ANNALES

DE

L'INSTITUT FOURIER

Dmitri I. PANYUSHEV & Oksana S. YAKIMOVA

A remarkable contraction of semisimple Lie algebras

Tome 62, n° 6 (2012), p. 2053-2068.

<http://aif.cedram.org/item?id=AIF_2012__62_6_2053_0>

Ann. Inst. Fourier, Grenoble
62, 6 (2012) 2053-2068

A REMARKABLE CONTRACTION OF SEMISIMPLE LIE ALGEBRAS

by Dmitri I. PANYUSHEV & Oksana S. YAKIMOVA

Abstract. — Recently, E.Feigin introduced a very interesting contraction $\mathfrak{q}$ of a semisimple Lie algebra $\mathfrak{g}$ (see arXiv:1007.0646 and arXiv:1101.1898). We prove that these non-reductive Lie algebras retain good invariant-theoretic properties of $\mathfrak{g}$. For instance, the algebras of invariants of both adjoint and coadjoint representations of $\mathfrak{q}$ are free, and also the enveloping algebra of $\mathfrak{q}$ is a free module over its centre.

Résumé. — E. Feigin a introduit la contraction $\mathfrak{q}$ d'une algèbre de Lie semi-simple $\mathfrak{g}$ dans arXiv :1007.0646 et arXiv :1101.1898. Nous démontrons que ces algèbres de Lie non-réductives conservent quelque unes des propriétés de $\mathfrak{g}$. En particulier, les algèbres des invariants des représentations adjointe et respectivement coadjointe de $\mathfrak{q}$ sont libres, et l'algèbre enveloppante de $\mathfrak{q}$ est un module libre sur son centre.

Introduction

The ground field $\mathbb{F}$ is algebraically closed and $\mathsf{char}\ \mathbb{F} = 0$. Let G be a connected semisimple algebraic group of rank l with Lie algebra $\mathfrak{g}$. Recently, E. Feigin introduced a very interesting contraction of $\mathfrak{g}$ [2]. His motivation came from some problems in Representation Theory [4], and making use of this contraction he also studied certain degenerations of flag varieties [3]. Our goal is to elaborate on invariant-theoretic properties of these contractions of semisimple Lie algebras.

Fix a triangular decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{t} \oplus \mathfrak{u}^-$, where $\mathfrak{t}$ is a Cartan subalgebra. Then $\mathfrak{b} = \mathfrak{u} \oplus \mathfrak{t}$ is the fixed Borel subalgebra of $\mathfrak{g}$. The corresponding subgroups of G are B, U, and T. Using the vector space isomorphism $\mathfrak{g}/\mathfrak{b} \simeq \mathfrak{u}^-$, we regard $\mathfrak{u}^-$ as a B-module. If $b \in \mathfrak{b}$ and $\eta \in \mathfrak{u}^-$, then

Keywords: Inönü-Wigner contraction, coadjoint representation, algebra of invariants, orbit.
Math. classification: 13A50, 14L30, 17B40, 22E46.

$(b,\eta)\mapsto b\circ\eta$ stands for the corresponding representation of $\mathfrak b$. That is, if $p_-\colon \mathfrak g\to\mathfrak u^-$ is the projection with kernel $\mathfrak b$, then $b\circ\eta=p_-([b,\eta])$.

Following [2, Sect. 2], consider the semi-direct product $\mathfrak q=\mathfrak b\ltimes(\mathfrak g/\mathfrak b)^a=\mathfrak b\ltimes(\mathfrak u^-)^a$, where the superscript 'a' means that the $\mathfrak b$-module $\mathfrak u^-$ is regarded as an abelian ideal in $\mathfrak q$. We may (and will) identify the vector spaces $\mathfrak g$ and $\mathfrak q$ using the decomposition $\mathfrak g=\mathfrak b\oplus\mathfrak u^-$. If $(b,\eta),(b',\eta')\in\mathfrak q$, then the Lie bracket in $\mathfrak q$ is given by

$$[(b,\eta),(b',\eta')]=([b,b'],b\circ\eta'-b'\circ\eta). \tag{0.1}$$

The corresponding connected algebraic group is $Q=B\ltimes N$, where $N=\exp((\mathfrak u^-)^a)$ is an abelian normal unipotent subgroup of Q. The exponential map $\exp\colon(\mathfrak u^-)^a\to N$ is an isomorphism of varieties, and elements of Q are written as product $s{\cdot}\exp(\eta)$, where $s\in B$ and $\eta\in\mathfrak u^-$. If $(s,\eta)\mapsto s{\centerdot}\eta$ is the representation of B in $\mathfrak u^-$, then the adjoint representation of Q is given by

$$\mathrm{Ad}_Q(s{\cdot}\exp(\eta))(b,\eta')=(\mathrm{Ad}(s)b,s{\centerdot}(\eta'-b\circ\eta)). \tag{0.2}$$

In this note, we explicitly construct certain polynomials that generate the algebras of invariants $\mathbb F[\mathfrak q]^Q$ and $\mathbb F[\mathfrak q^*]^Q$, and thereby prove that these two algebras are free. Furthermore, we also show that these polynomials generate the corresponding fields of invariants, $\mathbb F(\mathfrak q)^Q$ and $\mathbb F(\mathfrak q^*)^Q$, and that $\mathbb F[\mathfrak q]$ is a free $\mathbb F[\mathfrak q]^Q$-module and $\mathbb F[\mathfrak q^*]$ is a free $\mathbb F[\mathfrak q^*]^Q$-module. The last assertion implies that the enveloping algebra of $\mathfrak q$, $\mathcal U(\mathfrak q)$, is a free module over its centre. The Lie algebra $\mathfrak q$ is an *Inönü-Wigner contraction* of $\mathfrak g$ (see [15, Ch. 7 § 2.5]), and we also discuss the corresponding relationship between the invariants of G and Q.

Certain classes of non-reductive algebraic Lie algebras $\mathfrak q$ such that $\mathbb F[\mathfrak q^*]^Q$ is a polynomial ring have been studied before. They include the centralisers of nilpotent elements in $\mathfrak{sl}_{l+1}$ and $\mathfrak{sp}_{2l}$ [9], $\mathbb Z_2$-contractions of $\mathfrak g$ [10], and the truncated seaweed (biparabolic) subalgebras of $\mathfrak{sl}_{l+1}$ and $\mathfrak{sp}_{2l}$ [7]. Our result enlarges this interesting family of Lie algebras.

Let $\mathfrak q^*_{\rm reg}$ denote the set of regular elements of $\mathfrak q^*$, *i.e.*, $x\in\mathfrak q^*_{\rm reg}$ if and only if $\dim Q{\cdot}x$ is maximal. For many problems related to coadjoint representations, it is vital to have that $\mathrm{codim}(\mathfrak q^*\setminus\mathfrak q^*_{\rm reg})\geqslant 2$ [10, 9]. However, we prove that if $\mathfrak g$ is simple and not of type $\mathbf A_l$, then $\mathfrak q^*\setminus\mathfrak q^*_{\rm reg}$ contains a divisor.

Notation.

- the centraliser in $\mathfrak g$ of $x\in\mathfrak g$ is denoted by $\mathfrak g^x$.
- $\varkappa$ is the Killing form on $\mathfrak g$.
- $\mathfrak g_{\rm reg}$ is the set of regular elements of $\mathfrak g$, *i.e.*, $x\in\mathfrak g_{\rm reg}$ if and only if $\dim\mathfrak g^x=l$.

- If X is an irreducible variety, then $\mathbb{F}[X]$ is the algebra of regular functions and $\mathbb{F}(X)$ is the field of rational functions on X. If X is acted upon by an algebraic group A, then $\mathbb{F}[X]^A$ and $\mathbb{F}(X)^A$ denote the subsets of respective A-invariant functions.
- If $\mathbb{F}[X]^A$ is finitely generated, then $X/\!\!/A := \mathrm{Spec}(\mathbb{F}[X]^A)$ and $\pi\colon X \to X/\!\!/A$ is determined by the inclusion $\mathbb{F}[X]^A \hookrightarrow \mathbb{F}[X]$. If $\mathbb{F}[X]^A$ is graded polynomial, then the elements of any set of algebraically independent homogeneous generators will be referred to as *basic invariants*.
- $\mathcal{S}^i(V)$ is the i-th symmetric power of the vector space V and $\mathcal{S}(V) = \oplus_{i\geqslant 0}\mathcal{S}^i(V)$ is the symmetric algebra of V.

