Olivier BRUNAT

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ON SEMISIMPLE CLASSES AND SEMISIMPLE CHARACTERS IN FINITE REDUCTIVE GROUPS

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ABSTRACT. — In this article, we study the elements with disconnected centralizer in the Brauer complex associated to a simple algebraic group $G$ defined over a finite field with corresponding Frobenius map $F$ and derive the number of $F$-stable semisimple classes of $G$ with disconnected centralizer when the order of the fundamental group has prime order. We also discuss extendibility of semisimple characters of the fixed point subgroup $G^F$ to their inertia group in the full automorphism group. As a consequence, we prove that “twisted” and “untwisted” simple groups of type $E_6$ are “good” in defining characteristic, which is a contribution to the general program initialized by Isaacs, Malle and Navarro to prove the McKay Conjecture in representation theory of finite groups.

1. Introduction

This article is concerned with the semisimple characters of finite reductive groups. A finite reductive group is the fixed-point subgroup $G^F$ of a connected reductive group $G$ defined over the finite field $\mathbb{F}_q$ of characteristic $p > 0$, where $F: G \to G$ is the Frobenius map corresponding to this
\(\mathbb{F}_q\)-structure. The semisimple characters of \(G^F\) are the constituents of the duals of Gelfand-Graev characters (for the Alvis-Curtis duality) and play an important role in the ordinary representation theory of \(G^F\), because, apart from a few exceptions, they are the \(p'\)-characters of \(G^F\) (that is the irreducible characters of \(G^F\) whose degree is prime to \(p\)). In the following, we will write \(\text{Irr}_s(G^F)\) for the set of semisimple characters of \(G^F\). One of the aims of this work is to study these characters, compute their number, understand the action of the automorphism group of \(G^F\) on \(\text{Irr}_s(G^F)\), and determine the extendibility of \(\chi \in \text{Irr}_s(G^F)\) to its inertia group in the full automorphism group. These questions are crucial, for example in order to prove that \(G^F\) satisfies the inductive McKay condition at the prime \(p\).

Using Deligne-Lusztig theory [10], Lusztig has shown that the irreducible characters of \(G^F\) can be partitioned into series (the so-called rational Lusztig series) labelled by the semisimple classes of \(G^*F^*\), where \((G^*, F^*)\) denotes a pair dual to \((G, F)\). If such a series is labelled by a semisimple class of \(G^*F^*\) with representative \(s\), then it contains \(|A_{G^*}(s)^{F^*}|\) semisimple characters, where \(A_{G^*}(s) = C_{G^*}(s)/C_\circ G^*(s)\) is the component group of \(s\). In fact, \(\text{Irr}_s(G^F)\) can be parametrized in a natural way by pairs \((s, \xi)\) where \(s\) runs over a set of representatives of the semisimple classes of \(G^*F^*\) and \(\xi \in \text{Irr}(A_{G^*}(s)^{F^*})\). So, in order to understand \(\text{Irr}_s(G^F)\), we have to particularly consider the semisimple classes of \(G^*F^*\) and their component groups.

In [7], we explicitly computed the number of semisimple classes of \(G^*F^*\) when \(G^*\) is simple and \(p\) is a good prime for \(G^*\). For that, we used the theory of Gelfand-Graev characters for connected reductive groups with disconnected center, developed by Digne-Lehrer-Michel in [12] and [13]. This method gives a lot of information on the \(F^*\)-stable semisimple \(G^*\)-classes of \(G^*\) with disconnected centralizer (by the centralizer of an \(F^*\)-stable class, we mean the centralizer of a fixed \(F^*\)-stable representative), which allows us to prove the inductive McKay condition in defining characteristic for simple groups coming from simple algebraic groups with fundamental group of order 2; see [5, Theorem 1.1]. However, we cannot derive from [7] all information that we need, especially phenomena appearing only in the algebraic group, as for example the description of the \(F^*\)-stable semisimple classes with disconnected centralizer such that the fixed-point subgroup \(A_{G^*}(s)^{F^*}\) is trivial (\(s\) is any \(F^*\)-stable representative); see Remark 2.14.

In this work, we will consider the Brauer complex, initially introduced by J. Humphreys in [18] for describing \(p\)-modular representation theory of \(G^F\). When \(G\) is a simple simply-connected group, Deriziotis proved in [11]
that the $F$-stable semisimple classes of $G$ (and thus, the semisimple classes of $G^F$) are parametrized by the faces of the Brauer complex of maximal dimension. We generalize here some results to any simple algebraic groups using an approach of Bonnafé [2].

Moreover, we are interested in the problem of the extendibility of semisimple characters of $G^F$ to their inertia groups in $\text{Aut}(G^F)$. Digne-Michel [15] and Malle [21] developed a theory of Deligne-Lusztig characters for finite disconnected reductive groups. Using this theory, Sorlin [23] constructed extensions of Gelfand-Graev characters of $G^F$ to $G^F \rtimes \langle \sigma \rangle$, where $\sigma$ is a quasi-central semisimple or unipotent automorphism of $G$. We will use results of [23] in order to prove that, under certain assumptions, the semisimple characters of $G^F$ are extendible to their inertia groups in the full automorphism group.

Finally, recall that the McKay Conjecture asserts that for any finite group $G$, if $\text{Irr}_{p'}(G)$ denotes the set of $p'$-characters of $G$, then $|\text{Irr}_{p'}(G)| = |\text{Irr}_{p'}(N_G(P))|$, where $P$ is a fixed $p$-Sylow subgroup of $G$. In [20], Isaacs-Malle-Navarro proved a reduction theorem of this conjecture to finite simple groups. They showed that if every simple group satisfies a refined property, the so-called inductive McKay condition (see [20, §10] for more details), then the McKay Conjecture holds for all finite groups. As an application of our results, we will prove that “twisted” and “untwisted” finite simple groups of type $E_6$ satisfy the inductive McKay condition in defining characteristic.

The paper is organized as follows. In Section 2, we introduce the Brauer complex of $G$ and describe the faces containing points with disconnected centralizer; see Theorem 2.5. Then we compute the number of $F$-stable semisimple classes of $G$ with disconnected centralizer when $G$ is not of type $D_{2n}$ and has fundamental group of prime order; see Proposition 2.11. Note that this result requires no condition on $q$. Furthermore, if $G$ is not of type $D_{2n}$, we describe the $F$-stable points of the Brauer complex in the case that $F$ acts trivially on the center of $G$; see Propositions 2.16 and 2.17. As first consequences we prove that if $p$ is odd, then the McKay Conjecture holds for $G^F$ at the prime $p$, where $G$ is a simple and simply-connected group of type $D_{2n+1}$; see Remark 2.18. It also holds for $p = 2$ (resp. $p = 3$) when $G$ is a simple and simply-connected group of type $E_6$ (resp. $E_7$); see Remark 2.15. In Section 3, we recall the construction of semisimple characters and give the action of automorphisms of $G^F$ on $\text{Irr}_{s}(G^F)$. Then we discuss extendibility in a special case of these characters to their inertia groups; see Proposition 3.13. Finally, in Section 4 we prove
the inductive McKay condition in defining characteristic (for \( p > 3 \)) for "untwisted" simple groups of type \( E_6 \) in Theorem 4.9, and for the "twisted" version in Theorem 4.10. Note that these methods also prove that the inductive McKay condition is satisfied at the prime \( p \) by simple groups of Lie type of type \( A_{2n} \) such that \( 2n + 1 \) is a prime number distinct from \( p \); see Proposition 4.12.

It should be noted that recently, after the submission of this paper, using a work of Maslowski [22], Späth showed that all finite simple groups of Lie type are "good" for the defining characteristic; see [24].

2. Semisimple classes with disconnected centralizers

2.1. Notation

Throughout this paper, \( G \) denotes a simple algebraic group over \( \overline{\mathbb{F}}_p \). We fix a maximal torus \( T \) contained in a Borel subgroup \( B \) of \( G \). Let \( \Phi \) be the root system of \( G \) relative to \( T \). We write \( \Phi^+ \) and \( \Delta \) for the set of positive roots and the set of simple roots of \( \Phi \) corresponding to \( B \). We denote by \( X(T) \) and \( Y(T) \) the groups of characters and cocharacters of \( T \) and write \( \langle , \rangle : X(T) \times Y(T) \to \mathbb{Z} \) for the duality pairing between \( X(T) \) and \( Y(T) \). For \( \alpha \in \Phi \), we denote by \( \alpha^\vee \) the coroot of \( \alpha \) and write \( \Phi^\vee = \{ \alpha^\vee | \alpha \in \Phi \} \).

We define \( V = \mathbb{Q} \otimes_{\mathbb{Z}} Y(T) \) and \( V^* = \mathbb{Q} \otimes_{\mathbb{Z}} X(T) \), and extend \( \langle , \rangle \) to a nondegenerate bilinear form

\[
\langle , \rangle : V^* \times V \to \mathbb{Q}.
\]

Let \( G_{sc} \)(resp. \( G_{ad} \)) be the simply-connected version (resp. adjoint version) of \( G \). We denote by \( \pi_{sc} : G_{sc} \to G \) and \( \pi_{ad} : G \to G_{ad} \) corresponding isogenies. Write \( T_{sc} = \pi_{sc}^{-1}(T) \) and \( T_{ad} = \pi_{ad}(T) \). Then \( T_{sc} \) and \( T_{ad} \) are maximal tori of \( G_{sc} \) and \( G_{ad} \), and the surjective homomorphisms \( \pi_{sc} : T_{sc} \to T \) and \( \pi_{ad} : T \to T_{ad} \) induce injective homomorphisms \( \pi_{sc,X} : T(T) \to T(T_{sc}), \chi \to \chi \circ \pi_{sc} \) and \( \pi_{ad,X} : X(T_{ad}) \to X(T), \chi \to \chi \circ \pi_{ad} \). Using \( \pi_{sc,X} \) and \( \pi_{ad,X} \), we identify \( X(T) \) with a subgroup of \( X(T_{sc}) \) containing \( X(T_{ad}) \), such that the root systems \( \pi_{sc}(\Phi) \) and \( \pi_{ad}(\Phi) \) of \( G_{sc} \) and \( G_{ad} \) are identified with \( \Phi \). Similarly, using the injective morphisms \( \pi_{sc,Y} : Y(T_{sc}) \to Y(T), \gamma \to \pi_{sc} \circ \gamma \) and \( \pi_{ad,Y} : Y(T) \to Y(T_{ad}), \gamma \to \pi_{ad} \circ \gamma \), the group \( Y(T) \) is viewed as a subgroup of \( Y(T_{ad}) \) containing \( Y(T_{sc}) \) such that \( \pi_{sc,Y}(\Phi^\vee) \) and \( \pi_{ad,Y}(\Phi^\vee) \) are identified with \( \Phi^\vee \). Note that \( V = \mathbb{Q} \otimes_{\mathbb{Z}} Y(T_{sc}) = \mathbb{Q} \otimes_{\mathbb{Z}} Y(T_{ad}), V^* = \mathbb{Q} \otimes_{\mathbb{Z}} X(T_{sc}) = \mathbb{Q} \otimes_{\mathbb{Z}} X(T_{ad}) \).
Recall that \( \Lambda = \{ \lambda \in V^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \quad \forall \alpha \in \Phi \} \).

We define the group of weights by

\[
(2.1) \quad \Lambda = \{ \lambda \in V^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \quad \forall \alpha \in \Phi \}. 
\]

Recall that \( X(T_{ad}) = \mathbb{Z}\Phi \) and \( X(T_{sc}) = \Lambda \), and the fundamental group of \( \Phi \) is the finite group \( \Lambda/\mathbb{Z}\Phi \). Now, we denote by \((\varpi^\vee_\alpha)_{\alpha \in \Delta} \) and \((\varpi_\alpha)_{\alpha \in \Delta} \) the dual bases with respect to \( , \) of \( \Delta \) and \( \Delta^\vee = \{ \alpha^\vee \mid \alpha \in \Delta \} \), respectively. Since \( X(T_{ad}) = \mathbb{Z}\Phi \) and \( X(T_{sc}) = \{ \varpi_\alpha, \alpha \in \Delta \} \), we deduce that

\[
(2.2) \quad Y(T_{sc}) = \bigoplus_{\alpha \in \Delta} \mathbb{Z}\alpha^\vee \quad \text{and} \quad Y(T_{ad}) = \bigoplus_{\alpha \in \Delta} \mathbb{Z}\varpi^\vee_\alpha.
\]

We denote by \( W = N_G(T)/T \) the Weyl group of \( G \). Then \( W \) acts on \( X(T) \) and on \( Y(T) \) by

\[
(2.3) \quad w^\ast \chi(t) = \chi(t^w) \quad \text{and} \quad \gamma^w(t) = \gamma(t)^w,
\]

for \( \gamma \in Y(T) \), \( \chi \in X(T) \) and \( t \in T \). In particular, we have \( \langle w^\ast \chi, \gamma \rangle = \langle \chi, \gamma^w \rangle \) and \( W(\Phi) = \Phi \) and \( W(\Phi^\vee) = \Phi^\vee \). Recall from [9, §1.9] that \( W = \langle s_\alpha \mid \alpha \in \Phi \rangle \), and for \( \chi \in X(T) \) and \( \gamma \in Y(T) \), we have \( s_\alpha(\chi) = \chi - \langle \chi, \alpha^\vee \rangle \alpha \) and \( s_\alpha(\gamma) = \gamma - \langle \alpha, \gamma \rangle \alpha^\vee \).

### 2.2. Semisimple classes with disconnected centralizers

We denote by \( \mathbb{Q}_{p^r} \) the additive subgroup of \( \mathbb{Q} \) of rational numbers of the form \( a/b \) with \( b \) not divisible by \( p \), and write \( \tilde{i} : \mathbb{Q} \to \mathbb{F}_p^\times \) for the composition of the morphisms \( \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}_{p^r}/\mathbb{Z} \to \mathbb{F}_p^\times \), where the second morphism is obtained by killing the \( p \)-torsion subgroup of \( \mathbb{Q}/\mathbb{Z} \), and the last morphism is a fixed isomorphism between \( \mathbb{Q}_{p^r}/\mathbb{Z} \) and \( \mathbb{F}_p^\times \) as in [9, 3.1.3]. We set \( \tilde{i}_T : \mathbb{Q} \otimes_{\mathbb{Z}} Y(T) \to T, r \otimes \gamma \to \gamma(\tilde{i}(r)) \), which induces a bijective morphism \( \mathbb{Q}_{p^r}/\mathbb{Z} \otimes \mathbb{Q}_{p^r}/\mathbb{Z} Y(T) \to T \); see [9, 3.1.2, 3.1.3]. Furthermore, the surjective homomorphism from \( \mathbb{Q}_{p^r}/\mathbb{Z} \otimes \mathbb{Q}_{p^r}/\mathbb{Z} Y(T) \) onto \( \mathbb{Q}_{p^r}/\mathbb{Z} \otimes \mathbb{Q}_{p^r}/\mathbb{Z} Y(T) \) with kernel \( Y(T) \) induces the isomorphism of groups

\[
(2.4) \quad T \simeq \mathbb{Q}_{p^r}/\mathbb{Z} \otimes \mathbb{Q}_{p^r}/\mathbb{Z} Y(T) \simeq \mathbb{Q}_{p^r}/\mathbb{Z} \otimes \mathbb{Q}_{p^r}/\mathbb{Z} Y(T) / Y(T).
\]

Note that the action of \( W \) on \( Y(T) \) defined in Equation (2.3) can be naturally extended to \( \mathbb{Q}_{p^r}/\mathbb{Z} \otimes \mathbb{Q}_{p^r}/\mathbb{Z} Y(T) \) and to \( \mathbb{Q}_{p^r}/\mathbb{Z} \otimes \mathbb{Q}_{p^r}/\mathbb{Z} Y(T) / Y(T) \) and is compatible with the isomorphisms of Equation (2.4). We define \( \overline{W}_a = Y(T) \rtimes W \).

