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WELL-POSEDNESS FOR DENSITY-DEPENDENT INCOMPRESSIBLE FLUIDS WITH NON-LIPSCHITZ VELOCITY

by Boris HASPOT

Abstract. — This paper is dedicated to the study of the initial value problem for density dependent incompressible viscous fluids in $\mathbb{R}^N$ with $N \geqslant 2$. We address the question of well-posedness for *large* and *small* initial data having critical Besov regularity in functional spaces as close as possible to the ones imposed in the incompressible Navier Stokes system by Cannone, Meyer and Planchon (where $u_0 \in B^{\frac{N}{p}-1}_{p,r}$ with $1 \leqslant p < +\infty, 1 \leqslant r \leqslant +\infty$). This improves the classical analysis where u_0 is considered belonging in $B^{\frac{N}{p}-1}_{p,1}$ such that the velocity u remains Lipschitz. Our result relies on a new a priori estimate for transport equation introduce by Bahouri, Chemin and Danchin when the velocity u is not necessary Lipschitz but only log Lipschitz. Furthermore it gives a first kind of answer to the problem of self-similar solution.

Résumé. — Ce papier est dédié à l'étude de Cauchy pour le système de Navier-Stokes non homogène dans $\mathbb{R}^N$ avec $N \geqslant 2$. Nous adressons la question du caractère bien posé pour des données initiales *grandes* et *petites* ayant une régularité critique dans des espaces de Besov aussi proches que possible de ceux utilisés par Cannone, Meyer et Planchon pour Navier Stokes incompressible (où $u_0 \in B^{\frac{N}{p}-1}_{p,r}$ avec $1 \leqslant p < +\infty, 1 \leqslant r \leqslant +\infty$). Cela améliore l'analyse classique où la vitesse initiale u_0 est supposée appartenir à $B^{\frac{N}{p}-1}_{p,1}$ de telle manière que la vitesse u reste Lipschitz. Notre résultat utilise de nouvelles estimées pour l'équation de transport introduites par Bahouri, Chemin et Danchin lorsque la vitesse u n'est pas nécessairement Lipschitz mais seulement log Lipschitz. De plus, cela donne une première réponse de résultat au problème des solutions autosimilaires.

Keywords: Navier-Stokes equations Cauchy problem, Littlewood-Paley theory, losing estimates for the transport equation.
Math. classification: 76D03, 76D05, 35S50.

1. Introduction

In this paper, we are concerned with the following model of incompressible viscous fluid with variable density:

$$(1.1)\qquad \begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u\otimes u) - \mathrm{div}(2\mu(\rho)Du) + \nabla\Pi = \rho f, \\ \mathrm{div}\, u = 0, \\ (\rho, u)_{/t=0} = (\rho_0, u_0). \end{cases}$$

Here $u = u(t,x) \in \mathbb{R}^N$ stands for the velocity field and $\rho = \rho(t,x) \in \mathbb{R}^+$ is the density, $Du = \frac{1}{2}(\nabla u +^t \nabla u)$ is the strain tensor. We denote by μ the viscosity coefficients of the fluid, which is assumed to satisfy $\mu > 0$. The term $\nabla\Pi$ (namely the gradient of the pressure) may be seen as the Lagrange multiplier associated to the constraint $\mathrm{div}\, u = 0$. We supplement the problem with initial condition (ρ_0, u_0) and an outer force f. Throughout the paper, we assume that the space variable $x \in \mathbb{R}^N$ or to the periodic box $\mathbb{T}^N_a$ with period a_i, in the i-th direction. We restrict ourselves to the case $N \geqslant 2$.

The existence of global weak solution for (1.1) under the assumption that $\rho_0 \in L^\infty$ is nonnegative and that $\sqrt{\rho_0}u_0 \in L^2$ has been studied by different authors. It is based on the energy equality:

$$(1.2)\qquad \|\sqrt{\rho}u(t)\|^2_{L^2} + \int_0^t \|\sqrt{\mu(\rho)}Du(\tau)\|^2_{L^2} d\tau = \|\sqrt{\rho_0}u_0\|^2_{L^2} + \int 2(\rho f\cdot u)(\tau,x) d\tau\, dx.$$

Using (1.2) and the fact that the density is advected by the flow of u so that the L^p norms of ρ are (at least formally) conserved during the evolution, it is then possible to use compactness methods for proving the existence of global weak solution. This approach has been introduced by J. Leray in 1934 in the homogeneous case (*i.e.*, $\rho = 1$) and no external force. For the non-homogeneous equation (1.1), we refer to [3] and to [25] for an overview of results on weak solution. Some recent improvements have been obtained by B. Desjardins in [16]. In the sequel we shall only consider the case of constant viscosity coefficients.

The question of unique resolvability for (1.1) has been first addressed by O. Ladyzhenskaya and V. Solonnikov in the late seventies (see [23]). The authors consider system (1.1) in a bounded domain Ω with homogeneous Dirichlet boundary conditions for u. Under the assumption that $u_0 \in W^{2-\frac{2}{q},q}$ $(q > N)$ is divergence-free and vanishes on $\partial\Omega$ and that $\rho_0 \in C^1(\bar{\Omega})$ is bounded away from zero, the results are the following:

- global well-posedness in dimension $N=2$,
- local well-posedness in dimension $N=3$. If in addition u_0 is small in $W^{2-\frac{2}{q},q}$, then global well-posedness holds true.

Let us mention by passing that for the dimension $N=2$, O. Ladyzhenskaya and V. Solonnikov use a quasi-conservation law for the H^1 norm of the velocity and get global H^1 solutions. We would like also to point out that the problem of the existence of global strong solution in dimension $N=2$ is open when the viscosity coefficients are variable.

The case of unbounded domains has been investigate by S. Itoh and A. Tani in [21]. In this framework, they show that the system (1.1) is locally well-posed. In the present paper, we aim at proving similar qualitative results in the whole space $\mathbb{R}^N$ or in the torus $\mathbb{T}^N$ under weaker regularity assumptions.

Guided in our approach by numerous works dedicated to the incompressible Navier-Stokes equation (see z.g [26]):

$$(NS)\qquad \begin{cases} \partial_t v + v\cdot\nabla v - \mu\Delta v + \nabla\Pi = 0,\\ \operatorname{div} v = 0,\end{cases}$$

we aim at solving in the case where the initial data (ρ_0,u_0,f) have critical regularity for the scaling of the equations and in particular when the initial velocity belongs to the same Besov spaces than Cannone, Meyer and Planchon in [6] for the incompressible Navier-Stokes system. It means that we would like to obtain strong solutions results when u_0 is in $B^{\frac{N}{p}-1}_{p,r}$ with $1\leqslant p<+\infty$, $1\leqslant r\leqslant+\infty$ (we refer to the Section 2 for the definition of Besov spaces). By critical, we mean that we want to solve the system in functional spaces with norm is invariant by the changes of scales which leaves (1.1) invariant. That approach has been initiated by H. Fujita and T. Kato in [17]. In the case of incompressible fluids, it is easy to see that the transformations:

$$v(t,x)\longrightarrow lv(l^2t,lx),\quad \forall\, l\in\mathbb{R}$$

have that property.

For density-dependent incompressible fluids, one can check that the appropriate transformations are:

$$\begin{aligned}&(\rho_0(x),u_0(x))\longrightarrow(\rho_0(lx),lu_0(lx)),\quad \forall\, l\in\mathbb{R}.\\ &(\rho(t,x),u(t,x),\Pi(t,x))\longrightarrow(\rho(l^2t,lx),lu(l^2t,lx),l^2\Pi(l^2t,lx)).\end{aligned}\tag{1.3}$$

The use of critical functional frameworks led to several new well-posedness results for incompressible fluids (see [6], [22]). In the case of the density dependent incompressible fluids we would like to cite recent improvements by R. Danchin in [10], [9], [13] , H. Abidi in [1] (when the viscosity coefficients are variable) and H. Abidi, M. Paicu in [2]. All these works deal with the existence of strong solutions in critical spaces for the scaling of the equations. More precisely R. Danchin shows the existence of strong solution in finite time in [10], [9], [13] when the initial data check $(\rho_0^{-1}-1,u_0)\in\big(B^{\frac{N}{2}}_{2,\infty}\cap L^\infty\big)\times B^{\frac{N}{2}-1}_{2,1}$ or $(\rho_0^{-1}-1,u_0)\in B^{\frac{N}{p}}_{p,1}\times B^{\frac{N}{p}}_{p,1}$ with $1\leqslant p\leqslant N$. In addition R. Danchin needs a condition of smallness on the initial density, it means that $\|\rho_0^{-1}-1\|_{B^{\frac{N}{p}}_{p,1}}$ is assumed small. More recently H. Abidi and M. Paicu in [2] improved these results by working with initial data in Besov space with different Lebesgue index for the velocity and the density (we would like to point out that this idea has also been used in the context of compressible Navier-Stokes equations, see [20]), in particular (ρ_0-1) and u_0 belong respectively to $B^{\frac{N}{p_1}}_{p_1,1}$ and $B^{\frac{N}{p_2}-1}_{p_2,1}$ with p_1 and p_2 suitably chosen. This enables them to get strong solution for initial data u_0 in $B^{\frac{N}{p_2}-1}_{p_2,1}$ with $1\leqslant p_2<2N$ which extends the results of R. Danchin. In the same way, they obtain in the same functional spaces the existence of global strong solution with small initial data. All these results use in a crucial way the fact that the solution are Lipschitz. In particular, it explains the choice of the third index $r=1$ for these different Besov space, indeed it entails a Lipschitz control on the velocity u, more precisely ∇u belongs in $L^1_T(B^{\frac{N}{p}}_{p,1})$ which is embedded in $L^1_T(L^\infty)$. This control is imperative in these works in order to estimate via the transport equation the density.

However the scaling of (1.3) suggests to choose initial data (ρ_0,u_0) in $B^{\frac{N}{p_1}}_{p_1,r'}\times B^{\frac{N}{p_2}-1}_{p_2,r}$ with $(p_1,p_2)\in[1,+\infty)^2$ and $(r,r')\in[1,+\infty]^2$. Indeed it seems that it is not mandatory just by some scaling considerations to impose a condition of type $r,r'=1$ as in the works of H. Abidi, R. Danchin and M. Paicu. The goal of this article is to reach the critical case with a general third index for the Besov spaces r and r'. More precisely in the sequel we will restrict our study to the case where the initial data (ρ_0,u_0) and external force f are such that, for some positive constant $\bar\rho>0$:

$$(\rho_0-\bar\rho)\in B^{\frac{N}{p_1}+\varepsilon}_{p_1,\infty}\cap L^\infty,\ u_0\in B^{\frac{N}{p_2}}_{p_2,r}\ \text{ and }\ f\in\widetilde{L}^1_{\text{loc}}\Big(\mathbb{R}^+,\in B^{\frac{N}{p_2}-1}_{p_2,r}\Big).$$

with $r\in[1,+\infty]$, $\varepsilon>0$ and with p_1, p_2 suitably chosen (for a definition of $\widetilde{L}^1$ we refer to the Section 2).

In this article we extend the result of H. Abidi, R. Danchin and M. Paicu by working with initial data in $B^{\frac{N}{p_2}}_{p_2,r}$ with the third index r in$[1,+\infty]$. In particular we generalize to the case of the Navier-Stokes incompressible dependent density the well-known result of existence of strong solution of Cannone-Meyer-Planchon for incompressible Navier-Stokes equations (see [6]) when u_0 is assumed belonging in$B^{\frac{N}{p}-1}_{p,r}$ with $1 \leqslant p < +\infty$, $1 \leqslant r \leqslant +\infty$). To do this, we need new estimates in order to control the density via the transport equation when the velocity is not Lipschitz. We then use some new a priori estimates on the transport equation when the velocity is only assumed log Lipschitz. One of the main difficulty is to deal in this case with the loss of regularity on the density, that is why to compensate this loss we shall work with a bit more regularity on the density ρ_0. The crucial point consists in obtaining sufficient regularity on the density inasmuch as this density remains in a good multiplier space for the velocity (indeed we recall that the momentum equation is close from a Stoke equation with variable coefficient in the density).

Furthermore we also extends the results of H. Abidi, R. Danchin and M. Paicu inasmuch as we do not need to assume any condition of smallness on the initial density. In [10], [9], [13], [1] and [2] , it is mandatory to make the additional assumption that $\rho - \bar{\rho}$ is *small* in $B^{\frac{N}{p}}_{p,1}$. To do this, we follow an idea of R. Danchin in [14] used for the case of compressible Navier-Stokes equations, it consists in handling the elliptic operator in the momentum equation of (1.1) as a constant coefficient second order operator plus a perturbation introduced by $\rho - \bar{\rho}$ which, may be treated as a harmless source term. It is precisely at this point of the proof that we need to control the vacuum (it means $\frac{1}{\rho}$) in the space $L^\infty_T\big(B^{\frac{N}{p_1}}_{p_1,\infty} \cap L^\infty\big)$ (which is embedded in the multiplier space $\mathcal{M}\big(B^{\frac{N}{p}-1}_{p,r}\big)$ with p_1 and p suitably chosen) in order to obtain regularizing effects on the velocity u. We recall that our choice of $(\rho_0 - 1) \in B^{\frac{N}{p_1}+\varepsilon}_{p_1,\infty}$ (with $\varepsilon > 0$) allows to compensate the eventual loss of regularity on the density when the velocity is only assumed log Lipschitz. Indeed by this way we are able to conserve the $L^\infty_T\big(B^{\frac{N}{p_1}}_{p_1,\infty} \cap L^\infty\big)$ of the density.

As long as ρ does not vanish, the equations for $(a = \rho^{-1} - 1, u)$ read:

$$
\begin{cases}
\partial_t a + u \cdot \nabla a = 0, \\
\partial_t u + u \cdot \nabla u + (1 + a)(\nabla \Pi - \mu \Delta u) = f, \\
\operatorname{div} u = 0, \\
(a, u)_{/t=0} = (a_0, u_0).
\end{cases}
\tag{1.4}
$$

One can now state our main result which generalizes the work of Cannone, Meyer, Planchon (see [6]) on the existence of strong solution for Navier-Stokes equations to the density-dependent incompressible Navier-Stokes equations.

Theorem 1.1. — *Let $1 \leqslant r < \infty$, $1 \leqslant p_1 < \infty$, $1 < p_2 < \infty$ and $\varepsilon > 0$ such that:*

$$\frac{N}{p_1} + \varepsilon < \frac{N}{p_2} + 1 \quad \textit{and} \quad \frac{N}{p_2} - 1 \leqslant \frac{N}{p_1}.$$

Assume that $u_0 \in B^{\frac{N}{p_2}-1}_{p_2,r}$ with $\operatorname{div} u_0 = 0$, $f \in \widetilde{L}^1_{\mathrm{loc}}\Big(\mathbb{R}^+, B^{\frac{N}{p_2}-1}_{p_2,r}\Big)$ and $a_0 \in B^{\frac{N}{p_1}+\varepsilon}_{p_1,\infty} \cap L^\infty$, with $1 + a_0$ bounded away from zero and it exists $c > 0$ such that:

$$\|a_0\|_{B^{\frac{N}{p_1}}_{p_1,\infty} \cap L^\infty} \leqslant c.$$

If $\frac{1}{p_1} + \frac{1}{p_2} > \frac{1}{N}$, there exists a positive time T such that system (1.4) has a solution (a, u) with $1 + a$ bounded away from zero and:

$$a \in \widetilde{C}\Big([0,T], B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty}\Big), \;\; u \in \Big(\widetilde{C}\Big([0,T]; B^{\frac{N}{p_2}-1}_{p_2,r}\Big) \cap \widetilde{L}^1\Big(0,T, B^{\frac{N}{p_2}+1}_{p_2,r}\Big)\Big)^N$$
$$\textit{and} \;\; \nabla\Pi \in \widetilde{L}^1\Big(0,T, B^{\frac{N}{p_2}-1}_{p_2,r}\Big).$$

If in addition we assume that $a_0 \in L^{p_1}$, it exists a constant c such that if:

$$\|u_0\|_{\dot{B}^{\frac{N}{p_2}-1}_{p_2,r}} + \|a_0\|_{\dot{B}^{\frac{N}{p_1}+\varepsilon}_{p_1,\infty} \cap L^\infty \cap L^{p_1}} + \|f\|_{\widetilde{L}^1\big(\dot{B}^{\frac{N}{p_2}-1}_{p_2,r}\big)} \leqslant c\mu,$$

then $T = +\infty$. This solution is unique when $\frac{2}{N} \leqslant \frac{1}{p_1} + \frac{1}{p_2}$.

Remark 1.2. — We can observe that we need additional information in low frequencies for getting the global existence of strong solution, indeed we ask that a_0 belongs in L^{p_1}. Furthermore we are working with homogeneous Besov spaces which is absolutely mandatory in order to obtain global solution. The fact that we add the condition $a_0 \in L^{p_1}$ is due to the different behavior in low and high frequencies when we are dealing with estimates on the transport equation with loss of regularity. We will give additional details on this fact in the proof of the Theorem 1.1 (see Section 8).

Remark 1.3. — In the present paper we did not strive for unnecessary generality which may hide the new ideas of our analysis. Hence we focused on the case of constant viscosity coefficients. We believe that our analysis may be generalized to the case of variable viscosity coefficients.

Remark 1.4. — As in the work of H. Abidi and M. Paicu in [2], we are able to get strong solution when $u_0 \in B^{\frac{N}{p_2}-1}_{p_2,r}$ with $1 < p_2 \leqslant 2N$, it improves the result of R. Danchin in [10, 9] and [13].

Moreover we get weak solution with initial data very close from $(a_0, u_0) \in B^{1+\varepsilon}_{N,\infty} \times B^{-1}_{\infty,r}$ and $(a_0, u_0) \in B^{\varepsilon}_{\infty,\infty} \times B^{0}_{N,r}$. It means that in the first case we are not far away from the results of Koch-Tataru in [22] when the initial data for the velocity u_0 is belonging in BMO^{-1}; in the second case the initial density a_0 is close from being only in L^∞, which is of great interest for the system multifluids. We refer also to the interesting work of P. Germain in [18].

Remark 1.5. — It would be possible to improve in dimension $N = 2$ the existence of global strong solution by working close to a solution u_L of incompressible Navier-Stokes equations when the initial data is u_0. Indeed in our case for simplicity we are working close to a solution u_L of the heat equation (see the proof for more details). Indeed we would be able in this case to obtain global strong solution in dimension $N = 2$ without assuming smallness on the initial velocity. We could proceed similarly in order to obtain global strong solution in dimension $N = 3$ with a family of large initial velocity for the critical Besov norms (however there would be a condition of smallness in low frequencies). We refer to the works of J.-Y. Chemin, I. Gallagher and M. Paicu (see [7]) in the case of incompressible Navier-Stokes equations. The idea of the proof consists in choosing initial data such that $u_L \cdot \nabla u_L$ is small in $\widetilde{L}^1(B^{\frac{N}{p_2}-1}_{p_2,1})$ in order to "cancel out" in some sense the nonlinearity which requires in general smallness condition on the initial data. Here u_L is solution of the Stokes equateion.

Remark 1.6. — In the previous theorem, we need a condition of smallness, because when $p_2 \neq 2$, we have extra term in our Proposition 4.1 which requires a condition of smallness on a.

In the following theorem, we improve the previous result in the specific case where $p_2 = 2$. In this case we don't need to impose condition of smallness on the initial data.

