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THE DEHN FUNCTIONS OF $\text{Out}(F_n)$ AND $\text{Aut}(F_n)$

by Martin R. BRIDSON & Karen VOGTMANN (*)

Abstract. — For $n$ at least 3, the Dehn functions of $\text{Out}(F_n)$ and $\text{Aut}(F_n)$ are exponential. Hatcher and Vogtmann proved that they are at most exponential, and the complementary lower bound in the case $n = 3$ was established by Bridson and Vogtmann. Handel and Mosher completed the proof by reducing the lower bound for $n$ bigger than 3 to the case $n = 3$. In this note we give a shorter, more direct proof of this last reduction.

Résumé. — Pour $n$ au moins 3, les fonctions de Dehn de $\text{Out}(F_n)$ et $\text{Aut}(F_n)$ sont exponentielles. Hatcher et Vogtmann ont montré qu’elles étaient au plus exponentielles, et la borne inférieure a été établie par Bridson et Vogtmann dans le cas $n = 3$. Handel et Mosher ont complété la démonstration en ramenant la preuve de la borne inférieure pour $n$ au moins 4 au cas $n = 3$. Dans cet article, nous donnons un argument plus direct permettant de passer du cas $n = 3$ au cas général.

Dehn functions provide upper bounds on the complexity of the word problem in finitely presented groups. They are examples of filling functions: if a group $G$ acts properly and cocompactly on a simplicial complex $X$, then the Dehn function of $G$ is asymptotically equivalent to the function that provides the optimal upper bound on the area of least-area discs in $X$, where the bound is expressed as a function of the length of the boundary of the disc. This article is concerned with the Dehn functions of automorphism groups of finitely-generated free groups.

Much of the contemporary study of $\text{Out}(F_n)$ and $\text{Aut}(F_n)$ is based on the deep analogy between these groups, mapping class groups, and lattices in semisimple Lie groups, particularly $\text{SL}(n, \mathbb{Z})$. The Dehn functions of mapping class groups are quadratic [9], as is the Dehn function of $\text{SL}(n, \mathbb{Z})$ if $n \geq 5$ (see [10]). In contrast, Epstein et al. [6] proved that the Dehn function of $\text{SL}(3, \mathbb{Z})$ is exponential. Building on their result, we proved

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in [3] that $\text{Aut}(F_3)$ and $\text{Out}(F_3)$ also have exponential Dehn functions. Hatcher and Vogtmann [8] established an exponential upper bound on the Dehn function of $\text{Aut}(F_n)$ and $\text{Out}(F_n)$ for all $n \geq 3$. The comparison with $\text{SL}(n, \mathbb{Z})$ might lead one to suspect that this last result is not optimal for large $n$, but recent work of Handel and Mosher [7] shows that in fact it is: they establish an exponential lower bound by using their general results on quasi-retractions to reduce to the case $n = 3$.

**Theorem.** — For $n \geq 3$, the Dehn functions of $\text{Aut}(F_n)$ and $\text{Out}(F_n)$ are exponential.

This theorem answers Questions 35 and 37 of [4].

We learned the contents of [7] from Lee Mosher at Luminy in June 2010 and realized that one can also reduce the Theorem to the case $n = 3$ using a simple observation about natural maps between different-rank Outer spaces and Auter spaces (Lemma 3). The purpose of this note is record this observation and the resulting proof of the Theorem.