Note that \( \overline{W}_a \) acts on \( V \) as a group of affine transformations by

\[
(\gamma \cdot w)(\lambda \otimes \gamma') = \lambda \otimes \gamma'^w + \gamma,
\]
for $\gamma, \gamma' \in Y(T), \lambda \in \mathbb{Q}$ and $w \in W$. So, we have
\[
(Q_{p'} \otimes_{\mathbb{Z}} Y(T)/Y(T))/W = Q_{p'} \otimes_{\mathbb{Z}} Y(T)/W_a.
\]
Since the set of semisimple classes $s(G)$ of $G$ is in bijection with the set $T/W$ of $W$-orbits on $T$ (see [9, 3.7.1]), we deduce that $s(G)$ is in bijection with 
\[
(Q_{p'} \otimes_{\mathbb{Z}} Y(T))/W_a,
\]
Now, we write $\alpha_0$ for the highest root of $\Phi$ (with respect to the height defined by $\Delta$) and put 
\[
(2.6) \quad \alpha_0 = \sum_{\alpha \in \Delta} n_{\alpha} \alpha,
\]
with $n_{\alpha} \in \mathbb{N}^*$. We define the affine Weyl group $W_a$ of $V$ as the subgroup of affine transformations of $V$ generated by $s_{\alpha}$ (for $\alpha \in \Delta$) and $s_{\alpha_0,1} = s_{\alpha_0} + \alpha_0^\vee$. Then by [4, p.174], the alcove 
\[
\mathcal{C} = \{ \lambda \in V \mid \langle \lambda, \alpha \rangle \geq 0 \forall \alpha \in \Delta, \langle \alpha_0, \lambda \rangle \leq 1 \}
\]
is a fundamental domain for the action of $W_a$ on $V$. Recall that $W_a = Y(T_{sc}) \rtimes W$ (see [4, VI §2, Prop 1]). In particular, $W_a \leq W_a^\prime$, which implies that every $W_a$-orbit of $V$ contains at least one element of $\mathcal{C}$. We write 
\[
\tilde{\Delta} = \Delta \cup \{-\alpha_0\}
\]
for the affine Dynkin diagram of $G$ and $\tilde{\Delta}_{min} = \{ \alpha \in \tilde{\Delta} | n_{\alpha} = 1 \}$, with the convention that $n_{-\alpha_0} = 1$. For $\alpha \in \tilde{\Delta}_{min}$, we set $z_{\alpha} = w_{\alpha} w_0$, where $w_0$ and $w_{\alpha}$ are the longest elements of $W$ and $W_{\Delta \setminus \{a\}}$, respectively (note that $w_{-\alpha_0} = w_0$ and $z_{-\alpha_0} = 1$). Then $z_{\alpha}$ induces an automorphism of the extended Dynkin diagram $\tilde{\Delta}$. We define 
\[
\mathcal{A} = \{ z_{\alpha} \mid \alpha \in \tilde{\Delta}_{min} \}.
\]
Recall that $\mathcal{A}$ is isomorphic to the center $Z(G_{sc})$ as follows. By [4, Corollaire p.177], we have $Z(G_{sc}) = \{ i_T(\varpi_{\alpha}^\vee) \mid \alpha \in \tilde{\Delta}_{min} \}$. Moreover, for $z \in Z(G_{sc})$, there is a unique element $w_z \in W$ (obtained as the projection on $W$ of any element $w$ of $W_a$ satisfying $\omega(\mathcal{C}) = y + \mathcal{C}$ for $y \in Q_{p'} \otimes Y(T)$ such that $i_T(y) = z$; note that $w_z$ is well-defined because it does not depend on $y$). Then [4, Proposition 6 p.176] implies that $z_{\alpha} = w_{i_T(\varpi_{\alpha}^\vee)}$ for $\alpha \in \tilde{\Delta}_{min}$. Furthermore, the map 
\[
(2.7) \quad Z(G_{sc}) \to \mathcal{A}, \quad i_T(\varpi_{\alpha}^\vee) \mapsto z_{\alpha},
\]
is an isomorphism of groups; see [4, VI §2.3].

Now, we write $\widehat{W}_a = Y(T_{ad}) \rtimes W$ and $\Gamma_{\mathcal{C}}$ for the subgroup of $\widehat{W}_a$ which stabilizes $\mathcal{C}$. Then in [4, VI §2 Prop 6], the following result is proven.
Proposition 2.1. — The group of automorphisms of \( \tilde{\Delta} \) induced by elements of \( W \) is \( A \). For \( \alpha \in \tilde{\Delta}_{\text{min}} \), we set \( f_\alpha = z_\alpha + \varpi_\alpha \). Then \( f_\alpha \in \tilde{W}_a \) and we have \( \Gamma_{\varphi} = \{ f_\alpha \mid \alpha \in \tilde{\Delta}_{\text{min}} \} \).

Moreover, the map
\[
\varpi^\vee : A \to Y(T_{\text{ad}})/Y(T_{\text{sc}}), \quad z_\alpha \mapsto \varpi_\alpha + Y(T_{\text{sc}}),
\]
is an isomorphism of groups.

Note that, by composition of \( \varpi^\vee \) with the isomorphism defined in Equation (2.7), we can identified the quotient \( Y(T_{\text{ad}})/Y(T_{\text{sc}}) \) with \( Z(G_{\text{sc}}) \).

For \( \lambda \in \mathcal{C} \), we will denote by \( (\lambda_\alpha)_{\alpha \in \tilde{\Delta}} \) its affine coordinates, that is, the unique family of rational numbers such that \( \sum_{\alpha \in \tilde{\Delta}} \lambda_\alpha = 1 \) and \( \lambda = \sum_{\alpha \in \tilde{\Delta}} n_\alpha \varpi_\alpha \), where \( n_\alpha \) are the integers defined in Equation (2.6) and \( \varpi_{-\alpha} = 0 \). Note that \( \lambda \in \mathcal{C} \) if and only if \( \lambda_\alpha \geq 0 \) for every \( \alpha \in \tilde{\Delta} \); see [4, Corollaire p.175].

Now, following [2], we define the subgroup \( A_G \) of \( A \) to be the inverse image of \( Y(T_{\text{ad}})/Y(T_{\text{sc}}) \) under \( \varpi^\vee \). We also define
\[
(2.8) \quad \tilde{\Delta}_{\text{min},G} = \{ \alpha \in \tilde{\Delta}_{\text{min}} \mid z_\alpha \in A_G \} \quad \text{and} \quad \Gamma_G = \{ f_\alpha \mid \alpha \in \tilde{\Delta}_{\text{min},G} \}.
\]
Then Bonnafé proves in [2, (3.10), Cor. 3.12, Prop. 3.13]

Theorem 2.2. —

1. For \( \alpha \in \tilde{\Delta}_{\text{min}} \) and \( \lambda = (\lambda_\beta)_{\beta \in \tilde{\Delta}} \in \mathcal{C} \), we have \( f_\alpha(\lambda) = \lambda_{z_\alpha^{-1}(\beta)} \) for \( \beta \in \tilde{\Delta} \).

2. The points \( \lambda, \mu \in \mathcal{C} \) are in the same \( \tilde{W}_a \)-orbit if and only if there is \( z \in A_G \) such that \( z(\lambda) - \mu \in Y(T) \).

3. Let \( [s]_G \in s(G) \) be a semisimple class with representative \( s \in G \) corresponding to a \( \tilde{W}_a \)-orbit \( [\lambda] \) (here, \( \lambda \) denotes a representative in \( \mathcal{C} \)) on \( \mathcal{C} \). Then \( I_\lambda = \{ \alpha \in \tilde{\Delta} \mid \lambda_\alpha = 0 \} \) is a basis of the root system of \( C_G(s)^\circ \) and the component group \( A_G(s) = C_G(s)/C_G(s)^\circ \) is isomorphic to
\[
A_G(\lambda) = \{ z \in A_G \mid \forall \alpha \in \tilde{\Delta}, \lambda_{z(\alpha)} = \lambda_\alpha \}.
\]

For \( \alpha \in \tilde{\Delta}_{\text{min}} \), we write
\[
(2.9) \quad V_\alpha = \{ v \in V \mid f_\alpha(v) = v \}
\]
for the affine subspace of \( V \) of the invariants under \( f_\alpha \).
LEMMA 2.3. — Let $\alpha, \alpha' \in \tilde{\Delta}_{\mathrm{min}}$. If $\langle z_\alpha \rangle = \langle z_{\alpha'} \rangle$, then $V_\alpha = V_{\alpha'}$. Moreover, if $\Omega_\alpha$ denotes the set of $\langle z_\alpha \rangle$-orbits on $\tilde{\Delta}$, then $\dim(V_\alpha) = |\Omega_\alpha| - 1$.

In Table 2.1, we explicitly give the dimension of $V_\alpha$ for every irreducible extended affine Dynkin diagram. For the notation, we use the labelling of Bourbaki [4, Planche I-IX]. In particular, note that $z_{\alpha_i}(-\alpha_0) = \alpha_i$.

| Type     | $|A|$ | $A$            | $z_\alpha$ | $\dim(V_\alpha)$ |
|----------|------|----------------|------------|------------------|
| $A_n$    | $n + 1$ | $\langle z_{\alpha_1} \rangle$ | $d|n + 1, z_{\alpha_1}^d$ | $d - 1$ |
| $B_n$    | 2    | $\langle z_{\alpha_1} \rangle$ | $z_{\alpha_1}$ | $n - 1$ |
| $C_n$    | 2    | $\langle z_{\alpha_n} \rangle$ | $z_{\alpha_n}$ | $\lfloor \frac{n}{2} \rfloor$ |
| $D_{2n+1}$ | 4    | $\langle z_{\alpha_n} \rangle$ | $z_{\alpha_n}, z_{\alpha_n-1} = z_{\alpha_n}^{-1}$ | $n - 1$ |
| $D_{2n}$ | 4    | $\langle z_{\alpha_n-1} \rangle \times \langle z_{\alpha_n} \rangle$ | $z_{\alpha_n}, z_{\alpha_n-1}$ | $n$ |
| $E_6$    | 3    | $\langle z_{\alpha_1} \rangle$ | $z_{\alpha_1}$ | $z_{\alpha_1}^{-1}$ | 2 |
| $E_7$    | 2    | $\langle z_{\alpha_1} \rangle$ | $z_{\alpha_1}$ | 4 |
| $E_8$    | 1    | $\{1\}$ | 1 | 8 |
| $G_2$    | 1    | $\{1\}$ | 1 | 2 |
| $F_4$    | 1    | $\{1\}$ | 1 | 4 |

Table 2.1. The dimension of the invariant subspace

2.3. Fixed-points under a Frobenius map

Now, we suppose that $G$ is defined over the finite field $\mathbb{F}_q$ (for $q$ a $p$-power) and we denote by $F : G \to G$ the corresponding Frobenius map. We assume that $F$ is the composition of a split Frobenius with a graph automorphism coming from a symmetry $\rho$ of $\Delta$ (which could be trivial). We suppose that the maximal torus $T$ of $G$ is chosen to be $F$-stable and
contained in an $F$-stable Borel subgroup $B$. Moreover, $F$ induces Frobenius maps on $G_{sc}$ and $G_{ad}$ (also denoted by $F$) such that $F$ and the isogenies $\pi_{sc}$ and $\pi_{ad}$ commute. We can define an $F$-action on $X(T)$ and $Y(T)$ by setting

$$F(\chi)(t) = \chi(F(t)) \quad \text{and} \quad F(\gamma)(x) = F(\gamma(x)),$$

for $\chi \in X(T)$, $\gamma \in Y(T)$, $t \in T$ and $x \in \mathbb{F}_p^X$. The $F$-action on $Y(T)$ extends naturally to $\mathbb{Q}_p'/\mathbb{Z} \otimes_{\mathbb{Z}} Y(T)$ and to $(\mathbb{Q}_p' \otimes_{\mathbb{Z}} Y(T))/Y(T)$ and is compatible with the isomorphisms of Equation (2.4). Since the set of $F$-stable semisimple classes of $G$ is in bijection with the set $(T/W)^F$; see [9, 3.7.2], it follows that it is in bijection with the set $(\mathbb{Q}_p' \otimes_{\mathbb{Z}} Y(T)/W_a)^F$.

We put $W_{a,q} = F^{-1}(Y(T)) \rtimes W$. Then the map

$$F : W_{a,q} \rightarrow W_a, \ yw \mapsto F(y)F(w),$$

is an isomorphism of groups. Note that $W_a$ is a subgroup of $W_{a,q}$. Furthermore, for $v \in V$ and $g \in W_{a,q}$, we have

$$(2.10) \quad F(g)(F(v)) = F(g(v)).$$

We define $W_{a,q} = F^{-1}(Y(T_{sc})) \rtimes W$ and $\hat{W}_{a,q} = F^{-1}(Y(T_{ad})) \rtimes W$. In particular, we have $W_{a,q} \leq W_{a,q} \leq \hat{W}_{a,q}$. Moreover, Equation (2.10) implies that the set

$$\mathcal{C}_q = \{v \in V \mid F(v) \in \mathcal{C}\}$$

is a fundamental region for the $W_{a,q}$-action on $V$. We denote by $h_q : V \rightarrow V$ the homothety with origin 0 and ratio $\frac{1}{q}$. Recall that $F$ acts on $V$ by $h_q^{-1}F_0$, where $F_0 : V \rightarrow V$ is a linear transformation defined by $F_0(\alpha) = \rho^{-1}(\alpha)$ for all $\alpha \in \Delta$. Note that $V$ can be regarded as a euclidean space on which the elements of $\hat{W}_a$ and $F_0$ act as isometries. In the following, we will denote by $d_0$ the associated metric. Since $\langle F_0(\chi), \gamma \rangle = \langle \chi, F_0(\gamma) \rangle$, we deduce that $F_0(\varpi^\vee_\alpha) = \varpi^\vee_{\rho^{-1}(\alpha)}$ for every $\alpha \in \Delta$. For $\alpha \in \Delta$, we set

$$\varpi^\vee_{\alpha,q} = F^{-1}(\varpi^\vee_\alpha) = \frac{1}{q} \cdot \varpi^\vee_{\rho(\alpha)}.$$ 

Note that $n_{\rho^{-1}(\alpha)} = n_\alpha$. Hence, $\tilde{\Delta}_{\min}$ is $\rho^{-1}$-stable. For $\alpha \in \tilde{\Delta}_{\min}$, we define

$$f_{\alpha,q} = z_{\rho(\alpha)} + \varpi^\vee_{\alpha,q}.$$
Lemma 2.4. — Let $\alpha \in \tilde{\Delta}_{\min}$. Then the following diagram is commutative.