Theorem 1.7. — *Let $1 \leqslant r < \infty$, $1 \leqslant p_1 < \infty$ and $\varepsilon > 0$ such that:*

$$\frac{N}{p_1} + \varepsilon < \frac{N}{2} + 1 \quad \text{and} \quad \frac{N}{2} \leqslant 1 + \frac{N}{p_1}.$$

Assume that $u_0 \in B^{\frac{N}{2}-1}_{2,r}$ with $\operatorname{div} u_0 = 0$, $f \in \widetilde{L}^1_{\text{loc}}\Big(\mathbb{R}^+, B^{\frac{N}{2}-1}_{2,r}\Big)$ and $a_0 \in B^{\frac{N}{p_1}+\varepsilon}_{p_1,\infty} \cap L^\infty$, with $1 + a_0$ bounded away from zero. There exists a positive *time T such that system (1.4) has a solution (a,u) with $1+a$ bounded away from zero and:*

$$a\in\widetilde{C}\Big([0,T],B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty}\Big),\;\; u\in\Big(\widetilde{C}\Big([0,T];B^{\frac{N}{2}-1}_{2,r}\Big)\cap\widetilde{L}^1\Big(0,T,B^{\frac{N}{2}+1}_{2,r}\Big)\Big)^N$$
$$\text{and}\;\;\nabla\Pi\in\widetilde{L}^1\Big(0,T,B^{\frac{N}{2}-1}_{2,r}\Big).$$

If in addition we assume that $a_0\in L^{p_1}$, it exists a constant c such that if:

$$\|u_0\|_{\dot{B}^{\frac{N}{2}-1}_{2,r}}+\|a_0\|_{\dot{B}^{\frac{N}{p_1}+\varepsilon}_{p_1,\infty}\cap L^\infty\cap L^{p_1}}+\|f\|_{\widetilde{L}^1(\dot{B}^{\frac{N}{p_2}-1}_{p_2,r})}\leqslant c\mu,$$

then $T=+\infty$. This solution is unique when $\frac{2}{N}\leqslant\frac{1}{p_1}+\frac{1}{2}$.

In the following theorems we want to deal with the case $r=+\infty$, we have to treat the case of a linear loss of regularity on the density ρ which depends on the behavior of the velocity u when u belongs in $\widetilde{L}^1_T\big(B^{\frac{N}{p}+1}_{p,\infty}\big)$.

Theorem 1.8. — *Let $1\leqslant p_1<\infty$, $1<p_2<\infty$, and $\varepsilon>0$ such that:*

$$\frac{N}{p_1}+\varepsilon<\frac{N}{p_2}+1\quad\text{and}\quad\frac{N}{p_2}\leqslant 1+\frac{N}{p_1}.$$

Assume that $u_0\in B^{\frac{N}{p_2}-1}_{p_2,\infty}$ with $\operatorname{div}u_0=0$, $f\in\widetilde{L}^1_{\rm loc}\big(\mathbb{R}^+,B^{\frac{N}{p_2}-1}_{p_2,\infty}\big)$ and $a_0\in B^{\frac{N}{p_1}+\varepsilon}_{p_1,\infty}\cap L^\infty$, with $1+a_0$ bounded away from zero and it exists $c>0$ such that:

$$\|a_0\|_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty}\leqslant c.$$

If $\frac{1}{p_1}+\frac{1}{p_2}>\frac{1}{N}$, there exists a positive time T such that system (1.4) has a solution (a,u) with $1+a$ bounded away from zero and:

$$a\in\widetilde{C}\Big([0,T],B^{\sigma(T)}_{p_1,\infty}\Big),\;\; u\in\Big(\widetilde{C}\Big([0,T];B^{\frac{N}{p_2}-1}_{p_2,\infty}\cap\widetilde{L}^1\Big(0,T,B^{\frac{N}{p_2}+1}_{p_2,\infty}\Big)\Big)^N$$
$$\text{and}\;\;\nabla\Pi\in\widetilde{L}^1\Big(0,T,B^{\frac{N}{p_2}-1}_{p_2,\infty}\Big),$$

with:

$$\sigma(T)=\frac{N}{p_1}+\varepsilon-\lambda\|u\|_{\widetilde{L}^1_T\big(B^{\frac{N}{p_2}+1}_{p_2,\infty}\big)}$$

for any $\lambda>0$ depending only on N, p_1 and p_2. If in addition we assume that $a_0\in L^{p_1}$, it exists a constant c such that if:

$$\|u_0\|_{\dot{B}^{\frac{N}{p_2}-1}_{p_2,\infty}}+\|a_0\|_{\dot{B}^{\frac{N}{p_1}+\varepsilon}_{p_1,\infty}\cap L^\infty\cap L^{p_1}}+\|f\|_{\widetilde{L}^1\big(\dot{B}^{\frac{N}{p_2}-1}_{p_2,\infty}\big)}\leqslant c\mu,$$

then $T=+\infty$. This solution is unique when $\frac{2}{N}\leqslant\frac{1}{p_1}+\frac{1}{p_2}$.

Remark 1.9. — We would like to point out that in the previous theorem we can choose initial velocity such that u_0 is homogeneous of degree -1. Indeed an important open problem concerns the existence of self similar solutions, it means solutions such that ρ_0 is homogeneous of degree 0 and u_0 homogeneous of degree -1. Via the scaling of the equation, if we have existence of strong solutions for such initial data, we have then for all $l \in \mathbb{R}$:

$$(\rho(t,x), u(t,x), \Pi(t,x)) = (\rho(l^2 t, lx), lu(l^2 t, lx), l^2\Pi(l^2 t, lx)).$$

The existence of such solution for incompressible Navier-Stokes equation with dependent density is actually open (indeed the main difficulty consists in dealing with "the a priori loss" of regularity on the initial density while we assumed the initial density critical). For the classical incompressible Navier-Stokes equations we refer to the book of Lemarié-Rieusset in [24] and the works of Cannone, Meyer and Planchon [6].

However our result gives a first kind of answer to this problem. Indeed we are able to choose initial homogeneous velocity of degree -1 (for example $u_0 = \frac{1}{|x|}$), more precisely we obtain a solution u which can be splitter in a self-semilar solution plus a small perturbative term. Indeed following the proof of Theorem 1.8, the solution is such that:

$$u = u_L + \bar{u},$$

with u_L solution of the Stokes equation with initial data u_0. u_L is then self-similar and $\bar{u}$ has to be considered as a smallperturbation. It means that the solution u remains close to a self similar solution along the time.

In the following theorem, we generalize the previous result with large initial data for the initial density when $p_2 = 2$.

THEOREM 1.10. — *Let $1 \leqslant p_1 < \infty$ and $\varepsilon > 0$ such that:*

$$\frac{N}{p_1} + \varepsilon < \frac{N}{2} + 1 \quad \text{and} \quad \frac{N}{2} \leqslant 1 + \frac{N}{p_1}.$$

Assume that $u_0 \in B^{\frac{N}{2}-1}_{2,\infty}$ with $\operatorname{div} u_0 = 0$, $f \in \widetilde{L}^1_{\text{loc}}(\mathbb{R}^+, B^{\frac{N}{2}-1}_{2,\infty})$ and $a_0 \in B^{\frac{N}{p_1}+\varepsilon}_{p_1,\infty} \cap L^\infty$, with $1 + a_0$ bounded away from zero. There exists a positive time T such that system (1.4) has a solution (a, u) with $1 + a$ bounded away from zero and:

$$a \in \widetilde{C}([0,T], B^{\sigma(T)}_{p_1,\infty}), \;\; u \in \Big(\widetilde{C}\Big([0,T]; B^{\frac{N}{2}-1}_{2,\infty}\Big) \cap \widetilde{L}^1\Big(0,T, B^{\frac{N}{2}+1}_{2,\infty}\Big)\Big)^N$$
$$\text{and} \;\; \nabla\Pi \in \widetilde{L}^1\Big(0,T, B^{\frac{N}{2}-1}_{2,\infty}\Big),$$

with:

$$\sigma(T) = \frac{N}{p_1} + \varepsilon - \lambda \|u\|_{\widetilde{L}^1_T\left(B^{\frac{N}{2}+1}_{2,\infty}\right)},$$

for $\lambda > 0$ depending only on N and p_1. If in addition we assume that $a_0 \in L^{p_1}$, it exists a constant c such that if:

$$\|u_0\|_{\dot{B}^{\frac{N}{2}-1}_{2,\infty}} + \|a_0\|_{\dot{B}^{\frac{N}{p_1}+\varepsilon}_{p_1,\infty} \cap L^\infty \cap L^{p_1}} + \|f\|_{\widetilde{L}^1\left(\dot{B}^{\frac{N}{2}-1}_{2,\infty}\right)} \leqslant c\mu,$$

then $T = +\infty$. This solution is unique when $\frac{2}{N} \leqslant \frac{1}{p_1} + \frac{1}{2}$.

The key of the Theorems 1.1, 1.7, 1.8 and 1.10 is based on new estimates for transport equation on the velocity u when it is is not considered Lipschitz. In this case we have to pay a loss of regularity on the density ρ. The basic idea to deal with this loss of regularity is to add a little bit regularity on the initial density a_0 in order to conserve a on a small interval $(0,T)$ in $\widetilde{C}_T\left(B^{\frac{N}{p_1}}_{p_1,+\infty}\right) \cap L^\infty$ which has good properties of multiplier for the term Δu.

Our paper is structured as follows. In the Section 2, we give a few notation and briefly introduce the basic Fourier analysis techniques needed to prove our result. Section 4 and 5 are devoted to the proof of key estimates for the linearized system (1.4) in particular the elliptic operator of the momentum equation with variable coefficients and the transport equation when the velocity is not assumed Lipschitz. In Section 6, we prove the existence of solutions for Theorem 1.1 whereas Section 7 is devoted to the proof of uniqueness. In Section 8, we prove the part of Theorem 1.1 concerning the global existence and the Theorem 1.7. Finally in Section 9, we briefly show how to prove Theorem 1.8 and 1.10. Elliptic and technical estimates commutator are postponed in an appendix.

2. Littlewood-Paley theory and Besov spaces

Throughout the paper, C stands for a constant whose exact meaning depends on the context. The notation $A \lesssim B$ means that $A \leqslant CB$. For all Banach space X, we denote by $C([0,T],X)$ the set of continuous functions on $[0,T]$ with values in X. For $p \in [1,+\infty]$, the notation $L^p(0,T,X)$ or $L^p_T(X)$ stands for the set of measurable functions on $(0,T)$ with values in X such that $t \to \|f(t)\|_X$ belongs to $L^p(0,T)$.

2.1. Littlewood-Paley decomposition

Littlewood-Paley decomposition corresponds to a dyadic decomposition of the space in Fourier variables. Let $\alpha > 1$ and (φ, χ) be a couple of smooth functions valued in $[0,1]$, such that φ is supported in the shell supported in $\{\xi \in \mathbb{R}^N / \alpha^{-1} \leqslant |\xi| \leqslant 2\alpha\}$, χ is supported in the ball $\{\xi \in \mathbb{R}^N / |\xi| \leqslant \alpha\}$ such that:

$$\forall \xi \in \mathbb{R}^N, \quad \chi(\xi) + \sum_{l \in \mathbb{N}} \varphi(2^{-l}\xi) = 1.$$

Denoting $h = \mathcal{F}^{-1}\varphi$, we then define the dyadic blocks by:

$$\begin{aligned}
&\Delta_l u = 0 \quad \text{if} \;\; l \leqslant -2,\\
&\Delta_{-1} u = \chi(D)u = \widetilde{h} * u \quad \text{with} \;\; \widetilde{h} = \mathcal{F}^{-1}\chi,\\
&\Delta_l u = \varphi(2^{-l}D)u = 2^{lN}\int_{\mathbb{R}^N} h(2^l y)u(x-y)dy \;\; \text{with}\; h = \mathcal{F}^{-1}\chi, \;\; \text{if} \;\; l \geqslant 0,\\
&S_l u = \sum_{k \leqslant l-1} \Delta_k u.
\end{aligned}$$

Formally, one can write that: $u = \sum_{k\in\mathbb{Z}} \Delta_k u$. This decomposition is called nonhomogeneous Littlewood-Paley decomposition.

2.2. Nonhomogeneous Besov spaces and first properties

Definition 2.1. — *For $s \in \mathbb{R}$, $p \in [1,+\infty]$, $q \in [1,+\infty]$, and $u \in \mathcal{S}'(\mathbb{R}^N)$ we set:*

$$\|u\|_{B^s_{p,q}} = \Big(\sum_{l\in\mathbb{Z}} (2^{ls}\|\Delta_l u\|_{L^p})^q\Big)^{\frac{1}{q}}.$$

The Besov space $B^s_{p,q}$ is the set of temperate distribution u such that $\|u\|_{B^s_{p,q}} < +\infty$.

Remark 2.2. — The above definition is a natural generalization of the nonhomogeneous Sobolev and Hölder spaces: one can show that $B^s_{\infty,\infty}$ is the nonhomogeneous Hölder space C^s and that $B^s_{2,2}$ is the nonhomogeneous space H^s.

Proposition 2.3. — *The following properties holds:*

(1) *there exists a constant universal C such that:*
$C^{-1}\|u\|_{B^s_{p,r}} \leqslant \|\nabla u\|_{B^{s-1}_{p,r}} \leqslant C\|u\|_{B^s_{p,r}}$.

(2) *If $p_1 < p_2$ and $r_1 \leqslant r_2$ then $B^s_{p_1,r_1} \hookrightarrow B^{s-N(1/p_1-1/p_2)}_{p_2,r_2}$.*

(3) $(B^{s_1}_{p,r}, B^{s_2}_{p,r})_{\theta,r'} = B^{\theta s_1+(1-\theta)s_2}_{p,r'}$.

Let now recall a few product laws in Besov spaces coming directly from the paradifferential calculus of J-M. Bony (see [5]) and rewrite on a generalized form in [2] by H. Abidi and M. Paicu (in this article the results are written in the case of homogeneous sapces but it can easily generalize for the nonhomogeneous Besov spaces).

Proposition 2.4. — *We have the following laws of product:*

- *For all $s \in \mathbb{R}$, $(p,r) \in [1,+\infty]^2$ we have:*

$$\|uv\|_{\widetilde{B}^s_{p,r}} \leqslant C\big(\|u\|_{L^\infty}\|v\|_{B^s_{p,r}} + \|v\|_{L^\infty}\|u\|_{B^s_{p,r}}\big). \tag{2.1}$$

- *Let $(p, p_1, p_2, r, \lambda_1, \lambda_2) \in [1,+\infty]^2$ such that: $\frac{1}{p} \leqslant \frac{1}{p_1} + \frac{1}{p_2}$, $p_1 \leqslant \lambda_2$, $p_2 \leqslant \lambda_1$, $\frac{1}{p} \leqslant \frac{1}{p_1} + \frac{1}{\lambda_1}$ and $\frac{1}{p} \leqslant \frac{1}{p_2} + \frac{1}{\lambda_2}$. We have then the following inequalities:*
if $s_1 + s_2 + N \inf\big(0, 1 - \frac{1}{p_1} - \frac{1}{p_2}\big) > 0$, $s_1 + \frac{N}{\lambda_2} < \frac{N}{p_1}$ and $s_2 + \frac{N}{\lambda_1} < \frac{N}{p_2}$ then:

$$\|uv\|_{B^{s_1+s_2-N(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})}_{p,r}} \lesssim \|u\|_{B^{s_1}_{p_1,r}}\|v\|_{B^{s_2}_{p_2,\infty}}, \tag{2.2}$$

when $s_1 + \frac{N}{\lambda_2} = \frac{N}{p_1}$ (resp. $s_2 + \frac{N}{\lambda_1} = \frac{N}{p_2}$) we replace $\|u\|_{B^{s_1}_{p_1,r}}\|v\|_{B^{s_2}_{p_2,\infty}}$ (resp. $\|v\|_{B^{s_2}_{p_2,\infty}}$) by $\|u\|_{B^{s_1}_{p_1,1}}\|v\|_{B^{s_2}_{p_2,r}}$ (resp. $\|v\|_{B^{s_2}_{p_2,\infty}\cap L^\infty}$), if $s_1 + \frac{N}{\lambda_2} = \frac{N}{p_1}$ and $s_2 + \frac{N}{\lambda_1} = \frac{N}{p_2}$ we take $r = 1$.
If $s_1 + s_2 = 0$, $s_1 \in \big(\frac{N}{\lambda_1} - \frac{N}{p_2}, \frac{N}{p_1} - \frac{N}{\lambda_2}\big]$ and $\frac{1}{p_1} + \frac{1}{p_2} \leqslant 1$ then:

$$\|uv\|_{B^{-N(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})}_{p,\infty}} \lesssim \|u\|_{B^{s_1}_{p_1,1}}\|v\|_{B^{s_2}_{p_2,\infty}}. \tag{2.3}$$

If $|s| < \frac{N}{p}$ for $p \geqslant 2$ and $-\frac{N}{p'} < s < \frac{N}{p}$ else, we have:

$$\|uv\|_{B^s_{p,r}} \leqslant C\|u\|_{B^s_{p,r}}\|v\|_{B^{\frac{N}{p}}_{p,\infty}\cap L^\infty}. \tag{2.4}$$

Remark 2.5. — In the sequel p will be either p_1 or p_2 and in this case $\frac{1}{\lambda} = \frac{1}{p_1} - \frac{1}{p_2}$ if $p_1 \leqslant p_2$, resp $\frac{1}{\lambda} = \frac{1}{p_2} - \frac{1}{p_1}$ if $p_2 \leqslant p_1$.

Corollary 2.6. — *Let $r \in [1,+\infty]$, $1 \leqslant p \leqslant p_1 \leqslant +\infty$ and s such that:*

- *$s \in \big(-\frac{N}{p_1}, \frac{N}{p_1}\big)$ if $\frac{1}{p} + \frac{1}{p_1} \leqslant 1$,*
- *$s \in \big(-\frac{N}{p_1} + N\big(\frac{1}{p} + \frac{1}{p_1} - 1\big), \frac{N}{p_1}\big)$ if $\frac{1}{p} + \frac{1}{p_1} > 1$,*

then we have if $u \in B^s_{p,r}$ and $v \in B^{\frac{N}{p_1}}_{p_1,\infty} \cap L^\infty$:

$$\|uv\|_{B^s_{p,r}} \leqslant C\|u\|_{B^s_{p,r}}\|v\|_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty}.$$

The study of non stationary PDE's requires space of type $L^\rho(0,T,X)$ for appropriate Banach spaces X. In our case, we expect X to be a Besov space, so that it is natural to localize the equation through Littlewood-Payley decomposition. But, in doing so, we obtain bounds in spaces which are not type $L^\rho(0,T,X)$ (except if $r=p$). We are now going to define the spaces of Chemin-Lerner in which we will work, which are a refinement of the spaces $L^\rho_T(B^s_{p,r})$.

Definition 2.7. — *Let $\rho\in[1,+\infty]$, $T\in[1,+\infty]$ and $s_1\in\mathbb{R}$. We set:*

$$\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})}=\Big(\sum_{l\in\mathbb{Z}}2^{lrs_1}\|\Delta_l u(t)\|^r_{L^\rho(L^p)}\Big)^{\frac{1}{r}}.$$

We then define the space $\widetilde{L}^\rho_T(B^{s_1}_{p,r})$ as the set of temperate distribution u over $(0,T)\times\mathbb{R}^N$ such that $\mathcal{S}'((0,T)\times\mathbb{R}^N)$ $\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})}<+\infty$.

We set $\widetilde{C}_T(\widetilde{B}^{s_1}_{p,r})=\widetilde{L}^\infty_T(\widetilde{B}^{s_1}_{p,r})\cap\mathcal{C}([0,T],B^{s_1}_{p,r})$.

Remark 2.8. — Let us emphasize that, according to Minkowski inequality, we have:

$$\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})}\leqslant\|u\|_{L^\rho_T(B^{s_1}_{p,r})}\ \text{ if }\ r\geqslant\rho,\quad \|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})}\geqslant\|u\|_{L^\rho_T(B^{s_1}_{p,r})}\ \text{ if }\ r\leqslant\rho.$$

Remark 2.9. — It is easy to generalize Proposition 2.4, to $\widetilde{L}^\rho_T(B^{s_1}_{p,r})$ spaces. The indices s_1, p, r behave just as in the stationary case whereas the time exponent ρ behaves according to Hölder inequality.