1. Definitions

Let $A$ be a 1-connected simplicial complex. We consider simplicial loops $\ell: S \to A^{(1)}$, where $S$ is a simplicial subdivision of the circle. A *simplicial filling* of $\ell$ is a simplicial map $L: D \to A^{(2)}$, where $D$ is a triangulation of the 2-disc and $L|_{\partial D} = \ell$. Such fillings always exist, by simplicial approximation. The filling area of $\ell$, denoted $\text{Area}_A(\ell)$, is the least number of triangles in the domain of any simplicial filling of $\ell$. The *Dehn function* $\delta_A: \mathbb{N} \to \mathbb{N}$ of $A$ is the least function $\delta_A: \mathbb{N} \to \mathbb{N}$ such that $\text{Area}_A(\ell) \leq \delta_A(n)$ for all loops of length $\leq n$ in $A^{(1)}$. The Dehn function of a finitely presented group $G$ is the Dehn function of any 1-connected 2-complex on which $G$ acts simplicially with finite stabilizers and compact quotient. This is well-defined up to the following equivalence relation: functions $f, g: \mathbb{N} \to \mathbb{N}$ are equivalent if $f \preceq g$ and $g \preceq f$, where $f \preceq g$ means that there is a constant $a > 1$ such that $f(n) \leq a g(an + a) + an + a$. The Dehn function can be interpreted as a measure of the complexity of the word problem for $G$ — see [2].

**Lemma 1.** — If $A$ and $B$ are 1-connected simplicial complexes, $F: A \to B$ is a simplicial map, and $\ell$ is a loop in the 1-skeleton of $A$, then $\text{Area}_A(\ell) \geq \text{Area}_B(F \circ \ell)$.

(1) The standard definition of area and Dehn function are phrased in terms of singular discs, but this version is $\simeq$ equivalent.
Proof. — If $L : D \to A$ is a simplicial filling of $\ell$, then $F \circ L$ is a simplicial filling of $F \circ \ell$, with the same number of triangles in the domain $D$. \qed

Corollary. — Let $A, B$ and $C$ be 1-connected simplicial complexes with simplicial maps $A \to B \to C$. Let $\ell_n$ be a sequence of simplicial loops in $A$ whose length is bounded above by a linear function of $n$, let $\overline{\ell}_n$ be the image loops in $C$ and let $\alpha(n) = \text{Area}_C(\overline{\ell}_n)$. Then the Dehn function of $B$ satisfies $\delta_B(n) \geq \alpha(n)$.

Proof. — This follows from Lemma 1 together with the observation that a simplicial map does not increase the length of any loop in the 1-skeleton. \qed

2. Simplicial complexes associated to $\text{Out}(F_n)$ and $\text{Aut}(F_n)$

Let $K_n$ denote the spine of Outer space, as defined in [5], and $L_n$ the spine of Auter space, as defined in [8]. These are contractible simplicial complexes with cocompact proper actions by $\text{Out}(F_n)$ and $\text{Aut}(F_n)$ respectively, so we may use them to compute the Dehn functions for these groups.

Recall from [5] that a marked graph is a finite metric graph $\Gamma$ together with a homotopy equivalence $g : R_n \to \Gamma$, where $R_n$ is a fixed graph with one vertex and $n$ loops. A vertex of $K_n$ can be represented either as a marked graph $(g, \Gamma)$ with all vertices of valence at least three, or as a free minimal action of $F_n$ on a simplicial tree (namely the universal cover of $\Gamma$). A vertex of $L_n$ has the same descriptions except that there is a chosen basepoint in the marked graph (respected by the marking) or in the simplicial tree. Note that we allow marked graphs to have separating edges. Both $K_n$ and $L_n$ are flag complexes, so to define them it suffices to describe what it means for vertices to be adjacent. In the marked-graph description, vertices of $K_n$ (or $L_n$) are adjacent if one can be obtained from the other by a forest collapse (i.e. collapsing each component of a forest to a point).

3. Three natural maps

There is a forgetful map $\phi_n : L_n \to K_n$ which simply forgets the basepoint; this map is simplicial.

Let $m < n$. We fix an ordered basis for $F_n$, identify $F_m$ with the subgroup generated by the first $m$ elements of the basis, and identify $\text{Aut}(F_m)$ with
the subgroup of $\text{Aut}(F_n)$ that leaves $F_m < F_n$ invariant and fixes the last $n - m$ basis elements. We consider two maps associated to this choice of basis.

First, there is an equivariant augmentation map $\iota: L_m \to L_n$ which attaches a bouquet of $n - m$ circles to the basepoint of each marked graph and marks them with the last $n - m$ basis elements of $F_n$. This map is simplicial, since a forest collapse has no effect on the bouquet of circles at the basepoint.