Acknowledgments. — During the preparation of this paper, the second author benefited from an inspiring environment of the trimester program "On the Interaction of Representation Theory with Geometry and Combinatorics" at HIM (Bonn). She is grateful to P. Littelmann for the invitation. We would like to thank the anonymous referee for valuable comments.

1. On adjoint and coadjoint invariants of Inönü-Wigner contractions

The algebra $\mathfrak{q} = \mathfrak{b} \ltimes (\mathfrak{u}^-)^a$ is an Inönü-Wigner contraction of $\mathfrak{g}$. For this reason, we recall the relevant setting and then describe a general procedure for constructing adjoint and coadjoint invariants of Inönü-Wigner contractions. The $\mathbb{Z}_2$-*contractions* of $\mathfrak{g}$ (considered in [10, 11]) are special cases of Inönü-Wigner contractions, and for them such a procedure is exposed in [10, Prop. 3.1]. However, the more general situation considered here requires another proof.

For a while, we assume that G is any connected algebraic group. Let H be an arbitrary connected subgroup of G and let $\mathfrak{m}$ be a complementary subspace to $\mathfrak{h} = \mathrm{Lie}\, H$ in $\mathfrak{g}$. Using the vector space isomorphism $\mathfrak{g}/\mathfrak{h} \simeq \mathfrak{m}$, we regard $\mathfrak{m}$ as H-module. Consider the invertible linear map $\mathsf{c}_t\colon \mathfrak{g} \to \mathfrak{g}$, $t \in \mathbb{F} \setminus \{0\}$, such that $c_t(h+m) = h + tm$ $(h \in \mathfrak{h}, m \in \mathfrak{m})$ and define the Lie algebra multiplication $[\ ,\ ]_{(t)}$ on the vector space $\mathfrak{g}$ by the rule

$$[x,y]_{(t)} := \mathsf{c}_t^{-1}\big([\mathsf{c}_t(x), \mathsf{c}_t(y)]\big), \quad x, y \in \mathfrak{g}\ .$$

Write $\mathfrak{g}_{(t)}$ for the corresponding Lie algebra. The operator $(\mathsf{c}_t)^{-1} = \mathsf{c}_{t^{-1}}\colon \mathfrak{g} \to \mathfrak{g}_{(t)}$ yields an isomorphism between the Lie algebras $\mathfrak{g} = \mathfrak{g}_{(1)}$ and $\mathfrak{g}_{(t)}$, hence all algebras $\mathfrak{g}_{(t)}$ are isomorphic. It is easily seen that $\lim_{t\to 0} \mathfrak{g}_{(t)} \simeq \mathfrak{h} \ltimes (\mathfrak{g}/\mathfrak{h})^a = \mathfrak{h} \ltimes \mathfrak{m}^a$.

The resulting Lie algebra $\mathfrak{k} := \mathfrak{h} \ltimes \mathfrak{m}^a$ is called an *Inönü-Wigner contraction* of $\mathfrak{g}$, *cf.* Example 7 in [15, Chapter 7, § 2]. The corresponding connected algebraic group is $K = H \ltimes \exp(\mathfrak{m}^a)$. We identify the vector spaces $\mathfrak{g}$ and $\mathfrak{k}$ using the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

Remark. — For $\mathfrak{g}$ semisimple, the contraction $\mathfrak{g} \rightsquigarrow \mathfrak{b} \ltimes \mathfrak{u}^-$ is presented in a more lengthy way, using structure constants, in [2, Remark 2.3].

1.1. To construct invariants of the coadjoint representation of $\mathfrak{k}$, we proceed as follows. Let $f \in \mathcal{S}(\mathfrak{g}) = \mathbb{F}[\mathfrak{g}^*]$ be a homogeneous polynomial of degree n. Using the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, we consider the bi-homogeneous components of f:

$$f = \sum_{a \leqslant i \leqslant b} f^{(n-i,i)},$$

where $f^{(n-i,i)} \in \mathcal{S}^{n-i}(\mathfrak{h}) \otimes \mathcal{S}^i(\mathfrak{m}) \subset \mathcal{S}^n(\mathfrak{g})$, and both $f^{(n-a,a)}$ and $f^{(n-b,b)}$ are assumed to be nonzero. In particular, $f^{(n-b,b)}$ is the bi-homogeneous component having the maximal degree relative to $\mathfrak{m}$. Since $\mathfrak{g}_{(t)}$ and $\mathfrak{k}$ are just the same vector spaces, we also can regard each $f^{(n-i,i)}$ as an element of $\mathcal{S}^n(\mathfrak{g}_{(t)})$ or $\mathcal{S}^n(\mathfrak{k})$.

Theorem 1.1. — *If $f \in \mathcal{S}^n(\mathfrak{g})^G = \mathbb{F}[\mathfrak{g}^*]^G_n$, then $f^{(n-b,b)} \in \mathcal{S}^n(\mathfrak{k})^K = \mathbb{F}[\mathfrak{k}^*]^K_n$.*

Proof. — The isomorphism of Lie algebras $\mathsf{c}_{t^{-1}} \colon \mathfrak{g} \to \mathfrak{g}_{(t)}$ implies that $\sum_{a\leqslant i\leqslant b} t^{-i} f^{(n-i,i)} \in \mathcal{S}(\mathfrak{g}_{(t)})^{G_{(t)}}$ for all $t \neq 0$. It is harmless to replace the last expression with the $G_{(t)}$-invariant $f_{(t)} := \sum_{a\leqslant i\leqslant b} t^{n-i} f^{(n-i,i)}$. Since $f_{(t)}$ is killed by $\mathfrak{g}_{(t)}$ for all $t \neq 0$, its limit at 0, which is $f^{(n-b,b)}$, is killed by $\lim_{t\to 0} \mathfrak{g}_{(t)} = \mathfrak{k}$. Hence $f^{(n-b,b)}$ is K-invariant. □

Let us say that $f^\bullet := f^{(n-b,b)}$ is the *highest component* of $f \in \mathbb{F}[\mathfrak{g}^*]^G_n$ (with respect to the contraction $\mathfrak{g} \rightsquigarrow \mathfrak{k}$). Denote by $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}^*]^G)$ the linear span of $\{f^\bullet \mid f \in \mathbb{F}[\mathfrak{g}^*]^G$ is homogeneous$\}$. Clearly, it is a graded algebra, and Theorem 1.1 implies that $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}^*]^G) \subset \mathbb{F}[\mathfrak{k}^*]^K$. We say that $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}^*]^G)$ is the *algebra of highest components* for $\mathbb{F}[\mathfrak{g}^*]^G$.

Invariants of the adjoint representation of $\mathfrak{k}$ can be constructed in a similar way. Set $\mathfrak{m}^* := \mathfrak{h}^\perp$, the annihilator of $\mathfrak{h}$ in $\mathfrak{g}^*$. Likewise, $\mathfrak{h}^* = \mathfrak{m}^\perp$. Then $\mathfrak{g}^* = \mathfrak{m}^* \oplus \mathfrak{h}^*$, and the adjoint operator $\mathsf{c}_t^* \colon \mathfrak{g}^* \to \mathfrak{g}^*$ is given by $\mathsf{c}_t^*(m^* + h^*) = t^{-1}m^* + h^*$ ($m^* \in \mathfrak{m}^*$, $h^* \in \mathfrak{h}^*$). Having identified $\mathfrak{q}^*$ and $\mathfrak{k}^*$, we can play the same game with homogeneous elements of $\mathcal{S}(\mathfrak{g}^*) = \mathbb{F}[\mathfrak{g}]$. If $\tilde{f} \in \mathcal{S}^n(\mathfrak{g}^*)$, then $\tilde{f}^{(i,n-i)}$ denotes its bi-homogeneous component that belongs to $\mathcal{S}^i(\mathfrak{m}^*) \otimes \mathcal{S}^{n-i}(\mathfrak{h}^*)$. The resulting assertion is the following:

THEOREM 1.2. — *For $\tilde f \in \mathcal{S}^n(\mathfrak{g}^*)^G$, let $\tilde f^{(a,n-a)}$ be the bi-homogeneous component with minimal a, i.e., having the maximal degree relative to $\mathfrak{h}^* = \mathfrak{m}^\perp$. Then $\tilde f^{(a,n-a)} \in \mathcal{S}^n(\mathfrak{k}^*)^K$.*

Likewise, we write $\tilde f^\bullet := \tilde f^{(a,n-a)}$ and consider the algebra of highest components, $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}]^G)$, which can be regarded as a graded subalgebra of $\mathbb{F}[\mathfrak{k}]^K$.