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{f_\alpha} & \mathcal{C} \\
\downarrow F^{-1} & & \downarrow F^{-1} \\
\mathcal{C}_q & \xrightarrow{f_{\alpha,q}} & \mathcal{C}_q
\end{array}
\]

Proof. — For every $\alpha \in \tilde{\Delta}_{\min}$, we set $I_\alpha = \Delta \setminus \{\alpha\}$ and $\Phi_\alpha = W_{I_\alpha}(I_\alpha)$ the corresponding root subsystem with basis $I_\alpha$. Since $f_{-\alpha_0} = f_{-\alpha_0,q} = 1$, the lemma is trivially true for $-\alpha_0$. Fix now $\alpha \in \Delta$ with $n_\alpha = 1$. For every $x \in V$, we have

$$F^{-1}f_\alpha F(x) = \frac{1}{q} \cdot w_{\rho(\alpha)}^\vee + F^{-1}_0 z_\alpha F_0(x).$$

Moreover, we have $\rho(I_\alpha) = I_{\rho_\alpha}$ and $\Phi_{\rho(\alpha)} = \rho(\Phi_\alpha)$. If $w_\alpha$ and $w_{\rho(\alpha)}$ are the longest elements of $W_{I_\alpha}$ and $W_{I_{\rho(\alpha)}}$, then $w_{\rho(\alpha)} = \rho(w_\alpha)$. Indeed, for every $\beta \in \Delta$, we have $s_{\rho(\beta)} = F^{-1}_0 s_\beta F_0$ which implies that $\rho(w_\alpha) = F^{-1}_0 w_\alpha F_0$. Now, if $\beta \in \Phi^+_{\rho(\alpha)}$, then $\rho(w_\alpha)(\beta) = F^{-1}_0 (-F_0(\beta)) = -\beta$, as required. So, we have $F^{-1}_0 z_\alpha F_0 = z_{\rho(\alpha)}$, and the result follows. \[\Box\]

Note that $W_{a,q}$ is the affine Weyl group generated by $s_\alpha$ for $\alpha \in \Delta$ and by the affine reflection $s_{\alpha_0,1/q} = s_{\alpha_0} + \frac{1}{q} \alpha_0^\vee$. We denote by $\mathcal{H}_q$ the collection of all hyperplanes defined by the affine reflections of $W_{a,q}$. Moreover, $W_a$ is a subgroup of $W_{a,q}$. It follows that $\mathcal{C}$ is a union of certain transforms of $\mathcal{C}_q$ under $W_{a,q}$. We write $E_q$ for the set of elements $\omega \in W_{a,q}$ such that $\omega(\mathcal{C}_q) \subseteq \mathcal{C}$ and define

\[(2.11) \quad \Omega_q = \{\omega(\mathcal{C}_q) \mid \omega \in E_q\}.\]

We now can prove

Theorem 2.5. — Let $\alpha \in \tilde{\Delta}_{\min}$. We define

$$M_{\alpha,q} = \{\omega \in E_q \mid f_\alpha(\omega(\mathcal{C}_q)) = \omega(\mathcal{C}_q)\}.$$

Let $V_\alpha$ be the subspace of invariants as in Equation (2.9). If $V_\alpha$ is contained in some hyperplane of $\mathcal{H}_q$, then $|M_{\alpha,q}| = 0$. Otherwise, we have $|M_{\alpha,q}| = q^{\dim V_\alpha}$.

Proof. — Since $\hat{W}_a$ is a subgroup of $\hat{W}_{a,q}$, it follows that $f_\alpha \in \hat{W}_{a,q}$. In particular, $f_\alpha$ permutes the elements of $\mathcal{H}_q$ and also the set of alcoves $\omega(\mathcal{C}_q)$ for $\omega \in W_{a,q}$. Hence, $f_\alpha$ permutes the elements of $\Omega_q$ (because for
\( \omega \in E_q \), we have \( f_\alpha(\omega(\mathcal{C}_q)) \subseteq \mathcal{C} \). We denote by \( r_\alpha \in W_{a,q} \) the element such that

\[
(2.12) \quad r_\alpha(\mathcal{C}_q) = f_\alpha(\mathcal{C}_q).
\]

Let \( \omega \in E_q \). Then we have

\[
\omega(\mathcal{C}_q) = f_\alpha(\omega(\mathcal{C}_q)) = f_\alpha \omega f_\alpha^{-1}(f_\alpha(\mathcal{C}_q)) = f_\alpha \omega f_\alpha^{-1}r_\alpha(\mathcal{C}_q),
\]

Furthermore, the group \( \hat{W}_{a,q} \) normalizes \( W_{a,q} \), which implies that \( f_\alpha \omega f_\alpha^{-1} \in W_{a,q} \). In particular, we have \( f_\alpha \omega f_\alpha^{-1}r_\alpha \in W_{a,q} \). However, by [4, VI. §2.1], the group \( W_{a,q} \) acts simply transitively on the set of alcoves \( \{ \omega(\mathcal{C}_q) \mid \omega \in W_{a,q} \} \). It follows that \( f_\alpha \omega f_\alpha^{-1}r_\alpha = \omega \), which implies

\[
(2.13) \quad \omega^{-1}f_\alpha \omega = r_\alpha^{-1}f_\alpha
\]

Note that \( r_\alpha^{-1}f_\alpha(\mathcal{C}_q) = \mathcal{C}_q \) and \( r_\alpha^{-1}f_\alpha \in \hat{W}_{a,q} \). Proposition 2.1 implies that there is \( \tilde{\alpha} \in \Delta_{\text{min}} \) such that \( f_{\tilde{\alpha},q} = r_\alpha^{-1}f_\alpha \). We define

\[
(2.14) \quad m : \tilde{\Delta}_{\text{min}} \to \tilde{\Delta}_{\text{min}}, \alpha \mapsto \tilde{\alpha}.
\]

Hence, the following diagram is commutative

\[
\begin{array}{ccc}
\mathcal{C}_q & \xrightarrow{\omega} & \omega(\mathcal{C}_q) \\
\downarrow f_{\tilde{\alpha},q} & & \downarrow f_\alpha \\
\mathcal{C}_q & \xrightarrow{\omega} & \omega(\mathcal{C}_q)
\end{array}
\]

For \( A \subseteq V \) and \( f : V \to V \), we denote by \( A^f \) the subset of elements of \( A \) invariant under \( f \). So, Equation (2.13) implies that \( \omega : \mathcal{C}_q^{f_{\tilde{\alpha},q}} \to \omega(\mathcal{C}_q)^{f_\alpha} \) is bijective. Thus the sets \( \mathcal{C}_q^{f_{\tilde{\alpha},q}} \) and \( \omega(\mathcal{C}_q)^{f_\alpha} \) are isometric. Let \( x \in V_\alpha \). Suppose that \( x \) lies in the interior of some \( \omega(\mathcal{C}_q) \) for some \( \omega \in E_q \). Then \( x = f_\alpha(x) \) lies in the interior of \( f(\omega(\mathcal{C}_q)) \), which implies that \( f_\alpha(\omega(\mathcal{C}_q)) = \omega(\mathcal{C}_q) \). Conversely, if \( \omega(\mathcal{C}_q) \) is \( f_\alpha \)-invariant, then the interior of \( \omega(\mathcal{C}_q) \) contains elements of \( V_\alpha \) (for example, the isobarycentre of the simplex \( \omega(\mathcal{C}_q) \), because \( f_\alpha \) is an affine map). It follows that \( M_{a,q} = \emptyset \) if and only if \( V_\alpha \) is contained in some hyperplane of \( \mathcal{H}_q \).

Suppose that \( V_\alpha \) is not contained in some hyperplane of \( \mathcal{H}_q \). For \( H \in \mathcal{H}_q \), consider the affine space \( V_\alpha \cap H \). Then \( \dim(V_\alpha \cap H) < \dim V_\alpha \) (otherwise,
$V_{\alpha}$ would be contained in $H$). This implies that

$$\text{(2.15)} \quad \text{Vol}(\mathcal{C}^{f_{\alpha}}) = \text{Vol}\left(\mathcal{C}^{f_{\alpha}} \setminus \bigcup_{H \in H_q, H \cap \mathcal{C} \neq \emptyset} H\right).$$

Furthermore, we have

$$\text{(2.16)} \quad \mathcal{C}^{f_{\alpha}} \setminus \bigcup_{H \in H_q, H \cap \mathcal{C} \neq \emptyset} H = \bigcup_{\omega \in M_{\alpha,q}} \omega^\circ(\mathcal{C}_q^{f_{\alpha}}).$$

Since $\text{Vol}(\omega^\circ(\mathcal{C}_q^{f_{\alpha}})) = \text{Vol}(\omega^\circ(\mathcal{C}_q^{f_{\alpha}}))$ for every $\omega \in M_{\alpha,q}$, Equations (2.15) and (2.16) imply that

$$\text{(2.17)} \quad \text{Vol}(\mathcal{C}^{f_{\alpha}}) = \sum_{\omega \in M_{\alpha,q}} \text{Vol}(\omega(\mathcal{C}_q^{f_{\alpha}})).$$

Moreover, the sets $\mathcal{C}^{f_{\alpha}}$ and $\omega(\mathcal{C}_q^{f_{\alpha}})$ are isometric. Thus they have the same volume, and Equation (2.17) implies that

$$\text{(2.18)} \quad |M_{\alpha,q}| = \frac{\text{Vol}(\mathcal{C}^{f_{\alpha}})}{\text{Vol}(\mathcal{C}_q^{f_{\alpha}})}.$$

Thanks to Lemma 2.4, we have $F(\mathcal{C}_q^{f_{\alpha}}) = \mathcal{C}^{f_{\tilde{\alpha}}}$. By Equation (2.13) and Lemma 2.4, $f_{\alpha}$ and $f_{\tilde{\alpha}}$ are conjugate (in the group of affine transformations of $V$). Hence, $z_{\alpha}$ and $z_{\tilde{\alpha}}$ have the same order. If $G$ is not of type $D_{2n}$, then $A$ is cyclic. This implies that $\langle z_{\alpha} \rangle = \langle z_{\tilde{\alpha}} \rangle$ and thanks to Lemma 2.3, we have $V_{\alpha} = V_{\tilde{\alpha}}$. Hence, we have $\mathcal{C}^{f_{\alpha}} = \mathcal{C}^{f_{\tilde{\alpha}}}$. If $G$ is of type $D_{2n}$, then the invariant subspaces of $f_{\alpha}$ and $f_{\tilde{\alpha}}$ have the same dimension (because $f_{\alpha}$ and $f_{\tilde{\alpha}}$ are conjugate). Table 2.1 implies that there is $i \geq 0$ such that $\tilde{\alpha} = \rho^i(\alpha)$. Then $f_{\tilde{\alpha}} = F_0^{-i}f_\alpha F_0^i$, and $\mathcal{C}^{f_{\tilde{\alpha}}}$ and $\mathcal{C}^{f_{\alpha}}$ are isometric. We have proven that

$$\text{Vol}\left(F(\mathcal{C}_q^{f_{\tilde{\alpha}}})\right) = \text{Vol}(\mathcal{C}^{f_{\alpha}}).$$

Since $F = h_q^{-1}F_0$, we deduce that $\text{Vol}\left(F(\mathcal{C}_q^{f_{\tilde{\alpha}}})\right) = q^{\dim(V_{\alpha})} \text{Vol}\left(\mathcal{C}_q^{f_{\tilde{\alpha}}}ight)$, and it follows that $\text{Vol}\left(\mathcal{C}_q^{f_{\alpha}}\right) = \frac{1}{q^{\dim(V_{\alpha})}} \text{Vol}(\mathcal{C}^{f_{\alpha}})$. Now, the result follows from Equation (2.18). □

Remark 2.6. — If $V_{\alpha}$ is contained in some hyperplane $H$ of $H_q$, then $H$ is not a wall of $\mathcal{C}$. Indeed, the walls of $\mathcal{C}$ are the hyperplanes $H_\beta = \{\lambda \in V \mid \langle \beta, \lambda \rangle = m_\beta\}$ for $\beta \in \tilde{\Delta}$, where $m_\beta = 0$ for $\beta \in \Delta$ and $m_{-\alpha_0} = -1$. 

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Let $\beta \in \tilde{\Delta}$. We then define the element $\lambda^\alpha$ by setting $\lambda^\alpha_{z_{\alpha}(\beta)} = \frac{1}{|\langle (z_{\alpha(\beta)} ) \rangle |}$ for all $i \geq 0$ and $\lambda^\alpha_0 = 0$ for $\gamma \notin \langle (z_{\alpha(\beta)} ) \rangle \cdot \beta$. Note that $\lambda^\alpha \in \mathcal{C}$ and $f_\alpha(\lambda^\alpha) = \lambda^\alpha$.

Moreover, $\langle \beta, \lambda^\alpha \rangle = m_\beta + \frac{1}{n_\beta |\langle (z_{\alpha(\beta)} ) \rangle |} \neq m_\beta$, which implies that $\lambda^\alpha \notin H_\beta$.

**Lemma 2.7.** — Let $\lambda \in \omega(\mathcal{C}_q)$ for some $\omega \in E_q$. We suppose that $\lambda \notin \tilde{\Delta}_{\text{min}}$. Then, $\lambda$ lies in an $F$-stable $\bar{W}_a$-orbit if and only if there is $\alpha \in \tilde{\Delta}_{\text{min}}$ with $z_\alpha \in A_G$ such that $F(\lambda) = F(\omega)f_\alpha(\lambda)$.

**Proof.** — Since $F(\omega(\mathcal{C}_q))$ is an alcove for $W_a$, there is a unique $v \in W_a$ such that $vF(\omega(\mathcal{C}_q)) = \mathcal{C}$. In particular, $vF(\lambda) \in \mathcal{C}$. If $F(\lambda)$ and $\lambda$ lie in the same $\bar{W}_a$-orbit, then $vF(\lambda)$ and $\lambda$ too. Then, by Theorem 2.2(2), there is $z_\alpha \in A_G$ (for $\alpha \in \tilde{\Delta}_{\text{min}}$) such that $vF(\lambda) - z_\alpha(\lambda) \in Y(T)$. Since $\omega_\alpha^\vee \in Y(T)$, we deduce that $vF(\lambda) - f_\alpha(\lambda) \in Y(T)$. But $vF(\lambda)$ and $f_\alpha(\lambda)$ lie in $\mathcal{C}$. We denote by $(\mu_\beta)_{\beta \in \Delta}$ and $(\lambda'_\beta)_{\beta \in \Delta}$ the affine coordinate of $vF(\lambda)$ and $f_\alpha(\lambda)$. We have

$$vF(\lambda) - f_\alpha(\lambda) = \sum_{\beta \in \Delta} \frac{\mu_\beta - \lambda'_\beta}{n_\beta} \omega_\beta^\vee.$$ 

We have $0 \leq \mu_\beta \leq 1$ and $0 \leq \lambda'_\beta \leq 1$ for all $\beta \in \Delta$ (because $vF(\lambda)$ and $f_\alpha(\lambda)$ lie in $\mathcal{C}$). Thus, for every $\beta \in \Delta$, we have

$$-\frac{1}{n_\beta} \leq \frac{\mu_\beta - \lambda'_\beta}{n_\beta} \leq \frac{1}{n_\beta}.$$

Since $Y(T) \leq Y(T_{\text{ad}})$, we deduce from Equation (2.2) that $\mu_\beta = \lambda'_\beta$ if $\beta \notin \tilde{\Delta}_{\text{min}}$. Suppose that $\beta \in \tilde{\Delta}_{\text{min}}$. Then $n_\beta = 1$ and $\mu_\beta - \lambda'_\beta \in \{-1, 0, 1\}$. If $\mu_\beta - \lambda'_\beta \neq 0$, then we can assume that $\mu_\beta = 1$ and $\lambda'_\beta = 0$. Then $\mu_\beta = 0$ for all $\beta' \neq \beta$, and $vF(\lambda) = \omega_\beta^\vee$. Similarly, we deduce that $f_\alpha(\lambda) = \omega_\beta^\vee$ for some $\beta' \in \tilde{\Delta}_{\text{min}}$ (because $f_\alpha(\lambda) \neq vF(\lambda)$, which implies that there is $\beta' \in \tilde{\Delta}_{\text{min}}$ with $\lambda'_\beta'(1) = 1$). It follows that $\lambda = \omega_{z_{\alpha(\beta')}}^\vee \in \tilde{\Delta}_{\text{min}}$.