Here we recall a result of interpolation which explains the link of the space $B^s_{p,1}$ with the space $B^s_{p,\infty}$, see [4].

Proposition 2.10. — *There exists a constant C such that for all $s\in\mathbb{R}$, $\varepsilon>0$ and $1\leqslant p<+\infty$,*

$$\|u\|_{\widetilde{L}^\rho_T(B^s_{p,1})}\leqslant C\frac{1+\varepsilon}{\varepsilon}\|u\|_{\widetilde{L}^\rho_T(B^s_{p,\infty})}\left(1+\log\frac{\|u\|_{\widetilde{L}^\rho_T(B^{s+\varepsilon}_{p,\infty})}}{\|u\|_{\widetilde{L}^\rho_T(B^s_{p,\infty})}}\right).$$

Definition 2.11. — *Let Γ be an increasing function on $[1,+\infty[$. We denote by $B_\Gamma(\mathbb{R}^N)$ the set of bounded real valued functions u over $\mathbb{R}^N$ such that:*

$$\|u\|_{B_\Gamma}=\|u\|_{L^\infty}+\sup_{j\geqslant0}\frac{\|\nabla S_j u\|_{L^\infty}}{\Gamma(2^j)}<+\infty.$$

We give here a proposition concerning these spaces showed by J-Y. Chemin (it can be found in [4]).

Proposition 2.12. — *Let* $\varepsilon > 0$ *and* $u \in \widetilde{L}^1_T\big(B^{\frac{N}{p}+1}_{p,r}\big)$ *then we have* $u \in L^1_T(B_\Gamma(\mathbb{R}^N))$ *with* $\Gamma(t) = (-\log t)^{1+\varepsilon-\frac{1}{r}}$ *for* $0 \leqslant t \leqslant 1$ *.*

3. Homogeneous Besov spaces

Littlewood-Paley decomposition corresponds to a dyadic decomposition of the space in Fourier variables. We have then:

$$\sum_{l\in\mathbb{Z}} \varphi(2^{-l}\xi) = 1 \quad \text{if} \quad \xi \neq 0.$$

Denoting $h = \mathcal{F}^{-1}\varphi$, we then define the dyadic blocks for $l \in \mathbb{Z}$ by:

$$\dot{\Delta}_l u = \varphi(2^{-l}D)u = 2^{lN}\int_{\mathbb{R}^N} h(2^l y)u(x-y)dy \quad \text{and} \quad \dot{S}_l u = \sum_{k\leqslant l-1} \Delta_k u.$$

Formally, one can write that:

$$u = \sum_{k\in\mathbb{Z}} \dot{\Delta}_k u.$$

This decomposition is called homogeneous Littlewood-Paley decomposition. Let us observe that the above formal equality does not hold in $\mathcal{S}^{'}(\mathbb{R}^N)$ for two reasons:

(1) The right hand-side does not necessarily converge in $\mathcal{S}^{'}(\mathbb{R}^N)$.
(2) Even if it does, the equality is not always true in $\mathcal{S}^{'}(\mathbb{R}^N)$ (consider the case of the polynomials).

This motivates the following definition:

Definition 3.1. — *We note by* $\mathcal{S}^{'}_h$ *the space of temperate distributions* u *such that:*

$$\lim_{j\to-\infty} \dot{S}_j u = 0 \ \ in \ \ \mathcal{S}^{'}.$$

Definition 3.2. — *For* $s \in \mathbb{R}$, $p \in [1,+\infty]$, $q \in [1,+\infty]$, *and* $u \in \mathcal{S}^{'}(\mathbb{R}^N)$ *we set:*

$$\|u\|_{\dot{B}^s_{p,q}} = \Big(\sum_{l\in\mathbb{Z}} (2^{ls}\|\Delta_l u\|_{L^p})^q\Big)^{\frac{1}{q}}.$$

The Besov space $\dot{B}^s_{p,q}$ *is the set of temperate distribution* $u \in \mathcal{S}^{'}_h$ *such that* $\|u\|_{\dot{B}^s_{p,q}} < +\infty$.

The properties of homogeneous Besov spaces are essentially the same than in the case of the nonhomogeneous Besov spaces. For more details we refer to [4].

4. Estimates for parabolic system with variable coefficients

In this section, the following linearization of the momentum equation is studied:

$$\begin{cases} \partial_t u + b(\nabla\Pi - \mu\Delta u) = f + g, \\ \operatorname{div} u = 0, \\ u_{/t=0} = u_0 \end{cases} \tag{4.1}$$

where b, f, g and u_0 are given. Above u is the unknown function. We assume that $u_0 \in B^s_{p,r}$ and $f \in \widetilde{L}^1(0,T;B^s_{p,r})$, that b is bounded by below by a positive constant $\underline{b}$ and that $a = b-1$ belongs to $\widetilde{L}^\infty\big(0,T;B^{\frac{N}{p_1}+\alpha}_{p_1,\infty}\big) \cap L^\infty$. In the present subsection, we aim at proving a priori estimates for (4.1) in the framework of nonhomogeneous Besov spaces. Before stating our results let us introduce the following notation:

$$\mathcal{A}_T = 1 + \underline{b}^{-1}\|\nabla b\|_{\widetilde{L}^\infty\left(B^{\frac{N}{p_1}+\alpha-1}_{p_1,\infty}\right)} \quad \text{with} \;\; \alpha > 0. \tag{4.2}$$

Proposition 4.1. — *Let $\underline{\nu} = \underline{b}\mu$ and $(p,p_1) \in [1,+\infty]$.*

- *If $p_1 > p$ we assume that $s \in \big(-\frac{N}{p_1}, \frac{N}{p_1}\big)$ if $\frac{1}{p}+\frac{1}{p_1} \leqslant 1$ and $s \in \big(-\frac{N}{p_1} + N\big(\frac{1}{p}+\frac{1}{p_1}-1\big), \frac{N}{p_1}\big)$ if $\frac{1}{p}+\frac{1}{p_1} > 1$.*
- *If $p_1 \leqslant p$ then we suppose that $s \in \big(-\frac{N}{p}, \frac{N}{p}\big)$ if $p \geqslant 2$ and $s \in \big(-\frac{N}{p'}, \frac{N}{p}\big)$ if $p < 2$.*

If $p \neq 2$ we need to assume than there exists $c > 0$ such that:

$$\|\nabla a\|_{\widetilde{L}^\infty\left(B^{\frac{N}{p_1}+\alpha-1}_{p_1,\infty}\right)} \leqslant c.$$

Let $m \in \mathbb{Z}$ be such that $b_m = 1 + S_m a$ satisfies:

$$\inf_{(t,x)\in[0,T)\times\mathbb{R}^N} b_m(t,x) \geqslant \frac{\underline{b}}{2}. \tag{4.3}$$

There exist three constants c, C and κ (with c, C, depending only on N and on s, and κ universal) such that if in addition we have:

$$\|a - S_m a\|_{\widetilde{L}^\infty(0,T;B^{\frac{N}{p_1}}_{p_1,\infty})\cap L^\infty} \leqslant c\frac{\underline{\nu}}{\mu} \tag{4.4}$$

then setting:

$$Z_m(t) = 2^{2m\alpha}\mu^2\underline{\nu}^{-1}\int_0^t \|a\|^2_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty}\, d\tau,$$

Let $\alpha' > 0$ *checking* $\alpha' \leqslant \min\left(1, \alpha, \frac{s-2+\frac{2}{m}}{2}\right)$*. We have for all* $t \in [0,T]$ *and* $\kappa = \frac{s}{\alpha'}$*:*

(4.5)

$$\|u\|_{\widetilde{L}^\infty_T(B^s_{p,r})} + \kappa\underline{\nu}\|u\|_{\widetilde{L}^1_T(B^{s+2}_{p,r})} \leqslant e^{CZ_m(T)}\Bigg(\|u_0\|_{B^s_{p,r}} + \mathcal{A}^\kappa_T\Big(\|\mathcal{P}f\|_{\widetilde{L}^1_T(B^s_{p,r})} + \underline{\mu}^{\frac{1}{m}}\|\mathcal{P}g\|_{\widetilde{L}^m_T\left(B^{s-2+\frac{2}{m}}_{p,r}\right)} + \underline{\mu}^{\frac{1}{m}}\left(\frac{\underline{\nu}(p-1)}{p}\right)\mathcal{A}_T\|u\|_{\widetilde{L}^1_T(B^{s+2-\alpha'}_{p,r})}\Big)\Bigg).$$

Moreover we have $\nabla\Pi = \nabla\Pi_1 + \nabla\Pi_2$ *with:*

(4.6)

$$\underline{b}\|\nabla\Pi_1\|_{\widetilde{L}^1_T(B^s_{p,r})} \leqslant \mathcal{A}^\kappa_T\|\mathcal{P}f\|_{\widetilde{L}^1_T(B^s_{p,r})},$$

$$\underline{b}\|\nabla\Pi_2\|_{\widetilde{L}^1_T(B^s_{p,r})} \leqslant \mathcal{A}^\kappa_T\Big(\|\mathcal{Q}g\|_{\widetilde{L}^m_T\left(B^{s-2+\frac{2}{m}}_{p,r}\right)} + \mu\|a\|_{\widetilde{L}^\infty_T\left(B^{\frac{N}{p_1}+\alpha}_{p_1,+\infty}\right)}\|\Delta u\|_{\widetilde{L}^m_T\left(B^{s-2+\frac{2}{m}}_{p,r}\right)}\Big).$$

Remark 4.2. — Let us stress the fact that if $a \in \widetilde{L}^\infty\big((0,T) \times B^{\frac{N}{p_1}}_{p_1,\infty}\big)$ then Assumption (4.3) and (4.4) are satisfied for m large enough. This will be used in the proof of Theorem 1.7 and 1.10. Indeed, according to Bernstein inequality for m large enough 9 (4.3) and (4.4) are satisfied.

Proving Proposition 4.1 in the case $b = cste$ is not too involved as one can easily get rid of the pressure by taking advantage of the Leray projector $\mathcal{P}$ on solenoidal vector-fields. Then system (4.1) reduce to a linear ψDO which may be easily solved by mean of energy estimates. In our case where b is not assumed to be a constant, getting rid of the pressure will still be an appropriate strategy. This may be achieved by applying the operator div to (4.1). Indeed by doing so, we see that the pressure solves the elliptic equation:

$$\operatorname{div}(b\nabla\Pi) = \operatorname{div} F \tag{4.7}$$

with $F = f + g + \mu a\Delta u$. Therefore denoting by $\mathcal{H}_b$ the linear operator $F \to \nabla\Pi$, system (4.1) reduces to a linear ODE in Banach spaces. Actually, due to the consideration of two forcing terms f and g with different regularities, the pressure has to be split into two parts, namely $\Pi = \Pi_1 + \Pi_2$ with:

$$\operatorname{div}(b\Pi_1) = \operatorname{div} f \tag{4.8}$$

$$\operatorname{div}(b\Pi_2) = \operatorname{div} H \quad \text{and} \quad H = g + \mu a\Delta u. \tag{4.9}$$

Proof of Proposition 4.1. — Let us first rewrite (4.1) as follows:

$$\begin{cases} \partial_t u - b_m \mu \Delta u + b \nabla \Pi = f + g + E_m, \\ \operatorname{div} u = 0, \\ u_{t=0} = u_0 \end{cases} \tag{4.10}$$

with $E_m = \mu \Delta u \, (\mathrm{Id} - S_m) a$ and $b_m = 1 + S_m a$. Note that by using Corollary 2.6 and as $-\frac{N}{p_1} < s < \frac{N}{p_1}$ for $p \geqslant 2$ or $\frac{N}{p_1'} < s < \frac{N}{p_1}$ else, the error term E_m may be estimated by:

$$\|E_m\|_{B^s_{p,r}} \lesssim \|a - S_m a\|_{B^{\frac{N}{p_1}}_{p_1,\infty} \cap L^\infty} \|D^2 u\|_{B^s_{p,r}}. \tag{4.11}$$

Now applying operator Δ_q and next operator of free divergence yield $\mathcal{P}$ to momentum equation (4.10) yields:

$$\frac{d}{dt} u_q - \mu \operatorname{div}(b_m \nabla u_q) = \mathcal{P} f_q + \mathcal{P} g_q + \Delta_q \mathcal{P} E_m + \widetilde{R}_q - \Delta_q \mathcal{P}(a \nabla \Pi), \tag{4.12}$$

where we denote by $u_q = \Delta_q u$ and with:

$$\widetilde{R}_q = \widetilde{R}^1_q + \widetilde{R}^2_q$$

where:

$$\begin{aligned} \widetilde{R}^1_q &= \mu\big(\mathcal{P} \Delta_q (b_m \Delta u) - \mathcal{P} \operatorname{div}(b_m \nabla u_q)\big), \\ \widetilde{R}^2_q &= \mu\big(\mathcal{P} \operatorname{div}(b_m \nabla u_q) - \operatorname{div}(b_m \nabla u_q)\big) = -\mu \mathcal{Q} \operatorname{div}(S_m a \nabla u_q) \end{aligned}$$

where $\mathcal{Q}$ is the gradient yield projector. □

Case $p \neq 2$

Next multiplying both sides by $|u_q|^{p-2} u_q$, and integrating by parts in the second, third and last term in the left-hand side, we get by using Bony decomposition (for the notation see [4]):

(4.13)

$$\begin{aligned} &\frac{1}{p} \frac{d}{dt} \|u_q\|^p_{L^p} + \mu \int_{\mathbb{R}^N} b_m |\nabla u_q|^2 |u_q|^{p-2} dx + \mu \int_{\mathbb{R}^N} b_m |u_q|^{p-4} |\nabla |u|^2|^2 dx \\ &\leqslant \|u_q\|^{p-1}_{L^p} \big(\|\mathcal{P} f_q\|_{L^p} + \|\mathcal{P} g_q\|_{L^p} + \|\widetilde{R}_q\|_{L^p} + \|\Delta_q (T_{\nabla a} \Pi)\|_{L^p} \\ &\qquad + 2^q \|\Delta_q (T_a \Pi)\|_{L^p} + \|\Delta_q (T'_{\nabla \Pi} a)\|_{L^p} + \|\mathcal{P} \Delta_q E_m\|_{L^p}\big). \end{aligned}$$

Indeed we have as $\operatorname{div} u = 0$ and by using Bony's decomposition and by performing an integration by parts:

$$\begin{aligned}\int_{\mathbb{R}^N} \Delta_q(a\nabla\Pi)|u_q|^{p-2}u_q &= \int_{\mathbb{R}^N} \Delta_q(T'_{\nabla\Pi}a)|u_q|^{p-2}u_q dx\\ &- \int_{\mathbb{R}^N} \Delta_q(T_{\nabla a}\Pi)|u_q|^{p-2}u_q dx - \int_{\mathbb{R}^N} \Delta_q(T_a\Pi)\operatorname{div}(|u_q|^{p-2}u_q)dx.\end{aligned}$$

Next we have:

$$\nabla(|u_q|^{p-2})\cdot u_q = (p-2)|u_q|^{p-4}\sum_{i,k} u_q^k\partial_i u_q^k u_q^i,$$

and by Hölder's and Berstein's inequalities:

$$\|\nabla(|u_q|^{p-2})\cdot u_q\|_{L^{\frac{p}{p-1}}} \leqslant C(p-2)2^q\|u_q\|_{L^p}^{p-1}.$$

Next from inequality (4.13), we get by using classical lemma on the heat equation (see [4]):

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|u_q\|_{L^p}^p + \frac{\underline{\nu}(p-1)}{p^2}2^{2q}\|u_q\|_{L^p}^p\\ &\leqslant \|u_q\|_{L^p}^{p-1}\big(\|\mathcal{P}f_q\|_{L^p} + \|\mathcal{P}g_q\|_{L^p} + \|\mathcal{P}\Delta_q E_m\|_{L^p}\\ &\quad + \|\Delta_q(T_{\nabla a}\Pi)\|_{L^p} + 2^q\|\Delta_q(T_a\Pi)\|_{L^p} + \|\Delta_q(T'_{\nabla\Pi}a)\|_{L^p} + \|\widetilde{R}_q\|_{L^p}\big).\end{aligned}$$

Therefore, elementary computation yield (at least formally):

$$\begin{aligned}e^{-\frac{\underline{\nu}(p-1)}{p^2}2^{2q}t}\frac{d}{dt}\Big(e^{\frac{\underline{\nu}(p-1)}{p^2}2^{2q}t}\|u_q\|_{L^p}\Big) &\lesssim \|\mathcal{P}f_q\|_{L^p} + \|\mathcal{P}g_q\|_{L^p} + \|\mathcal{P}\Delta_q E_m\|_{L^p}\\ &+ \|\Delta_q(T_{\nabla a}\Pi)\|_{L^p} + 2^q\|\Delta_q(T_a\Pi)\|_{L^p} + \|\Delta_q(T'_{\nabla\Pi}a)\|_{L^p} + \|\widetilde{R}_q\|_{L^p}.\end{aligned}$$

We thus have:

$$\begin{aligned}&\|u_q(t)\|_{L^p} \lesssim e^{-\frac{\underline{\nu}(p-1)}{p^2}2^{2q}t}\|\Delta_q u_0\|_{L^p} + \int_0^t e^{-\frac{\underline{\nu}(p-1)}{p^2}2^{2q}(t-\tau)}\\ &\big(\|\mathcal{P}f_q\|_{L^p} + \|\mathcal{P}g_q\|_{L^p} + \|\mathcal{P}\Delta_q E_m\|_{L^p} + \|\Delta_q(T_{\nabla a}\Pi)\|_{L^p} + 2^q\|\Delta_q(T_a\Pi)\|_{L^p}\\ &+ \|\Delta_q(T'_{\nabla\Pi}a)\|_{L^p} + \|\widetilde{R}_q\|_{L^p}\big)(\tau)d\tau,\end{aligned}$$

which leads for all $q \geqslant -1$, after performing a time integration and using convolution inequalities to:

(4.14)

$$\begin{aligned}
&\Big(\frac{\underline{\nu}(p-1)}{p^2}\Big)^{\frac{1}{m}} 2^{\frac{2q}{m}} \|u_q\|_{L^m_T(L^p)} \lesssim \|\Delta_q u_0\|_{L^p} + \|\mathcal{P} f_q\|_{L^1_T(L^p)} \\
&+ \|\Delta_q(T_{\nabla a}\Pi_1)\|_{L^1_T(L^p)} + 2^q \|\Delta_q(T_a \Pi_1)\|_{L^1_T(L^p)} + \|\Delta_q(T'_{\nabla \Pi_1} a)\|_{L^1_T(L^p)} \\
&+ \|\widetilde{R}_q\|_{L^1_T(L^p)} + \|\mathcal{P}\Delta_q E_m\|_{L^1_T(L^p)} + \Big(\frac{\underline{\nu}(p-1)}{p^2}\Big)^{\frac{1}{m}-1} \\
&\quad 2^{q(\frac{2}{m}-2)}\big(\|\Delta_q(T_{\nabla a}\Pi_2)\|_{L^1_T(L^p)} + 2^q\|\Delta_q(T_a\Pi_2)\|_{L^1_T(L^p)} \\
&+ \|\Delta_q(T'_{\nabla \Pi_2} a)\|_{L^1_T(L^p)} + \|\mathcal{P} g_q\|_{L^m_T(L^p)}\big).
\end{aligned}$$

We are now interested by treating the commutator term $\widetilde{R}^1_q$, we have then by using Lemma 10.2 in the appendix the following estimates with $\alpha < 1$:

$$\|\widetilde{R}^1_q\|_{L^p} \lesssim c_q \bar{\nu} 2^{(-1+\alpha)qs} \|S_m a\|_{B^{\frac{N}{p_1}+\alpha}_{p_1,\infty}} \|Du\|_{B^s_{p,r}}, \tag{4.15}$$

where $(c_q)_{q\in\mathbb{Z}}$ is a positive sequence such that $c_q \in l^r$, and $\bar{\nu} = \mu$. Note that, owing to Bernstein inequality, we have:

$$\|S_m a\|_{B^{\frac{N}{p_1}+\alpha}_{p_1,\infty}} \lesssim 2^{m\alpha} \|a\|_{B^{\frac{N}{p_1}}_{p_1,\infty}}.$$