LEMMA 1.3. — *The graded algebras $\mathbb{F}[\mathfrak{g}^*]^G$ and $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}^*]^G)$ have the same Poincaré series, i.e., $\dim \mathbb{F}[\mathfrak{g}^*]^G_n = \dim \mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}^*]^G_n)$ for all $n \in \mathbb{N}$; and likewise for $\mathbb{F}[\mathfrak{g}]^G$ and $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}]^G)$.*

Proof. — Actually, the assertion concerns vector spaces. Let $\tilde V = \oplus_{i\in\mathbb{Z}} \tilde V_i$ be a finite-dimensional $\mathbb{Z}$-graded vector space and V an arbitrary subspace of $\tilde V$. For $v \in V$, let $v^\bullet$ denote the highest component of v with respect to the $\mathbb{Z}$-grading. Set $\mathcal{L}^\bullet(V) = \mathrm{span}\{v^\bullet \mid v \in V\}$. We claim that there is a basis for V, say $(v_1, \dots, v_m)$, such that $(v_1^\bullet, \dots, v_m^\bullet)$ is a basis for $\mathcal{L}^\bullet(V)$. (Left to the reader.) In particular, $\dim V = \dim \mathcal{L}^\bullet(V)$.

Now, apply this claim to $\tilde V = \mathbb{F}[\mathfrak{g}^*]_n = \bigoplus_i \mathbb{F}[\mathfrak{g}^*]_{(i,n-i)}$ and $V = \mathbb{F}[\mathfrak{g}^*]^G_n$. □

It is not always the case that $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}^*]^G) = \mathbb{F}[\mathfrak{k}^*]^K$ or $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}]^G) = \mathbb{F}[\mathfrak{k}]^K$. For instance, we will see below that, for $\mathfrak{g}$ semisimple and $\mathfrak{q} = \mathfrak{b} \ltimes (\mathfrak{u}^-)^a$, such an equality holds only for the invariants of the coadjoint representation. By the very construction, the algebras $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}^*]^G)$ and $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}]^G)$ are bi-graded. Moreover, it follows from [10, Theorem 2.7] that the algebras $\mathbb{F}[\mathfrak{k}^*]^K$ and $\mathbb{F}[\mathfrak{k}]^K$ are always bi-graded.

1.2. If $\mathfrak{g}$ is semisimple, then we may identify $\mathfrak{g}$ and $\mathfrak{g}^*$ (and hence $\mathcal{S}(\mathfrak{g})$ and $\mathcal{S}(\mathfrak{g}^*)$) using the Killing form $\varkappa$. If $\mathfrak{h}$ is also reductive, then $\varkappa$ is non-degenerated on $\mathfrak{h}$ and one can take $\mathfrak{m}$ to be the orthocomplement of $\mathfrak{h}$ with respect to $\varkappa$. Then $\mathfrak{h}^\perp \simeq \mathfrak{m}$ and the decompositions of $\mathfrak{g}$ and $\mathfrak{g}^*$ considered in the general setting of Inönü-Wigner contractions coincide. Moreover, we can also identify the vector spaces $\mathfrak{k}$ and $\mathfrak{k}^*$. However, to obtain invariants of the adjoint and coadjoint representations of $\mathfrak{q}$, one has to take the bi-homogeneous components of maximal degree with respect to *different* summands in the sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. In this situation, Theorems 1.1 and 1.2 admit the following simultaneous formulation:

Suppose that $f \in \mathbb{F}[\mathfrak{g}]^G_n \simeq \mathcal{S}(\mathfrak{g})^G_n$ and $f = \sum_{a\leqslant i\leqslant b} f^{(n-i,i)}$ is the bi-homogeneous decomposition relative to the sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. (That is, $\deg_{\mathfrak{h}} f^{(n-i,i)} = n - i$, etc.) Then, upon identifications of vector spaces $\mathfrak{g}$,$\mathfrak{k}$, and $\mathfrak{k}^*$, we have $f^{(n-a,a)} \in \mathbb{F}[\mathfrak{k}]^K$ and $f^{(n-b,b)} \in \mathbb{F}[\mathfrak{k}^*]^K$.

Such a phenomenon was already observed in the case of $\mathbb{Z}_2$-contractions of semisimple Lie algebras, *i.e.*, if $\mathfrak{h}$ is the fixed-point subalgebra of an involution, see [10, Prop. 3.1].

2. Invariants of the adjoint representation of Q

In this section, we describe the algebra of invariants of the adjoint representation of Q.

To prove that a certain set of invariants generates the whole algebra of invariants, we use the following lemma of Igusa [6].

Lemma 2.1 (Igusa). — *Let A be an algebraic group acting regularly on an irreducible affine variety X. Suppose that S is an integrally closed finitely generated subalgebra of $\mathbb{F}[X]^A$ and the morphism $\pi\colon X \to \operatorname{Spec} S =: Y$ has the properties:*

(i) *the fibres of π over a dense open subset of Y contain a dense A-orbit;*

(ii) $\operatorname{Im}\pi$ *contains an open subset Ω of Y such that* $\operatorname{codim}(Y \setminus \Omega) \geqslant 2$.

Then $S = \mathbb{F}[X]^A$. In particular, the algebra of A-invariants is finitely generated.

Remark 2.2. — A proof of the Igusa lemma is given, for example, in [11, Lemma 6.1]. This proof shows that the above condition (i) can be replaced with the condition that $S \subset \mathbb{F}[X]^A$ generates the field $\mathbb{F}(X)^A$. (In fact, it is not hard to prove that (i) holds *if and only if* S separates A-orbits in a dense open subset of X *if and only if* S generates $\mathbb{F}(X)^A$.)

Lemma 2.3. — *If $t \in \mathfrak{t}$ is regular and $u \in \mathfrak{u}$ is arbitrary, then* (i) *$t + u$ and t belong to the same* Ad *U-orbit;* (ii) $(t + u) \circ \mathfrak{u}^- = \mathfrak{u}^-$.

Proof.

(i) Clearly, $(\operatorname{Ad} U)t \subset t + \mathfrak{u}$ for all $t \in \mathfrak{t}$. If t is regular, then $\dim(\operatorname{Ad} U)t = \dim \mathfrak{u}$. It is also known that the orbits of a unipotent group acting on an affine variety are closed. Hence $(\operatorname{Ad} U)t = t + \mathfrak{u}$.

(ii) This is obvious if $u = 0$. In general, this follows from (i).

□

Theorem 2.4. — *We have $\mathbb{F}[\mathfrak{q}]^Q \simeq \mathbb{F}[\mathfrak{t}]$, and the quotient morphism $\pi_Q\colon \mathfrak{q} \to \mathfrak{t}$ is given by $(t + u, \eta) \mapsto t$.*

Proof. — Clearly, $\mathbb{F}[\mathfrak{q}]^Q = (\mathbb{F}[\mathfrak{q}]^N)^B$. We prove that 1) $\mathbb{F}[\mathfrak{q}]^N \simeq \mathbb{F}[\mathfrak{b}]$ and 2) $\mathbb{F}[\mathfrak{b}]^B \simeq \mathbb{F}[\mathfrak{t}]$.

1) Consider the projection $\pi_N : \mathfrak{q} \to \mathfrak{q}/(\mathfrak{u}^-)^a \simeq \mathfrak{b}$. Clearly, N acts trivially on $\mathfrak{q}/(\mathfrak{u}^-)^a$ and π_N is a surjective N-equivariant morphism. Hence $\mathbb{F}[\mathfrak{b}] \subset \mathbb{F}[\mathfrak{q}]^N$. By Lemma 2.1, the equality $\mathbb{F}[\mathfrak{b}] = \mathbb{F}[\mathfrak{q}]^N$ will follow from the fact that general fibres of π_N are N-orbits.

If $t \in \mathfrak{t}$ is regular and $u \in \mathfrak{u}$ is arbitrary, then $b = t + u$ is a regular semisimple element of $\mathfrak{g}$. By (0.2) with $s = 1$, we have

$$\mathrm{Ad}_Q(N)(b, \eta) = (b, \eta + b \circ \mathfrak{u}^-).$$

It then follows from Lemma 2.3 that $\mathrm{Ad}_Q(N)(b, \eta) = (b, \mathfrak{u}^-)$. On the other hand, $\pi_N^{-1}(b) = (b, \mathfrak{u}^-)$, *i.e.*, $\pi_N^{-1}(b)$ is a single N-orbit whenever b is regular semisimple.