So, if we assume that $\lambda \notin \tilde{\Delta}_{\text{min}}$, we deduce that $vF(\lambda) = f_\alpha(\lambda)$. Moreover, $vF(\omega(\mathcal{C}_q)) = \mathcal{C}$ implies that $F(\omega)(\mathcal{C}) = v^{-1}(\mathcal{C})$. Since $F(\omega) \in W_a$, it follows from [4, VI.§2.1] that $F(\omega) = v^{-1}$. Conversely, if $F(\lambda) = F(\omega)f_\alpha(\lambda)$, then $F(\omega^{-1})F(\lambda)$ and $f_\alpha(\lambda)$ are in $\mathcal{C}$. So, we have

$$F(\omega^{-1})F(\lambda) - z_\alpha(\lambda) = -\omega_\alpha^\vee \in Y(T),$$

and the result comes from Theorem 2.2(2). \hfill \Box

**Lemma 2.8.** — Suppose that $A$ is cyclic. For $\alpha \in \tilde{\Delta}_{\text{min}}$, $\alpha \neq -\alpha_0$ and $\omega \in M_{\alpha,q}$, if $\lambda \in \omega(\mathcal{C}_q)$ lies in an $F$-stable $\bar{W}_a$-orbit, then $\lambda \in V_a$. 

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Proof. — Suppose there is $\varpi_\beta \in \omega(\mathcal{C}_q)$ with $\beta \in \tilde{\Delta}_{\min}$. Then $f_\alpha(\varpi_\beta) = \varpi_\beta$. Moreover, $\varpi_\beta = f_\beta(0)$ and $f_\alpha(0) = \varpi_\alpha$. It follows that

$$f_\beta(\varpi_\alpha) = f_\beta f_\alpha(0) = f_\alpha f_\beta(0) = f_\alpha(\varpi_\beta) = \varpi_\beta = f_\beta(0),$$

which implies that $\varpi_\alpha = 0$, i.e. $\alpha = -\alpha_0$.

So, this proves that $\lambda \not\in \tilde{\Delta}_{\min}$ and by Lemma 2.7, there is $\beta \in \tilde{\Delta}_{\min}$ such that $\omega F^{-1} f_\beta(\lambda) = \lambda$. Then $\lambda$ is a fixed-point of the map $\omega F^{-1} f_\beta : \mathcal{C} \to \omega(\mathcal{C}_q)$, which is a contraction map with rapport $1/q$ with respect to the metric $d_0$. By the contraction mapping theorem, $\lambda$ is the unique fixed-point of this map. We write $\tilde{\alpha} = m(\alpha)$ where $m$ is the map defined in Equation (2.14). Then, using Lemma 2.4 and Equation (2.13), we deduce that

$$\omega F^{-1} f_\beta(\alpha \varpi) = \omega F^{-1} f_\beta f_\alpha(\lambda) = \omega F^{-1} f_\beta,$$

In particular, we have $\omega F^{-1} f_\beta(\mathcal{C}_\alpha^\tilde{\alpha}) = \omega(\mathcal{C}_q)^\tilde{\alpha}$. Moreover, in the proof of Theorem 2.5 we have seen that $\mathcal{A}$ cyclic implies that $\mathcal{C}_\alpha^\tilde{\alpha} = \mathcal{C}^\tilde{\alpha}$. Hence, it follows that

$$\omega F^{-1} f_\beta(\mathcal{C}_\alpha^\tilde{\alpha}) \subseteq \mathcal{C}^\tilde{\alpha}.$$ 

So, $\lambda$ is the limit of the sequence defined by $x_0 \in \mathcal{C}^\tilde{\alpha}$ and $x_k = \omega F^{-1} f_\beta(x_{k-1})$ for $k \geq 1$. Since $\mathcal{C}^\tilde{\alpha}$ is closed, we deduce that $\lambda \in \mathcal{C}^\tilde{\alpha}$, as required. \qed

The proof of Lemma 2.8 shows that the map $\omega F^{-1} f_\alpha : \mathcal{C} \to \omega(\mathcal{C}_q)$ for $\omega \in E_q$ and $\alpha \in \tilde{\Delta}_{\min}$ has a unique fixed-point, denoted by $\lambda_{\omega,\alpha}$ in the following. Moreover, we define

$$(2.19) \quad S_q,\alpha = \{\lambda_{\omega,\alpha} \mid \omega \in E_q\}.$$ 

To simplify, we will write $S_q,\alpha_0 = S_q$ and $\lambda_{\omega,-\alpha_0} = \lambda_\omega$.

**Lemma 2.9.** — Let $\alpha \in \tilde{\Delta}_{\min}$. If $m : \tilde{\Delta}_{\min} \to \tilde{\Delta}_{\min}$ denotes the map defined in Equation (2.14), then $\varpi_\alpha^\gamma \in S_q$ if and only if $m(\alpha) = \alpha$. Moreover, if $\varpi_\alpha^\gamma \in S_q$, then for $\omega \in E_q$ and $\beta \in \tilde{\Delta}_{\min}$ we have

$$f_\alpha(\lambda_{\omega,\beta}) = \lambda_{\omega',\beta},$$

where $\omega' = f_\alpha \omega f_\alpha^{-1} r_\alpha$ and $r_\alpha$ is defined in Equation (2.13).
Proof. — We have
\[
    r_\alpha F^{-1}(\bar{\omega}_\alpha^\vee) = r_\alpha F^{-1} f_\alpha(0) \\
    = r_\alpha f_{m(\alpha),q} F^{-1}(0) \\
    = f_{m(\alpha)}(0) \\
    = \bar{\omega}_{m(\alpha)}^\vee.
\]
Hence, \(m(\alpha) = \alpha\) if and only if \(\bar{\omega}_\alpha^\vee \in S_q\). For \(\omega \in E_q\), we denote by \(\omega' \in E_q\) the element such that \(f_\alpha(\omega(C_q)) = \omega(C_q)\). We have
\[
    f_\alpha \omega f^{-1}_\alpha r_\alpha(C_q) = \omega'(C_q).
\]
Since \(f_\alpha \omega f^{-1}_\alpha r_\alpha\) and \(\omega'\) lie in \(W_{a,q}\), we deduce from [4, VI §2.1] that \(w' = f_\alpha \omega f^{-1}_\alpha r_\alpha\). Now, if \(m(\alpha) = \alpha\), for \(\beta \in \Delta_{\min}\), we deduce that
\[
    f_\alpha(\lambda_{\omega,\beta}) = f_\alpha \omega F^{-1} f_\beta(\lambda_{\omega,\beta}) \\
    = f_\alpha \omega f^{-1}_\alpha r_\alpha f_{m(\alpha),q} F^{-1} f_\beta(\lambda_{\omega,\beta}) \\
    = w' F^{-1} f_\beta(f_{m(\alpha)}(\lambda_{\omega,\beta})) \\
    = w' F^{-1} f_\beta(f_\alpha(\lambda_{\omega,\beta})).
\]
By unicity, we deduce that \(f_\alpha(\lambda_{\omega,\beta}) = \lambda_{\omega',\beta}\).

Remark 2.10. — In [9, 3.8], it is proved that \(S_q \subset \mathbb{Q}_p^\prime \otimes \mathbf{Z} Y(T_{sc})\). Since \(Y(T_{sc}) \leq Y(T)\), the \(\overline{W}_a\)-orbits on \(S_q\) parametrize some \(F\)-stable semisimple classes of \(G\).

**Proposition 2.11.** — Assume there is \(\alpha \in \tilde{\Delta}_{\min,G}\), where \(\Delta_{\min,G}\) is the set defined in Equation (2.8), such that \(A_G = \langle z_\alpha \rangle\) has prime order and assume that \(G\) is not of type \(D_{2n}\). If \(V_\alpha\) is not contained in some hyperplane of \(H_q\), then the number of \(F\)-stable semisimple classes of \(G\) with disconnected centralizer is \(q^{\dim(V_\alpha)}\). Otherwise, there are no \(F\)-stable classes with disconnected centralizer.

Proof. — Since \(A_G\) has prime order, the \(\overline{W}_a\)-orbit of \(\lambda \in \mathcal{C}\) corresponding to (see Equation (2.5)) a semisimple class of \(G\) with disconnected centralizer lies in \(V_\alpha\), by Theorem 2.2(3). First suppose that \(V_\alpha\) is not contained in some hyperplane of \(H_q\). Then, \(M_{\alpha,q}\) is non-empty. In the proof of Lemma 2.8, we showed that if \(\lambda \in \omega(C_q)\) with \(\omega \in M_{\alpha,q}\) is such that its \(\overline{W}_a\)-orbit is \(F\)-stable, then \(\lambda = \lambda_{\omega,\beta}\) for some \(\beta \in \tilde{\Delta}_{\min,G}\). Moreover, by Lemma 2.8, we have \(\lambda \in V_\alpha\). Since \(A_G\) is generated by \(z_\alpha\), we have \(V_\alpha \subset V_\beta\). This implies that \(f_\beta(\lambda) = \lambda\). Thus, \(\lambda = \lambda_{\omega,\beta}\) for all \(\beta \in \tilde{\Delta}_{\min,G}\) and \(\lambda\) is the unique element of \(\omega(C_q)\) lying in an \(F\)-stable \(\overline{W}_a\)-orbit. So, in particular, \(\lambda = \lambda_\omega\). Moreover, Theorem 2.2(2) implies that for \(\omega' \in M_{\alpha,q}\) such that
\[ \omega \neq \omega', \text{ the elements } \lambda_\omega \text{ and } \lambda_{\omega'} \text{ are not } W_a\text{-conjugate. Then there are } |M_{\alpha,q}| \text{ such elements. By Remark 2.10, these points lie in } \mathbb{Q}_{\rho'} \otimes \mathbb{Z} Y(T) \cap \mathcal{C} \text{ and correspond to the } F\text{-stable semisimple classes of } G \text{ with disconnected centralizer. The claim then follows from Theorem 2.5.} \]

Suppose now that \( V_\alpha \) is contained in some hyperplane of \( \mathcal{H}_q \), and assume there is \( \lambda \in V_\alpha \) with \( F\)-stable \( W_a\)-orbit. Then \( \lambda \notin \tilde{\Delta}_{\min} \) (because \( \alpha \neq -\alpha_0 \) and no element of \( \tilde{\Delta}_{\min} \) is fixed by \( f_\alpha \)). Hence, Lemma 2.7 implies that \( F(\lambda) = F(\omega)(\lambda) \) with \( \omega \in E_q \) such that \( \lambda \in \omega(\mathcal{C}_q) \), i.e., \( \lambda = \omega F^{-1}(\lambda) \). Then, \( \lambda \in S_q \). However, by [9, 3.8.2], \( \lambda \) lies in a unique element of \( \Omega_q \), which contradicts the fact that \( \lambda \) lies in a hyperplane which is not a wall of \( \mathcal{C} \) (see Remark 2.6).

**Lemma 2.12.** — If \( p \) does not divide \( |\mathcal{A}_G| \), then for every \( \alpha \in \tilde{\Delta}_{\min,G} \), the invariant subspace \( V_\alpha \) is contained in no hyperplane of \( \mathcal{H}_q \).

**Proof.** — As we remarked at the end of the proof of Proposition 2.11, if there is \( \lambda \in S_q \) such that \( f_\alpha(\lambda) = \lambda \) for \( \alpha \in \tilde{\Delta}_{\min,G} \), then \( V_\alpha \) is contained in no hyperplane of \( \mathcal{H}_q \). Suppose there is \( \lambda \in \mathcal{C} \) such that:

1. We have \( \lambda \in \mathbb{Q}_{\rho'} \otimes \mathbb{Z} Y(T_{sc}) \cap \mathcal{C} \).
2. The type \( I_\lambda \) is not equal to the type of \( I_\mu \) for every \( \mu \neq \lambda \).
3. We have \( f_\alpha(\lambda) = \lambda \).

Let \( \lambda \) be such an element. Property (1) implies that there is a semisimple class \( [s]_{G_{sc}} \) in \( G_{sc} \) corresponding to \( \lambda \). Then \( F(s) \) is a semisimple element whose centralizer is of the same type as the one of \( s \) (because \( F \) is an isogeny). Let \( \mu \) be the point of \( \mathcal{C} \) which is \( W_a\)-conjugate to \( F(\lambda) \), i.e., \( \mu \) corresponds to the class of \( F(s) \) in the identification given in Equation (2.5). By Theorem 2.2(3), we have \( I_\mu = I_\lambda \) and it follows from Property (2) that \( \lambda = \mu \). Hence, the \( W_a\)-orbit of \( \lambda \) is \( F\)-stable. We conclude using Property (3).

Suppose that \( p \) is a prime that satisfies the condition in Table 2.2. For types \( B_n, C_{2n}, D_{2n}, E_6 \) and \( E_7 \), the corresponding elements given in Table 2.2 satisfy Properties (1), (2), and (3). The result follows in these cases. Furthermore, for the type \( C_{2n+1} \), if we denotes by \( \lambda_n \) the corresponding element of Table 2.2, we have

\[ F(\lambda_n) = (q - \varepsilon_0) \cdot \lambda_n + \varepsilon_0 \cdot \lambda_n, \]

where \( \varepsilon_0 \in \{-1, 1\} \) is such that \( q \equiv \varepsilon_0 \mod 4 \). Put \( \delta = 0 \) if \( \varepsilon_0 = 1 \) and \( \delta = 1 \) otherwise, and define \( r_n = t_n w_0^\delta \), with \( t_n \) the translation of vector \((q - \varepsilon_0) \cdot \lambda_n \in Y(T_{sc}) \) and \( w_0 \) is the longest element of \( W \). By [4, Ch. VI, §4.8], \( w_0 \) acts on \( V \) as \(-1\), which implies that \( w_0(\lambda_n) = -\lambda_n \). Thus,
we have \( r_n \in W_a \) and \( F(\lambda_n) = r_n(\lambda_n) \). It follows that \( \lambda_n \in S_q \) and we conclude as above.