Next we have by Corollary 2.6:

$$\|\widetilde{R}^2_q\|_{L^p} \lesssim c_q \bar{\nu} 2^{-qs} \|S_m a\|_{B^{\frac{N}{p_1}}_{p_1,r}\cap L^\infty} \|u\|_{B^{s+2}_{p,r}}. \tag{4.16}$$

Hence, plugging (4.15), (4.16) and (4.11) in (4.14), then multiplying by 2^{qs} and summing up on $q \in \mathbb{Z}$ in l^r, we discover that, for all $t \in [0, T]$:

(4.17)

$$\begin{aligned}
&\|u\|_{\widetilde{L}^\infty_T(B^s_{p,r})} + \Big(\frac{\underline{\nu}(p-1)}{p}\Big)^{\frac{1}{m}} \|u\|_{\widetilde{L}^m_T\big(B^{s+\frac{2}{m}}_{p,r}\big)} \leqslant \|u_0\|_{B^s_{p,r}} + \|\mathcal{P} f\|_{\widetilde{L}^1_T(B^s_{p,r})} \\
&+ \|T_a \Pi_1\|_{\widetilde{L}^1_T(B^{s+1}_{p,r})} + \|T_{\nabla a}\Pi_1\|_{\widetilde{L}^1_T(B^s_{p,r})} + \|T'_{\nabla\Pi_1} a\|_{\widetilde{L}^1_T(B^s_{p,r})} \\
&+ C\bar{\nu}\|a - S_m a\|_{\widetilde{L}^\infty\big(B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty\big)} \|u\|_{\widetilde{L}^1(B^{s+2}_{p,r})} + 2^{m\alpha}\int_0^T \|a\|_{B^{\frac{N}{p_1}}_{p_1,\infty}}(\tau)\|u\|_{B^{s+1}_{p,r}}(\tau)d\tau \\
&+ \Big(\frac{\underline{\nu}(p-1)}{p^2}\Big)^{\frac{1}{m}} \Big(\|\mathcal{P} g\|_{\widetilde{L}^m_T\big(B^{s-2+\frac{2}{m}}_{p,r}\big)} + \|T_a\Pi_2\|_{\widetilde{L}^m_T\big(B^{s-1+\frac{2}{m}}_{p,r}\big)} \\
&+ \|T_{\nabla a}\Pi_2\|_{\widetilde{L}^m_T\big(B^{s-2+\frac{2}{m}}_{p,r}\big)} + \|T'_{\nabla\Pi_2} a\|_{\widetilde{L}^m_T\big(B^{s-2+\frac{2}{m}}_{p,r}\big)}\Big),
\end{aligned}$$

for a constant C depending only on N and s. With our assumption on α, $\alpha^{'}$ and s, the terms $\|T_{\nabla a}\Pi_1\|_{\widetilde{L}^1_T(B^s_{p,r})}$ and $\|T^{'}_{\nabla\Pi_1}a\|_{\widetilde{L}^1_T(B^s_{p,r})}$ may be bounded by:

$$\|a\|_{\widetilde{L}^\infty\left(B^{\frac{N}{p_1}+\alpha}_{p,\infty}\cap L^\infty\right)}\|\nabla\Pi_1\|_{\widetilde{L}^1_T(B^{s-\alpha'}_{p,r})}$$

whereas $\|T_{\nabla a}\Pi_2\|_{\widetilde{L}^m_T\left(B^{s-2+\frac{2}{m}}_{p,r}\right)}$ and $\|T^{'}_{\nabla\Pi_2}a\|_{\widetilde{L}^m_T\left(B^{s-2+\frac{2}{m}}_{p,r}\right)}$ may be bounded by:

$$\|a\|_{\widetilde{L}^\infty\left(B^{\frac{N}{p_1}+\alpha}_{p,\infty}\cap L^\infty\right)}\|\nabla\Pi_2\|_{\widetilde{L}^m_T\left(B^{s-2+\frac{2}{m}-\alpha'}_{p,r}\right)}.$$

Moreover we control $\|T_a\Pi_1\|_{\widetilde{L}^1_T(B^{s+1}_{p,r})}$ and $\|T_a\Pi_2\|_{\widetilde{L}^m_T\left(B^{s-1+\frac{2}{m}}_{p,r}\right)}$ by respectively:

$$\|a\|_{\widetilde{L}^\infty\left(B^{\frac{N}{p_1}}_{p,\infty}\cap L^\infty\right)}\|\nabla\Pi_1\|_{\widetilde{L}^1_T(B^s_{p,r})}.$$
$$\|a\|_{\widetilde{L}^\infty\left(B^{\frac{N}{p_1}}_{p,\infty}\cap L^\infty\right)}\|\nabla\Pi_2\|_{\widetilde{L}^m_T\left(B^{s-2+\frac{2}{m}}_{p,r}\right)}.$$

Hence in view of Proposition 10.1 and provided that $0<\alpha^{'}<\min(1,\alpha,\frac{s}{2})$ and $s<\frac{N}{p_1}$ (which is assumed in the statement of Proposition 4.1) and $\alpha^{''}\in[0,\alpha^{'}]$,

$$\underline{b}\|\nabla\Pi_1\|_{\widetilde{L}^1_T(B^{s-\alpha''}_{p,r})}\lesssim\mathcal{A}_T^{\frac{s-\alpha''}{\alpha'}}\|\mathcal{Q}f\|_{\widetilde{L}^1(B^s_{p,r})}. \tag{4.18}$$

On the other hand, by virtue of Proposition 2.4, and of assumption on α, $\alpha^{'}$, $\alpha^{''}$ and s, we have:

$$\|\mathcal{Q}H\|_{\widetilde{L}^m_T\left(B^{s-2+\frac{2}{m}-\alpha''}_{p,r}\right)}\lesssim\|\mathcal{Q}g\|_{\widetilde{L}^m_T\left(B^{s-2+\frac{2}{m}-\alpha''}_{p,r}\right)}$$
$$+\mu\|a\|_{\widetilde{L}^\infty\left(B^{\frac{N}{p_1}+\alpha}_{p_1,\infty}\cap L^\infty\right)}\|\Delta u\|_{\widetilde{L}^m_T\left(B^{s-2+\frac{2}{m}-\alpha''}_{p,r}\right)}.$$

As $\alpha^{'}\leqslant\min(1,\alpha,\frac{1}{2}(s-2+\frac{2}{m}))$, Proposition 10.1 with $\sigma=s-2+\frac{2}{m}-\alpha^{''}$ (here comes $s>2-\frac{2}{m}$) applies, from which we get for all $\varepsilon>0$ ($\varepsilon=0$ does if $m\geqslant 2$),

(4.19)

$$\underline{b}\|\nabla\Pi_2\|_{\widetilde{L}^m_T\left(B^{s-2+\frac{2}{m}-\alpha''}_{p,r}\right)}\lesssim\mathcal{A}_T^{\frac{s-2+\frac{2}{m}+\varepsilon}{\alpha'}}\Big(\|\mathcal{Q}g\|_{\widetilde{L}^m_T\left(B^{s-2+\frac{2}{m}-\alpha''}_{p,r}\right)}$$
$$+\mu\|a\|_{\widetilde{L}^\infty\left(B^{\frac{N}{p_1}+\alpha}_{p_1,\infty}\cap L^\infty\right)}\|\Delta u\|_{\widetilde{L}^m_T\left(B^{s-2+\frac{2}{m}-\alpha''}_{p,r}\right)}\Big).$$

Let $X(t) = \|u\|_{L_t^\infty(B^s_{p,r})} + \nu\underline{b}\|u\|_{L^1_t(B^{s+2}_{p,r})}$. Assuming that m has been chosen so large as to satisfy:

$$C\bar{\nu}\|a - S_m a\|_{L_T^\infty(B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty)} \leqslant \underline{\nu}\,,$$

and by interpolation we get:

$$C\bar{\nu}2^{m\alpha}\|a\|_{B^{\frac{N}{p_1}}_{p_1,\infty}}\|u\|_{B^{s+2}_{p,r}} \leqslant \kappa\underline{\nu} + \frac{C^2\bar{\nu}^2 2^{2m\alpha}}{4\kappa\underline{\nu}}\|a\|^2_{B^{\frac{N}{p_1}}_{p_1,\infty}}\|u\|_{B^s_{p,r}}. \tag{4.20}$$

Plugging (4.18), (4.19) and (4.20) in (4.17), we end up with:

$$\begin{aligned} X(T) \leqslant \|u_0\|_{B^s_{p,r}} + \mathcal{A}_T^{\frac{s}{\alpha'}}\Big(&\|\mathcal{P}f\|_{\widetilde{L}^1_t(B^s_{p,r})} + \|\mathcal{P}g\|_{\widetilde{L}^m_t\left(B^{s-2+\frac{2}{m}}_{p,r}\right)} \\ &+ C\int_0^t \Big(\frac{\bar{\nu}^2}{\underline{\nu}}2^{2m\alpha}\|a\|^2_{B^{\frac{N}{p_1}}_{p_1,\infty}}(\tau)\times X(\tau)\Big)d\tau \\ &+ \Big(\frac{\underline{\nu}(p-1)}{p}\Big)^{\frac{1}{m}}\mathcal{A}_T\|u\|_{\widetilde{L}^1_T(B^{s+2-\alpha'}_{p,r})}\Big). \end{aligned}$$

Grönwall lemma then leads to the desired inequality.

Case $p = 2$

In this case we do not need any condition of smallness on $\|a\|_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty}$, indeed the bad terms as $\widetilde{R}_q^2$ or $2^q\|\Delta_q(T_a\Pi)\|_{L^2}$ disappear in the integration by parts as $\operatorname{div} u_q = 0$. So we can follow the same procedure and conclude.

5. The mass conservation equation

5.1. Losing estimates for transport equation

We now focus on the mass equation associated to (1.1):

$$\begin{cases} \partial_t a + v\cdot\nabla a = g, \\ a_{/t=0} = a_0. \end{cases} \tag{5.1}$$

We will precise in the sequel the regularity of a_0, v and g. In this section we intend to recall some result on transport equation associated to vector fields which are not Lipschitz with respect to the space variable. Since we still have in mind to get regularity theorems, those vector field cannot be to rough. In order to measure precisely the regularity of the vector field v, we shall introduce the following notation:

$$V'_{p_1,\alpha}(t) = \sup_{j\geqslant 0} \frac{2^{j\frac{N}{p_1}}\|\nabla S_j v(t)\|_{L^{p_1}}}{(j+1)^\alpha} < +\infty. \tag{5.2}$$

Let us remark that if $p_1 = +\infty$ then $V'_{p_1,\alpha}$ is exactly the norm $\|B_\Gamma\|$ of Definition 2.11.

5.1.1. Limited loss of regularity

In this section, we make the assumption that there exists some $\alpha \in]0,1[$ such that the function $V'_{p_1,\alpha}$ defined in (5.2) be locally integrable. We will show that in the case $\alpha = 1$, then a linear loss of regularity may occur. In the theorem below, Bahouri, Chemin and Danchin show in [4] that if $\alpha \in]0,1[$ then the loss of regularity in the estimate is arbitrarily small.

Theorem 5.1. — *Let (p,p_1) be in $[1,+\infty]^2$ such that $1 \leqslant p \leqslant p_1$ and σ satisfying $\sigma > -1 - N\min(\frac{1}{p_1},\frac{1}{p'})$. Assume that $\sigma < 1 + \frac{N}{p_1}$ and that $V'_{p_1,\alpha} \in]0,1[$ is in $L^1([0,T])$. Let $a_0 \in B^\sigma_{p,\infty}$ and $g \in \widetilde{L}^1_T(B^\sigma_{p,\infty})$. Then the equation (5.1) has a unique solution $a \in C([0,T],\cap_{\sigma'<\sigma} B^{\sigma'}_{p,\infty})$ and the following estimate holds for all small enough ε:*

$$\|a\|_{\widetilde{L}^\infty_T(B^{\sigma-\varepsilon}_{p,\infty})} \leqslant C\Big(\|a_0\|_{B^\sigma_{p,\infty}} + \|g\|_{\widetilde{L}^1_T(B^\sigma_{p,\infty})}\Big)\exp\Big(\frac{C}{\varepsilon^{\frac{\alpha}{1-\alpha}}}(V_{p_1,\alpha}(T))^{\frac{1}{1-\alpha}}\Big), \tag{5.3}$$

where C depends only on α, p, p_1, σ and N.

In the following proposition, we are interested in showing a control of the high frequencies on the density when u is not Lipschitz. Indeed we recall that in the Proposition 4.1 when $p=2$, we need to control the high frequencies of the density. In particular the following proposition is useful only in the case of Theorem 1.7.

Proposition 5.2. — *Let (p,p_1) be in $[1,+\infty]^2$ such that $1 \leqslant p \leqslant p_1$ and σ satisfying $\sigma > -1 - N\min\big(\frac{1}{p_1},\frac{1}{p'}\big)$. Assume that $\sigma < 1 + \frac{N}{p_1}$ and that $V'_{p_1,\alpha} \in]0,1[$ is in $L^1([0,T])$. Let $a_0 \in B^\sigma_{p,\infty}$ and $g \in \widetilde{L}^1_T(B^\sigma_{p,\infty})$, the equation (5.1) has a unique solution $a \in C([0,T],\cap_{\sigma'<\sigma} B^{\sigma'}_{p,\infty})$ and the following estimate holds for all small enough ε:*

$$\begin{aligned}\sup_{l\geqslant m} 2^{(\sigma-\varepsilon)l}\|\Delta_l a(t')\|_{L^\infty(L^p)} \lesssim \sup_{l\geqslant m}(2^{\sigma l}\|\Delta_l a_0\|_{L^p}) + C\eta^{\frac{\alpha}{1-\alpha}}\int_0^t V'_{p_1,\alpha}(t')\\ \times\Big(\|a_0\|_{B^\sigma_{p,\infty}} + \|g\|_{\widetilde{L}^1_{t'}(B^\sigma_{p,\infty})}\Big)\exp\Big(\frac{C}{\varepsilon^{\frac{\alpha}{1-\alpha}}}(V_{p_1,\alpha}(t'))^{\frac{1}{1-\alpha}}\Big)dt',\end{aligned}$$

where C depends only on α, p, p_1, σ and N.

Proof. — By using the proof of Bahouri, Chemin and Danchin in [4] one can write:

$$(5.4)\quad 2^{(2+l)\sigma_t}\|\Delta_l a(t)\|_{L^p} \leqslant 2^{(2+l)\sigma}\|\Delta_l a_0\|_{L^p} + C\Big(\frac{2C}{\eta\log 2}\Big)^{\frac{\alpha}{1-\alpha}}\int_0^t V'_{p_1,\alpha}(t')\|a(t')\|_{B^{\sigma_{t'}}_{p,\infty}}\,dt'.$$

Whence taking the supremum over $l \geqslant m$, we get

$$\sup_{t'\in[0,t]}\sup_{l\geqslant m}(2^{\sigma_{t'}l}\|\Delta_l a(t')\|_{L^p}) \lesssim \sup_{l\geqslant m}(2^{\sigma l}\|\Delta_l a_0\|_{L^p}) + C\eta^{\frac{\alpha}{1-\alpha}}\int_0^t V'_{p_1,\alpha}(t')\|a(t')\|_{B^{\sigma_{t'}}_{p,\infty}}\,dt'.$$

We now insert in previous inequality (5.3) which leads to the proposition. □

Remark 5.3. — In the sequel, we will use the Theorem 5.1 and the Proposition 5.2 when $p_1 = \infty$ and $\alpha = 1 + \varepsilon' - \frac{1}{r}$. Indeed we will have u is in $\widetilde{L}^1\big(B^{\frac{N}{p_2}+1}_{p_2,r}\big)$ (with $1 \leqslant r < \infty$) and according Proposition 2.12 and Definition 2.11 we get:

$$\int_0^t V'_{\infty,\alpha}(t')dt' \lesssim \|u\|_{\widetilde{L}^1\big(B^{\frac{N}{p_2}+1}_{p_2,r}\big)}.$$

So it will allow to get estimates on the density with an arbitrarly small loss of regularity.

5.1.2. Loss of regularity in Besov spaces in terms of $\|v\|_{\widetilde{L}^1_T\big(B^{\frac{N}{p}+1}_{p,\infty}\big)}$ with $1 \leqslant p < +\infty$

This section is devoted to the estimates with loss of regularity on the density a when we assume only that the velocity u belongs to $\|v\|_{\widetilde{L}^1_T\big(B^{\frac{N}{p}+1}_{p,\infty}\big)}$. In this particular case, we remind a very interesting result due to Danchin and Paicu in [15] p. 286.

Proposition 5.4. — *Let $s \in]-1-\frac{N}{2}, 1+\frac{N}{2}[$ and a a solution of (5.1). Then there exists $N_0 \in \mathbb{N}$ depending on φ, C_0 a universal constant and two constants c, C such that if:*

$$\|v\|_{\widetilde{L}^1_T\big(B^{1+\frac{N}{2}}_{2,\infty}\big)} \leqslant c,$$

then we have the following estimate $\forall t\in[0,T]$*:*

$$\sup_{l\geqslant -1,\tau\in[0,t]} 2^{ls-\epsilon_l(\tau)}\|\Delta_l a(\tau)\|_{L^2} \leqslant C_0\Big(\|a_0\|_{B^s_{2,\infty}}+\sup_{l\geqslant -1}\int_0^t 2^{ls-\varepsilon_l(\tau)}\|\Delta_l g(\tau)\|_{L^2}d\tau\Big)$$

where:

$$\varepsilon_l(t)=C\sum_{l'=-1}^{l}2^{l'(1+\frac{N}{2})}\int_0^t\|\widetilde{\Delta}_{l'}v\|_{L^2}d\tau,$$

with $\widetilde{\Delta}_{l'}=\sum_{|\alpha|\leqslant N_0}\Delta_{l'+\alpha}$.

Corollary 5.5. — *Let* $1\leqslant p\leqslant p_1\leqslant\infty$ *and* $s_1\in\mathbb{R}$ *satisfies* $s_1>-N\min\big(\frac{1}{p_1},\frac{1}{p'}\big)$. *Let* σ *in* $]s_1,1+\frac{N}{p_1}[$ *and* v *a vector field such that* $v\in\widetilde{L}^1\big(B^{\frac{N}{p_1}+1}_{p_1,\infty}\big)$ *and:*

$$\|v\|_{\widetilde{L}^1_T\big(B^{1+\frac{N}{2}}_{2,\infty}\big)}\leqslant c,$$

(as in the previous proposition). There exists constants C *and* λ *depending only on* p, p_1, σ, s_1 *and* N *such that for any* $T>0$ *and and if* $\sigma_T\geqslant s_1$ *with:*

$$\sigma_t=\sigma-\lambda\|v\|_{\widetilde{L}^1_t\big(B^{\frac{N}{p_1}+1}_{p_1,\infty}\big)}$$

then the following property holds true.