Secondly, there is a restriction map $\rho: K_n \to K_m$ which is easiest to describe using trees. A point in $K_n$ is given by a minimal free simplicial action of $F_n$ on a tree $T$ with no vertices of valence 2. We define $\rho(T)$ to be the minimal invariant subtree for $F_m < F_n$; more explicitly, $\rho(T)$ is the union of the axes in $T$ of all elements of $F_m$. (Vertices of $T$ that have valence 2 in $\rho(T)$ are no longer considered to be vertices.)

One can also describe $\rho$ in terms of marked graphs. The chosen embedding $F_m < F_n$ corresponds to choosing an $m$-petal subrose $R_m \subset R_n$. A vertex in $K_n$ is given by a graph $\Gamma$ marked with a homotopy equivalence $g: R_n \to \Gamma$, and the restriction of $g$ to $R_m$ lifts to a homotopy equivalence $\hat{g}: R_m \to \hat{\Gamma}$, where $\hat{\Gamma}$ is the covering space corresponding to $g_\ast(F_m)$. There is a canonical retraction $r$ of $\hat{\Gamma}$ onto its compact core, i.e. the smallest connected subgraph containing all nontrivial embedded loops in $\Gamma$. Let $\hat{\Gamma}_0$ be the graph obtained by erasing all vertices of valence 2 from the compact core and define $\rho(g, \Gamma) = (r \circ \hat{g}, \hat{\Gamma}_0)$.

**Lemma 2.** — For $m < n$, the restriction map $\rho: K_n \to K_m$ is simplicial.

**Proof.** Any forest collapse in $\Gamma$ is covered by a forest collapse in $\hat{\Gamma}$ that preserves the compact core, so $\rho$ preserves adjacency. □

**Lemma 3.** — For $m < n$, the following diagram of simplicial maps commutes:

$$
\begin{array}{ccc}
L_m & \xrightarrow{\iota} & L_n \\
\phi_m & \downarrow & \downarrow \phi_n \\
K_m & \xleftarrow{\rho} & K_n
\end{array}
$$

**Proof.** Given a marked graph with basepoint $(g, \Gamma; v) \in L_n$, the marked graph $\iota(g, \Gamma; v)$ is obtained by attaching $n - m$ loops at $v$ labelled by the elements $a_{m+1}, \ldots, a_n$ of our fixed basis for $F_n$. Then $(g_n, \Gamma_n) := \phi_n \circ \iota(g, \Gamma; v)$ is obtained by forgetting the basepoint, and the cover of $(g_n, \Gamma_n)$ corresponding to $F_m < F_n$ is obtained from a copy of $(g, \Gamma)$ (with its labels) by attaching $2(n - m)$ trees. (These trees are obtained from the Cayley graph of $F_n$ as follows: one cuts at an edge labelled $a_i^\epsilon$, with
\[ i \in \{m + 1, \ldots, n\} \text{ and } \varepsilon = \pm 1, \text{ takes one component of the result, and then attaches the hanging edge to the basepoint } v \text{ of } \Gamma. \] The effect of \( \rho \) is to delete these trees.)

4. Proof of the Theorem

In the light of the Corollary and Lemma 3, it suffices to exhibit a sequence of loops \( \ell_i \) in the 1-skeleton of \( L_3 \) whose lengths are bounded by a linear function of \( i \) and whose filling area when projected to \( K_3 \) grows exponentially as a function of \( i \). Such a sequence of loops is essentially described in [3]. What we actually described there were words in the generators of \( \text{Aut}(F_3) \) rather than loops in \( L_3 \), but standard quasi-isometric arguments show that this is equivalent. More explicitly, the words we considered were \( w_i = T^i A T^{-i} B T^i A^{-1} T^{-i} B^{-1} \) where

\[
T: \begin{cases}
a_1 \mapsto a_1^2 a_2 \\
a_2 \mapsto a_1 a_2 \\
a_3 \mapsto a_3
\end{cases}, \quad
A: \begin{cases}
a_1 \mapsto a_1 \\
a_2 \mapsto a_2 \\
a_3 \mapsto a_1 a_3
\end{cases}, \quad
B: \begin{cases}
a_1 \mapsto a_1 \\
a_2 \mapsto a_2 \\
a_3 \mapsto a_3 a_2
\end{cases}.
\]