Note that if \( G \) is of type \( A_n \), there is no element \( \lambda \in \mathcal{C} \) which satisfies the above properties (1), (2), (3). So, in order to prove the result for the type \( A_n \), we will more precisely describe the hyperplanes of \( H_q \). For this, we write \( \Delta = \{ \alpha_i \mid 1 \leq i \leq n \} \) as in [4, Planche I] and recall that the positive roots are the elements \( \alpha_{ij} = \sum_{i \leq r < j} \alpha_r \) for \( 1 \leq i < j \leq n + 1 \). Hence, the hyperplanes of \( H_q \) are given by

\[
H_{ij,k} = \left\{ \lambda \in V \mid \langle \alpha_{ij}, \lambda \rangle = \frac{k}{q} \right\},
\]

for \( 1 \leq i < j \leq n + 1 \) and \( k \in \mathbb{Z} \). In particular, an element \( \lambda = (\lambda_i)_{1 \leq i \leq n} \) lies in \( H_{ij,k} \) if and only if

\[
(2.20) \quad \sum_{i \leq r < j} \lambda_r = \frac{k}{q}.
\]

Let \( z_{\alpha} \in A \). Write \( d \) for the order of \( z_{\alpha} \) and define the element \( \lambda_d \in \mathcal{C} \) such that \( \lambda_{d,z_{\alpha}} = 1/d \) for all \( i \geq 0 \) and \( \lambda_{d,\beta} = 0 \) for \( \beta \in \tilde{\Delta} \) which is not in \( \langle z_{\alpha} \rangle \cdot \alpha_1 \). In particular, thanks to Equation (2.20), \( \lambda_d \in H_{ij,k} \) for some \( 1 \leq i < j \leq n + 1 \) and \( 1 \leq k \leq (q - 1) \) (we indeed can suppose that \( H_{ij,k} \) is not a wall of \( \mathcal{C} \) by Remark 2.6) if and only if \( qn_d = kd \), where \( n_d = |\{ \alpha_r \mid i \leq r < j \} \cap \langle z_{\alpha} \rangle \cdot \alpha_1| \). Since \( q > k \geq 0 \), we deduce that \( p \) divides \( d \). It follows that if \( p \) does not divide \( |A_G| \), then \( V_\alpha \) is not contained in some hyperplane, as required.

Finally, if \( G \) is of type \( D_{2n+1} \), then we show using equations of hyperplanes derived from [4, Planche III] and an argument similar to type \( A_n \), that the \( f_{\alpha} \)-stable element \( \frac{1}{2}(\varpi_1 + \varpi_{2n} + \varpi_{2n+1}) \) lies in no hyperplane of \( H_q \) which are not walls of \( \mathcal{C} \).

**Corollary 2.13.** — In table 2.3, we give the number \( n(q) \) of \( F \)-stable semisimple classes with disconnected centralizer for simple algebraic groups such that \( A \) has prime order. If \( s \) is a representative of such a class, we write \( A_G(s) = C_G(s)/C_G(s) \circ \) for the component group of \( C_G(s) \).

**Proof.** — This is a direct consequence of Proposition 2.11 and Table 2.1. The condition on \( p \) comes from Lemma 2.12.

**Remark 2.14.** — Recall that the Lang-Steinberg theorem implies that the number \( |s(G^F)| \) of semisimple classes of the finite fixed-point subgroup
Table 2.2. Some invariant elements

\( \mathbf{G}^F \) is given by

\[
|s(\mathbf{G}^F)| = \sum_{[s]_{A_G(s)^F} \in s(\mathbf{G})^F} |A_G(s)^F|, \quad F(s)=s
\]

where the sum is over the set of \( F \)-stable semisimple classes of \( \mathbf{G} \) and the representative \( s \) is chosen to be \( F \)-stable, which is possible by the Lang-Steinberg theorem. Suppose that \( A_G \) has prime order. Then every semisimple element \( s \) with disconnected centralizer has a component group \( A_G(s) \) isomorphic to \( A_G \) and so to a subgroup \( H \) of \( Z(\mathbf{G}_{sc}) \) of order \( |A_G| \) (using the isomorphism of Equation (2.7)), such that the actions of \( F \) on \( A_G(s) \) and on \( H \) are equivalent. In particular, we deduce that the actions of \( F \) on the groups \( A_G(s) \) for all \( s \) such that \( A_G(s) \) is not trivial, are equivalent. We denote by \( c_1 \) (resp. \( c_2 \)) a set of representatives of the \( F \)-stable semisimple classes of \( \mathbf{G} \) with connected centralizer (resp. a disconnected centralizer) and we suppose that the elements of \( c_1 \) and \( c_2 \) are chosen to be \( F \)-stable. Then Equation (2.21) gives

\[
|s(\mathbf{G}^F)| = |c_1| + |H^F| \cdot |c_2|. \tag{2.22}
\]

In [7], we computed \( |s(\mathbf{G}^F)| \) for every simple algebraic group \( \mathbf{G} \) defined over \( \mathbb{F}_q \); see [7, Table 1]. Moreover, it is well-known [9, 3.7.6] that

\[
q^{|\Delta|} = |c_1| + |c_2|. \tag{2.23}
\]

In particular, if \( |H^F| \) is not trivial (this condition is for example related in [7, Table 1]), we can deduce \( n(q) = |c_2| \) from Equations (2.22), (2.23) and [7, Table 1]. We retrieve the results of Table 2.3. But, when \( H^F \) is trivial
(for example, for \( q \equiv 2 \mod 3 \) for \( G \) of type \( E_6 \)), we get in Equation (2.22) no new information, and \( n(q) \) cannot be computed using [7].

**Remark 2.15.** — In [7], the prime \( p \) is supposed to be a good prime for \( G \), i.e. \( p \) does not divide any of the numbers \( n_\alpha \) (for \( \alpha \in \Delta \)) defined in Equation (2.6). Note that Proposition 2.11 applies for any prime \( p \) which does not divide \( |A_G| \); see Lemma 2.12. In particular, we deduce from Equations (2.22), (2.23), and Table 2.3 that

\[
|s (E_{6,\text{ad}}(2^f))| = 2^{6f} + 2^{2f+1} \quad \text{and} \quad |s (E_{7,\text{ad}}(3^f))| = 3^{7f} + 3^{4f},
\]

where \( \epsilon = 1 \) if \( F \) is a split Frobenius map and \( \epsilon = -1 \) otherwise. Moreover, thanks to [7, Prop. 4.1 and Prop. 5.9], the ordinary McKay Conjecture holds in defining characteristic for these groups.

| Type | \( n(q) \) | \( |A_G(s)| \) |
|------|-------------|-------------|
| \( A_{n,\text{ad}} \) | \( n+1 \text{ prime} \) | \( p \neq n+1 \) | 1 | \( n+1 \) |
| \( B_{n,\text{ad}} \) | \( p \neq 2 \) | \( q^{n-1} \) | 2 |
| \( C_{n,\text{ad}} \) | \( p \neq 2 \) | \( q^{\lfloor \frac{n}{2} \rfloor} \) | 2 |
| \( E_{6,\text{ad}} \) | \( p \neq 3 \) | \( q^2 \) | 3 |
| \( E_{7,\text{ad}} \) | \( p \neq 2 \) | \( q^4 \) | 2 |

**Table 2.3. Number of semisimple classes with disconnected centralizer**

Now, we define

\[
\Theta_q = \bigsqcup_{\alpha \in \Delta_{\text{min, }G}} S_{q,\alpha}.
\]

**Proposition 2.16.** — Suppose that \( F_0 \) is trivial and that \( q \equiv 1 \mod |A_G| \).

Then the \( \Gamma_G \)-orbits of \( \Theta_q \) correspond to the \( F \)-stable semisimple classes of \( G \) in the bijection given in Equation (2.5). Moreover, if \( A_G \) is cyclic and \( G \) is not of type \( D_{2n} \), and if for \( \alpha \in \Delta_{\text{min, }G} \), the invariant space \( V_\alpha \) is not contained in some hyperplane of \( H_q \), then we have

\[
|\Theta_{q,\alpha}| = q^{\dim(V_\alpha)},
\]

where \( \Theta_{q,\alpha} \) denotes the set of \( \Gamma_G \)-orbits of \( \Theta_q \) whose representatives are contained in \( V_\alpha \).
Proof. — We have \(|\mathcal{A}_G| = |Y(T)/Y(T_{sc})|\). In particular, \(|\mathcal{A}_G|\) is the product of the elementary divisors of \(Y(T_{sc}) \leq Y(T)\) (viewed as \(\mathbb{Z}\)-modules). Hence, it follows that \(|\mathcal{A}_G| \cdot Y(T) \leq Y(T_{sc})\) and \(|\mathcal{A}_G| \cdot F^{-1}(Y(T)) \leq F^{-1}(Y(T_{sc}))\). For \(\alpha \in \Delta_{\min}\), we have \(\omega_{\alpha,q} = \frac{1}{q} \cdot \omega_{\alpha}\) (because \(F_0\) is trivial) and \(\omega_{\alpha} = (q-1)\omega_{\alpha,q} + \omega_{\alpha,q}\). Denote by \(r_{\alpha}\) the translation of vector \((q-1)\omega_{\alpha,q}\). So, \(\omega_{\alpha} = q\omega_{\alpha,q} = r_{\alpha}(\omega_{\alpha,q})\). Since \(|\mathcal{A}_G|\) divides \(q-1\), the translation \(r_{\alpha}\) lies in \(F^{-1}(Y(T))\), and we have \(r_{\alpha}(\mathcal{C}) \subset \mathcal{C}\). Thus, \(f_{\alpha}(\mathcal{C}) = r_{\alpha}(\mathcal{C})\). Furthermore, we have
\[
r_{\alpha}F^{-1}(\omega_{\alpha}) = \omega_{\alpha}.
\]
This proves that \(\omega_{\alpha} \in S_{\alpha}\), and by Lemma 2.9, we deduce that \(m(\alpha) = \alpha\) and that \(\Gamma_G\) acts on \(\Theta_q\). Now, since \(\omega_{\alpha} \in \Theta_q\), Lemma 2.7 implies that the elements of \(\Theta_q\) are the elements of \(\mathcal{C}\) whose \(\mathbb{W}_a\)-orbit is \(F\)-stable. Moreover, by Theorem 2.2(2), two elements of \(\Theta_q\) are \(\mathbb{W}_a\)-conjugate if and only if they lie in the same \(\Gamma_G\)-orbit. As we have seen in the proof of Theorem 2.5, \(\Gamma_G\) acts on the set \(\Omega_q\). We denote by \(S\) a system of representatives of \(\Omega_q/\Gamma_G\). Then we have
\[
\Theta_q/\Gamma_G = \bigcup_{\omega(\mathcal{C}) \in S} \{\Gamma_G \cdot \lambda_{\omega,\beta} | \beta \in \tilde{\Delta}_{\min,G}\}.
\]
Let \(\omega(\mathcal{C}) \in S\). Then we have \(|\{\Gamma_G \cdot \lambda_{\omega,\beta} | \beta \in \tilde{\Delta}_{\min,G}\}| = |S_{\omega,q}|\), where \(S_{\omega,q} = \{\lambda_{\omega,\beta} | \beta \in \tilde{\Delta}_{\min,G}\}\). Furthermore, the proof of Lemma 2.9 implies that \(\lambda_{\omega,\beta} = \lambda_{\omega,\beta'}\) if and only if there is \(f \in \Gamma_G\) with \(f(\omega(\mathcal{C})) = \omega(\mathcal{C})\) and \(\beta'\) is the element of \(\tilde{\Delta}_{\min,G}\) such that \(f_{\beta'} = f_{\alpha}\). This proves that
\[
|S_{\omega,q}| = |\Gamma_G|/|\text{Stab}_{\Gamma_G}(\omega(\mathcal{C}))| = |\Gamma_G \cdot \omega(\mathcal{C})|.
\]
It follows that
\[
|\Theta_q/\Gamma_G| = \sum_{\omega(\mathcal{C}) \in S} |\Gamma_G \cdot \omega(\mathcal{C})| = |E_q| = q^{\Delta_j}.
\]
The last equality comes from \([11, \S 3]\). However, the number of \(F\)-stable semisimple classes of \(G\) is \(q^{\Delta}\); see \([9, 3.6.7]\). Hence, there are at most \(q^{\Delta_j}\) orbits under \(\mathbb{W}_a\) in \(\mathbb{Q}_p' \otimes_{\mathbb{Z}} Y(T)\) which are \(F\)-stable. It follows that the elements of \(\Theta_q/\Gamma_G\) have to contain points in \(\mathbb{Q}_p' \otimes_{\mathbb{Z}} Y(T) \cap \mathcal{C}\), and correspond to the \(F\)-stable semisimple classes of \(G\).

Now, we remark that \(\Gamma_G\) stabilizes \(\Theta_q \cap V_{\alpha}\). Hence, we have
\[
\Theta_{q,\alpha} = \Theta_q \cap V_{\alpha}/\Gamma_G.
\]
We set \(S_{\alpha} = \{\omega(\mathcal{C}) \in S | \omega \in M_{\alpha,q}\}\). Note that the elements of \(S_{\alpha}\) do not depend on the choice of representatives \(S\) of \(\Omega_q/\Gamma_G\). Indeed, if \(\omega \in M_{\alpha,q}\) and \(\omega' \in E_q\) is such that there is \(f \in \Gamma_G\) with \(\omega'(\mathcal{C}) = f(\omega(\mathcal{C}))\), then
\( \omega' \in M_{\alpha,q} \) because \( f \) and \( f_\alpha \) commute. Moreover, for \( \omega \in M_{\alpha,q} \), every \( \lambda_{\omega,\beta} \) with \( \beta \in \tilde{\Delta}_{\min,G} \) lies in \( V_\alpha \) (by Lemma 2.8, because \( \mathcal{A}_G \) is cyclic). Therefore, we have

\[
\Theta_{q,\alpha} = \bigcup_{\omega(\epsilon_q) \in S_\alpha} \{ \Gamma_G \cdot \lambda_{\omega,\beta} \mid \beta \in \tilde{\Delta}_{\min,G} \}.
\]

As above, we conclude that \( |\Theta_{q,\alpha}| = |M_{\alpha,q}| \), and the result comes from Theorem 2.5. \( \square \)

**Proposition 2.17.** — Suppose that \( F_0 \) is not trivial and that \( q \equiv -1 \mod |\mathcal{A}_G| \). Moreover, assume that \( G \) is not of type \( D_{2n} \). Then the conclusion of Proposition 2.16 holds.