Let $a_0\in B^{\sigma}_{p,\infty}$ *ans* $a\in\mathcal{C}([0,T];B^{s_1}_{p,\infty})$ *be a solution of (5.1), then the following estimate holds:*

$$\|a(t)\|_{\widetilde{L}^\infty_t(B^{\sigma_t}_{p,\infty})}\leqslant\frac{\lambda}{\lambda-C}\Big(\|a_0\|_{B^{\sigma}_{p,\infty}}+\sup_{l\geqslant -1}\int_0^t 2^{ls-\varepsilon_l(\tau)}\|\Delta_l g(t)\|_{L^p}dt\Big),$$

with $\varepsilon_l(t)=C\sum_{l'=-1}^{l}2^{l'(1+\frac{N}{p_1})}\int_0^t\|\widetilde{\Delta}_{l'}v\|_{L^{p_1}}d\tau$, *where* $\widetilde{\Delta}_{l'}=\sum_{|\alpha|\leqslant N_0}\Delta_{l'+\alpha}$ *(*N_0 *an universal constant).*

Proof. — We will just prove this result in the case where $p=p_1=2$ (the more general case follows the same lines than the proof of Proposition 5.4 in [15] and of the case $p=p_1=2$). By using the Proposition 5.4, we have:

$$\sup_{l\geqslant -1,\tau\in[0,t]} 2^{ls-\epsilon_l(\tau)}\|\Delta_l a(\tau)\|_{L^2} \leqslant C_0\Big(\|a_0\|_{B^s_{2,\infty}}+\sup_{l\leqslant -1}\int_0^t 2^{ls-\varepsilon_l(\tau)}\|\Delta_l g(\tau)\|_{L^2}d\tau\Big), \tag{5.5}$$

where:

$$\varepsilon_l(t) = C \sum_{l'=-1}^{l} 2^{l'(1+\frac{N}{2})} \int_0^t \|\widetilde{\Delta}_{l'} v\|_{L^2} d\tau,$$

with $\widetilde{\Delta}_{l'} = \sum_{|\alpha|\leqslant N_0} \Delta_{l'+\alpha}$. We now have for $\tau \in [0,t]$:

$$\begin{aligned}
\varepsilon_l(\tau) = C \sum_{l'=-1}^{l} 2^{l'(1+\frac{N}{2})} \int_0^\tau \|\widetilde{\Delta}_{l'} v\|_{L^2} d\tau &\leqslant Cl \sup_{-1\leqslant l'\leqslant l} 2^{l'(1+\frac{N}{2})} \|\widetilde{\Delta}_{l'} v\|_{L^1_t(L^2)},\\
&\leqslant Cl \sum_{|\alpha|\leqslant N_0} 2^{(l'+\alpha)(1+\frac{N}{2})} \|\widetilde{\Delta}_{l'} v\|_{L^1_t(L^2)} 2^{-\alpha(1+\frac{N}{2})},\\
&\leqslant Cl\, 2^{N_0(1+\frac{N}{2})} \sum_{|\alpha|\leqslant N_0} 2^{(l'+\alpha)(1+\frac{N}{2})} \|\widetilde{\Delta}_{l'+\alpha} v\|_{L^1_t(L^2)},\\
&\leqslant C' l \|v\|_{\widetilde{L}^1_t\left(B^{\frac{N}{2}+1}_{2,\infty}\right)}.
\end{aligned}$$

We have then from (5.7) and the previous inequality:

$$\begin{aligned}
\sup_{l\geqslant -1,\tau\in[0,t]} 2^{l\left(s-C'\|v\|_{\widetilde{L}^1_t\left(B^{\frac{N}{2}+1}_{2,\infty}\right)}\right)} \|\Delta_l a(\tau)\|_{L^2}\\
\leqslant C_0(\|a_0\|_{B^s_{2,\infty}} + \sup_{l\leqslant -1} \int_0^t 2^{ls-\varepsilon_l(\tau)} \|\Delta_l g(\tau)\|_{L^2} d\tau).
\end{aligned} \tag{5.6}$$

We have then obtained that:

$$\|a\|_{\widetilde{L}^\infty_t\left(B_{2,\infty}^{s-C'\|v\|_{\widetilde{L}^1_t\left(B^{\frac{N}{2}+1}_{2,\infty}\right)}}\right)} \leqslant C\Big(\|a_0\|_{B^s_{2,\infty}} + \sup_{l\leqslant -1} \int_0^t 2^{ls-\varepsilon_l(\tau)} \|\Delta_l g(\tau)\|_{L^2} d\tau\Big). \tag{5.7}$$

□

Remark 5.6. — In the sequel, we will use the Corollary 5.5 when we will control only the velocity u in $\widetilde{L}^1(B^{\frac{N}{p_2}+1}_{p_2,\infty})$.

6. Proof of the existence for Theorem 1.1

We use a standard scheme:

(1) We smooth out the data and get a sequence of smooth solutions $(a^n, u^n)_{n\in\mathbb{N}}$ to (1.4) on a bounded interval $[0, T^n]$ which may depend on n.

(2) We exhibit a positive lower bound T for T^n, and prove uniform estimates in the space:

$$E_T = \widetilde{C}_T\Big(B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty}\Big) \times \Big(\widetilde{C}_T\big(B^{\frac{N}{p_2}-1}_{p_2,r}\big) \cap \widetilde{L}^1\big(B^{\frac{N}{p_2}+1}_{p_2,r}\big)\Big)^N,$$

for the smooth solution (a^n, u^n) with $1 \leqslant r < +\infty$.

(3) We use compactness to prove that the sequence converges, up to extraction, to a solution of (1.4).

Construction of approximate solutions

We smooth on the data as follows:

$$a_0^n = S_n a_0, \ \ u_0^n = S_n u_0 \ \ \text{and} \ \ f^n = S_n f.$$

Note that we have:

$$\forall l \in \mathbb{Z}, \ \ \|\Delta_l a_0^n\|_{L^{p_1}} \leqslant \|\Delta_l a_0\|_{L^{p_1}} \quad \text{and} \quad \|a_0^n\|_{B^{\frac{N}{p_1}+\varepsilon'}_{p,\infty}} \leqslant \|a_0\|_{B^{\frac{N}{p_1}+\varepsilon'}_{p,\infty}},$$

and similar properties for u_0^n and f^n, a fact which will be used repeatedly during the next steps. Now, according [2], one can solve (1.4) with the smooth data (a_0^n, u_0^n, f^n). We get a solution (a^n, u^n) on a non trivial time interval $[0, T_n]$ such that:

$$a^n \in \widetilde{C}([0,T_n), H^{N+\varepsilon}), \ \ u^n \in \mathcal{C}([0,T_n), H^N) \cap \widetilde{L}^1_{T_n}(H^{N+2}) \quad \text{and} \ \ \nabla\Pi^n \in \widetilde{L}^1_{T_n}(H^N). \tag{6.1}$$

Uniform bounds

Let T_n be the lifespan of (a^n, un), that is the supremum of all $T > 0$ such that (1.4) with initial data (a_0^n, u_0^n) has a solution which satisfies (6.1). Let T be in $(0, T_n)$, we aim at getting uniform estimates in E_T for T small enough. For that, we need to introduce the solution $(u_L^n, \nabla\Pi_L^n)$ to the nonstationary Stokes system:

$$(L) \qquad \begin{cases} \partial_t u_L^n - \mu\Delta u_L^n + \nabla\Pi_L^n = f^n, \\ \operatorname{div} u_L^n = 0, \\ (u_L^n)_{t=0} = u_0^n. \end{cases}$$

Now, the vectorfields $\tilde{u}^n = u^n - u_L^n$ and $\nabla\Pi^n = \nabla\Pi_L^n + \nabla\tilde{\Pi}^n$ satisfy the parabolic system:

$$\begin{cases} \partial_t \tilde{u}^n - \mu(1+a^n)\Delta\tilde{u}^n + (1+a^n)\nabla\tilde{\Pi}^n = H(a^n, u^n, \nabla\Pi^n), \\ \operatorname{div}\tilde{u}^n = 0, \\ \tilde{u}^n(0) = 0, \end{cases} \tag{6.2}$$

which has been studied in Proposition 4.1 with:

$$H(a^n, u^n, \nabla\Pi^n) = a^n(\mu\Delta u_L^n - \nabla\Pi_L^n) - u^n \cdot \nabla u^n.$$

Define $m \in \mathbb{Z}$ by:

$$m = \inf\Big\{p \in \mathbb{Z}/\ \ 2\bar{\nu}\sum_{l\geqslant q} 2^{l\frac{N}{p_1}}\|\Delta_l a_0\|_{L^{p_1}} \leqslant c\bar{\nu}\Big\} \tag{6.3}$$

where c is small enough positive constant (depending only N) to be fixed hereafter.

Let:

$$\bar{b} = 1+\sup_{x\in\mathbb{R}^N} a_0(x),\ \ A_0 = 1+2\|a_0\|_{B^{\frac{N}{p_1}+\varepsilon'}_{p_1,\infty}},\ U_0 = \|u_0\|_{B^{\frac{N}{p_2}-1}_{p_2,r}} + \|f\|_{\widetilde{L}^1\big(B^{\frac{N}{p_2}-1}_{p_2,r}\big)},$$

and $\widetilde{U}_0 = 2CU_0 + 4C\bar{\nu}A_0$ (where C stands for a large enough constant depending only N which will be determined when applying Proposition 2.4, 4.1 and 5.2 in the following computations.) We assume that the following inequalities are fulfilled for some $\eta > 0$, $\alpha > 0$:

$$\|a^n - S_m a^n\|_{\widetilde{L}^\infty_T\big(B^{\frac{N}{p_1}}_{p_1,\infty}\big)\cap L^\infty} \leqslant c\underline{\nu}\bar{\nu}^{-1}, \tag{$\mathcal{H}_1$}$$

$$C\bar{\nu}^2 T\|a^n\|^2_{\widetilde{L}^\infty_T\big(B^{\frac{N}{p_1}}_{p_1,\infty}\big)\cap L^\infty} \leqslant 2^{-2m}\underline{\nu}, \tag{$\mathcal{H}_2$}$$

$$\frac{1}{2}\underline{b} \leqslant 1 + a^n(t,x) \leqslant 2\bar{b}\ \text{ for all }\ (t,x) \in [0,T]\times\mathbb{R}^N, \tag{$\mathcal{H}_3$}$$

$$\|a^n\|_{\widetilde{L}^\infty_T\big(B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty}\big)\cap L^\infty} \leqslant A_0, \tag{$\mathcal{H}_4$}$$

$$\|u_L^n\|_{\widetilde{L}^1\big(B^{\frac{N}{p_2}+1}_{p_2,r}\big)} \leqslant \eta, \tag{$\mathcal{H}_5$}$$

$$\|\widetilde{u}^n\|_{\widetilde{L}^\infty\big(B^{\frac{N}{p_2}-1}_{p_2,r}\big)} + \underline{\nu}\|\widetilde{u}^n\|_{\widetilde{L}^1\big(B^{\frac{N}{p_2}+1}_{p_2,r}\big)} \leqslant \widetilde{U}_0\eta, \tag{$\mathcal{H}_6$}$$

$$\|\nabla\Pi_L^n\|_{\widetilde{L}^1\big(B^{\frac{N}{p_2}-1}_{p_2,r}\big)} \leqslant \eta, \tag{$\mathcal{H}_7$}$$

$$\|\nabla\widetilde{\Pi}^n\|_{\widetilde{L}^1\big(B^{\frac{N}{p_2}-1}_{p_2,r}\big)} \leqslant \widetilde{\Pi}_0\eta. \tag{$\mathcal{H}_8$}$$

We just want to mention here that in fact when $p_2 \neq 2$, the inequality $(\mathcal{H}_1)$ is not necessary (it is just important in the proof of Theorem 1.7 as we do not assume any condition of smallness on a_0). Indeed in this case we can choose A_0 small in $(\mathcal{H}_4)$ and we can then easily apply the Proposition 4.1. However we prefer to keep the condition $(\mathcal{H}_1)$ because it gives an idea of how to prove Theorem 1.7.

Remark that since:

$$1 + S_m a^n = 1 + a^n + (S_m a^n - a^n),$$

assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$ insure that:

$$\inf_{(t,x)\in[0,T]\times\mathbb{R}^N}(1+S_m a^n)(t,x)\geqslant\frac{1}{4}\underline{b}\,, \tag{6.4}$$

provided c has been chosen small enough (note that $\frac{\underline{\nu}}{\bar{\nu}}\leqslant\bar{b}$).

We are going to prove that under suitable assumptions on T and η (to be specified below) if condition $(\mathcal{H}_1)$ to $(\mathcal{H}_8)$ are satisfied, then they are actually satisfied with strict inequalities. Since all those conditions depend continuously on the time variable and are strictly satisfied initially, a basic bootstrap argument insures that $(\mathcal{H}_1)$ to $(\mathcal{H}_8)$ are indeed satisfied for a small $T_n^{'}\leqslant T^n$. First we shall assume that η satisfies:

$$C(1+\underline{\nu}^{-1}\widetilde{U}_0)\eta\leqslant\log 2 \tag{6.5}$$

so that denoting:

$$(\tilde{V}^n)_{p_2,1-\frac{1}{r}}(t)=\int_0^t(\tilde{V}^n)^{'}_{p_2,1-\frac{1}{r}}(s)ds \text{ and } (V^n_L)_{p_2,1-\frac{1}{r}}(t)=\int_0^t(V^n_L)^{'}_{p_2,1-\frac{1}{r}}(s)ds,$$

with:

$$(\tilde{V}^n)^{'}_{p_2,1-\frac{1}{r}}(s)=\sup_{l\geqslant 0}\Big(\frac{2^{l\frac{N}{p_2}}\|\nabla S_l\tilde{u}^n(s)\|_{L^{p_2}}}{(l+1)^{1-\frac{1}{r}}}\Big)ds$$

$$\text{and } (V^n_L)^{'}_{p_2,1-\frac{1}{r}}(s)=\sup_{l\geqslant 0}\Big(\frac{2^{l\frac{N}{p_2}}\|\nabla S_l u^n_L(s)\|_{L^{p_2}}}{(l+1)^{1-\frac{1}{r}}}\Big)ds.$$

We recall now that according Proposition 2.12:

$$(\tilde{V}^n)_{p_2,1-\frac{1}{r}}(t)\leqslant C\|\tilde{u}^n\|_{\widetilde{L}^1_t\big(B^{\frac{N}{p_2}+1}_{p_2,r}\big)} \text{ and } (V^n_L)_{p_2,1-\frac{1}{r}}(t)\leqslant C\|u^n_L\|_{\widetilde{L}^1_t\big(B^{\frac{N}{p_2}+1}_{p_2,r}\big)},$$

we have, according to $(\mathcal{H}_5)$ and $(\mathcal{H}_6)$:

$$e^{C(\frac{\varepsilon}{2})^{1-r}\Big(\|\tilde{u}^n\|_{\widetilde{L}^1_t\big(B^{\frac{N}{p_2}+1}_{p_2,r}\big)}+\|u^n_L\|_{\widetilde{L}^1_t\big(B^{\frac{N}{p_2}+1}_{p_2,r}\big)}\Big)}<2. \tag{6.6}$$

In order to bound a^n in $\widetilde{L}^\infty_T\big(B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty}\big)$, we apply Theorem 5.1 and get:

$$\|a^n\|_{\widetilde{L}^\infty_T\big(B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty}\big)}<e^{C(\frac{\varepsilon}{2})^{1-r}}\Big(\|\tilde{u}^n\|_{\widetilde{L}^1_t\big(B^{\frac{N}{p_2}}_{p_2,r}\big)}+\|u^n_L\|_{\widetilde{L}^1_t\big(B^{\frac{N}{p_2}}_{p_2,r}\big)}\Big)\|a_0\|_{B^{\frac{N}{p_1}+\varepsilon}_{p_1,\infty}}. \tag{6.7}$$

Moreover as we know that $\|a^n\|_{L^\infty}\leqslant\|a_0\|_{L^\infty}$, $(\mathcal{H}_4)$ is satisfied with a strict inequality. Next, applying classical proposition on heat equation (see [11]) yields:

$$\|u^n_L\|_{\widetilde{L}^\infty_T\big(B^{\frac{N}{p_2}-1}_{p_2,r}\big)}\leqslant U_0, \tag{6.8}$$

(6.9)
$$\kappa\nu\|u_L^n\|_{\widetilde{L}^1_T(B^{\frac{N}{p_2}+1}_{p_2,r})} \leqslant \Big(\sum_{l\in\mathbb{Z}} 2^{lr(\frac{N}{p_2}-1)}(1-e^{-\kappa\nu 2^{2l}T})^r(\|\Delta_l u_0\|^r_{L^{p_2}} + \|\Delta_l f\|^r_{L^1(\mathbb{R}^+,L^{p_2})})\Big)^{\frac{1}{r}}.$$

Hence taking T such that:

(6.10)
$$\Big(\sum_{l\in\mathbb{Z}} 2^{lr\left(\frac{N}{p_2}-1\right)}(1-e^{-\kappa\nu 2^{2l}T})^r\big(\|\Delta_l u_0\|^r_{L^2} + \|\Delta_l f\|^r_{L^1(\mathbb{R}^+,L^2)}\big)\Big)^{\frac{1}{r}} < \kappa\eta\nu,$$

insures that $(\mathcal{H}_5)$ is strictly verified.

Since $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and (6.4) are satisfied, Proposition 4.1 may be applied with $\alpha = \frac{\varepsilon}{2}$. We get:

$$\begin{aligned}&\|\widetilde{u}^n\|_{\widetilde{L}^\infty_T\left(B^{\frac{N}{p_2}-1}_{p_2,r}\right)} + \underline{\nu}\|\widetilde{u}^n\|_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}+1}_{p_2,r}\right)} \\ &\leqslant Ce^{CZ^n_m(T)}\Big(\|u_0\|_{B^{\frac{N}{p_2}-1}_{p_2,r}} + \mathcal{A}^\kappa_{T,n} \times \big(\|a^n(\Delta u^n_L - \nabla\Pi^n_L)\|_{\widetilde{L}^1\left(B^{\frac{N}{p_2}-1}_{p_2,r}\right)} \\ &\qquad + \|u^n\cdot\nabla u^n\|_{\widetilde{L}^1\left(B^{\frac{N}{p_2}-1}_{p_2,r}\right)} + \mathcal{A}_{T,n}\|u\|_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}+1-\alpha'}_{p_2,r}\right)}\big)\Big).\end{aligned}$$

with $Z^n_m(T) = 2^{m\varepsilon}\bar{\nu}^2\underline{\nu}^{-1}\int_0^T \|a^n\|^2_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty} d\tau$. Next using Bony's decomposition and $\operatorname{div} u^n = 0$, one can write:

$$\operatorname{div}(u^n\otimes u^n) = T_{\partial_j v}(u^n)^j + T_{(u^n)^j}\partial_j u^n + \partial_j R(u^n,(u^n)^j),$$

with the summation convention over repeated indices.

