (4.22) and provided k is large enough, we will show consecutively that $\partial_y \rho \in C^0$, $\partial_y^2 u \in C^0$, $\partial_y^2 \rho \in C^0$, $\partial_y^3 u \in C^0$, etc.. With (4.22) this in turn will imply $\rho \in C^1$, $u \in C^2$, $\rho \in C^2$, $u \in C^3$, etc..

Ignoring the boundary conditions, we obtain from (1.3) that on $y \geqslant 0$ w satisfies

(4.23)

$$(a)\ D_0(w)w_t + D_1(w)w_x + D_2(w)w_y - (B_{11}w_{xx} + B_{12}w_{xy} + B_{22}w_{yy}) = 0$$

$$(b)\ w|_{t=0} = (\rho^0, u^0).$$

The first component of (4.23)(a) in $y \geqslant 0$ has the form

$$\mathbb{M}_u \rho := (\partial_t + u \cdot \nabla)\rho = -\rho \mathrm{div} u, \tag{4.24}$$

We now solve the pure initial value problem for ρ^*:

$$\begin{aligned} &\mathbb{M}_u \rho^* = -\rho \mathrm{div} u \\ &\rho^*|_{t=0} = \rho^0 \end{aligned} \tag{4.25}$$

by integrating along characteristics of the tangential vector field $\mathbb{M}_u$. Set $X = (x, y)$ and $X_0 = (x_0, y_0)$ for $y_0 \geqslant 0$. The characteristics are the curves $t \to (t, X(t, X_0))$ where

$$\begin{aligned} &\dot{X} = u(t, X) \\ &X(0, X_0) = X_0. \end{aligned} \tag{4.26}$$

For $y_0 \geqslant 0$ these curves remain in $y \geqslant 0$ since $\mathbb{M}_u$ is a tangential vector field. The solution ρ^* is C^1 in t, but even though u is C^1, ρ^* is not obviously better than C^0 in (x, y) since the right side of (4.25) is just continuous in (x, y). On the other hand $\rho = \rho^0$ at $t = 0$ so (4.24) implies $\rho^* = \rho$.

Next apply ∂_y to (4.24) to obtain

$$\mathbb{M}_u \rho_y + (\mathrm{div} u + \partial_y u_2)\rho_y = -\rho(\partial^2_{xy} u_1 + \partial^2_{yy} u_2) - \rho_x \partial_y u_1. \tag{4.27}$$

The term $\partial^2_{yy} u_2$ on the right is not known to be C^0, but from the third component of (4.23)(a) we see that the combination

$$\frac{p'(\rho)}{\lambda + \mu}\rho_y - \partial^2_{yy} u_2 \tag{4.28}$$

is continuous. So we rewrite (4.27) as

$$\begin{aligned} \mathbb{M}_u \rho_y + \left(\mathrm{div} u + \partial_y u_2 + \frac{p'(\rho)\rho}{\lambda + \mu}\right)\rho_y \\ = -\rho \partial^2_{xy} u_1 - \rho_x \partial_y u_1 + \left(\frac{p'(\rho)\rho}{\lambda + \mu}\rho_y - \rho \partial^2_{yy} u_2\right). \end{aligned} \tag{4.29}$$

The right side of (4.29) and the coefficient of ρ_y are C^0, and we can again solve an initial value problem by integrating along characteristics, this time with initial data $\partial_y(\rho^0, u^0)$, to conclude $\rho_y \in C^0$, and hence $\rho \in C^1$. Continuity of (4.28) then implies $\partial^2_y u_2 \in C^0$, so $u \in C^2$. As long as k in Theorem 4.2 is large enough, one can apply ∂_y to (4.29) (and to the third component of (4.23)(a)) and repeat the argument to deduce $\rho \in C^2$, $u \in C^3$, etc.. More precisely, we have shown:

THEOREM 4.5. — *Let $k \geqslant 5$ be as in Theorem 4.2. The solution $v = (p,u)$ to the initial boundary value problem* (4.2) *obtained there satisfies $p \in C^{k-5}$, $u \in C^{k-4}$. The solution (ρ, u) to the problem in the original variables has the same regularity.*

4.2. A linear Prandtl-type equation with Neumann boundary conditions.

Here we state and prove Proposition 4.7, which was used to solve (2.44) for $u_1^{B,1}$. We first consider the case of nonzero forcing and zero boundary data. The next Proposition is a small modification of Theorem 4.1 of [22]. The main difference is that here we have Neumann boundary conditions, whereas [22] treated Dirichlet boundary conditions.