**Proof.** — Let \( \alpha \in \tilde{\Delta}_{\min} \). For every \( \beta \in \Delta \), we define \( \tilde{s}_\beta = s_\beta - \frac{\delta_{s_\beta}}{q} \cdot \alpha^\vee \). Write \( z_\alpha = \prod_{\beta \in I} \tilde{s}_\beta \) for some index subset \( I \) of \( \Delta \) and define

\[
\tilde{z}_\alpha = \prod_{\beta \in I} \tilde{s}_\beta \in W_{a,q}.
\]

Note that \( \tilde{z}_\alpha(\mathcal{C}_q) = \mathcal{C}_q - \frac{1}{q} \cdot \varpi_\alpha^\vee \) (by Proposition 2.1) and \( \tilde{z}_\alpha(\mathcal{C}_q - \frac{1}{q} \cdot \varpi_\alpha^\vee) = \mathcal{C}_q - \frac{2}{q} \cdot \varpi_\alpha^\vee \). Furthermore, we have \( F_0 z_\alpha F_0^{-1} = z_{p^{-1}(\alpha)} \) (see the proof of Lemma 2.4). Moreover, \( \rho \) acts on \( \mathcal{A} \) by \( x \mapsto x^{-1} \) (because \( \mathcal{A} \) is cyclic; see [4, Planche I-IX]). This implies that \( f_{p^{-1}(\alpha)} \cdot f_\alpha = \text{Id} \). Hence

\[
F_0 z_\alpha F_0^{-1}\left(\frac{1}{q} \cdot \varpi_\alpha^\vee\right) = -\frac{1}{q} \cdot \varpi_{\rho(\alpha)}^\vee + f_{p^{-1}(\alpha)}(0),
\]

and we deduce that \( z_\alpha(F_0^{-1}(\frac{1}{q} \cdot \varpi_\alpha^\vee)) = -\frac{1}{q} \cdot \varpi_\alpha^\vee \). Note that \( \tilde{z}_\alpha(\frac{1}{q} \cdot \varpi_\alpha^\vee) = -\frac{1}{q} \cdot \varpi_\alpha^\vee \). Since \( |\mathcal{A}_G| \) divides \((q+1)\) the translation \( t \) of vector \( (q+1) \cdot \frac{1}{q} \varpi_\alpha^\vee \) lies in \( W_\alpha \). We set \( r_\alpha = t \tilde{z}_\alpha z_\alpha \in W_\alpha \). Then \( r_\alpha(\mathcal{C}_q) \subset \mathcal{C} \) and \( \varpi_\alpha^\vee \) lies in \( r_\alpha(\mathcal{C}_q) \). Thus \( f_\alpha(\mathcal{C}_q) = r_\alpha(\mathcal{C}_q) \). Moreover, we have

\[
r_\alpha F_0^{-1}\left(\frac{1}{q} \cdot \varpi_\alpha\right) = \varpi_\alpha^\vee.
\]

This proves that \( \varpi_\alpha^\vee \in S_\alpha \) and we conclude as in the proof of Proposition 2.16. \( \square \)

**Remark 2.18.** — Suppose that \( G \) is adjoint of type \( D_{2n+1} \), that \( p \) is odd and that \( F^* \) acts trivially on \( Z(G^*) \) (that is, \( q \equiv 1 \mod 4 \) if \( F \) is split, and \( q \equiv -1 \mod 4 \) otherwise). Thanks to Propositions 2.16 and 2.17, we have

\[
\begin{array}{c|cccc}
\alpha & -\alpha_0 & \alpha_1 & \alpha_{2n} & \alpha_{2n+1} \\
|\Theta_{q,\alpha}| & q^{2n+1} & q^{2n-1} & q^{n-1} & q^n
\end{array}
\]

Note that we have in fact \( \Theta_{q,\alpha_{2n}} = \Theta_{q,\alpha_{2n+1}} \). We denote by \( c_d(q) \) a set of representatives (chosen to be \( F \)-stable) of \( F \)-stable semisimple classes of \( G \).
whose centralizer component group has order $d$. Since $G$ is of adjoint type, we have $A_G = A \simeq \mathbb{Z}_4$ and $d|4$. Furthermore, since $V_{\alpha_2} \subseteq V_{\alpha_1} \subseteq V_{-\alpha_0}$, we have

\begin{equation}
|c_4(q)| = q^{n-1}, \quad |c_2(q)| = q^{2n-1} - q^{n-1} \quad \text{and} \quad |c_1(q)| = q^{2n+1} - q^{2n-1}.
\end{equation}

Therefore, Equation (2.21) implies that

$$|s(G^F)| = q^{2n+1} + q^{2n-1} + 2q^{n-1}.$$ 

and we retrieve the result of [7, Table 1]. Now, using [7, Prop. 4.1] (note that $G^*$ is simply-connected), if we denote by $\text{Irr}_{p'}(G^*F^*)$ the set of irreducible $p'$-characters of $G^*F^*$, then we deduce that

$$|\text{Irr}_{p'}(G^*F^*)| = q^{2n+1} + 3q^{2n-1} + 12q^{n-1}.$$ 

Comparing with [7, Prop. 5.10], this proves that the ordinary McKay Conjecture holds for $G^*F^*$ at the prime $p$.

3. Semisimple characters

3.1. Stable semisimple and regular characters

In this section we keep the notation of Section 2.3 and suppose that the Frobenius map $F : G \to G$ is split (i.e the map $F_0 : V \to V$ defined on p.1679 is trivial). Moreover, we assume that $p$ is a good prime for $G$. For every $\alpha \in \Phi$, we write $X_\alpha$ for the corresponding one-dimensional subgroup of $G$ normalized by $T$ and choose an isomorphism $x_\alpha : \mathbb{F}_p \to X_\alpha$ in such a way that $F(x_\alpha(u)) = x_\alpha(u^q)$. Let $\rho$ be a symmetry of the Dynkin diagram. Then we write $\sigma : G \to G$ for the graph automorphism on $G$ defined for all $\alpha \in \Phi$ and $u \in \mathbb{F}_p$ by $\sigma(x_\alpha(u)) = x_{\rho(\alpha)}(\gamma_\alpha u)$ where $\gamma_\alpha = \pm 1$ is chosen such that $\gamma_{\pm \alpha} = 1$ for all $\alpha \in \Delta$; see [8, 12.2.3]. Note that $F$ and $\sigma$ commute.

We denote by $U$ the unipotent radical of $B$. Recall that $B = U \rtimes T$ and that $U = \prod_{\alpha \in \Phi^+} X_\alpha$. Note that the product in the last equation is the inner product of $G$. Now, we define the normal subgroup

$$U_0 = \prod_{\alpha \in \Phi^+ \setminus \Delta} X_\alpha \subseteq U$$

and the quotient $U_1 = U/U_0$ (with canonical projection map $\pi_{U_0} : U \to U_1$). Then we have $U_1 \simeq \prod_{\alpha \in \Delta} X_\alpha$ (as direct product), and $U_0$ is $F$-stable and connected, which implies

$$U_1^F \simeq \prod_{\alpha \in \Delta} X_\alpha^F,$$

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(as direct product), because $X_\alpha$ is $F$-stable for every $\alpha \in \Delta$. Fix $u_1 \in U^F$ such that $\pi_{U_0}(u_1)\alpha \neq 1$ for all $\alpha \in \Delta$ (such an element is regular) and recall that $A_G(u_1) = Z(G)$, because $p$ is a good prime for $G$; see [14, 14.15, 14.18]. Then by the Lang-Steinberg theorem, we can parametrize the $G^F$-classes of regular elements by $H^1(F, Z(G))$ [14, 14.24]. For $z \in H^1(F, Z(G))$, we denote by $U_z$ the corresponding class of $G^F$. Furthermore, a linear character $\phi \in \text{Irr}(U^F)$ is regular if it has $U_0^F$ in its kernel, and if the induced character on $U_1^F$ (also denoted by the same symbol) satisfies $\text{Res}_{X_{k_0}^F}(\phi) \neq 1_{X_{k_0}^F}$ for all $\alpha \in \Delta$. By [14, 14.28], we also can parametrize the $T^F$-orbits of regular characters of $U^F$ by $H^1(F, Z(G))$. For this, we fix a regular character $\phi_1$ of $U^F$. Then, for every $z \in H^1(F, Z(G))$, the regular character $\phi_z = t_z \phi_1$, where $t_z$ is an element of $T$ such that $t_z^{-1}F(t_z) \in z$, is a representative of the $T^F$-orbit corresponding to $z$.

Now, for $z \in H^1(F, Z(G))$, we define the corresponding Gelfand-Graev character by setting

$$\Gamma_z = \text{Ind}_{U^F}^{G^F}(\phi_z).$$

We denote by $D_{G^F}$ the duality of Alvis-Curtis and define $\text{Irr}_r(G^F) = \{ \chi \in \text{Irr}(G^F) \mid \exists z \in H^1(F, Z(G)), \langle \chi, \Gamma_z \rangle \neq 0 \}$ and

$$\text{Irr}_s(G^F) = \{ \epsilon \chi D_{G^F}(\chi) \mid \chi \in \text{Irr}_r(G^F) \},$$

where $\epsilon$ is a sign chosen to be such that $\epsilon \chi D_{G^F}(\chi) \in \text{Irr}(G^F)$. The elements of $\text{Irr}_r(G^F)$ (resp. of $\text{Irr}_s(G^F)$) are the so-called regular characters (resp. semisimple characters) of $G^F$. In order to describe more precisely the sets $\text{Irr}_s(G^F)$ and $\text{Irr}_r(G^F)$, we first introduce further notation. We choose a $\sigma$- and $F$-stable torus $T_0$ containing $Z(G)$ and we consider the connected reductive group

$$\tilde{G} = T_0 \times Z(G) G,$$

where $Z(G)$ acts on $G$ and on $T_0$ by translation. We extend $\sigma$ and $F$ to $\tilde{G}$. Note that $\tilde{G}$ has connected center and the derived subgroup of $\tilde{G}$ contains $G$. Furthermore, $\tilde{T} = T_0 T$ is an $F$-stable maximal torus of $\tilde{G}$ contained in the $F$-stable Borel subgroup $B = U \times \tilde{T}$ of $G$. Moreover, we write $(G^*, F^*)$ and $(\tilde{G}^*, F^*)$ for pairs dual to $(G, F)$ and $(\tilde{G}, F)$, respectively. Then the embedding $i : G \rightarrow \tilde{G}$ induces a surjective homomorphism $i^* : \tilde{G}^* \rightarrow G^*$. Now, we write $\mathcal{T}$ and $\tilde{\mathcal{T}}$ for a set of $F^*$-stable representatives of $s(G^*)^{F^*}$ and $s(\tilde{G}^*)^{F^*}$. Note that $s(\tilde{G})^{F^*} = s(\tilde{G}^*)^{F^*}$ because the center of $G$ is connected, and $\tilde{\mathcal{T}}$ is then a system of representatives of the semisimple classes of $\tilde{G}^{F^*}$. Furthermore, for $s \in \mathcal{T}$ and $a \in H^1(F^*, A_{G^*}(s))$, we choose $g_a \in G^*$ such that the $F^*$-class of $g_a^{-1}F^*(g_a)$ in $A_{G^*}(s)$ is $a$ and write
$s_a = g_a s g_a^{-1} \in G^s F^s$. Then the elements $s_a$ for $a \in H^1(F^s, A_{G^s}(s))$ are a system of representatives of the $G^s F^s$ classes of $[s]_{G^s} \cap G^s F^s$. In particular, the set

$$\mathcal{S} = \bigcup_{s \in T} \{ s_a | a \in H^1(F^s, A_{G^s}(s)) \}$$

is a set of representatives of $s(G^s F^s)$. Note that the elements of $\widetilde{T}$ are chosen such that, if $s \in \mathcal{S}$, there is $\tilde{s} \in \widetilde{T}$ with $i^*(\tilde{s}) = s$. Now, for any semisimple element $s \in G^s F^s$ and $\tilde{s} \in G^s F^s$, we denote by $E(G^s, s) \subseteq \text{Irr}(G^s)$ and $E(\widetilde{G}^s, \tilde{s}) \subseteq \text{Irr}(\widetilde{G}^s)$ the corresponding rational Lusztig series. Recall that $E(G^s, s)$ consists of the irreducible constituents of Deligne-Lusztig characters $R_{T^w}(s)$, where $T^w$ is any $F^s$-stable maximal torus of $G^s$ containing $s$, and we have

$$\text{Irr}(G^s) = \bigcup_{s \in S} E(G^s, s) \quad \text{and} \quad \text{Irr}(\widetilde{G}^s) = \bigcup_{\tilde{s} \in \widetilde{T}} E(\widetilde{G}^s, \tilde{s}).$$

For $\tilde{s} \in \widetilde{T}$, let $W(\tilde{s}) \subseteq W$ be the Weyl group of $C_{G^s}(\tilde{s})$. Denote by $\widetilde{T}^s_1$ an $F$-stable maximal torus of $C_{G^s}(\tilde{s})$ containing $s$ and for $w \in W(s)$ we write $T^w_s$ for the $F^s$-stable maximal torus of $C_{G^s}(\tilde{s})$ obtained by twisting $\widetilde{T}^s_1$ by $w \in W$. We define

$$\rho_s = \sum_{w \in W(\tilde{s})} R_{T^w}(\tilde{s}),$$

$$\chi_s = \sum_{w \in W(\tilde{s})} \varepsilon(w) R_{T^w}(\tilde{s}),$$

where $\varepsilon$ is the sign character of $W$ and $\varepsilon_{G^s} = (-1)^{rk_{F}(\tilde{G})}$. Here, $rk_{F}(\tilde{G})$ denotes the $\mathbb{F}_q$-rank of $\tilde{G}$; see [14, 8.3]. Then we have $\text{Irr}_s(\widetilde{G}^s) = \{ \rho_s | \tilde{s} \in \widetilde{T} \}$ and $\text{Irr}_r(\widetilde{G}^s) = \chi_s | \tilde{s} \in \widetilde{T}$. Let $s \in T$. Write $\tilde{s} \in \widetilde{T}$ such that $i^*(\tilde{s}) = s$ and define

$$\chi_s = \text{Res}_{G^s}^{\widetilde{G}^s}(\chi_{\tilde{s}}) \quad \text{and} \quad \rho_s = \text{Res}_{G^s}^{\widetilde{G}^s}(\rho_{\tilde{s}}).$$

Furthermore, for $s \in \mathcal{S}$, we recall that there is a surjective group homomorphism [3, (8.4)]

$$\hat{\omega}_s^0 : H^1(F, Z(G)) \to \text{Irr} \left( A_{G^s}(s) F^s \right).$$

We now can recall the following result [3, Proposition 15.3, Corollaire 15.14].
Theorem 3.1. — For every $s \in S$, we have $\langle \Gamma_1, \chi_s \rangle_{G^F} = 1$. We write $\chi_{s,1}$ for the common constituent and put $\rho_{s,1} = \varepsilon G, \varepsilon c_{G^G, (s)} D_G (\chi_{s,1})$. Moreover, for $\xi \in \mathrm{Irr}(A_{G^*} (s) F^* )$, we define
\[
\chi_{s, \xi} = t_z \chi_{s,1} \quad \text{and} \quad \rho_{s, \xi} = t_z \rho_{s,1},
\]
where $z$ is any elements of $H^1 (F, \mathcal{Z}(G))$ such that $\hat{\omega}_s (z) = \xi$ and $t_z \in T$ with $t_z^{-1} F (t_z) \in z$. Then
\begin{enumerate}
\item For $z \in H^1 (F, \mathcal{Z}(G))$ and $\xi \in \mathrm{Irr}(A_{G^*} (s) F^* )$, the character $\chi_{s, \xi}$ (resp. $\rho_{s, \xi}$) is an irreducible constituent of $\Gamma_z$ (resp. of $D_G (\Gamma_z)$), if and only if $\xi = \hat{\omega}_s (z)$.
\item We have
\[
\chi_s = \sum_{\xi \in \mathrm{Irr}(A_{G^*} (s) F^* )} \chi_{s, \xi} \quad \text{and} \quad \rho_s = \sum_{\xi \in \mathrm{Irr}(A_{G^*} (s) F^* )} \rho_{s, \xi}.
\]
\item We have $\mathrm{Irr}_s (G^F) = \{ \rho_{s, \xi} \mid s \in S, \xi \in \mathrm{Irr}(A_{G^*} (s) F^* ) \}$ and $\mathrm{Irr}_r (G^F) = \{ \chi_{s, \xi} \mid s \in S, \xi \in \mathrm{Irr}(A_{G^*} (s) F^* ) \}$.
\end{enumerate}

Convention 3.2. — The regular character $\phi_1 \in \mathrm{Irr}(U^F)$ will be chosen to be $\sigma$-stable. This choice is possible by [5, 3.1] (note that Lemma 3.1 of [5] is stated for a Frobenius map $F'$ which commutes with $F$. But the argument is still valuable for a graph automorphism commuting with $F$).