Hence combining Proposition 1.4.1 and 1.4.2 in [11] with the fact that $\widetilde{L}^\rho_T\big(B^{\frac{N}{p_2}-\frac{1}{2}}_{p_2,r}\big) \hookrightarrow \widetilde{L}^\rho_T\big(B^{-\frac{1}{2}}_{\infty,\infty}\big)$ for $\rho = \frac{4}{3}$ or $\rho = 4$, we get:

$$\|\operatorname{div}(u^n\otimes u^n)\|_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}-1}_{p_2,r}\right)} \leqslant C\|u^n\|_{\widetilde{L}^{\frac{4}{3}}_T\left(B^{\frac{N}{p_2}+\frac{1}{2}}_{p_2,r}\right)}\|u^n\|_{\widetilde{L}^4_T\left(B^{\frac{N}{p_2}-\frac{1}{2}}_{p_2,r}\right)}.$$

By taking advantage of Proposition 4.1, 2.4, 2.3 and Young'inequality, we end up with:

$$\begin{aligned}
&\|\widetilde{u}^n\|_{\widetilde{L}^\infty_T\left(B^{\frac{N}{p_2}-1}_{p_2,r}\right)}+\underline{\nu}\|\widetilde{u}^n\|_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}+1}_{p_2,r}\right)}\\
&\leqslant e^{CZ^n_m(T)}\bigg(\|u_0\|_{B^{\frac{N}{p_2}-1}_{p_2,r}}+\Big(\|a^n\|_{\widetilde{L}^\infty_T\left(B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty}\cap L^\infty\right)}+1\Big)^\kappa\\
&\qquad\times\Big(C\|u^n_L\|_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}+1}_{p_2,r}\right)}\Big(\bar{\nu}\|a^n\|_{L^\infty_T\left(B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty\right)}+\|u^n_L\|_{\widetilde{L}^\infty_T\left(B^{\frac{N}{p_2}-1}_{p_2,r}\right)}\Big)\\
&\qquad\quad+\|\nabla\Pi^n_L\|_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}-1}_{p_2,r}\right)}\times\|a^n\|_{L^\infty_T\left(B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty\right)}\Big)+\Big(\|a^n\|_{\widetilde{L}^\infty_T\left(B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty}\cap L^\infty\right)}+1\Big)\\
&\qquad\quad T^{\frac{\varepsilon}{2}}\|\widetilde{u}^n\|_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}+1}_{p_2,r}\right)}\bigg)
\end{aligned}$$

with $C = C(N)$ and $C_g = (N, g, \underline{b}, \bar{b})$. Now, using assumptions $(\mathcal{H}_2)$, $(\mathcal{H}_4)$, $(\mathcal{H}_5)$, $(\mathcal{H}_6)$ and $(\mathcal{H}_7)$, and inserting (6.6) in the previous inequality and choosing T enough small gives:

$$\|\widetilde{u}^n\|_{\widetilde{L}^\infty_T\left(B^{\frac{N}{p_2}-1}_{p_2,r}\right)}+\|\widetilde{u}^n\|_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}+1}_{p_2,r}\right)}\leqslant 2C(\bar{\nu}A_0+U_0)\eta+2C_gTA_0,$$

hence $(\mathcal{H}_6)$ is satisfied with a strict inequality provided:

$$C_gT < C\bar{\nu}\eta. \tag{6.11}$$

To show that $(\mathcal{H}_7)$ and $(\mathcal{H}_8)$ are strictly verified on $(0, T_n^{'})$, we proceed similarly as for $(\mathcal{H}_5)$ and $(\mathcal{H}_6)$. We now have to check whether $(\mathcal{H}_1)$ is satisfied with strict inequality. For that we apply Proposition 5.2 which yields for all $m \in \mathbb{Z}$,

$$\begin{aligned}
&\sup_{l\geqslant m} 2^{l\frac{N}{p_1}}\|\Delta_l a^n\|_{L^\infty_T(L^{p_1})}\\
&\leqslant C\Big(\sup_{l\geqslant m} 2^{l\frac{N}{p_1}}\|\Delta_l a_0\|_{L^{p_1}}\Big)\Big(1+\|\tilde{u}^n\|_{\widetilde{L}^1_t\left(B^{\frac{N}{p_2}}_{p_2,r}\right)}+\|u^n_L\|_{\widetilde{L}^1_t\left(B^{\frac{N}{p_2}}_{p_2,r}\right)}\Big).
\end{aligned}\tag{6.12}$$

Using (6.5) and $(\mathcal{H}_5)$, $(\mathcal{H}_6)$, we thus get:

$$\begin{aligned}
&\|a^n-S_ma^n\|_{L^\infty_T\left(B^{\frac{N}{p_1}}_{p_1,\infty}\right)\cap L^\infty}\\
&\leqslant \sup_{l\geqslant m} 2^{l\frac{N}{p_1}}\|\Delta_l a_0\|_{L^{p_1}}+\frac{C}{\log 2}\Big(1+\|a_0\|_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty}\Big)(1+\underline{\nu}^{-1}\widetilde{L}_0)\eta.
\end{aligned}$$

Hence $(\mathcal{H}_1)$ is strictly satisfied provided that η further satisfies:

$$\frac{C}{\log 2}\Big(1+\|a_0\|_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty}\Big)(1+\underline{\nu}^{-1}\widetilde{L}_0)\eta<\frac{c\underline{\nu}}{2\bar{\nu}}. \tag{6.13}$$

So $\mathcal{H}_1$ is strictly verified.

$(\mathcal{H}_3)$ is trivially verified by the transport equation as we assume that $1+a_0$ is bounded away and that $a_0 \in L^\infty$.

Next, according to $(\mathcal{H}_4)$ condition $(\mathcal{H}_2)$ is satisfied provide:

$$T < \frac{2^{-2m}\underline{\nu}}{C\bar{\nu}^2 A_0^2}. \tag{6.14}$$

One can now conclude that if $T < T^n$ has been choosen so that conditions (6.10), (6.11) and (6.14) are satisfied (with η verifying (6.5) and (6.13), and m defined in (6.3) and $n \geqslant m$ then (a^n, u^n, Π^n) satisfies $(\mathcal{H}_1)$ to $(\mathcal{H}_8)$, thus is bounded independently of n on $[0,T]$.

We still have to state that T^n may be bounded by below by the supremum $\bar{T}$ of all times T such that (6.10), (6.11) and (6.14) are satisfied. This is actually a consequence of the uniform bounds we have just obtained, and of a theorem of blow-up of R. Danchin in [13]. Indeed, by combining all these informations, one can prove that if $T^n < \bar{T}$ then $(a^n, u^n, \nabla\Pi^n)$ is actually in:

$$\widetilde{L}^\infty_{T^n}\left(B^{\frac{N}{p_1}+\varepsilon'}_{p_1,\infty} \cap B^{\frac{N}{2}+1}_{2,1}\right) \times \left(\widetilde{L}^\infty_{T^n}\left(B^{\frac{N}{p_2}-1}_{p_2,r} \cap B^{\frac{N}{2}}_{2,1}\right) \cap \widetilde{L}^1_{T^n}\left(B^{\frac{N}{p_2}+1}_{p_2,r} \cap B^{\frac{N}{2}+2}_{2,1}\right)\right)^N \times \widetilde{L}^1_{T^n}\left(B^{\frac{N}{p_2}-1}_{p_2,r} \cap B^{\frac{N}{2}}_{2,1}\right).$$

hence may be continued beyond $\bar{T}$ (see the remark on the lifespan in [13] where a control of ∇u in $\widetilde{L}^1(B^0_{\infty,\infty})$ is required). We thus have $T^n \geqslant \bar{T}$.

Compactness arguments

We now have to prove that $(a^n, u^n)_{n\in\mathbb{N}}$ tends (up to a subsequence) to some function (a,u) which belongs to E_T and satisfies (1.4). The proof is based on Ascoli's theorem and compact embedding for Besov spaces. As similar arguments have been employed in [10] or [9], we only give the outlines of the proof.

- Convergence of $(a^n)_{n\in\mathbb{N}}$:
  We use the fact that $\widetilde{a}^n = a^n - a_0^n$ satisfies:
  $$\partial_t \widetilde{a}^n = -u^n \cdot \nabla a^n.$$
  Since $(u^n)_{n\in\mathbb{N}}$ is uniformly bounded in $\widetilde{L}^1_T\left(B^{\frac{N}{p_2}+1}_{p_2,r}\right) \cap \widetilde{L}^\infty_T\left(B^{\frac{N}{p_2}-1}_{p_2,r}\right)$, it is, by interpolation, also bounded in $\widetilde{L}^{r'}_T\left(B^{\frac{N}{p_2}-1+\frac{2}{r'}}_{p_2,r}\right)$ for any $r' \in [1,+\infty]$. By taking $r=2$ and using the standard product

laws in Besov spaces, we thus easily gather that $(\partial_t\widetilde{a}^n)$ is uniformly bounded in $\widetilde{L}^2_T\Big(B^{\frac{N}{p_1}-1}_{p_1,\infty}\Big)$.

$$\|\partial_t\widetilde{a}^n\|_{\widetilde{L}^2_T\left(B^{\frac{N}{p_1}-1}_{p_1,\infty}\right)} \lesssim \|u^n\|_{\widetilde{L}^2_T\left(B^{\frac{N}{p_2}}_{p_2,r}\right)}\|\nabla a^n\|_{\widetilde{L}^\infty_T\left(B^{\frac{N}{p_1}-1}_{p_1,\infty}\right)}.$$

Hence $(\widetilde{a}^n)_{n\in\mathbb{N}}$ is bounded in $\widetilde{L}^\infty_T\Big(B^{\frac{N}{p_1}-1}_{p_1,\infty}\cap B^{\frac{N}{p_1}}_{p_1,\infty}\Big)$ and equicontinuous on $[0,T]$ with values in $B^{\frac{N}{p_1}-1}_{p_1,\infty}$. Since the embedding $B^{\frac{N}{p_1}-1}_{p_1,\infty}\cap B^{\frac{N}{p_1}}_{p_1,\infty}\hookrightarrow B^{\frac{N}{p_1}-1}_{p_1,\infty}$ is (locally) compact, and $(a^n_0)_{n\in\mathbb{N}}$ tends to a_0 in $B^{\frac{N}{p_1}}_{p_1,\infty}$, we conclude that $(a^n)_{n\in\mathbb{N}}$ tends (up to extraction) to some distribution a. Given that $(a^n)_{n\in\mathbb{N}}$ is bounded in $\widetilde{L}^\infty_T\Big(B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,r}\Big)$, we actually have $a\in\widetilde{L}^\infty_T\Big(B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,r}\Big)$.

- Convergence of $(u^n_L)_{n\in\mathbb{N}}$:
From the definition of u^n_L and classical proposition on Stokes equation, it is clear that $(u^n_L)_{n\in\mathbb{N}}$ and $(\nabla\Pi^n_L)_{n\in\mathbb{N}}$ tend to solution u_L and $\nabla\Pi_L$ to:

$$\partial_t u_L-\mu\Delta u_l+\nabla\Pi_L=f,\ \ u_L(0)=u_0$$

in $\widetilde{L}^\infty_T\Big(B^{\frac{N}{p_2}-1}_{p_2,r}\Big)\cap\widetilde{L}^1_T\Big(B^{\frac{N}{p_2}+1}_{p_2,r}\Big)$ for $(u^n_L)_{n\in\mathbb{N}}$ and $\widetilde{L}^1_T\Big(B^{\frac{N}{p_2}-1}_{p_2,r}\Big)$ for $(\nabla\Pi^n_L)_{n\in\mathbb{N}}$.

- Convergence of $(\widetilde{u}^n)_{n\in\mathbb{N}}$:
We use the fact that:

$$\partial_t\widetilde{u}^n=-u^n_L\cdot\nabla\widetilde{u}^n-\widetilde{u}^n\cdot\nabla u^n+(1+a^n)\Delta\widetilde{u}^n+a^n\Delta u^n_L-u^n_L\cdot\nabla u^n_L-\nabla\widetilde{\Pi}^n.$$

As $(a^n)_{n\in\mathbb{N}}$ is uniformly bounded in $\widetilde{L}^\infty_T\Big(B^{\frac{N}{p_1}}_{p_1,\infty}\Big)$ and $(u^n)_{n\in\mathbb{N}}$ is uniformly bounded in $\widetilde{L}^\infty_T\Big(B^{\frac{N}{p_2}-1}_{p_2,r}\Big)\cap\widetilde{L}^{\frac{4}{3}}\Big(B^{\frac{N}{p_2}+\frac{1}{2}}_{p_2,r}\Big)$, it is easy to see that the last term of the right-hand side is uniformly bounded in $\widetilde{L}^\infty_T\Big(B^{\frac{N}{p_2}-1}_{p_2,r}\Big)$ and that the other terms are uniformly bounded in $\widetilde{L}^{\frac{4}{3}}\Big(B^{\frac{N}{p_2}-\frac{3}{2}}_{p_2,r}\Big)$.

Hence $(\widetilde{u}^n)_{n\in\mathbb{N}}$ is bounded in $\widetilde{L}^\infty_T\Big(B^{\frac{N}{p_2}-1}_{p_2,r}\Big)$ and equicontinuous on $[0,T]$ with values in $B^{\frac{N}{p_2}-1}_{p_2,r}+B^{\frac{N}{p_2}-\frac{3}{2}}_{p_2,r}$. This enables to conclude that $(\widetilde{u}^n)_{n\in\mathbb{N}}$ converges (up to extraction) to some function $\widetilde{u}\in\widetilde{L}^\infty_T\Big(B^{\frac{N}{p_2}-1}_{p_2,r}\Big)\cap\widetilde{L}^1_T\Big(B^{\frac{N}{p_2}+1}_{p_2,r}\Big)$.

We proceed similarly for $(\Pi_L^n)_{n\in\mathbb{N}}$ and $(\widetilde{\Pi}^n)_{n\in\mathbb{N}}$. By interpolating with the bounds provided by the previous step, one obtains better results of convergence so that one can pass to the limit in the mass equation and in (6.2). Finally by setting $u=\widetilde{u}+u_L$ and $\Pi=\widetilde{\Pi}+\Pi_L$, we conclude that (a,u,Π) satisfies (1.4).

In order to prove continuity in time for a it suffices to make use of Proposition 5.1. Indeed, a_0 is in $B^{\frac{N}{p_1}+\varepsilon}_{p_1,\infty}\cap L^\infty$, and having $a\in\widetilde{L}^\infty_T\Big(B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty}\Big)\cap L^\infty$ and $u\in\widetilde{L}^1_T\Big(B^{\frac{N}{p_2}+1}_{p_2,r}\Big)$ insure that $\partial_t a+u\cdot\nabla a$ belongs to $\widetilde{L}^1_T\Big(B^{\frac{N}{p_1}}_{p_1,\infty}\Big)$. Similarly, continuity for u may be proved by using that $u_0\in B^{\frac{N}{p_2}}_{p_2,r}$ and that $(\partial_t u-\mu\Delta u)\in\widetilde{L}^1_T\Big(B^{\frac{N}{p_2}-1}_{p_2,r}\Big)$.

7. The proof of the uniqueness

The proof of uniqueness is classical now and we inspire us of the works of R. Danchin in [12].

7.1. Uniqueness when $1\leqslant p_2<2N$, $\frac{2}{N}<\frac{1}{p_1}+\frac{1}{p_2}$ and $N\geqslant 3$

In this section, we focus on the case $N\geqslant 3$ and postpone the analysis of the other cases (which turns out to be critical) to the next section. Throughout the proof, we assume that we are given two solutions (a^1,u^1) and (a^2,u^2) of (1.4) which belongs to:

$$\Big(\widetilde{C}\Big([0,T];B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty}\Big)\cap L^\infty\Big)\times\Big(\widetilde{C}\Big([0,T];B^{\frac{N}{p_2}-1}_{p_2,r}\Big)\cap\widetilde{L}^1\Big(0,T;B^{\frac{N}{p_2}+1}_{p_2,r}\Big)\Big)^N.$$

Let $\delta a=a^2-a^1$, $\delta u=u^2-u^1$ and $\delta\Pi=\Pi^2-\Pi^1$. The system for $(\delta a,\delta u)$ reads:

$$\begin{cases}\partial_t\delta a+u^2\cdot\nabla\delta a=-\delta u\cdot\nabla a^1,\\ \partial_t\delta u-(1+a^2)(\mu\Delta\delta u-\nabla\delta\Pi)=F(a^i,u^i,\Pi^i)\end{cases}\tag{7.1}$$

with:

$$F(a^i,u^i,\Pi^i)=u^1\cdot\nabla\delta u+\delta u\cdot\nabla u^2+\delta a(\mu\Delta u^1-\nabla\Pi^1).$$

The function δa may be estimated by taking advantage of Proposition 5.1 with $s=\frac{N}{p_1}-1+\frac{\varepsilon}{2}$. We get for all $t\in[0,T]$,

$$\|\delta a(t)\|_{B^{\frac{N}{p_1}-1}_{p_1,\infty}}\leqslant C\|\delta u\cdot\nabla a^1\|_{\widetilde{L}^1_T\big(B^{\frac{N}{p_1}-1+\frac{\varepsilon}{4}}_{p_1,\infty}\big)}\exp\Big(\frac{C}{\varepsilon^{r-1}}(V_{p_1,1-\frac{1}{r}}(t))^r\Big).$$

We have then by Proposition 2.4 and 2.12:

$$\begin{aligned}\|\delta a(t)\|_{B^{\frac{N}{p_1}-1}_{p_1,\infty}} &\leqslant C\|\delta u\|_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}-\frac{\varepsilon}{4}}_{p_2,r}\right)}\|\nabla a^1\|_{\widetilde{L}^\infty_T\left(B^{\frac{N}{p_1}-1+\frac{\varepsilon}{2}}_{p_1,\infty}\right)} \\ &\qquad \exp\bigg(\frac{C}{\varepsilon^{r-1}}\Big(\|u^2\|_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}+1}_{p_2,r}\right)}\Big)^r\bigg), \\ &\leqslant C\|\delta u\|_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}}_{p_2,r}\right)}\|a^1\|_{\widetilde{L}^\infty_T\left(B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty}\right)} \\ &\qquad \exp\bigg(\frac{C}{\varepsilon^{r-1}}\Big(\|u^2\|_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}+1}_{p_2,r}\right)}\Big)^r\bigg).\end{aligned} \tag{7.2}$$

For bounding δu, we aim at applying Proposition 4.1 to the second equation of (7.1). So let us fix an integer m such that:

$$1+\inf_{(t,x)\in[0,T]\times\mathbb{R}^N} S_m a^2 \geqslant \frac{b}{2} \ \text{ and } \ \|a^2 - S_m a^2\|_{L^\infty_T\left(B^{\frac{N}{p_1}}_{p_1,\infty}\right)} \leqslant c\frac{\underline{\nu}}{\overline{\nu}}. \tag{7.3}$$

Now applying Proposition 4.1 with $s=\frac{N}{p_2}-2$ insures that for all time $t\in[0,T]$, we have:

(7.4)

$$\begin{aligned}&\|u\|_{\widetilde{L}^\infty_T\left(B^{\frac{N}{p_2}-2}_{p_2,r}\right)} + \kappa\underline{\nu}\|u\|_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}}_{p_2,r}\right)} + \|\nabla\delta\Pi\|_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}-2}_{p_2,r}\right)} \\ &\leqslant e^{CZ_m(T)}\times\bigg(\|\mathcal{P}F(a^i,u^i,\Pi^i)\|_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}-2}_{p_2,r}\right)} + \Big(\frac{\underline{\nu}(p_2-1)}{p_2}\Big)\mathcal{A}_T\|u\|_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}-\alpha'}_{p,r}\right)}\bigg)\end{aligned}$$

with $Z_m(t)=2^m\mu^2\underline{\nu}^{-1}\int_0^t\|a(\tau)\|^2_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty}\,d\tau$.