2) Consider the projection $\pi_B : \mathfrak{b} \to \mathfrak{b}/\mathfrak{u} \simeq \mathfrak{t}$. Clearly, B acts trivially on $\mathfrak{b}/\mathfrak{u}$ and π_B is a surjective B-equivariant morphism. Hence $\mathbb{F}[\mathfrak{t}] \subset \mathbb{F}[\mathfrak{b}]^B$. By Lemma 2.1, the equality $\mathbb{F}[\mathfrak{t}] = \mathbb{F}[\mathfrak{b}]^B$ will follow from the fact that general fibres of π_B are B-orbits. Again, it follows from Lemma 2.3 that if $t \in \mathfrak{t}$ is regular, then $(\mathrm{Ad}\ B)t = t + \mathfrak{u} = \pi_B^{-1}(t)$. □

Remark 2.5. — Theorem 2.4 can be proved in a less informative way. Notice that $[\mathfrak{q}, \mathfrak{q}] = \mathfrak{u} \ltimes (\mathfrak{u}^-)^a$ and therefore $\mathbb{F}[\mathfrak{t}] \subset \mathbb{F}[\mathfrak{q}]^Q$. Let $x \in \mathfrak{t}$ be regular semisimple. Then $\mathfrak{q}^x \simeq \mathfrak{g}^x = \mathfrak{t}$, since $\mathfrak{g}$ and $\mathfrak{q}$ are isomorphic as T-modules. The fibres of the morphism $\pi_Q : \mathfrak{q} \to \mathfrak{t}$, defined in Theorem 2.4, are linear spaces of dimension $\dim \mathfrak{q} - \dim \mathfrak{t} = \dim(\mathrm{Ad}\ Q)x$. Hence a general fibre contains a dense Q-orbit and Lemma 2.1 applies. We also see that the algebra $\mathbb{F}[\mathfrak{t}]$ separates Q-orbits in $\mathfrak{q}$ in general position and therefore $\mathbb{F}(\mathfrak{q})^Q = \mathbb{F}(\mathfrak{t})$.

Comparing with the adjoint representation of $\mathfrak{g}$, we see that, for $\mathfrak{q}$, the algebra of invariants remains polynomial, but the degrees of basic invariants drastically decrease! All the basic invariants in $\mathbb{F}[\mathfrak{q}]^Q$ are of degree 1. This clearly means that here $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}]^G) \subsetneq \mathbb{F}[\mathfrak{q}]^Q$.

3. Invariants of the coadjoint representation of Q

In this section, we describe the algebra of invariants of the coadjoint representation of Q. The coadjoint representation is much more interesting since $\mathbb{F}[\mathfrak{q}^*] = \mathcal{S}(\mathfrak{q})$ is a Poisson algebra, $\mathcal{S}(\mathfrak{q})^Q$ is the centre of this Poisson algebra, and $\mathcal{S}(\mathfrak{q})$ is related to the enveloping algebra of $\mathfrak{q}$ via the Poincaré-Birkhoff-Witt theorem.

Since $\mathfrak{q}$ is isomorphic to $\mathfrak{b}\oplus\mathfrak{g}/\mathfrak{b}\simeq\mathfrak{b}\oplus\mathfrak{u}^-$ as vector space, the dual vector space $\mathfrak{q}^*$ is isomorphic to $(\mathfrak{g}/\mathfrak{b})^*\oplus\mathfrak{b}^*$. Using $\varkappa$, we identify $\mathfrak{b}^*$ with $\mathfrak{b}^-:=\mathfrak{t}\oplus\mathfrak{u}^-$ and $(\mathfrak{g}/\mathfrak{b})^*$ with $\mathfrak{u}$. To stress that $\mathfrak{q}^*$ is regarded as a Q-module and $\mathfrak{b}^-$ appears to be a Q-stable subspace, we write $\mathfrak{q}^*=\mathfrak{u}\propto\mathfrak{b}^-$. If $(b,\eta)\in\mathfrak{q}$ and $(u,\xi)\in\mathfrak{q}^*$, i.e., $u\in\mathfrak{u}$ and $\xi\in\mathfrak{b}^-$, then the coadjoint representation of $\mathfrak{q}$ is given by the formula:

$$(b,\eta)\star(u,\xi)=([b,u],\phi(u,\eta)+b\star\xi). \tag{3.1}$$

Here $(b,\xi)\mapsto b\star\xi$ is the coadjoint representation of $\mathfrak{b}$, and

$$\phi\colon\mathfrak{u}\times\mathfrak{u}^-\simeq\mathfrak{u}\times\mathfrak{u}^*\xrightarrow{\psi}\mathfrak{b}^*\simeq\mathfrak{b}^-,$$

where ψ is the *moment map* associated with the $\mathfrak{b}$-module $\mathfrak{u}$. Upon our identifications, the mapping ϕ is directly defined by

$$\varkappa(b,\phi(u,\eta)):=\varkappa([b,u],\eta)=-\varkappa(u,b\circ\eta).$$

Recall some well-known properties of the B-module $\mathfrak{u}$:

- If $\tilde e\in\mathfrak{u}$ is regular nilpotent, then $\mathfrak{g}^{\tilde e}\subset\mathfrak{u}$ [8] and hence $(\mathrm{Ad}\,B)\tilde e$ is dense in $\mathfrak{u}$.
- For any $e\in\mathfrak{u}$, the irreducible components of $(\mathrm{Ad}\,G)e\cap\mathfrak{u}$ are called *orbital varieties* and each of them has dimension $\frac12\dim(\mathrm{Ad}\,G)e$ [14, 4.3.11].

Let $\mathrm{Mor}_G(\mathfrak{g},\mathfrak{g})$ denote the $\mathbb{F}[\mathfrak{g}]^G$-module of polynomial G-equivariant morphisms $F\colon\mathfrak{g}\to\mathfrak{g}$. By work of Kostant [8], $\mathrm{Mor}_G(\mathfrak{g},\mathfrak{g})$ is a free graded $\mathbb{F}[\mathfrak{g}]^G$-module of rank l. It was noticed by Th. Vust [16, Char. III, § 2] (see also [12]) that a homogeneous basis of this module is obtained as follows. Let $f_1,\dots,f_l$ be homogeneous algebraically independent generators of $\mathbb{F}[\mathfrak{g}]^G$. Each differential df_i determines a polynomial G-equivariant morphism (covariant) from $\mathfrak{g}$ to $\mathfrak{g}^*$. Identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ via κ yields a homogeneous covariant (or, vector field) $F_i=\mathrm{grad} f_i\colon\mathfrak{g}\to\mathfrak{g}$. Then $F_1,\dots,F_l$ form a homogeneous basis for $\mathrm{Mor}_G(\mathfrak{g},\mathfrak{g})$. If $\deg f_i=d_i$, then $\deg F_i=d_i-1=:m_i$. It is customary to say that $\{m_1,\dots,m_l\}$ are the *exponents* of (the Weyl group of) $\mathfrak{g}$. Recall that if $\mathfrak{g}$ is simple and $m_1\leqslant\cdots\leqslant m_l$, then $m_1=1$, $m_2\geqslant 2$, and m_i+m_{l-i+1} is the Coxeter number of $\mathfrak{g}$.

The covariants F_i have the following properties:

(i) $F_i(x)\in\mathfrak{g}^x$ for all $i\in\{1,2,\dots,l\}$ and $x\in\mathfrak{g}$;

(ii) The vectors $F_1(x),\dots,F_l(x)\in\mathfrak{g}$ are linearly independent if and only if $x\in\mathfrak{g}_{\mathrm{reg}}$ [8, Theorem 9].

It follows that $(F_1(x),\dots,F_l(x))$ is a basis for $\mathfrak{g}^x$ if and only if $x\in\mathfrak{g}_{\mathrm{reg}}$.

Lemma 3.1. — *If $x\in\mathfrak{b}$, then $F_i(x)\in\mathfrak{b}$. If $y\in\mathfrak{u}$, then $F_i(y)\in\mathfrak{u}$.*

Proof. — If $x \in \mathfrak{b} \cap \mathfrak{g}_{\rm reg}$, then $\mathfrak{g}^x \subset \mathfrak{b}$. (Indeed, $[\mathfrak{b}, x] \subset \mathfrak{u}$, hence $\dim \mathfrak{b}^x \geqslant \operatorname{rk} \mathfrak{g}$. On the other hand, $\mathfrak{b}^x \subset \mathfrak{g}^x$ and $\dim \mathfrak{g}^x = \operatorname{rk} \mathfrak{g}$.) Hence $F_i(x) \in \mathfrak{g}^x \subset \mathfrak{b}$. Since $\mathfrak{b} \cap \mathfrak{g}_{\rm reg}$ is open and dense in $\mathfrak{b}$, the assertion follows.