Consider the problem on $\mathbb{O}_{T_0}$ (see Notation (2.2)):

$$\begin{aligned} &\partial_t f + b(t,x)\partial_x f + c(t,x)\eta\partial_\eta f + d(t,x)f - e(t,x)\partial_\eta^2 f = F(t,x,\eta) \\ &\partial_\eta f|_{\eta=0} = 0 \\ &f = 0 \text{ in } t \leqslant -\frac{T_0}{2}, \end{aligned} \tag{4.30}$$

PROPOSITION 4.6. — *Suppose that the coefficients $b(t,x),\dots,e(t,x)$ in* (4.30) *are continuous and bounded along with their derivatives up to order $m \geqslant 1$. Suppose also that $e(t,x) \geqslant C_e > 0$ for all (t,x). Then if $F \in P^m(\mathbb{O}_{T_0})$, there is a unique solution $f \in P^{m-1}(\mathbb{O}_{T_0})$ to* (4.30).

Proof. — **1.** We follow the proof of Theorem 4.1 of [22]. The idea is to replace the degenerate parabolic problem (4.30) by the fully parabolic problem on $\mathbb{O}_{T_0}$:

(4.31)

$$(a)\ \partial_t f + b(t,x)\partial_x f + c(t,x)\eta\partial_\eta f + d(t,x)f - e(t,x)\partial_\eta^2 f - \delta\partial_x^2 f = F(t,x,\eta)$$

$$(b)\ \partial_\eta f|_{\eta=0} = 0$$

$$(c)\ f = 0 \text{ in } t \leqslant -\frac{T_0}{2},$$

and obtain weighted estimates on the solution f^δ of (4.31) that are uniform in $\delta > 0$ small. We now drop the δ on f.

2. Set $\Omega := \{(x,\eta) : \eta \geqslant 0\}$. After multiplying (4.31)(a) by $\langle\eta\rangle^{2l} f$, $l \in \{0,1,2,...\}$, and integrating by parts one finds

(4.32)
$$\frac{1}{2}\frac{d}{dt}\int_\Omega \langle\eta\rangle^{2l}|f|^2 dxd\eta - \frac{1}{2}\int_\Omega \langle\eta\rangle^{2l}\partial_x b|f|^2 dxd\eta - \frac{1}{2}\partial_\eta\left(\eta\langle\eta\rangle^{2l}\right)c|f|^2 dxd\eta \\ + \int_\Omega \langle\eta\rangle^{2l} d|f|^2 dxd\eta + \int_\Omega e\partial_\eta\left(\langle\eta\rangle^{2l} f\right)\partial_\eta f dxd\eta + \delta\int_\Omega \langle\eta\rangle^{2l}|\partial_x f|^2 dxd\eta \\ = \int_\Omega \langle\eta\rangle^{2l} F f dxd\eta.$$

Here we have used the fact that the boundary terms in the integrals involving c and e vanish because, respectively, $\eta = 0$ and $\partial_\eta f = 0$ on the boundary. From (4.32) we obtain

(4.33)
$$\frac{1}{2}\frac{d}{dt}\int_\Omega \langle\eta\rangle^{2l}|f|^2 dxd\eta + \frac{C_e}{2}\int_\Omega \langle\eta\rangle^{2l}|\partial_\eta f|^2 dxd\eta + \delta\int_\Omega \langle\eta\rangle^{2l}|\partial_x f|^2 dxd\eta \leqslant \\ C_0\int_\Omega \langle\eta\rangle^{2l}|f|^2 dxd\eta + \int_\Omega \langle\eta\rangle^{2l}|F|^2 dxd\eta,$$

where C_0 depends on l and the sup norms of $\partial_x b$, c, d, and e. Integrating in t and applying Gronwall we obtain

(4.34)
$$\sup_{t\in[-T_0,T_0]} \|\langle\eta\rangle^{2l} f(t)\|^2 + \int_{-T_0}^{T_0} \|\langle\eta\rangle^{2l}\partial_\eta f(s)\|^2 ds \leqslant C_1\int_{-T_0}^{T_0} \|\langle\eta\rangle^{2l} F(s)\|^2 ds,$$

where $\|\cdot\|$ has the obvious meaning and C_1 depends just on T_0 and the sup norms of $\partial_x b$, c, d, and e.

3. Tangential higher derivatives may now be estimated after differentiating the equation using the fact that the boundary condition is preserved. Normal derivative estimates then follow readily by induction since the coefficient of $\partial_\eta^2 f$ in (4.31) satisfies $e(t,x) \geqslant C_e > 0$ for all (t,x). For this and the details on the passage to the limit as $\delta \to 0$, we refer to [22], p. 529-537. □