Proposition 3.3. — Assume that $\phi_1 \in \mathrm{Irr}(U^F)$ is chosen as in Convention 3.2. For every $z \in H^1 (F, \mathcal{Z}(G))$, we have
\[
\sigma \Gamma_z = \Gamma_{\sigma (z)} \quad \text{and} \quad \sigma D_{G^F} (\Gamma_z) = D_{G^F} (\Gamma_{\sigma (z)}).
\]
Moreover, the operation of $\langle \sigma \rangle$ on the set of constituents of $\Gamma_1$ and of $\langle \sigma^{-1} \rangle$ on $s(G^F \mathcal{F}^*)$ commute, and for $s \in S$, if the $G^F \mathcal{F}^*$-class of $s$ is $\sigma^*$-stable, where $\sigma^* : G^* \to G^*$ denotes the automorphism of $G^*$ obtained in dualizing $\sigma$, then for every $z \in H^1 (F, \mathcal{Z}(G))$, we have
\[
\sigma \chi_s, \hat{\omega}_s (z) = \chi_s, \hat{\omega}_s (\sigma (z)) \quad \text{and} \quad \sigma \rho_s, \hat{\omega}_s (z) = \rho_s, \hat{\omega}_s (\sigma (z)).
\]

Proof. — For $s \in S$, we have
\begin{equation}
\sigma \mathcal{E}(G^F, s) = \mathcal{E}(G^F, \sigma^{-1} (s)).
\end{equation}
The proof is similar to [6, Proposition 1.1] (because $F$ and $\sigma$ commute). In particular, one has $\sigma \chi_s = \chi_{\sigma^{-1} (s)}$ and $\sigma \rho_s = \rho_{\sigma^{-1} (s)}$. Since $\phi_1$ is $\sigma$-stable, it follows that $\sigma \Gamma_1 = \Gamma_1$. This implies that if $s$ is $\sigma^*$-stable, then $\chi_{s,1}$ and $\rho_{s,1}$ are $\sigma$-stable. We conclude as in the proof of [5, Theorem 3.6]. \qed
Remark 3.4. — Note that Proposition 3.3 shows that $H^1(F, Z(G))^\sigma$ parametrizes the $\sigma$-stable Gelfand-Graev characters of $G^F$.

Lemma 3.5. — Suppose that $H^1(F, Z(G))$ has prime order. Then every $\sigma$-stable regular (resp. semisimple) character of $G^F$ is a constituent of some $\sigma$-stable Gelfand-Graev character (resp. dual of Gelfand-Graev character) of $G^F$.

Proof. — Let $\chi$ be a $\sigma$-stable regular character of $G^F$. Thanks to Theorem 3.1(3), there is $s \in S$ and $\xi \in \text{Irr}(A_{G^*}(s)^F^*)$ such that $\chi = \chi_{s, \xi}$ and Proposition 3.3 implies that the $G^*F^*$-class of $s$ is $\sigma^*$-stable. Let $z \in H^1(F, Z(G))$ be any element such that $\hat{\omega}_s^0(z) = \xi$. In particular, $\chi \in \Gamma_z$ by Theorem 3.1(1). Since $H^1(F, Z(G))$ has prime order, we deduce that $\ker(\hat{\omega}_s^0)$ is either trivial or equal to $H^1(F, Z(G))$. If $\ker(\hat{\omega}_s^0) = H^1(F, Z(G))$, then $\xi = \hat{\omega}_s^0(1)$ and $\chi \in \Gamma_1$, which is $\sigma$-stable with our choice in Convention 3.2. Suppose now that $\ker(\hat{\omega}_s^0)$ is trivial. By Proposition 3.3, we also have $\sigma^\chi_{s, \xi} = \chi_{s, \hat{\omega}_s^0(\sigma(z))}$. It follows from Theorem 3.1(1) that $\chi$ is $\sigma$-stable, if and only if $\hat{\omega}_s^0(\sigma(z)) = \hat{\omega}_s^0(z)$, which is equivalent to $z^{-1}\sigma(z) \in \ker(\hat{\omega}_s^0)$. Then $\sigma(z) = z$ and Proposition 3.3 implies that $\sigma\Gamma_z = \Gamma_z$, as required.

Remark 3.6. — Note that in general, if $\chi$ is a $\sigma$-stable regular character of $G^F$, then $\chi$ is not necessarily a constituent of some $\sigma$-stable Gelfand-Graev character of $G^F$. For example, consider a simple simply-connected group $G$ of type $A_3$ defined over $\mathbb{F}_q$ (with $q \equiv 1 \text{ mod } 4$) and suppose that the corresponding Frobenius map $F$ is split. We denote by $\alpha_1$, $\alpha_2$ and $\alpha_3$ the simple roots of $G$ (relative to an $F$ and $\sigma$-stable maximal torus $T$ of $G$) that we label as in [4, Planche I]. Write $\sigma$ for the non-trivial graph automorphism of $A_3$. The condition on $q$ implies that $F$ acts trivially on $Z(G)$. Hence, $H^1(F, Z(G)) = Z(G)$. Write $z_0$ for a generator of $Z(G)$. With the choice of Convention 3.2, we write $\Gamma_i = \Gamma_{z_0^i}$ (with $0 \leq i \leq 3$) for the $4$ Gelfand-Graev characters of $G^F$. Furthermore, we will denote by $\varpi^\gamma$ the fundamental weight corresponding to $\alpha_i$. Recall that $G^* = G_{\text{ad}}$. Define $\lambda = \frac{1}{2}(\varpi^\gamma + \varpi^\gamma_3)$ and write $s = i_T(\lambda) \in T_{\text{ad}}$. Then $F(s) = s$ and $\sigma(s) = s$. Moreover, $\lambda$ is stable under $f_{z_0^i}$, but not under $f_{z_0}$. Thus, by Theorem 2.2, we have $A_{G^*}(\lambda) = \langle z_0^2 \rangle$, and $A_{G^*}(s)^F^* \simeq \mathbb{Z}_2$. Denote by $1$ and $\eta$ the irreducible characters of $A_{G^*}(s)^F^*$, and by $\chi_{s, 1}$ and $\chi_{s, \eta}$ the corresponding regular characters of $G^F$ as in Theorem 3.1(1). Since $\sigma$ acts as $x \rightarrow x^{-1}$ on $Z(G)$, we have

\[ \hat{\omega}_s^0(1) = \hat{\omega}_s^0(z_0^2) = 1 \quad \text{and} \quad \hat{\omega}_s^0(z_0) = \hat{\omega}_s^0(z_0^3) = \eta, \]
and Proposition 3.3 implies that $\chi_{s,1}$ and $\chi_{s,\eta}$ are $\sigma$-stable. Moreover, thanks to Theorem 3.1(1), the Gelfand-Graev characters which have $\chi_{s,1}$ (resp. $\chi_{s,\eta}$) as constituent are $\Gamma_0$ and $\Gamma_2$ (resp. $\Gamma_1$ and $\Gamma_3$). However, by Proposition 3.3, we have

$$\sigma \Gamma_0 = \Gamma_0, \quad \sigma \Gamma_2 = \Gamma_2 \quad \text{and} \quad \sigma \Gamma_1 = \Gamma_3,$$

and $\chi_{s,\eta}$ is a $\sigma$-stable regular character of $G^F$ which is constituent of no $\sigma$-stable Gelfand-Graev characters of $G^F$, as claimed.

### 3.2. Disconnected reductive groups

By Clifford theory, an irreducible character $\chi$ of $G^F$ is $\sigma$-stable, if and only if it extends to the group $G^F \rtimes \langle \sigma \rangle$. Moreover, if $E(\chi)$ denotes an extension of $\chi$, then Gallagher’s theorem [19, 6.17] implies that every extension of $\chi$ is obtained by tensoring $E(\chi)$ with a linear character of $G^F \rtimes \langle \sigma \rangle$ trivial on $G^F$. So, in order to obtain information about the sets $\text{Irr}_r(G^F)^{\sigma}$ and $\text{Irr}_s(G^F)^{\sigma}$, we aim to understand the extensions of these characters to $G^F \rtimes \langle \sigma \rangle$. For this, we will consider the group

$$H = G \rtimes \langle \sigma \rangle.$$

We extend $F$ to a Frobenius map on $H$ by setting $F(\sigma) = \sigma$ (to simplify notation, the extended map will also be denoted by $F$). Note that $H$ is a disconnected reductive group defined over $F_q$ (the rational structure is given by $F$), and $H^0 = G$. Moreover, $\sigma$ is a rational quasi-central element in the sense of [15, 1.15]. Now, for $i \geq 0$, we define a scalar product on the space of class functions on the coset $G^F \cdot \sigma^i$, by setting

$$\langle \chi, \chi' \rangle_{G^F \cdot \sigma^i} = \frac{1}{|G^F|} \sum_{g \in G^F} \chi(g\sigma^i)\overline{\chi'(g\sigma^i)}.$$

Recall that in [15, 4.10], Digne and Michel define a duality involution $D_{G^F \cdot \sigma^i}$ for $i \geq 0$ on the set of class functions defined over the coset $G^F \cdot \sigma^i$, and prove in [15, 4.13] that if $\chi \in \text{Irr}(H^F)$ restricts to an irreducible character on $G^F$, then the class function $D_{H^F}(\chi)$ defined for all $g \in G^F$ and $i \geq 0$ by

$$(3.9) \quad D_{H^F}(\chi)(g\sigma^i) = D_{G^F \cdot \sigma^i}(\chi|_{G^F \cdot \sigma^i})(g\sigma^i),$$

is (up to a sign) an irreducible character of $H^F$. 

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We suppose that \( \phi_1 \in \text{Irr}(U^F) \) is chosen as in Convention 3.2. In particular, \( \phi_1 \) is \( \sigma \)-stable and linear. Thus, \( \phi_1 \) extends to a linear character \( E(\phi_1) \) of \( U^F \rtimes \langle \sigma \rangle \) by setting
\[
E(\phi_1)(u\sigma) = \phi_1(u) \quad \forall u \in U^F.
\]
This extension is the so-called canonical extension of \( \phi_1 \). We define
\[
E(\Gamma_1) = \text{Ind}_{U^F \rtimes \langle \sigma \rangle}^{H^F}(E(\phi_1)).
\]
Note that, as direct consequence of Mackey’s theorem [19, (5.6) p.74], we have
\[
\text{Res}_{H^F \rtimes \langle \sigma \rangle}^{G^F}(E(\Gamma_1)) = \Gamma_1.
\]
Hence, \( E(\Gamma_1) \) extends \( \Gamma_1 \). We write \( \Gamma_{1,\sigma} = \text{Res}_{G^F \rtimes \langle \sigma \rangle}^{G^F}(E(\Gamma_1)) \).

Write \( C_1 \) for the set of irreducible constituents of \( \Gamma_1 \) and for \( \chi \in C_1^\sigma \), denote by \( E(\chi) \) the constituent of \( E(\Gamma_1) \) that extends \( \chi \). Define
\[
\Psi_1 = \sum_{\chi \in C_1^\sigma} D_{H^F}(E(\chi)).
\]

**Lemma 3.7.** — We suppose that Convention 3.2 holds. Then we have
\[
\text{Res}_{G^F \rtimes \langle \sigma \rangle}^{G^F}(\Psi_1) = D_{G^F \rtimes \langle \sigma \rangle}(\Gamma_{1,\sigma}),
\]
and \( \langle \Psi_1, \Psi_1 \rangle_{H^F} = \langle \Gamma_{1,\sigma}, \Gamma_{1,\sigma} \rangle_{G^F \rtimes \langle \sigma \rangle} \). In particular,
\[
\langle \Gamma_{1,\sigma}, \Gamma_{1,\sigma} \rangle_{G^F \rtimes \langle \sigma \rangle} = \left| s(G^F)^* \right|^2.
\]

In [23], Sorlin develops a theory of Gelfand-Graev characters for disconnected groups when \( \sigma \) is semisimple or unipotent. These characters are extensions of some \( \sigma \)-stable Gelfand-Graev characters of \( G^F \) to \( H^F \); see [23, §5]. In particular, the following result is proven [23, 8.3].

**Theorem 3.8.** — Suppose that \( \sigma \) is a unipotent or a semisimple element of \( H^F \) and that \( H^1(F, Z(G^\sigma)) \) is trivial. Then \( H^F \) has a unique Gelfand-Graev character \( \Gamma \) and we have
\[
\langle \Gamma_{\sigma}, \Gamma_{\sigma} \rangle_{G^F \rtimes \langle \sigma \rangle} = \left| Z(G^\sigma)^* \right|^2 q^l
\]
where \( l \) is the semisimple rank of \( G^\sigma \) and \( \Gamma_{\sigma} \) denotes the restriction of \( \Gamma \) to the coset \( G^F \cdot \sigma \).

**Remark 3.9.** — The character \( E(\Gamma_1) \) defined in Equation (3.11) is a Gelfand-Graev character of \( H^F \) in the sense of [23], because the linear character \( E(\phi_1) \) defined in Equation (3.10) is regular [23, Définition 4.1]). Note that by [8, 12.2.3] the graph automorphism \( \sigma \) that we consider here always satisfies the condition (RS) defined in [23, Notation 2.1].
3.3. A result of extendibility

Let $n$ be a positive integer. The map $F' = F^n$ is a Frobenius map of $G$, which gives a rational structure over $\mathbb{F}_{q^n}$. Note that $F$ and $\sigma$ commute with $F'$. Then restrictions of these endomorphisms to $G_{F'}$ induce automorphisms of $G_{F'}$, denoted by the same symbol in the following. Note that, viewed as an automorphism of $G_{F'}$, the automorphism $F$ has order $n$. We write $A = \langle F, \sigma \rangle$ and

$$N_{F'/F} : U_1^{F'} \to U_1^{F}, u \mapsto u F(u) \ldots F^{n-1}(u),$$

for the norm map of $U_1$, where $U_1$ is the group defined before Equation (3.1), and we set $N_{F'/F}^* : \text{Irr}(U_1^{F}) \to \text{Irr}(U_1^{F'})$, $\phi \mapsto \phi \circ N_{F'/F}$. Since $F$ and $\sigma$ commute, we have

$$\sigma \circ N_{F'/F} = N_{F'/F} \circ \sigma. \quad (3.14)$$

**Lemma 3.10.** — If $\phi$ is a $\sigma$-stable character of $U_1^{F}$, then the character $N_{F'/F}^*(\phi)$ is stable under $F$ and $\sigma$.