If $y \in \mathfrak{u} \cap \mathfrak{g}_{\rm reg}$, *i.e.*, y is regular nilpotent, then $\mathfrak{g}^y \subset \mathfrak{u}$ [8]. The rest is the same. □

Consequently, letting $P_i := F_i|_{\mathfrak{u}}$, we obtain the covariants $P_1, \dots, P_l \in \operatorname{Mor}_B(\mathfrak{u}, \mathfrak{u})$. Actually, we consider the P_i's as B-equivariant morphisms $P_i \colon \mathfrak{u} \to \mathfrak{u} \subset \mathfrak{b}$. Using these covariants, we define polynomials $\widehat{P}_i \in \mathbb{F}[\mathfrak{q}^*] = \mathbb{F}[\mathfrak{u} \propto \mathfrak{b}^-]$ by the formula

$$\widehat{P}_i(u, \xi) = \varkappa(P_i(u), \xi), \quad i = 1, \dots, l, \tag{3.2}$$

where $u \in \mathfrak{u}$ and $\xi \in \mathfrak{b}^-$.

LEMMA 3.2. — *We have $\widehat{P}_i \in \mathbb{F}[\mathfrak{q}^*]^Q$.*

Proof. — Since $Q = B \ltimes N$, it suffices to verify that $\widehat{P}_i$ is both B- and N-invariant.

1) $\widehat{P}_i$ is B-invariant, since P_i is B-equivariant.
2) For polynomials obtained from covariants P_i as in (3.2), the invariance with respect to the commutative unipotent group N is equivalent to that $[P_i(u), u] = 0$, $u \in \mathfrak{u}$. Indeed, for $\eta \in \mathfrak{u}^-$, the coadjoint action of $\exp(\eta) \in N$ is given by $\exp(\eta) \star (u, \xi) = (u, \xi + \phi(u, \eta))$. Then

$$\begin{aligned} \widehat{P}_i(\exp(\eta) \cdot (u, \xi)) &= \varkappa(P_i(u), \xi + \phi(u, \eta)) \\ &= \varkappa(P_i(u), \xi)) + \varkappa(P_i(u), \phi(u, \eta)) \\ &= \widehat{P}_i(u, \xi) + \varkappa([P_i(u), u], \eta). \end{aligned}$$

Hence $\widehat{P}_i(\exp(\eta) \cdot (u, \xi)) = \widehat{P}_i(u, \xi)$ for all η if and only if $[P_i(u), u] = 0$. The latter follows from the corresponding property (i) for F_i.

□

Remark. — We prove below that $\widehat{P}_i$ is the highest component of $f_i \in \mathbb{F}[\mathfrak{g}^*]^G$. In view of Theorem 1.1, this also implies that $\widehat{P}_i$ is Q-invariant.

THEOREM 3.3. — *The algebra $\mathbb{F}[\mathfrak{q}^*]^Q$ is freely generated by $\widehat{P}_1, \dots, \widehat{P}_l$, and $\mathbb{F}(\mathfrak{q}^*)^Q$ is the fraction field of $\mathbb{F}[\mathfrak{q}^*]^Q$.*

Proof. — Consider the morphism

$$\pi \colon \mathfrak{q}^* = \mathfrak{u} \propto \mathfrak{b}^- \to \mathbb{A}^l,$$

given by $\pi(u, \xi) = (\widehat{P}_1(u, \xi), \dots, \widehat{P}_l(u, \xi))$. As in Section 2, to prove that π is the quotient by Q, we are going to apply Lemma 2.1 to π.

If $e \in \mathfrak{u}$ is regular, then $P_1(e), \dots, P_l(e)$ are linearly independent and form a basis for $\mathfrak{g}^e = \mathfrak{u}^e$. Therefore, (3.2) implies that π is onto, and condition (ii) in Lemma 2.1 is satisfied.

Let us prove that $\mathbb{F}(\mathfrak{q}^*)^Q = \mathbb{F}(\widehat{P}_1, \dots, \widehat{P}_l)$. Consider the morphism

$$\tilde{\pi}\colon \mathfrak{q}^* \to (\mathfrak{q}^*/\mathfrak{b}^-) \times \mathbb{A}^l = \mathfrak{u} \times \mathbb{A}^l$$

defined by $\tilde{\pi}(u,\xi) = (u, \widehat{P}_1(u,\xi), \dots, \widehat{P}_l(u,\xi))$. If $e \in \mathfrak{u} \cap \mathfrak{g}_{\rm reg}$, then Eq. (3.2) shows that $\tilde{\pi}^{-1}(e,a)$ is an affine subspace of $\mathfrak{q}^*$ for any $a \in \mathbb{A}^l$, and $\dim \tilde{\pi}^{-1}(e,a) = \dim \mathfrak{b} - l = \dim \mathfrak{u}$. As in the proof of Theorem 2.4, this implies that $\tilde{\pi}^{-1}(e,a)$ is a sole N-orbit. Thus, the coordinate functions on $\mathfrak{u}$ and $\widehat{P}_1, \dots, \widehat{P}_l$ separate generic N-orbits of maximal dimension. By the Rosenlicht theorem [1, 1.6], this implies that all these functions generate the field of N-invariants on $\mathfrak{q}^*$, *i.e.*, $\mathbb{F}(\mathfrak{q}^*)^N = \mathbb{F}(\mathfrak{u})(\widehat{P}_1, \dots, \widehat{P}_l)$. Since B has an open orbit in $\mathfrak{u}$, we have $\mathbb{F}(\mathfrak{u})^B = \mathbb{F}$. Hence

$$\mathbb{F}(\mathfrak{q}^*)^Q = (\mathbb{F}(\mathfrak{u})(\widehat{P}_1, \dots, \widehat{P}_l))^B = \mathbb{F}(\widehat{P}_1, \dots, \widehat{P}_l).$$

In view of Remark 2.2, this is sufficient for using Lemma 2.1, and we conclude that $\widehat{P}_1, \dots, \widehat{P}_l$ generate the algebra of Q-invariants on $\mathfrak{q}^*$. □

Remark 3.4. — Although we have proved that $\mathbb{F}(\mathfrak{q}^*)^N = \mathbb{F}(\mathfrak{u})(\widehat{P}_1, \dots, \widehat{P}_l)$, it is not true that $\mathbb{F}[\mathfrak{q}^*]^N = \mathbb{F}[\mathfrak{u}][\widehat{P}_1, \dots, \widehat{P}_l]$. The reason is that the morphism $\tilde{\pi}$ defined in the previous proof does not satisfy condition (ii) of Lemma 2.1. That is, the closure of the complement of $\operatorname{Im}\tilde{\pi}$ contains a divisor. One can prove that this divisor is equal to $D \times \mathbb{A}^l$, where $D = \mathfrak{u} \setminus (\mathrm{Ad}\ B)\tilde{e} = \mathfrak{u} \setminus (\mathfrak{u} \cap \mathfrak{g}_{\rm reg})$. Actually, we can explicitly point out a function in $\mathbb{F}[\mathfrak{q}^*]^N \setminus \mathbb{F}[\mathfrak{u}][\widehat{P}_1, \dots, \widehat{P}_l]$. Let v be a non-zero vector in the one-dimensional space $\mathfrak{b}^U$. We can regard v as a linear function on $\mathfrak{b}^-$ and hence on $\mathfrak{q}^*$. Making use of Eq. (0.1), it is not hard to check that the subalgebra $(\mathfrak{u}^-)^a \subset \mathfrak{q}$ commutes with v, *i.e.*, v is a required N-invariant in the symmetric algebra $\mathcal{S}(\mathfrak{q})$.

Recall that, for an algebraic group A with Lie algebra $\mathfrak{a}$, the *index* of $\mathfrak{a}$, $\mathsf{ind}\ \mathfrak{a}$, is defined as the minimal codimension of an A-orbit in the coadjoint representation. By the Rosenlicht theorem, one has $\mathsf{ind}\ \mathfrak{a} = \operatorname{trdeg} \mathbb{F}(\mathfrak{a}^*)^A$. It is easily seen that the index cannot decrease under contractions, hence $\mathsf{ind}\ \mathfrak{q} \geqslant \mathsf{ind}\ \mathfrak{g} = l$. The above description of the field of Q-invariants implies that

Corollary 3.5. — $\mathsf{ind}\ \mathfrak{q} = l$.