Proposition 4.7. — *Suppose $m \geqslant 3$ and consider the problem* (4.31) *on $\mathbb{O}_{T_0}$, but now with the inhomogeneous Neumann condition*

$$\partial_\eta f|_{\eta=0} = g(t,x) \in H^m(\eta = 0). \tag{4.35}$$

Suppose that the coefficients $b(t,x),\dots,e(t,x)$ in (4.30) *are continuous and bounded along with their derivatives up to order $m-2$. Suppose also that $e(t,x) \geqslant C_e > 0$ for all (t,x). Then if $F \in P^{m-2}(\mathbb{O}_{T_0})$, there is a unique solution $f \in P^{m-3}(\mathbb{O}_{T_0})$.*

Proof. — Denote the left side of (4.30)(a) by $\mathbb{P}_L f$. We reduce to a problem with zero Neumann data by looking for f of the form

$$f = f^* + \eta e^{-\eta^2} g \in P^m(\mathbb{O}_{T_0}), \tag{4.36}$$

which yields the following problem for f^*:

$$\begin{aligned} &(a)\ \mathbb{P}_L f^* = F(t,x,\eta) - \mathbb{P}_L(\eta e^{-\eta^2} g) := g^* \in P^{m-2}(\mathbb{O}_{T_0}) \\ &(b)\ \partial_\eta f^*|_{\eta=0} = 0 \\ &(c)\ f^* = 0 \text{ in } t \leqslant -\frac{T_0}{2}. \end{aligned} \tag{4.37}$$

Applying Proposition 4.6 we obtain that $f^* \in P^{m-3}(\mathbb{O}_{T_0})$ and hence $f \in P^{m-3}(\mathbb{O}_{T_0})$. ☐

4.3. Profile equations in the cases $\alpha^\epsilon = \epsilon$, $\alpha^\epsilon = \epsilon^{\frac{1}{2}}$, and some open questions.

Case 1: $\alpha^\epsilon = \epsilon$. One can try to solve for profiles following the same pattern as in the case $\alpha = 1$. As in that case one obtains

$$\rho^{B,0} = 0, \rho^{B,1} = 0, u_2^{B,0} = 0, \text{ and } \overline{u_2^{I,0}} = 0, \tag{4.38}$$

and again one solves the Euler system (2.25) to get $(\rho^{I,0}, u^{I,0})$. A crucial difference appears as one tries to solve the nonlinear Prandtl system (2.12)-(2.13) for $(u_1^{B,0}, \widetilde{u_2^{B,1}})$. From (2.22) and the order ϵ^0 terms in (2.23) we obtain the boundary conditions

$$\begin{aligned} u_1^{B,0}|_{\eta=0} &= -\overline{u_1^{I,0}} \\ \widetilde{u_2^{B,1}}|_{\eta=0} &= 0. \end{aligned} \tag{4.39}$$

This problem is similar to the one that arises in the case of no-slip ($\alpha = 0$) boundary conditions studied by Oleinik [17] in the incompressible setting. When $d = 2$ and a monotonicity assumption, $\partial_\eta u_1^{B,0}|_{t=0} > 0$, is imposed as in Oleinik's work, it may be possible to adapt her methods to our case - in particular to use Crocco's transformation to change the Prandtl system into a scalar degenerate parabolic equation in the half-plane to which one can apply the maximum principle. Her results do not apply directly to our problem: for example, she works with divergence free solutions and solves the problem on a spatial domain $\{0 \leqslant x \leqslant X, \eta \geqslant 0\}$ instead of the domain $\{-\infty \leqslant x \leqslant \infty,\ \eta \geqslant 0\}$ considered here. If a solution exists, it is clear that $u_1^{B,0} \neq 0$, so a layer should appear now in the *leading term* of amplitude

$O(1)$ of the approximate solution. The functions $\overline{u_2^{I,1}}$ and $u_2^{B,1}$ could then be recovered in the usual way from $\widetilde{u_2^{B,1}}$.

Next the profiles $(\rho^{I,1}, u^{I,1})$ would be determined by solving the linearized Euler system (2.4) with $j = 1$ with boundary data

$$u_2^{I,1}|_{y=0} = \lim_{\eta\to\infty} \widetilde{u_2^{B,1}}. \tag{4.40}$$

As explained in section 2.1, the profiles $(u_1^{B,1}, \widetilde{u_2^{B,2}})$ should satisfy the linearized Prandtl equations (2.20). From the $O(\epsilon^{\frac{1}{2}})$ terms in (2.23) we find

$$\partial_\eta u_1^{B,0} = \overline{u_1^{I,1}} + u_1^{B,1} \text{ on } \eta = 0. \tag{4.41}$$

From this and (2.22) we obtain the boundary conditions for $(u_1^{B,1}, \widetilde{u_2^{B,2}})$:

$$\begin{aligned} &(a)\ u_1^{B,1}|_{\eta=0} = \partial_\eta u_1^{B,0}|_{\eta=0} - \overline{u_1^{I,1}} \\ &(b)\ \widetilde{u_2^{B,2}}|_{\eta=0} = 0. \end{aligned} \tag{4.42}$$