To interpret these as loops in the 1-skeleton of \( L_3 \) (and \( K_3 \)) we note that \( A = \lambda_{31} \) and \( B = \rho_{32} \) are elementary transvections and \( T \) is the composition of two elementary transvections: \( T = \lambda_{21} \circ \rho_{12} \). Thus \( w_i \) is the product of \( 8i + 4 \) elementary transvections. There is a (connected) subcomplex of the 1-skeleton of \( L_3 \) spanned by roses (graphs with a single vertex) and Nielsen graphs (which have \( (n - 2) \) loops at the base vertex and a further trivalent vertex). We say roses are adjacent if they have distance 2 in this graph.

Let \( I \in L_3 \) be the rose marked by the identity map \( R_3 \to R_3 \). Each elementary transvection \( \tau \) moves \( I \) to an adjacent rose \( \tau I \), which is connected to \( I \) by a Nielsen graph \( N_\tau \). A composition \( \tau_1 \ldots \tau_k \) of elementary transvections gives a path through adjacent roses \( I, \tau_1 I, \tau_1 \tau_2 I, \ldots, \tau_1 \tau_2 \ldots \tau_k I \); the Nielsen graph connecting \( \sigma I \) to \( \sigma \tau I \) is \( \sigma N_\tau \). Thus the word \( w_i \) corresponds to a loop \( \ell_i \) of length \( 16i + 8 \) in the 1-skeleton of \( L_3 \). Theorem A of [3] provides an exponential lower bound on the filling area of \( \phi \circ \ell_i \) in \( K_3 \).

The square of maps in Lemma 3 ought to have many uses beyond the one in this note (cf. [7]). We mention just one, for illustrative purposes. This is a special case of the fact that every infinite cyclic subgroup of \( \text{Out}(F_n) \) is quasi-isometrically embedded [1].

**Proposition. —** The cyclic subgroup of \( \text{Out}(F_n) \) generated by any Nielsen transformation (elementary transvection) is quasi-isometrically embedded.
Proof. — Each Nielsen transformation is in the image of the map

$$\Phi: \text{Aut}(F_2) \to \text{Aut}(F_n) \to \text{Out}(F_n)$$

given by the inclusion of a free factor $F_2 < F_n$. Thus it suffices to prove that if a cyclic subgroup $C = \langle c \rangle < \text{Aut}(F_2)$ has infinite image in $\text{Out}(F_2)$, then $t \mapsto \Phi(c^t)$ is a quasi-geodesic. This is equivalent to the assertion that some (hence any) $C$-orbit in $K_n$ is quasi-isometrically embedded, where $C$ acts on $K_n$ as $\Phi(C)$ and $K_n$ is given the piecewise Euclidean metric where all edges have length 1.

$K_2$ is a tree and $C$ acts on $K_2$ as a hyperbolic isometry, so the $C$-orbits in $K_2$ are quasi-isometrically embedded. For each $x \in L_2$, the $C$-orbit of $\phi_2(x)$ is the image of the quasi-geodesic $t \mapsto c^t.\phi_2(x) = \phi_2(c^tx)$. We factor $\phi_2$ as a composition of $C$-equivariant simplicial maps $L_2 \xrightarrow{\iota} K_n \xrightarrow{\phi_n} K_2$, as in Lemma 3, to deduce that the $C$-orbit of $\phi_n\iota(x)$ in $K_n$ is quasi-isometrically embedded. □

A slight variation on the above argument shows that if one lifts a free group of finite index $\Lambda < \text{Out}(F_2)$ to $\text{Aut}(F_2)$ and then maps it to $\text{Out}(F_n)$ by choosing a free factor $F_2 < F_n$, then the inclusion $\Lambda \hookrightarrow \text{Out}(F_n)$ will be a quasi-isometric embedding.

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