Theorem 3.6. — *The polynomial ring $\mathbb{F}[\mathfrak{q}^*]$ is a free $\mathbb{F}[\mathfrak{q}^*]^Q$-module.*

Proof. — Since it is already known that $\mathbb{F}[\mathfrak{q}^*]^Q$ is a polynomial algebra (of Krull dimension l), it suffices to prove that the quotient morphism $\pi\colon \mathfrak{q}^* \to \mathfrak{q}^*/\!\!/Q \simeq \mathbb{A}^l$ is equidimensional [13, Prop. 17.29]. This, in turn, will follow from the fact that the null-cone $\mathcal{N} = \pi^{-1}(\pi(0))$ is of dimension $\dim\mathfrak{q}-l$. To estimate the dimension of $\mathcal{N}$, consider the projection $p\colon \mathcal{N}\to\mathfrak{u}$ and partition $\mathfrak{u}$ into finitely many orbital varieties (the irreducible components of $(\mathrm{Ad}\ G)e_i\cap\mathfrak{u}$), where $\{e_i\}$ runs over a finite set of representatives of all nilpotent G-orbits. Let Z_i be an irreducible component of $(\mathrm{Ad}\ G)e_i\cap\mathfrak{u}$. Since $\pi = (\widehat{P}_1,\dots,\widehat{P}_l)$, Eq. (3.2) shows that

$$\dim p^{-1}(Z_i) = \dim Z_i + \dim\mathfrak{b} - \dim\operatorname{span}\{P_1(e_i),\dots,P_l(e_i)\}.$$

As $\dim Z_i = \frac{1}{2}\dim(\mathrm{Ad}\ G)e_i$, the condition that $\dim p^{-1}(Z_i)\leqslant\dim\mathfrak{q}-l$ can easily be transformed into

$$\dim\mathfrak{g}^{e_i} + 2\dim\operatorname{span}\{P_1(e_i),\dots,P_l(e_i)\}\geqslant 3l. \tag{3.3}$$

Recall that $P_1,\dots,P_l$ are just the restrictions to $\mathfrak{u}$ of basic covariants $F_1,\dots,F_l$, and $F_j = \operatorname{grad} f_j$. Consequently, $\dim\operatorname{span}\{P_1(e_i),\dots,P_l(e_i)\}$ equals the rank of the differential at e of the quotient morphism $\pi_{\mathfrak{g},G}\colon\mathfrak{g}\to\mathfrak{g}/\!\!/G$. Therefore, (3.3) is precisely the inequality proved in [11, Theorem 10.6]. □

Corollary 3.7. — *The enveloping algebra $\mathcal{U}(\mathfrak{q})$ is a free module over its centre $\mathcal{Z}(\mathfrak{q})$.*

Proof. — This is a standard consequence of the fact that $\mathbb{F}[\mathfrak{q}^*]=\mathcal{S}(\mathfrak{q})$ is a free module over $\mathcal{S}(\mathfrak{q})^Q$, $\mathcal{S}(\mathfrak{q})^Q$ is the centre of the Poisson algebra $\mathcal{S}(\mathfrak{q})$, and $\mathrm{gr}\ \mathcal{Z}(\mathfrak{q}) = \mathcal{S}(\mathfrak{q})^Q$, *cf.* [8, Theorem 21], [5, Theorem 3.3]. □

Remark 3.8. — By Theorem 3.6, the irreducible components of all fibres of $\pi\colon\mathfrak{q}^*\to\mathfrak{q}^*/\!\!/Q\simeq\mathbb{A}^l$ are of dimension $\dim\mathfrak{q}-l$. However, unlike the case of the (co)adjoint representation of $\mathfrak{g}$, the zero fibre of π is highly reducible. For, if $\dim\mathfrak{g}^{e_i}+2\dim\operatorname{span}\{P_1(e_i),\dots,P_l(e_i)\}=3l$, then every irreducible component of $(\mathrm{Ad}\ G)e_i\cap\mathfrak{u}$ gives rise to an irreducible component of $\pi^{-1}(\pi(0))$. A complete classification of nilpotent elements of $\mathfrak{g}$ satisfying this equality is contained in [11, § 10].

Theorem 3.9. — *We have $\mathcal{L}^\bullet(\mathcal{S}(\mathfrak{g})^G) = \mathcal{S}(\mathfrak{q})^Q$. The polynomials $\widehat{P}_1,\dots,\widehat{P}_l\in\mathbb{F}[\mathfrak{q}^*]^Q=\mathcal{S}(\mathfrak{q})^Q$ are the highest components of $f_1,\dots,f_l\in\mathcal{S}(\mathfrak{g})^G$ in the sense of Subsection 1.1.*

Proof.

1) Since $\deg \widehat{P}_i = \deg f_i$ for all i, it follows from Lemma 1.3 and Theorem 3.3 that $\mathcal{L}^\bullet(\mathcal{S}(\mathfrak{g})^G)$ and $\mathcal{S}(\mathfrak{q})^Q$ have the same Poincaré series. Hence these algebras coincide.

2) Recall that $\deg f_i = d_i = m_i + 1$. According to Theorem 1.1, we have to take the decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{u}^-$ and pick the bi-homogeneous component of f_i of maximal degree with respect to $\mathfrak{u}^-$.

If the component $f_i^{(0,d_i)} \in \mathcal{S}^{d_i}(\mathfrak{u}^-)$ were non-trivial, then it would be a Q-invariant in $\mathcal{S}(\mathfrak{q})$ and in particular a B-invariant (Theorem 1.1). Recall that if we work in $\mathfrak{q}$, then $\mathfrak{u}^- \simeq \mathfrak{g}/\mathfrak{b}$ as B-module. Since $\mathcal{S}(\mathfrak{g}/\mathfrak{b}) \simeq \mathbb{F}[\mathfrak{u}]$ and $\mathbb{F}[\mathfrak{u}]^B = \mathbb{F}$, we get a contradiction. Hence $f_i^{(0,d_i)} = 0$.

Then next possible component is $f_i^{(1,m_i)} \in \mathfrak{b} \otimes \mathcal{S}^{m_i}(\mathfrak{u}^-)$. Using the identifications $\mathfrak{b}^* \simeq \mathfrak{b}^-$ and $\mathfrak{u}^* \simeq \mathfrak{u}^-$, we have $f_i^{(1,m_i)} \in \mathbb{F}[\mathfrak{b}^-]_1 \otimes \mathbb{F}[\mathfrak{u}]_{m_i}$. That is, if considered as a function on $\mathfrak{g} = \mathfrak{b}^- \oplus \mathfrak{u}$, it can be written as $f_i^{(1,m_i)}(\xi, u) = \kappa(\bar{P}_i(u), \xi)$ for some morphism $\bar{P}_i \colon \mathfrak{u} \to \mathfrak{b}$ of degree m_i. As we have already proved that $f_i^{(0,d_i)} = 0$, $\bar{P}_i(u)$ is nothing but the value of $\operatorname{grad} f_i$ at u. Hence $\bar{P}_i = P_i$, and we are done. □

4. Further properties of the coadjoint representation

4.1. For the classical Lie algebras, the basic covariants $F_i \colon \mathfrak{g} \to \mathfrak{g}$ (and hence P_i) have a simple description:

- if $x \in \mathfrak{sl}_{l+1}$, then $F_i(x) = x^i$, $i = 1, 2, \ldots, l$;
- if $x \in \mathfrak{sp}_{2l}$ or $\mathfrak{so}_{2l+1}$, then $F_i(x) = x^{2i-1}$, $i = 1, 2, \ldots, l$;
- if $x \in \mathfrak{so}_{2l}$, then $F_i(x) = x^{2i-1}$, $i = 1, 2, \ldots, l-1$. The covariant F_l that is related to the pfaffian is described as follows. Let x be a skew-symmetric matrix of order $2l$. For $i \neq j$, let $x_{[ij]}$ be the skew-symmetric sub-matrix of order $2l-2$ obtained by deleting ith and jth row and column. Set $a_{ij} = \mathsf{Pf}(x_{[ij]})$ if $i \neq j$, and $a_{ii} = 0$. Then $F_l(x) = (a_{ij})_{i,j=1}^{2l}$. Clearly, $\deg F_l = l - 1$, as required.

Results of Sections 2 and 3 explicitly yield the bi-degrees of basic invariants for $\mathfrak{q} = \mathfrak{b} \ltimes (\mathfrak{u}^-)^a$. For $\mathbb{F}[\mathfrak{q}]^Q$, all the basic invariants have bi-degrees $(1, 0)$. For $\mathbb{F}[\mathfrak{q}^*]^Q$, the basic invariants have bi-degrees $(m_i, 1)$, *i.e.*, $\widehat{P}_i \in \mathcal{S}^{m_i}(\mathfrak{u}^-) \otimes \mathfrak{b}$. In particular, for the coadjoint representation, the total degrees of the basic Q-invariants remain the same as for G.