In the initial boundary value problem for the linearized Prandtl system (2.20), (4.42) the unknowns are now coupled, unlike the system (2.40) obtained when $\alpha = 1$, and the solution $(u_1^{B,0}, \widetilde{u_2^{B,1}})$ of the nonlinear Prandtl system appears in the coefficients. The expected form of the approximate solution in the case $\alpha^\epsilon = \epsilon$ is:

(4.43)
$$\begin{cases} \rho^\epsilon(t,x,y) = \rho^{I,0}(t,x,y) + \epsilon^{\frac{1}{2}}\rho^{I,1}(t,x,y) + o(\epsilon^{\frac{1}{2}}) \\ u_1^\epsilon(t,x,y) = u_1^{I,0}(t,x,y) + u_1^{B,0}(t,x,\frac{y}{\sqrt{\epsilon}}) + o(1) \\ u_2^\epsilon(t,x,y) = u_2^{I,0}(t,x,y) + \epsilon^{\frac{1}{2}}(u_2^{I,1}(t,x,y) + u_2^{B,1}(t,x,\frac{y}{\sqrt{\epsilon}})) + o(\epsilon^{\frac{1}{2}}). \end{cases}$$

Note that a layer appears in the leading term of the expansion for u_1^ϵ, just as in the no-slip case.

Case 2: $\alpha^\epsilon = \epsilon^{\frac{1}{2}}$. Again one follows the same pattern as in section 2.1. In particular (4.38) holds and slow profiles are determined as before. The profiles $(u_1^{B,0}, \widetilde{u_2^{B,1}})$ should satisfy the nonlinear Prandtl system (2.12)-(2.13). From (2.22) and the $O(\epsilon^0)$ terms in (2.23), we determine the boundary conditions:

$$\begin{aligned} &(\partial_\eta u_1^{B,0} - u_1^{B,0})|_{\eta=0} = \overline{u_1^{I,0}} \\ &\widetilde{u_2^{B,1}}|_{\eta=0} = 0 \end{aligned} \tag{4.44}$$

Beyond the reasons mentioned above, the Robin boundary condition on $u_1^{B,0}$ is another difference that calls into question the applicability of Oleinik's results to this setting.

The profiles $(u_1^{B,1}, \widetilde{u_2^{B,2}})$ should satisfy the linearized Prandtl system (2.20) with boundary conditions:

$$\begin{aligned} &(\partial_\eta u_1^{B,1} - u_1^{B,1})|_{\eta=0} = \overline{u_1^{I,1}} - \partial_y u_1^{I,0}|_{y=0}. \\ &\widetilde{u_2^{B,2}}|_{\eta=0} = 0 \end{aligned} \tag{4.45}$$

As with the linearized problem (2.20), (4.42), work remains to be done to understand the solvability of (2.20), (4.45). The expected form of the approximate solution is again an expansion like (4.43).

Remark 4.8. — In general when the slip length $\alpha^\epsilon = \epsilon^\delta$ for a fixed $\delta > \frac{1}{2}$, the boundary layer is expected to appear in the leading term of the expansion of u^ϵ, and the leading profiles are required to satisfy the same Prandtl boundary problems as in the no-slip case $\alpha^\epsilon = 0$. On the other hand in the case $\delta = \frac{1}{2}$ one obtains a Robin boundary condition on $u_1^{B,0}$. Similar observations were made for the incompressible case in Wang-Wang-Xin [21], which includes a description of the profile equations that arise for various choices of δ.

We conclude by stating two open problems suggested by the above discussion for compressible Navier-Stokes boundary layers with Navier boundary conditions. Under an appropriate monotonicity hypothesis similar to the one made by Oleinik, solve the nonlinear Prandtl system (2.12)-(2.13) with Dirichlet boundary conditions (4.39) (for the case $\alpha = \epsilon$) and with Robin-Dirichlet boundary conditions (4.44) (for the case $\alpha = \epsilon^{1/2}$), along with the subsequent linearized Prandtl problems for higher fast profiles. We would then have approximate solutions valid to all orders, and the remaining difficult problem would be to rigorously determine whether the approximate solutions are close (for ϵ small) to exact Navier-Stokes solutions in suitable spaces.

ACKNOWLEDGMENTS: This research was carried out when Ya-Guang Wang was visiting the Department of Mathematics of Northwestern University and the Department of Mathematics of the University of North Carolina at Chapel Hill. He would like to express the gratitude to these two departments for their hospitality. The work of Ya-Guang Wang is also partially supported by the NNSF of China under grants 10971134 and 11031001. The work of Mark Williams was supported by NSF grants DMS-0701201 and DMS-1001616.

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Manuscrit reçu le 10 novembre 2010,
accepté le 4 mai 2011.

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