Hence, applying Proposition 2.4, Corollary 2.6 and the fact that $\operatorname{div}\delta u=0$, we get as example:

$$\begin{aligned}\|\delta u\cdot\nabla u^2\|_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}-2}_{p_2,r}\right)} &\lesssim \|u^2\|^{\frac12}_{\widetilde{L}^\infty_T\left(B^{\frac{N}{p_2}-1}_{p_2,r}\right)}\|u^2\|^{\frac12}_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}+1}_{p_2,r}\right)} \\ &\qquad \bigg(\|\delta u\|_{\widetilde{L}^\infty_T\left(B^{\frac{N}{p_2}-2}_{p_2,r}\right)} + \|\delta u\|_{\widetilde{L}^\infty_T\left(B^{\frac{N}{p_2}}_{p_2,r}\right)}\bigg),\end{aligned}$$

and:

$$\begin{aligned}\|u^1\cdot\nabla\delta u\|_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}-2}_{p_2,r}\right)} &\lesssim \|u^1\|^{\frac12}_{\widetilde{L}^\infty_T\left(B^{\frac{N}{p_2}-1}_{p_2,r}\right)}\|u^1\|^{\frac12}_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}+1}_{p_2,r}\right)} \\ &\qquad \bigg(\|\delta u\|_{\widetilde{L}^\infty_T\left(B^{\frac{N}{p_2}-2}_{p_2,r}\right)} + \|\delta u\|_{\widetilde{L}^\infty_T\left(B^{\frac{N}{p_2}}_{p_2,r}\right)}\bigg),\end{aligned}$$

By the fact that $\frac{2}{N} < \frac{1}{p_1} + \frac{1}{p_2}$, $N \geqslant 3$ and $\frac{1}{p_2} \leqslant \frac{1}{N} + \frac{1}{p_1}$ imply that:

(7.5)
$$\|\delta a(\mu\Delta u_1 - \nabla\delta\Pi)\|_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}-2}_{p_2,r}\right)} \lesssim \|\delta a\|_{\widetilde{L}^\infty_T\left(B^{\frac{N}{p_1}-1}_{p_1,\infty}\right)} \left(\|\Delta u_1\|_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}-1}_{p_1,r}\right)} + \|\nabla\Pi^1\|_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}-1}_{p_2,r}\right)}\right).$$

Now let choose T_1 enough small to control $\mathcal{A}_T\|u\|_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}-\alpha'}_{p,r}\right)}$ in (7.4) and $\|\Delta u_1\|_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}-1}_{p_1,r}\right)} + \|\nabla\Pi^1\|_{\widetilde{L}^1_T\left(B^{\frac{N}{p_2}-1}_{p_2,r}\right)}$ in (7.5) and by the fact that $\|a_1\|_{\widetilde{L}^\infty_T\left(B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty}\right)} \leqslant c$ with c small, we obtain finally:

$$\|(\delta a, \delta u, \nabla\delta\Pi)\|_{F_{T_1}} \leqslant cC\|(\delta a, \delta u, \nabla\delta\Pi)\|_{F_{T_1}},$$

with:

$$F_T = \widetilde{C}_T([0,T], B^{\frac{N}{p_1}}_{p_1,\infty}) \times \left(\widetilde{L}^1_T\left(B^{\frac{N}{p_2}}_{p_2,r}\right) \cap \widetilde{L}^\infty_T\left(B^{\frac{N}{p_2}-2}_{p_2,r}\right)\right) \times \widetilde{L}^1_T\left(B^{\frac{N}{p_2}-2}_{p_2,r}\right).$$

We obtain so $(\delta a, \delta u, \nabla\delta\Pi) = 0$ on $[0, T_1]$ for T_1 enough small. By connexity we obtain that $(\delta a, \delta u, \delta\nabla\Pi) = 0$ on $[0, T]$. This conclude this case.

7.2. Uniqueness when: $\frac{2}{N} = \frac{1}{p_1} + \frac{1}{p_2}$ or $p_2 = 2N$ or $N = 2$

The above proof fails in dimension two. One of the reasons why is that the product of functions does not map $B^{\frac{N}{p_1}-1}_{p_1,\infty} \times B^{\frac{N}{p_2}-1}_{p_2,r}$ in $B^{\frac{N}{p_2}-2}_{p_2,r}$ but only in the larger space $B^{-1}_{2,\infty}$. This induces us to bound δa in $\widetilde{L}^\infty_T\left(B^{\frac{N}{p_1}-1}_{p_1,\infty}\right)$ and δu in $\widetilde{L}^\infty_T\left(B^{\frac{N}{p_2}-2}_{p_2,\infty}\right) \cap \widetilde{L}^1_T\left(B^{\frac{N}{p_2}}_{p_2,\infty}\right)$. In fact, it is enough to study only the case $\frac{2}{N} = \frac{1}{p_1} + \frac{1}{p_2}$. Indeed the other cases deduct from this case. If $p_2 = 2N$ then $p_1 = \frac{2N}{3}$ as $\frac{1}{p_1} \leqslant \frac{1}{N} + \frac{1}{p_2}$ and $\frac{2}{N} \leqslant \frac{1}{p_1} + \frac{1}{p_2}$. So it is a particular case of $\frac{2}{N} = \frac{1}{p_1} + \frac{1}{p_2}$. For $N = 2$, we begin with $p_2 = 4$ and $p_1 = \frac{4}{3}$ and by embedding we get the result for $1 \leqslant p_1 \leqslant \frac{4}{3}$, $1 \leqslant p_2 \leqslant 4$ and for $1 \leqslant p_1 \leqslant 4$, $1 \leqslant p_2 \leqslant \frac{4}{3}$.

Moreover in your case, it exists two possibilities, one when $1 < p_2 < 2$ and when $p_2 \geqslant 2$. The first case is resolved by embedding so we have just to treat the case $p_2 \geqslant 2$. We want show that $(\delta a, \delta u, \nabla\delta\Pi) \in G_T$ where:

$$G_T = \widetilde{C}_T\left([0,T], B^{\frac{N}{p_1}-1+\frac{\varepsilon}{2}}_{p_1,\infty}\right) \times \left(\widetilde{L}^1_T\left(B^{\frac{N}{p_2}}_{p_2,\infty}\right) \cap \widetilde{L}^\infty_T\left(B^{\frac{N}{p_2}-2}_{p_2,\infty}\right)\right) \times \widetilde{L}^1_T\left(B^{\frac{N}{p_2}-2}_{p_2,\infty}\right).$$

In fact we proceed exactly as in the previous proof but we take in account that a is in $\widetilde{C}_T\Big([0,T],B^{\frac{N}{p_1}-1+\frac{\varepsilon}{2}}_{p_1,\infty}\Big)$ which gives sense to the product $\delta a\Delta u$. It conclude the proof of uniqueness.

8. Global existence

In order to obtain global solution, we have to pay attention on the behavior in low frequencies of the velocity and of the density for two different reasons. The first concerns the velocity, indeed it is necessary to work with homogeneous Besov spaces because in other case the constant of estimates of Proposition 4.1 in nonhomogeneous Besov spaces depend on the time T, so it does not allow to conclude when we want to deal with global solution. The second reason is that the Proposition 5.1 is valid only for nonhomogeneous Besov spaces. Indeed the behavior in low frequencies is different, that is why we assume that a_0 belongs in L^{p_1}. It means that we are differentiate the behavior between low frequencies and high frequencies as in [19] in the context of compressible Navier-Stokes equations. By considering the transport equation we easily obtain that a is in $L^\infty(L^{p_1})$. If we split a in low and high frequencies, we have:

$$a=a_L+a_H,$$

where $a_L=\sum_{l<0}\Delta_l a$ and $a_H=\sum_{l\geqslant 0}\Delta_l a$. As a belongs in $L^\infty_T(L^{p_1})$ for every $T>0$, we then have that a_L is in $\widetilde{L}^\infty(\dot{B}^0_{p_1,\infty})$ which is embedded in low frequencies in $L^\infty\Big(\dot{B}^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty}\Big)$.

Combining this point with the fact that we can show by using Proposition 5.1 that $a_H=\sum_{l\geqslant 0}\Delta_l a$ is in $\widetilde{L}^\infty_T\Big(\dot{B}^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty}\Big)$, we finally are able to show that a is in $\widetilde{L}^\infty_T\Big(\dot{B}^{\frac{N}{p_1}}_{p_1,\infty}\Big)$. The rest of the proof follows the same lines that in Section 6.

More precisely we are going to prove that the sequel (a^n,u^n) constructed in the Section 6 exists on $(0,T)$ with $T=+\infty$ when we assume the initial data small enough. For the moment we just know that this solution exists on $(0,T_n)$. By using the Theorem 5.1 and the fact that a_0 is in L^{p_1} we have:

$$\|a^n\|_{\widetilde{L}^\infty_{T_n}\big(\dot{B}^{\frac{N}{p_1}}_{p_1,\infty}\big)}\leqslant C\bigg(e^{C\|u^n\|^{\frac{1}{1-\alpha}}_{\widetilde{L}^1_{T_n}\big(\dot{B}^{\frac{N}{p_2}}_{p_2,r}\big)}}\|a_0^H\|_{\dot{B}^{\frac{N}{p_1}+\varepsilon}_{p_1,r}}+\|a_0\|_{L^{p_1}}\bigg),$$

where α depends on r. By using the same estimations on u^n than in Section 6, we have:

$$U^n(T_n)) \leqslant C\Big(\|u_0\|_{\dot{B}^{\frac{N}{p_2}-1}_{p_2,r}} + \|u^n\|_{\widetilde{L}^{\frac{4}{3}}_{T_n}(\dot{B}^{\frac{N}{p_2}+1}_{p_2,r})}\|u^n\|_{\widetilde{L}^{4}_{T_n}(\dot{B}^{\frac{N}{p_2}-\frac{1}{2}}_{p_2,r})}$$
$$+ \|a^n\|_{\widetilde{L}^{\infty}_{T_n}(\dot{B}^{\frac{N}{p_1}}_{p_1,\infty})\cap L^\infty} \times \Big(\|u^n\|_{\widetilde{L}^{1}_{T_n}(\dot{B}^{\frac{N}{p_2}+1}_{p_2,r})} + \|\nabla\Pi^n\|_{\widetilde{L}^{1}_{T_n}(\dot{B}^{\frac{N}{p_2}-1}_{p_2,r})}\Big)\Big),$$

with:

$$U^n(T_n) = \|u^n\|_{\widetilde{L}^{\infty}_{T_n}(\dot{B}^{\frac{N}{p_2}-1}_{p_2,r})} + \|u^n\|_{\widetilde{L}^{1}_{T_n}(\dot{B}^{\frac{N}{p_2}+1}_{p_2,r})} + \|\nabla\Pi^n\|_{\widetilde{L}^{1}_{T_n}(\dot{B}^{\frac{N}{p_2}-1}_{p_2,r})}.$$

We have then for all $0 < T < T_n$:

$$U^n(T) \leqslant C\Big(\|u_0\|_{\dot{B}^{\frac{N}{p_2}-1}_{p_2,\infty}} + U^n(T)^2 + \Big(C\exp\Big(CU^n(T)^{\frac{1}{1-\alpha}}\Big)$$
$$\|a_0^H\|_{\dot{B}^{\frac{N}{p_1}+\varepsilon}_{p_1,r}} + \|a_0\|_{L^{p_1}}\Big)U^n(T)\Big).$$

Let $T_1 < T_n$ such that:

$$U^n(T_1) \leqslant M\theta,$$

with $\theta = \|a_0^H\|_{\dot{B}^{\frac{N}{p_1}+\varepsilon}_{p_1,r}} + \|a_0\|_{L^{p_1}} + \|u_0\|_{\dot{B}^{\frac{N}{p_2}-1}_{p_2,r}}$. We then have:

$$U^n(T_1) \leqslant C^2 e^{C(M\theta)^{\frac{1}{1-\alpha}}}(\theta + M^2\theta^2 + 2\theta M\theta).$$

By choosing $M = 4C^2$ and θ such that:

$$e^{C(M\theta)^{\frac{1}{1-\alpha}}} \leqslant 2,\ M^2\theta \leqslant \frac{1}{3} \text{ and } 2M\theta \leqslant \frac{1}{3}.$$

Then we have $U^n(T_1) \leqslant \frac{5}{6}M\theta$. By connectivity we can conclude easily that:

$$U^n(T_n) \leqslant M\theta.$$

By classical argument of blow up (as we control ∇u^n in $\widetilde{L}^1_{T_n}\big(\dot{B}^{\frac{N}{p_2}-1}_{p_2,r}\big)$) we show that T_n is not maximal and then $T_n = +\infty$. It is sufficient to conclude by using the same type of argument than in the Section 6 and 7.

8.1. Proof of Theorem 1.7

The proof follows strictly the same lines than the proof of Theorem 1.1 except that we take into account that in the Proposition 4.1 we do not need any condition of smallness on the initial density a_0 when $p_2 = 2$.

9. Proof of Theorems 1.8 and 1.10

9.1. Proof of Theorem 1.8 – Existence in finite time

In this case by using the same type of estimates than in Section 6, we have only a uniform control on u^n in $\widetilde{L}^1\Big(B^{\frac{N}{p_2}+1}_{p_2,\infty}\Big)$, that is why in order to estimate the density via the transport equation, we need to use the Corollary 5.5. We have then a loss of regularity depending on $\|u^n\|_{\widetilde{L}^1_T\big(B^{\frac{N}{p_2}+1}_{p_2,\infty}\big)}$ at the condition that $\|u^n\|_{\widetilde{L}^1_T\big(B^{\frac{N}{p_2}+1}_{p_2,\infty}\big)}$ be small enough (it is the case for a time T small enough).

We would like to point out that a^n verifies a transport equation without remainder g^n what is absolutely crucial in our proof. Indeed if we consider the Corollary 5.5, we are not able to give a precise sense in term of Besov space to the quantity:

$$\sup_{l\geqslant -1}\int_0^t 2^{ls-\varepsilon_l(\tau)}\|\Delta_l g(\tau)\|_{L^2}d\tau.$$

But by chance, in your context there is no reminder term such that we are able to obtain estimates on $a^n\in\widetilde{L}^\infty\big(B^{\sigma(t)}_{p_1,\infty}\big)$ with $\sigma(t)\geqslant\frac{N}{p_1}$ for T small enough (the situation would be delicate for some models like the Olroyd model). It means in particular that we control $a^n\in\widetilde{L}^\infty\big(B^{\frac{N}{p_1}}_{p_1,\infty}\big)\cap L^\infty$ what is enough to use the same arguments than in Section 6. The rest of the proof is exactly similar to proof of Theorem 1.1.

9.2. Proof of Theorem 1.8 – Uniqueness

In this situation the situation is more tricky than in Section 6, indeed as mentionned previously it is difficult to estimate a^n when there is a remainder term g^n in the transport equation. Unfortunately it is exactly the case when we are interested in dealing with the problem of uniqueness. Indeed we recall that whit the same notation than in Section 7, we have:

$$\begin{cases}\partial_t\delta a+u^2\cdot\nabla\delta a=-\delta u\cdot\nabla a^1,\\ \partial_t\delta u-(1+a^2)(\mu\Delta\delta u-\nabla\delta\Pi)=F(a^i,u^i,\Pi^i),\end{cases}\tag{9.1}$$

with:

$$F(a^i,u^i,\Pi^i)=u^1\cdot\nabla\delta u+\delta u\cdot\nabla u^2+\delta a(\mu\Delta u^1-\nabla\Pi^1).$$

We can observe than we have a remainder term in the transport equation of the form $-\delta u\cdot\nabla a^1$, it seems then difficult to conclude. However we are going to use the fact that we have surcritical regularity on the initial density in order to conclude. We set:

$$\delta\phi(t)=\|\delta a\|_{\widetilde{L}^\infty_t\big(B^{\frac{N}{p_1}-1}_{p_1,\infty}\big)},$$

and:

$$\delta U(t)=\|\delta u\|_{\widetilde{L}^\infty_t\big(B^{\frac{N}{p_2}-2}_{p_2,\infty}\big)}+\|\delta u\|_{\widetilde{L}^1_t\big(B^{\frac{N}{p_2}}_{p_2,\infty}\big)}+\|\nabla\delta\Pi\|_{\widetilde{L}^1_t\big(B^{\frac{N}{p_2}-2}_{p_2,\infty}\big)}.$$

We are going to use the same ideas than in the Section 7 except that we are going to work with $\delta\theta$ and δU. We then have when $1\leqslant p_2<2N$, $\frac{2}{N}<\frac{1}{p_1}+\frac{1}{p_2}$ and $N\geqslant 3$ by applying Proposition 4.1:

$$\begin{aligned}\delta U(t)\leqslant\bigg(&\|u^1\|^{\frac12}_{\widetilde{L}^\infty_t\big(B^{\frac{N}{p_2}-1}_{p_2,\infty}\big)}\|u^1\|^{\frac12}_{\widetilde{L}^1_t\big(B^{\frac{N}{p_2}+1}_{p_2,\infty}\big)}+\|u^2\|^{\frac12}_{\widetilde{L}^\infty_t\big(B^{\frac{N}{p_2}-1}_{p_2,\infty}\big)}\\ &\|u^2\|^{\frac12}_{\widetilde{L}^1_t\big(B^{\frac{N}{p_2}+1}_{p_2,\infty}\big)}\bigg)\delta U(t)+\phi(t)\bigg(\|\Delta u^2\|_{\widetilde{L}^1_t\big(B^{\frac{N}{p_2}-1}_{p_2,\infty}\big)}+\|\nabla\Pi^2\|_{\widetilde{L}^1_t\big(B^{\frac{N}{p_2}-1}_{p_2,\infty}\big)}\bigg).\end{aligned}$$

We now have to estimate $\delta\phi(t)$, to do this we are using the Proposition 5.4 which tells us that:

$$\delta\phi(t)\leqslant C\sup_{l\geqslant -1}\int t_0 2^{(\frac{N}{p_1}-1+\varepsilon)l-\varepsilon_l(\tau)}\|\Delta_l\,\mathrm{div}(a_2\delta u)\|_{L^{p_1}}(\tau)d\tau.$$

For t small enough (such that $\big(\frac{N}{p_1}-1+\frac{\varepsilon}{4}\big)l-\varepsilon_l(\tau)>\big(\frac{N}{p_1}-1\big)l$ with $l\geqslant 0$) by using the Proposition 5.4 we have:

$$\delta\phi(t)\leqslant C\sup_{l\geqslant -1}\int_0^t 2^{(\frac{N}{p_1}-1+\frac{\varepsilon}{4})l-\varepsilon_l(\tau)}\|\Delta_l\,\mathrm{div}(a_2\delta u)\|_{L^{p_1}}(\tau)d\tau.$$

And we have:

$$\begin{aligned}&\sup_{l\geqslant -1}\int_0^t 2^{\left(\frac{N}{p_1}-1+\frac{\varepsilon}{4}\right)l-\varepsilon_l(\tau)}\|\Delta_l\mathrm{div}(a_2\delta u)\|_{L^{p_1}}(\tau)d\tau\leqslant\|\delta u\cdot\nabla a^2\|_{\widetilde{L}^1_t\big(B^{\frac{N}{p_1}-1+\frac{\varepsilon}{4}}_{p_1,\infty}\big)}\\ &\leqslant\|\delta u\|_{\widetilde{L}^1_t\big(B^{\frac{N}{p_2}-\frac{\varepsilon}{4}}_{p_2,\infty}\big)}\|\nabla a_2\|_{\widetilde{L}^1_t\big(B^{\frac{N}{p_1}-1+\frac{\varepsilon}{2}}_{p_2,\infty}\big)}\leqslant\|\delta u\|_{\widetilde{L}^1_t\big(B^{\frac{N}{p_2}-\frac{\varepsilon}{4}}_{p_2,\infty}\big)}\|a_2\|_{\widetilde{L}^1_t\big(B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_2,\infty}\big)},\\ &\leqslant\|\delta u\|_{\widetilde{L}^1_t\big(B^{\frac{N}{p_2}-\frac{\varepsilon}{4}}_{p_2,\infty}\big)}\|a_2\|_{\widetilde{L}^1_t\bigg(B_{p_1,\infty}^{\frac{N}{p_1}+\varepsilon-\lambda\|u^2\|_{\widetilde{L}^1_t\big(B^{\frac{N}{p_2}+1}_{p_2,\infty}\big)}}\bigg)},\\ &\leqslant\|\delta u\|_{\widetilde{L}^1_t\big(B^{\frac{N}{p_2}-\frac{\varepsilon}{4}}_{p_2,\infty}\big)}\|a_2\|_{\big(B^{\frac{N}{p_1}+\varepsilon}_{p_1,\infty}\big)}.\end{aligned}$$

Here we have also assumed t small enough such that $\varepsilon-\lambda\|u^2\|_{\widetilde{L}^1_t\left(B^{\frac{N}{p2}+1}_{p_2,\infty}\right)}\geqslant\frac{\varepsilon}{2}$. We can conclude by the same way than in Section 7. It means that $(\delta a,\delta u)=0$ on $(0,t)$ and we show the property on $(0,T)$ by connexity.

9.3. Proof of Theorem 1.8 – global existence

It follows exactly the same lines than for the Theorem 1.1 except that we are going to use the Corollary 5.5 in order to control the density. We are giving some details for the comfort of the reader.