4.2. Hereafter we assume that $\mathfrak{g}$ is simple and the basic invariants $f_1, \ldots, f_l \in \mathbb{F}[\mathfrak{g}]^G$ are numbered such that $d_i \leqslant d_{i+1}$. Then $d_l = \mathsf{h}$ is the Coxeter number of $\mathfrak{g}$. We show that the corresponding Q-invariant $\widehat{P}_l$ has a rather simple form. In fact, it turns out to be a product of linear forms.

Let Δ be the root system of $(\mathfrak{g},\mathfrak{t})$ and Δ^+ the subset of positive roots corresponding to $\mathfrak{u}$. Let $\Pi=\{\alpha_1,\dots,\alpha_l\}$ (resp. θ) be the set of simple roots (resp. the highest root) in Δ^+. Then $\theta=\sum_{i=1}^l a_i\alpha_i$ and $\sum_{i=1}^l a_i=\mathsf{h}-1$. For any $\gamma\in\Delta$, $\mathfrak{g}_\gamma$ denotes the corresponding root subspace, and we fix a nonzero vector $e_\gamma\in\mathfrak{g}_\gamma$.

Lemma 4.1. — *Up to a scalar multiple, we have* $\widehat{P}_l=e_{-\alpha_1}^{a_1}\cdots e_{-\alpha_l}^{a_l}e_\theta\in\mathcal{S}(\mathfrak{q})^Q$.

Proof. — Recall that $\mathfrak{q}=\mathfrak{b}\oplus\mathfrak{u}^-$ as vector space, and here $e_\theta\in\mathfrak{b}$ and $e_{-\alpha_i}\in\mathfrak{u}^-$. By the very construction, $\widehat{P}:=e_{-\alpha_1}^{a_1}\cdots e_{-\alpha_l}^{a_l}e_\theta$ is a T-invariant in $\mathcal{S}(\mathfrak{q})$. Then, using Eq. (0.1), one readily verifies that $\widehat{P}$ is both U-invariant and N-invariant. Hence $\widehat{P}$ is a polynomial in $\widehat{P}_1,\dots,\widehat{P}_l$. Since $\textsf{bi-deg}\,\widehat{P}=(\mathsf{h}-1,1)$ and $m_i<m_l$ for $i<l$, the subspace of bi-degree $(m_l,1)=(\mathsf{h}-1,1)$ in $\mathcal{S}(\mathfrak{q})^Q$ is one-dimensional and spanned by $\widehat{P}_l$. Hence the assertion. □

Since $\dim\mathfrak{q}=\dim\mathfrak{g}$, $\textsf{ind}\,\mathfrak{q}=\textsf{ind}\,\mathfrak{g}$, and the (total) degrees of the basic invariants of the coadjoint representations for G and Q coincide, we have the equality

$$\sum_{i=1}^{l}\deg\widehat{P}_i=\frac{\dim\mathfrak{q}+\textsf{ind}\,\mathfrak{q}}{2}\ , \tag{4.1}$$

which is very useful in the study of the coadjoint representation, see *e.g.* [10, Theorem 1.2]. Unfortunately, $\mathfrak{q}$ does not always possess another important ingredient, the so-called *codim-2* property. Recall that $x\in\mathfrak{q}^*$ is said to be *regular* if $\dim Q{\cdot}x$ is maximal. The set of all regular elements is denoted by $\mathfrak{q}^*_{\rm reg}$. It is an open subset of $\mathfrak{q}^*$, and we say that $\mathfrak{q}$ has the *codim-2* property if $\operatorname{codim}(\mathfrak{q}^*\setminus\mathfrak{q}^*_{\rm reg})\geqslant 2$.

Theorem 4.2. — *The algebra $\mathfrak{q}$ does not have the codim-2 property if $\mathfrak{g}$ is not of type $\boldsymbol{A}_l$.*

Proof. — Suppose that $\mathfrak{q}$ has the *codim-2* property. Since (4.1) is satisfied, it follows from [10, Theorem 1.2] that the differentials $(d\widehat{P}_i)_x$, $i=1,\dots,l$, are linearly independent if and only if $x\in\mathfrak{q}^*_{\rm reg}$. In particular, any divisor $\tilde{D}\subset\mathfrak{q}^*$ contains a point where the differentials of $\widehat{P}_1,\dots,\widehat{P}_l$ are linearly independent.

On the other hand, Lemma 4.1 shows that if $a_i\geqslant 2$ for some i, then $d\widehat{P}_l$ vanishes at the hyperplane $\{e_{-\alpha_i}=0\}$, where $e_{-\alpha_i}$ is regarded as a linear function on $\mathfrak{u}$ and hence on $\mathfrak{q}^*$. Thus, $\mathfrak{q}$ cannot have the *codim-2* property unless $a_i = 1$ for all i, *i.e.*, $\mathfrak{g}$ is of type $\mathbf{A}_l$. □

To prove the converse of this theorem, we need some preparations. For $\alpha_i \in \Pi$, let $\mathfrak{u}_i \subset \mathfrak{u}$ denote the kernel of the linear form $u \mapsto \kappa(e_{-\alpha_i}, u)$. By [8], $\mathfrak{u} \setminus \mathfrak{u} \cap \mathfrak{g}_{\rm reg} = \cup_i \mathfrak{u}_i$. Set

(4.2)
$$\mathcal{Y} = \mathcal{Y}(\mathfrak{q}^*) = \Big\{ x \in \mathfrak{q}^* \mid (d\widehat{P}_1)_x, \dots, (d\widehat{P}_l)_x \ \text{ are linearly independent} \Big\}.$$

Proposition 4.3. — *If* $\mathfrak{g} = \mathfrak{sl}_{l+1}$, *then* $\operatorname{codim}(\mathfrak{q}^* \setminus \mathcal{Y}) \geqslant 2$.

Proof. — Let $a = (e, \xi)$ and $a' = (e', \xi')$ be typical elements of $\mathfrak{q}^*$, where $e, e' \in \mathfrak{u}$ and $\xi, \xi' \in \mathfrak{b}^-$. According to formulae of Subsection 4.1, $\widehat{P}_i(e,\xi) = \kappa(e^i, \xi)$. Recall that $(d\widehat{P}_i)_a \in \mathfrak{q}$ and $\langle (d\widehat{P}_i)_a, a'\rangle$ is the coefficient of t in the expansion of $\widehat{P}_i(a + ta')$. Consequently,

$$\big\langle (d\widehat{P}_i)_a, a' \big\rangle = \kappa(e^i, \xi') + \kappa\Bigg(\sum_{k+m=i-1} e^k e' e^m, \xi \Bigg).$$

The vector $(d\widehat{P}_i)_a$ has the $\mathfrak{b}$- and $\mathfrak{u}^-$-components, and this equality shows that:

- the $\mathfrak{b}$-component of $(d\widehat{P}_i)_a$ equals e^i;
- the $\mathfrak{u}^-$-component of $(d\widehat{P}_i)_a$, say $(d\widehat{P}_i)_a\{\mathfrak{u}^-\}$, is determined by the equation $\kappa((d\widehat{P}_i)_a\{\mathfrak{u}^-\}, e') = \kappa(\sum_{k+m=i-1} e^k e' e^m, \xi)$.

Let $\mathcal{O}^{\rm reg}$ and $\mathcal{O}^{\rm sub}$ denote the regular and subregular nilpotent orbits in $\mathfrak{sl}_{l+1}$, respectively. Then $\overline{\mathcal{O}^{\rm sub}} \cap \mathfrak{u} = \cup_j \mathfrak{u}_j$. If $e \in \mathcal{O}^{\rm reg} \cap \mathfrak{u}$, then the $\mathfrak{b}$-components of $(d\widehat{P}_i)_{(e,\xi)}$, $i = 1, \dots, l$, are linearly independent, regardless of ξ. Hence $(\mathcal{O}^{\rm reg} \cap \mathfrak{u}) \times \mathfrak{b}^- \subset \mathcal{Y}$.

If $e \in \mathcal{O}^{\rm sub} \cap \mathfrak{u}$, then the $\mathfrak{b}$-components of $(d\widehat{P}_i)_{(e,\xi)}$, $i = 1, \dots, l-1$, are still linearly independent for any ξ, but $e^l = 0$. However, if e is sufficiently general, then the $\mathfrak{u}^-$-component of $(d\widehat{P}_l)_{(e,\xi)}$ appears to be nonzero for all ξ that belong to a dense open subset of $\mathfrak{b}^-$. More precisely, suppose that $e \in \mathfrak{u}_j$ and $\kappa(e, e_{-\alpha_i}) \neq 0$ for $i \neq j$. Taking $e' = e_{\alpha_j}$, one readily computes that $\sum_{k+m=l-1} e^k e' e^m = e^{j-1} e_{\alpha_i} e^{l-j}$ is a nonzero multiple of e_θ. Hence, one can take any ξ such that $\kappa(\xi, e_\theta) \neq 0$.

Thus, there is a dense open subset $\Omega \subset \cup_i \mathfrak{u}_i \times \mathfrak{b}^-$ such that $\Omega \subset \mathcal{Y}$, and the assertion follows. □

It turns out that Proposition 4.3 together with (4.1) is sufficient to conclude that for $\mathfrak{g} = \mathfrak{sl}_{l+1}$, $\mathfrak{q}$ has the *codim-2* property. This follows from the following general assertion:

Theorem 4.4. — *Let R be a connected algebraic group with Lie algebra $\mathfrak{r}$. Suppose that* (i) $\mathbb{F}[\mathfrak{r}^*]^R = \mathbb{F}[p_1,\dots,p_m]$ *is a graded polynomial algebra,* (ii) $\operatorname{ind}\mathfrak{r} = m$*, and* (iii) $\sum_{i=1}^m \deg p_i = (\dim\mathfrak{r} + \operatorname{ind}\mathfrak{r})/2$*. Then the following conditions are equivalent:*

(1) $\operatorname{codim}(\mathfrak{r}^* \setminus \mathfrak{r}^*_{\rm reg}) \geqslant 2$*;*

(2) $\operatorname{codim}(\mathfrak{r}^* \setminus \mathcal{Y}(\mathfrak{r}^*)) \geqslant 2$*, where* $\mathcal{Y}(\mathfrak{r}^*)$ *is defined as in* (4.2) *via the* p_i*'s.*

If these conditions are satisfied, then actually $\mathfrak{r}^*_{\rm reg} = \mathcal{Y}(\mathfrak{r}^*)$*.*

Proof. — The implication (1) $\Rightarrow$ (2) is already proved in [10, Theorem 1.2].

To prove the converse, one can slightly adjust the proof given in [10], see also the proof of Theorem 1.2 in [9]. Set $n = \dim\mathfrak{r}$. Let $T(\mathfrak{r}^*)$ denote the tangent bundle of $\mathfrak{r}^*$. The main part of that proof consists in a construction of two homogeneous polynomial sections of $\wedge^{n-m}T(\mathfrak{r}^*)$, denoted $\mathfrak{V}_1$ and $\mathfrak{V}_2$. Write $(\mathfrak{V}_i)_x$ for the value of $\mathfrak{V}_i$ at $x \in \mathfrak{r}^*$. These sections have the following properties:

(a) There exist nonzero polynomials $F_1, F_2 \in \mathbb{F}[\mathfrak{r}^*]$ such that $F_1\mathfrak{V}_1 = F_2\mathfrak{V}_2$;

(b) $(\mathfrak{V}_1)_x \neq 0$ if and only if $x \in \mathfrak{r}^*_{\rm reg}$

(c) $(\mathfrak{V}_2)_x \neq 0$ if and only if $x \in \mathcal{Y}(\mathfrak{r}^*)$;

(d) $\deg\mathfrak{V}_1 = (n-m)/2$ and $\deg\mathfrak{V}_2 = \sum_{i=1}^m(\deg p_i - 1)$.

This only requires assumptions (i) and (ii). If (iii) is also satisfied, then $\deg\mathfrak{V}_1 = \deg\mathfrak{V}_2$. Therefore either of conditions (1),(2) implies the other. Moreover, properties **(a)** and **(b)** imply that if (1) is satisfied, then $\deg F_2 = 0$, *i.e.*, $F_2 \in \mathbb{F}^\times$. Likewise, **(a)** and **(c)** imply that if (2) is satisfied, then $\deg F_1 = 0$. This yields the last assertion. □

Since $\mathfrak{q} = \mathfrak{b} \ltimes \mathfrak{u}^-$ does not have the *codim-2* property if $\mathfrak{g}$ is not of type $\mathbf{A}_l$, we cannot immediately conclude that in all cases $x \in \mathfrak{q}^*_{\rm reg}$ if and only if $(d\widehat{P}_1)_x, \dots, (d\widehat{P}_l)_x$ are linearly independent. Nevertheless, the fact that $\widehat{P}_1, \dots, \widehat{P}_l$ are the highest components of the basic G-invariants $f_1, \dots, f_l$ allows to circumvent this difficulty. It can be shown in general (see [17]) that the coadjoint representation $(Q\colon \mathfrak{q}^*)$ has the following property:

Claim 4.5. — *For $x \in \mathfrak{q}^*$ the following conditions are equivalent:*

- *The orbit $Q{\cdot}x$ is of maximal dimension, which is* $\dim\mathfrak{q} - l$ *in this situation;*
- *The differentials $(d\widehat{P}_i)_x$, $i = 1,\dots,l$, are linearly independent.*

This generalise a result of Kostant obtained for semisimple Lie algebras [8, Theorem 9].

BIBLIOGRAPHY

[1] M. Brion, "Invariants et covariants des groupes algébriques réductifs", dans : "*Théorie des invariants et géometrie des variétés quotients*, Travaux en cours, t. **61**, 83–168, Paris: Hermann, 2000.

[2] E. Feigin, "$\mathbb{G}_a^M$-degeneration of flag varieties", *Selecta Math., New Series*, to appear.

[3] ———, "Degenerate flag varieties and the median Genocchi numbers", *Math. Research Letters* **18** (2011), no. 6, p. 1163-1178.

[4] E. Feigin, G. Fourier & P. Littelmann, "PBW filtration and bases for irreducible modules in type A_n", *Transform. Groups* **16** (2011), no. 1, p. 71-89.

[5] F. Geoffriau, "Sur le centre de l'algèbre enveloppante d'une algèbre de Takiff", *Ann. Math. Blaise Pascal* **1** (1994), no. 2, p. 15-31 (1995).

[6] J. I. Igusa, "Geometry of absolutely admissible representations", in *Number theory, algebraic geometry and commutative algebra, in honor of Yasuo Akizuki*, Kinokuniya, Tokyo, 1973, p. 373-452.

[7] A. Joseph, "On semi-invariants and index for biparabolic (seaweed) algebras. II", *J. Algebra* **312** (2007), no. 1, p. 158-193.

[8] B. Kostant, "Lie group representations on polynomial rings", *Amer. J. Math.* **85** (1963), p. 327-404.

[9] D. I. Panyushev, A. Premet & O. Yakimova, "On symmetric invariants of centralisers in reductive Lie algebras", *J. Algebra* **313** (2007), no. 1, p. 343-391.

[10] D. I. Panyushev, "On the coadjoint representation of $\mathbb{Z}_2$-contractions of reductive Lie algebras", *Adv. Math.* **213** (2007), no. 1, p. 380-404.

[11] ———, "Semi-direct products of Lie algebras and their invariants", *Publ. Res. Inst. Math. Sci.* **43** (2007), no. 4, p. 1199-1257.

[12] M. Raïs, "Champs de vecteurs invariants sur une algèbre de Lie réductive complexe", *J. Math. Soc. Japan* **40** (1988), no. 4, p. 615-628.

[13] G. W. Schwarz, "Lifting smooth homotopies of orbit spaces", *Inst. Hautes Études Sci. Publ. Math.* (1980), no. 51, p. 37-135.

[14] T. A. Springer, "Conjugacy classes in algebraic groups", in *Group theory, Beijing 1984*, Lecture Notes in Math., vol. 1185, Springer, Berlin, 1986, p. 175-209.

[15] È. B. Vinberg, V. V. Gorbatsevich & A. L. Onishchik, *Группы и алгебры Ли - 3*, Соврем. пробл. математики. Фундам. направл., т. 41, Москва: ВИНИТИ, Russian, 1990, English translation: "Lie Groups and Lie Algebras" III (Encyclopaedia Math. Sci., vol. **41**, Berlin: Springer 1994.

[16] T. Vust, "Covariants de groupes algébriques réductifs", Thèse n° 1671, Université de Genève, 1974.

[17] O. Yakimova, "One-parameter contractions of Lie-Poisson brackets", *J. Europ. Math. Soc.*, to appear; `arXiv:1202.3009`.

Manuscrit reçu le 4 juillet 2011,
accepté le 16 septembre 2011.

Dmitri I. PANYUSHEV
Institute for Information Transmission Problems
of the R.A.S.
B. Karetnyi per. 19
Moscow 127994 (Russia)
panyushev@iitp.ru

Oksana S. YAKIMOVA
Friedrich-Schiller-Universität Jena
Mathematisches Institut
Jena D-07737 (Deutschland)
oksana.yakimova@uni-jena.de