As we mentioned previously, we need to work with homogenous Besov space and in particular to distinguish the behavior of the density in low and high frequencies, the reason concerns the Corollary 5.5 which gives us in some sense only information in high frequencies (it means for $\Delta_l a$ with $l\geqslant 0$). As we assume that a_0 belongs in L^{p_1}, we are able to control a in $L^\infty(L^{p_1})\to\widetilde{L}^\infty(\dot{B}^0_{p_1,\infty})$). If we split a in low and high frequencies, we have:

$$a=a_L+a_H.$$

Combining the fact that $a_L\in\widetilde{L}^\infty(\dot{B}^0_{p_1,\infty})$ which is embedded in $\widetilde{L}^\infty\left(\dot{B}^{\frac{N}{p_1}}_{p_1,\infty}\right)$ in low frequencies and that we can show by using Corollary 5.5 that $a_H=\sum_{l\geqslant 0}\Delta_l a$ is in $\widetilde{L}^\infty_T\left(\dot{B}_{p_1,\infty}^{\frac{N}{p_1}+\varepsilon-\lambda\|u\|_{\widetilde{L}^1_T\left(\dot{B}^{\frac{N}{p2}+1}_{p_2,\infty}\right)}}\right)$, we finally are able to show that a is in $\widetilde{L}^\infty_T\left(\dot{B}^{\frac{N}{p_1}}_{p_1,\infty}\right)$ (if along the time we can show that $\lambda\|u\|_{\widetilde{L}^1_T\left(\dot{B}^{\frac{N}{p2}+1}_{p_2,\infty}\right)}$ remains small enough).

More precisely we are now going to prove that the sequel (a^n,u^n) constructed in the Section 6 exists on $(0,T)$ with $T=+\infty$ when we assume the initial data small enough. For the moment we just know that this solution exists on $(0,T_n)$. By using the Corollary 5.5 and the fact that a_0 is in L^{p_1} we have (when $\lambda\|u^n\|_{\widetilde{L}^1_T\left(\dot{B}^{\frac{N}{p2}+1}_{p_2,\infty}\right)}<\varepsilon$ and $\|u^n\|_{\widetilde{L}^1_T\left(\dot{B}^{\frac{N}{p2}+1}_{p_2,\infty}\right)}<c$, c is the constant of Corollary 5.5):

$$\|a^n\|_{\widetilde{L}^\infty_T\left(\dot{B}^{\frac{N}{p_1}}_{p_1,\infty}\right)}\leqslant C\Big(\|a_0^H\|_{\dot{B}^{\frac{N}{p_1}+\varepsilon}_{p_1,r}}+\|a_0\|_{L^{p_1}}\Big).$$

By using the same estimations on u^n than in Section 6, we have at least for $T<T_n$ such that $\lambda\|u^n\|_{\widetilde{L}^1_T\left(\dot{B}^{\frac{N}{p2}+1}_{p_2,\infty}\right)}<\varepsilon$ and $\|u^n\|_{\widetilde{L}^1_T\left(\dot{B}^{\frac{N}{p2}+1}_{p_2,\infty}\right)}<c$ (c is

the constant of Corollary 5.5):

$$U^n(T)) \leqslant C\Big(\|u_0\|_{\dot{B}^{\frac{N}{p_2}-1}_{p_2,\infty}} + \|u^n\|_{\widetilde{L}^{\frac{4}{3}}_T(\dot{B}^{\frac{N}{p_2}+1}_{p_2,\infty})} \|u^n\|_{\widetilde{L}^{4}_T(\dot{B}^{\frac{N}{p_2}-\frac{1}{2}}_{p_2,\infty})}$$
$$+ \|a^n\|_{\widetilde{L}^{\infty}_T(\dot{B}^{\frac{N}{p_1}}_{p_1,\infty})\cap L^\infty} \times \Big(\|u^n\|_{\widetilde{L}^{1}_T(\dot{B}^{\frac{N}{p_2}+1}_{p_2,\infty})} + \|\nabla\Pi^n\|_{\widetilde{L}^{1}_T(\dot{B}^{\frac{N}{p_2}-1}_{p_2,\infty})} \Big)\Big),$$

with:

$$U^n(T) = \|u^n\|_{\widetilde{L}^{\infty}_T(\dot{B}^{\frac{N}{p_2}-1}_{p_2,\infty})} + \|u^n\|_{\widetilde{L}^{1}_T(\dot{B}^{\frac{N}{p_2}+1}_{p_2,\infty})} + \|\nabla\Pi^n\|_{\widetilde{L}^{1}_T(\dot{B}^{\frac{N}{p_2}-1}_{p_2,\infty})}.$$

We have then for all $0 < T < T_n$ such that $\lambda\|u^n\|_{\widetilde{L}^{1}_T(\dot{B}^{\frac{N}{p_2}+1}_{p_2,\infty})} < \varepsilon$ and $\|u^n\|_{\widetilde{L}^{1}_T(\dot{B}^{\frac{N}{p_2}+1}_{p_2,\infty})} < c$:

$$U^n(T) \leqslant C\Big(\|u_0\|_{\dot{B}^{\frac{N}{p_2}-1}_{p_2,\infty}} + U^n(T)^2 + C\Big(\|a_0\|_{\dot{B}^{\frac{N}{p_1}}_{p_1,\infty}} + \|a_0\|_{L^{p_1}}\Big)U^n(T)\Big).$$

Let $T_1 < T_n$ such that:

$$U^n(T_1) \leqslant M\theta,$$

and with:

$$\lambda M\theta \leqslant \frac{\varepsilon}{2} \;\text{ and }\; M\theta \leqslant \frac{c}{2},$$

with $\theta = \|a_0\|_{\dot{B}^{\frac{N}{p_1}+\varepsilon}_{p_1,\infty}} + \|a_0\|_{L^{p_1}} + \|a_0\|_{L^{p_1}} + \|u_0\|_{\dot{B}^{\frac{N}{p_2}-1}_{p_2,\infty}}$. We then have:

$$U^n(T_1) \leqslant C(\theta + M^2\theta^2 + C\theta M\theta).$$

By choosing $M = 4C$ and θ such that:

$$\theta \leqslant \frac{\varepsilon}{2\lambda M},\; \theta \leqslant \frac{c}{2M},\; M^2\theta \leqslant 1 \;\text{ and }\; CM\theta \leqslant 1.$$

Then we have $U^n(T_1) \leqslant 3C\theta \leqslant \frac{3}{4}M\theta$. By connexity we can then conclude easily that:

$$U^n(T_n) \leqslant M\theta.$$

By classical argument of blow up (as we control ∇u^n in $\widetilde{L}^1_{T_n}\big(\dot{B}^{\frac{N}{p_2}-1}_{p_2,\infty}\big)$) we show that T_n is not maximal and then $T_n = +\infty$. It is sufficient to conclude by using the same type of argument than in the Section 6 and 7.

9.4. Proof of Theorem 1.10

The proof follows strictly the same lines than the proof of Theorem 1.8 except that we take into account that in the Proposition 4.1 we do not need any condition of smallness on the initial density a_0 when $p_2 = 2$.

10. Appendix

10.1. Elliptic estimates

This section is devoted to the study of the elliptic equation:

$$\operatorname{div}(b\nabla\Pi) = \operatorname{div} F \tag{10.1}$$

with $b = 1 + a$.

Let us first study the stationary case where F and b are independent of the time:

Proposition 10.1. — *Let $0 < \alpha < 1$, $(p, r) \in [1, +\infty]^2$ and $\sigma \in \mathbb{R}$ satisfy $\alpha \leqslant \sigma \leqslant \alpha + \frac{N}{p_1}$. Then the operator $\mathcal{H}_b : F \to \nabla\Pi$ is a linear bounded operator in $B^{\sigma}_{p,r}$ and the following estimate holds true:*

$$\underline{b}\|\nabla\Pi\|_{B^{\sigma}_{p,r}} \lesssim \mathcal{A}^{\frac{|\sigma|}{\min(1,\alpha)}} \|\mathcal{Q}F\|_{B^{\sigma}_{p,r}}, \tag{10.2}$$

with if $\alpha \neq 1$:

$$\mathcal{A} = 1 + \underline{b}^{-1}\|\nabla b\|_{B^{\frac{N}{p_1}+\alpha-1}_{p_1,r}}.$$

Proof. — Let us first rewrite (10.1) as follows:

$$\operatorname{div}(b_m\nabla\Pi) = \operatorname{div} F - E_m, \tag{10.3}$$

with $E_m = \operatorname{div}((Id - S_m)a\nabla\Pi)$.

Apply Δ_q to (10.3) we get:

$$\operatorname{div}(b_m\nabla\Delta_q\Pi) = \operatorname{div} F_q - \Delta_q(E_m) + R_q, \tag{10.4}$$

with $R_q{=}\operatorname{div}(b_m\nabla\Pi_q){-}\Delta_q \operatorname{div}(b_m\nabla\Pi)$. Multiplying (10.4) by $\Delta_q\Pi|\Delta_q\Pi|^{p-2}$ and integrate, we gather:

$$\begin{aligned}
&\int_{\mathbb{R}^N} b_m|\nabla\Pi_q|^2|\Pi_q|^{p-2}dx + \int_{\mathbb{R}^N} b_m|\nabla|\Pi_q|^2|^p dx \\
&\leqslant (\|\operatorname{div} F_q\|_{L^p} + \|R_q\|_{L^p})\|\Pi_q\|^{p-1}_{L^p} + \int_{\mathbb{R}^N} |(Id - S_m)a|\,|\nabla\Pi_q|^2|\Pi_q|^{p-2}dx \\
&\quad + \int_{\mathbb{R}^N} |(Id - S_m)a|\,|\nabla|\Pi_q|^2|^p dx.
\end{aligned} \tag{10.5}$$

Assuming that m has been choose so large as to satisfy:

$$\|a - S_m a\|_{L^\infty_T\left(B^{\frac{N}{p_1}+\alpha}_{p_1,\infty}\right)\cap L^\infty} \leqslant \frac{\underline{b}}{2},$$

an by using Lemma A5 in [8]:

$$2^{2q}\|\Pi_q\|_{L^p} \lesssim 2^q\|F_q\|_{L^p} + \|R_q\|_{L^p}.$$

By multiplying by $2^{q(s-1)}$ and by integrating on l^r we get:

$$\underline{b}\|\nabla\Pi\|_{B^s_{p,r}} \lesssim \|\mathcal{Q}f\|_{B^s_{p,r}} + \|R_q\|_{B^s_{p,r}}.$$

The commutator may be bounded thanks to Lemma 10.2 with $0<\alpha<1$, $\sigma=s-1$. We have get:

$$\underline{b}\|\nabla\Pi\|_{B^s_{p,r}} \lesssim \|\mathcal{Q}F\|_{B^s_{p,r}} + \|\nabla S_m a\|_{B^{\frac{N}{p_1}+\alpha}_{p_1,r}} \|\nabla\Pi\|_{B^{s-\alpha}_{p,r}}.$$

Therefore complex interpolation entails:

$$\|\nabla\Pi\|_{B^{s-\alpha}_{p,r}} \leqslant \|\nabla\Pi\|^{\frac{s-\alpha}{s}}_{B^s_{p,r}} \|\nabla\Pi\|^{\frac{\alpha}{s}}_{B^0_{p,\infty}}.$$

Note that, owing to Bersntein inequality, we have:

$$\|\nabla S_m a\|_{B^{\frac{N}{p_1}+\alpha}_{p_1,r}} \lesssim 2^{m+\alpha}\|a\|_{B^{\frac{N}{p_1}}_{p_1,r}}.$$

We have then:

$$\underline{b}\|\nabla\Pi\|_{B^s_{p,r}} \lesssim \|\mathcal{Q}f\|_{B^s_{p,r}} + 2^{m+\alpha}\|a\|_{B^{\frac{N}{p_1}}_{p_1,\infty}} \|\nabla\Pi\|^{\frac{s-\alpha}{s}}_{B^s_{p,r}} \|\nabla\Pi\|^{\frac{\alpha}{s}}_{B^0_{p,\infty}}.$$

And we conclude by Young's inequality with $p_1=\frac{s}{s-\alpha}$ and $p_2=\frac{s}{\alpha}$. And we recall □

10.2. Commutator estimates

This section is devoted to the proof of commutator estimates which have been used in Section 2 and 4. They are based on paradifferentiel calculus, a tool introduced by J.-M. Bony in [5]. The basic idea of paradifferential calculus is that any product of two distributions u and v can be formally decomposed into:

$$uv = T_u v + T_v u + R(u,v) = T_u v + T'_v u$$

where the paraproduct operator is defined by $T_u v = \sum_q S_{q-1}u\Delta_q v$, the remainder operator R, by $R(u,v)=\sum_q \Delta_q u(\Delta_{q-1}v+\Delta_q v+\Delta_{q+1}v)$ and $T'_v u = T_v u + R(u,v)$. Inequality (4.15) is a consequence of the following lemma:

Lemma 10.2. — *Let $p_1\in[1,+\infty]$, $p\in[1,+\infty]$, $\alpha\in(1-\frac{N}{p},1[$, $k\in\{1,\cdots,N\}$ and $R_q=\Delta_q(a\partial_k w)-\partial_k(a\Delta_q w)$. There exists $c=c(\alpha,N,\sigma)$ such that:*

$$2^{q\sigma}\|\widetilde{R}_q\|_{L^p} \leqslant Cc_q\|a\|_{B^{\frac{N}{p_1}+\alpha}_{p_1,r}}\|w\|_{B^{\sigma+1-\alpha}_{p,r}} \tag{10.6}$$

whenever $-\frac{N}{p_1}<\sigma\leqslant\frac{N}{p_1}+\alpha$ and where $c_q\in L^r$.

In the limit case $\sigma = -\frac{N}{p_1}$, we have for some constant $C = C(\alpha, N)$:

$$2^{-q\frac{N}{p_1}} \|\widetilde{R}_q\|_{L^p} \leqslant C \|a\|_{B_{p,1}^{\alpha+\frac{N}{p_1}}} \|w\|_{B_{p,\infty}^{-\frac{N}{p_1}+1-\alpha}}. \tag{10.7}$$

Proof. — The proof is based on Bony's decomposition which enables us to split R_q into:

$$R_q = \underbrace{\partial_k[\Delta_q, T_a]w}_{R_q^1} - \underbrace{\Delta_q T_{\partial_k a} w}_{R_q^2} + \underbrace{\Delta_q T_{\partial_k w} a}_{R_q^3} + \underbrace{\Delta_q R(\partial_k w, a)}_{R_q^4} - \underbrace{\partial_k T'_{\Delta_q w} a}_{R_q^5}.$$

By using the fact that:

$$R_q^1 = \sum_{q'=q-4}^{q+4} \partial_k[\Delta_q, S_{q'-1}a]\Delta_{q'} w.$$

Using the definition of the operator Δ_q leads to:

$$[\Delta_q, S_{q'-1}a]\Delta_{q'} w(x) = -\int h(y)\big(S_{q'-1}a(x) - S_{q'-1}a(x - 2^{-q}y)\big)\Delta_{q'} w(x - 2^{-q}y)dy$$

and:

$$\begin{aligned}\big|[\Delta_q, S_{q'-1}a]\Delta_{q'}w(x)\big| &\leqslant \|\nabla S_{q'-1}a\|_{L^\infty} 2^{-q}\int 2^{qN}|h(2^q u)||2^q u||\Delta_{q'} w|(x-u)du,\\ &\leqslant 2^{qN}\|\nabla S_{q'-1}a\|_{L^\infty}|\big(h(2^q\cdot)| \cdot | * \Delta_{q'} w\big)|(x).\end{aligned}$$

So we get:

$$\|[\Delta_q, S_{q'-1}a]\Delta_{q'} w\|_{L^p} \leqslant \|\nabla S_{q'-1}a\|_{L^\infty} \|\Delta_{q'} w\|_{L^p}$$

we readily get under the hypothesis that $\alpha < 1$,

$$2^{q\sigma}\|R_q^1\|_{L^p} \lesssim \sum_{q'=q-4}^{q+4} 2^{q\sigma}\|\nabla S_{q'-1}a\|_{L^\infty}\|\Delta_{q'} w\|_{L^p}. \tag{10.8}$$

We have then:

$$2^{q\sigma}\|R_q^1\|_{L^p} \lesssim c_q \|\nabla a\|_{B_{\infty,\infty}^{\alpha-1}} \|w\|_{B_{p,r}^{\sigma+1-\alpha}}. \tag{10.9}$$

In the case $\alpha = 1$, we get:

$$2^{q\sigma}\|R_q^1\|_{L^p} \lesssim c_q \|\nabla a\|_{B_{\infty,1}^0} \|w\|_{B_{p,r}^{\sigma+1-\alpha}}. \tag{10.10}$$

For bounding R_q^2, standard continuity results for the paraproduct insure that if $\alpha < 1$, R_q^2 satisfies that:

$$2^{q\sigma}\|R_q^2\|_{L^p} \leqslant c_q \|\nabla a\|_{B_{\infty,\infty}^{\alpha-1}} \|w\|_{B_{p,r}^{\sigma+1-\alpha}}.$$

and if $\alpha = 1$

$$2^{q\sigma}\|R_q^2\|_{L^p} \leqslant c_q\|\nabla a\|_{B^{\alpha-1}_{\infty,1}}\|w\|_{B^{\sigma+1-\alpha}_{p,r}}.$$

Standard continuity results for the paraproduct insure that R_q^3 satisfies:

$$2^{q\sigma}\|R_q^3\|_{L^p} \lesssim c_q\|\nabla w\|_{B^{\sigma-\alpha-\frac{N}{p}}_{\infty,\infty}}\|a\|_{B^{\frac{N}{p}+\alpha}_{p,r}}. \tag{10.11}$$

provided $\sigma - \alpha - \frac{N}{p} < 0$.

If $\sigma - \alpha - \frac{N}{p} = 0$ then:

$$2^{q\sigma}\|R_q^3\|_{L^p} \lesssim c_q\|\nabla w\|_{B^0_{\infty,1}}\|a\|_{B^{\frac{N}{p}+\alpha}_{p,r}}. \tag{10.12}$$

Next, standard continuity result for the remainder insure that under the hypothesis $\sigma > -\frac{N}{p}$, we have:

$$2^{q\sigma}\|R_q^4\|_{L^p} \lesssim c_q\|\nabla w\|_{B^{\sigma-\alpha}_{p,r}}\|a\|_{B^{\frac{N}{p}+\alpha}_{p,\infty}}. \tag{10.13}$$

For bounding R_q^5 we use the decomposition:

$$R_q^5 = \sum_{q' \geqslant q-3} \partial_k(S_{q'+2}\Delta_q w \Delta_{q'} a),$$

which leads (after a suitable use of Bernstein and Hölder inequalities) to:

$$\begin{aligned} 2^{q\sigma}\|R_q^5\|_{L^p} &\lesssim \sum_{q' \geqslant q-3} 2^{q\sigma}2^{q'}\|\Delta_{q'} a\|_{L^\infty}\|S_{q'+2}\Delta_q w\|_{L^p} \\ &\lesssim \sum_{q' \geqslant q-2} 2^{(q-q')\left(\alpha+\frac{N}{p}-1\right)}2^{q(\sigma+1-\alpha)}\|\Delta_q w\|_{L^p}2^{q'\left(\frac{N}{p}+\alpha\right)}\|\Delta_{q'} a\|_{L^p}. \end{aligned}$$

Hence, since $\alpha + \frac{N}{p} - 1 > 0$, we have:

$$2^{q\sigma}\|R_q^5\|_{L^p} \lesssim c_q\|\nabla w\|_{B^{\sigma+1-\alpha}_{p,r}}\|a\|_{B^{\frac{N}{p}+\alpha}_{p,\infty}}.$$

Combining this latter inequality with (10.9), (10.11) and (10.13), and using the embedding $B^{\frac{N}{p}}_{p,r} \hookrightarrow B^{r-\frac{N}{p}}_{\infty,\infty}$ for $r = \frac{N}{p} + \alpha - 1$, $\sigma - \alpha$ completes the proof of (10.6).

The proof of (10.7) is almost the same: for bounding R_q^1, R_q^2, R_q^3 and R_q^5, it is just a matter of changing $\sum_q$ into $\sup_q$. We proceed similarly for R_q^4. □

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