**Proof.** — Since $U_1$ is abelian and connected, the map $N_{F'/F}$ is surjective [5, §2.4], and $N_{F'/F}^*$ is a bijection between $\text{Irr}(U_1^{F})$ and $\text{Irr}(U_1^{F'})$. Moreover, for every $\phi \in \text{Irr}(U_1^{F})$, Equation (3.14) implies that $N_{F'/F}^*(\phi)$ is $\sigma$-stable, as required. $\square$

**Remark 3.11.** — If $\phi \in \text{Irr}(U_1^{F})$ is regular and $\sigma$-stable, then the corresponding character of $U_1^{F'}$ is $\sigma$-stable. Applying Lemma 3.10 to this character, we obtain a character of $U_1^{F'}$ stable under $F$ and $\sigma$. Denote by $\tilde{\phi}$ the corresponding character of $U_1^{F'}$ (with $U_0^{F'}$ in its kernel). Then $\tilde{\phi}$ is a regular character of $U_1^{F'}$ stable under $F$ and $\sigma$. Thus, $\tilde{\phi}$ extends to $U_1^{F'} \rtimes (\langle \sigma \rangle)$. Now, it follows from Equation (3.10) that $E(\tilde{\phi})$ is $F$-stable.

**Convention 3.12.** — The character $\phi_1$ of $U_1^{F'}$ used to parametrize the Gelfand-Graev characters of $G_{F'}$ is chosen to be $\sigma$ and $F$-stable. This is possible by Remark 3.11.

**Proposition 3.13.** — Assume that $\phi_1 \in \text{Irr}(U_1^{F'})$ is chosen as in Convention 3.12. Suppose that $\sigma$ is semisimple and that the characteristic $p$ is a good prime of $(G^\sigma)^\circ$. If $H^1(F', Z(G^\sigma))$ is trivial, then the constituents of $\Psi_1$ are $F$-stable.

**Proof.** — Denote by $U_\sigma$ the set of regular elements of $H$ which are $G$-conjugate to an element of the coset $U \cdot \sigma$. In [23, §8], Sorlin defines
a family of subsets \((U_z)_{z \in H^1(F', \mathbb{Z}(G^*)})\) of \(U^F_{\sigma}\) which form a partition of \(U^F_{\sigma}\) (see [23, 8.1]). Furthermore, we define

\[
\gamma_u(g) = \begin{cases} |G^{F'}|/|U^F_{\sigma}| & \text{if } g \in U^F_{\sigma} \\ 0 & \text{otherwise} \end{cases}
\]

Remark 3.9 and the proof of [23, Théorème 8.4] imply that

\[
(D_{G^{F'}, \sigma}(\Gamma_1, \sigma)) = \gamma_u,
\]

because \(H^1(F', \mathbb{Z}(G^*))\) is trivial. Recall that the irreducible characters of \(\langle \sigma \rangle\) are described as follows. We fix a primitive \(|\langle \sigma \rangle|\)-complex root of unity \(\sigma_0\), and recall that the linear characters of \(\langle \sigma \rangle\) are the morphisms \(\varepsilon_i : \langle \sigma \rangle \to \mathbb{C}^\times\) such that \(\varepsilon_i(\sigma) = \sigma_i^0\). Let \(\rho_{s,1}\) be a \(\sigma\)-stable constituent of \(D_{G^{F'}}(\Gamma_1)\). Then the set \(\text{Irr}(H^{F'}, \rho_{s,1})\) of extensions of \(\rho_{s,1}\) to \(H^{F'}\) consists of the characters

\[
\rho_{s,1,i} = E(\rho_{s,1}) \otimes \varepsilon_i \in \text{Irr}(H^{F'}),
\]

for any \(i \geq 0\), where \(E(\rho_{s,1})\) denotes an extension of \(\rho_{s,1}\) to \(H^{F'}\) (such extensions exist by [19, 11.22]). Now, [23, Proposition 8.1] implies that \(U^F_{\sigma}\) is an \(H^{F'}\)-class (because \(p\) is good for \((G^*)^0\), the group \(H^1(F', \mathbb{Z}(G^*))\) is trivial and \(\sigma\) is semisimple). Hence, by Lemma 3.7 and Equation (3.15), for any \(h \in U^F_{\sigma}\), we have

\[
\rho_{s,1,i}(h) = \sigma_0^i \rho_{s,1,i}(h) = \sigma_0^i \gamma_u(\rho_{s,1,i}(h)) = \sigma_0^i \varepsilon_G \varepsilon_C_{s}(s).
\]

In particular, since \(\sigma_0\) has order \(|\langle \sigma \rangle|\), we deduce that

\[
(3.16) \quad \rho_{s,1,i} = \rho_{s,1,j} \iff \rho_{s,1,i}(h) = \rho_{s,1,j}(h) \quad \text{for } h \in U^F_{\sigma}.
\]

Suppose now that \(\rho_{s,1}\) is \(F\)-stable. Then \(F \rho_{s,1,i} = \rho_{s,1,j}\) for some \(j \geq 0\), because \(\text{Irr}(H^{F'}, \rho_{s,1})\) is \(F\)-stable, and for \(h \in U^F_{\sigma}\), we have

\[
\rho_{s,1,j}(h) = F \rho_{s,1,i}(h) = \rho_{s,1,i}(F(h)) = \rho_{s,1,i}(h),
\]

because \(F(h) \in U^F_{\sigma}\). Therefore, Equation (3.16) implies that \(F \rho_{s,1,i} = \rho_{s,1,i}\). \(\square\)

Remark 3.14. — In fact, in the proof of Proposition 3.13 we proved that every extension to \(H^{F'}\) of an \(F\)- and \(\sigma\)-stable constituent of \(D_{G^F}(\Gamma_1)\) is \(F\)-stable.
4. Application to finite groups of type $E_6$

4.1. Preliminaries

In this section, $G$ denotes a simple simply-connected group of type $E_6$ over $\mathbb{F}_p$. We suppose that $p$ is a good prime for $G$ (i.e., $p \neq 2,3$). Let $T$ be a maximal torus of $G$ contained in a Borel subgroup $B$ of $G$. We denote by $\Phi$ the root system of $G$ relative to $T$, and by $\Phi^+$ and $\Delta$ the sets of positive roots and simple roots corresponding to $B$. For $\alpha \in \Phi$, we write $X_\alpha$ for the corresponding root subgroup and choose an isomorphism $x_\alpha : \mathbb{F}_p \to X_\alpha$. Since $G$ is simple, we have $G = \langle x_\alpha(u) \mid \alpha \in \Phi, u \in \mathbb{F}_p \rangle$. For $\alpha \in \Phi$ and $t \in \mathbb{F}_p$, we set $n_\alpha(t) = x_\alpha(t)x^{-1}_{-\alpha}(-t^{-1})x_\alpha(t)$. Recall that the Weyl group $W$ of $G$ is generated by the coset $n_\alpha(1) \cdot T$ for all $\alpha \in \Phi$. Moreover, for $\alpha \in \Phi$ one has $\alpha^\vee(t) = n_\alpha(t)n_\alpha(1)^{-1}$ for all $t \in \mathbb{F}_p$. Then, we have $T = \langle \alpha^\vee(t) \mid \alpha \in \Phi, t \in \mathbb{F}_p^\times \rangle$ and $B = \langle T, x_\alpha(u), \alpha \in \Phi^+, u \in \mathbb{F}_p \rangle$; see [17, 1.12.1].

We write $\Delta = \{\alpha_1, \ldots, \alpha_6\}$ as in [4, Planche V] and denote by $\rho$ the symmetry of $\Delta$ of order 2. As in §3.1, we define the corresponding graph automorphism $\sigma : G \to G$ and a split Frobenius map by setting $F(x_\alpha(u)) = x_\alpha(u^p)$ for $\alpha \in \Phi$ and $u \in \mathbb{F}_p$, which commute with $\sigma$ and defines an $\mathbb{F}_p$-structure on $G$. Note that $T$ and $B$ are stable under $F$ and $\sigma$. Moreover, by [4, Planche V], we have

$$\tilde{\iota}_T(\omega_{\alpha_1}^\vee) = \alpha_1^\vee(\xi)\alpha_3^\vee(\xi^2)\alpha_5^\vee(\xi)\alpha_6^\vee(\xi^2),$$

where $\xi \in \mathbb{F}_p$ has order 3 (such an element exists because $p \neq 3$). Define

$$(4.1) \quad T_0 = \{\alpha_1^\vee(t)\alpha_3^\vee(t^2)\alpha_5^\vee(t)\alpha_6^\vee(t^2) \mid t \in \mathbb{F}_p^\times \}.$$  

Then $T_0$ is a subtorus of $T$ which contains $Z(G) = \langle \tilde{\iota}_T(\omega_{\alpha_1}^\vee) \rangle$, and is stable under $\sigma$ and $F$. In the following, we will use this torus for the construction of $\tilde{G}$ as in Equation (3.3).

Recall that $G^\sigma$ is a simple group of type $F_4$ (by [17, 1.15.2]) and $T^\sigma$ is a maximal $F$-stable torus of $G^\sigma$, contained in the $F$-stable Borel subgroup $B^\sigma$ of $G^\sigma$ (see [17, 4.1.4(c)]). In particular, $Z(G^\sigma)$ is trivial (for example, by Proposition 2.1, because $Y(T^\sigma_{ad})/Y(T^\sigma_{sc})$ is trivial).

Lemma 4.1. — With the above notation, the group $(\tilde{G}^\sigma)^\circ$ is a simple group of type $F_4$.

Proof. — The automorphism $\sigma$ of $\tilde{G}$ stabilizes $\tilde{T}$ and $\tilde{B}$. Then it is quasi-semisimple (see [15, 1.1]) and by [16, 0.1], the group $\tilde{G}^\sigma$ is reductive and the root system of $(\tilde{G}^\sigma)^\circ$ only depends on $\Phi$ (the root system of $\tilde{G}$) and
on $\sigma$. Hence, $(\tilde{G}^\sigma)^{\circ}$ and $G^\sigma$ have the same type, i.e., an irreducible root system of type $F_4$. Furthermore, by [15, 1.8], $T' = (T^\sigma)^{\circ}$ is a maximal torus of $(\tilde{G}^\sigma)^{\circ}$. Now the exact sequence
\begin{equation}
0 \to (X(T') \cap \mathbb{Q}\Phi)/\mathbb{Z}\Phi \to X(T')/\mathbb{Z}\Phi \to X(T')/(X(T') \cap \mathbb{Q}\Phi)
\end{equation}
induces an exact sequence for the $p'$-torsion subgroups of these groups. Since $X(T')/(X(T') \cap \mathbb{Q}\Phi)$ has no torsion, we deduce that
\begin{equation}
((X(T') \cap \mathbb{Q}\Phi)/\mathbb{Z}\Phi)_{p'} \simeq (X(T')/\mathbb{Z}\Phi)_{p'}
\end{equation}
However, the group $X(T') \cap \mathbb{Q}\Phi$ is a subgroup of the weight lattice $\Lambda$. So, it follows that $(X(T') \cap \mathbb{Q}\Phi)/\mathbb{Z}\Phi$ is a subgroup of $X(T^\sigma_c)/X(T^\sigma_{ad}) = \{1\}$ (because $G^\sigma$ is a simple group of type $F_4$, which implies that its fundamental group is trivial). It follows from Equation (4.3) and [3, 4.1] that $Z((\tilde{G}^\sigma)^{\circ})$ is connected. Denote by $\chi_0 : \tilde{T} \to \mathbb{F}_p^\times$ the character of $\tilde{T}$ induced by the character $T_0 \to \mathbb{F}_p^\times : t \mapsto t^3$ in $X(T_0)$ (the character $\chi_0$ is well-defined because it is trivial on $Z(G)$). Note that $X(\tilde{T}) = \langle \alpha, \alpha \in \Delta; \chi_0 \rangle$. Moreover, $\sigma\chi_0 = -\chi_0$ implies that
\begin{equation}
\text{Rk}_Z((1 - \sigma)X(\tilde{T})) = \text{Rk}_Z((1 - \sigma)X(T)) + 1.
\end{equation}
Now, the proof of [15, 1.28] implies that
\begin{equation}
\text{dim}(T') = \text{Rk}_Z\left(X(\tilde{T})/(1 - \sigma)X(\tilde{T})\right) = \text{Rk}_Z\left(X(\tilde{T})\right) - \text{Rk}_Z\left((1 - \sigma)X(\tilde{T})\right) = \text{Rk}_Z(X(T)) - \text{Rk}_Z((1 - \sigma)X(T)) = \text{dim}(T^\sigma).
\end{equation}
Hence, if $\Phi_{\sigma}$ denotes the root system of $G^\sigma$, we deduce from [25, 8.1.3] that
\begin{equation}
\text{dim}((\tilde{G}^\sigma)^{\circ}) = \text{dim}(T') + |\Phi_{\sigma}| = \text{dim}(T^\sigma) + |\Phi_{\sigma}| = \text{dim}(G^\sigma).
\end{equation}
The result follows. \hfill \Box

Let $n$ be a positive integer, $F' = F^n$ and $A = \langle F, \sigma \rangle$ as in §3.3. We consider the finite group $G^{F'}$.

**Lemma 4.2.** — The subgroups of $A$ are $\langle \sigma^iF^j \rangle$ (for $i \in \{1, 3\}$ and a divisor $j$ of $n$) and $\langle \sigma \rangle \times \langle F^j \rangle$ (for a divisor $j$ of $n$).
Proof. — Note that the elements of order 2 of $A$ are $σ, F^\frac{n}{2}$ and $σF^\frac{n}{2}$ if $n$ is even and $σ$ otherwise. Let $H$ be a subgroup of $A$. Then $H = H_2 \times H_2'$ with $H_2$ the 2-Sylow subgroup of $H$ and $H_2' \leq \langle F \rangle$. Then $H$ is cyclic if and only if $H_2$ is cyclic if and only $H_2$ contains a unique element of order 2. The result comes from the fact that if $H_2$ contains more than one element of order 2, then $σ \in H_2$.

Denote by $\text{Irr}_l(U^{F'})$ the set of linear characters of $U^{F'}$ and by $\text{Irr}_s(B^{F'})$ the set of irreducible characters $χ$ of $B^{F'}$ such that $\text{Res}_{U^{F'}}(χ)$ has constituents in $\text{Irr}_l(U^{F'})$. The characters of $\text{Irr}_s(B^{F'})$ can be described as follows. The $\tilde{T}^{F'}$-orbits of $\text{Irr}_l(U^{F'})$ are parametrized by the subsets of $Δ$. For $J \subset Δ$, we denote by $ω_J$ the corresponding $\tilde{T}^{F'}$-orbit, and write $L_J$ for the standard Levi subgroup (which is $F$-stable) with set of simple roots $J$. Note that $ω_J$ corresponds to the regular characters of $\text{Irr}(U^{F'})$.

CONVENTION 4.3. — Write $A_J = \text{Stab}_A(ω_J)$. Then by [5, Lemma 3.1] and Remark 3.11 there is $φ_J$ in $ω_J$ an $A_J$-stable character. Moreover, $\text{Stab}_A(φ_J) = A_J$ and if $τ \in A$, then $\text{Stab}_A(τφ_J) = \text{Stab}_A(φ_J)$, because $A$ is abelian. Let $Ω$ be an $A$-orbit of $\text{Irr}_l(U^{F'}/\tilde{T}^{F'})$ and we fix $ω_J \in Ω$. In the following, we will fix a $A_J$-stable character $φ_J \in ω_J$, and if $J' \subset Δ$ is such that there is $τ \in Ω$ with $ω_{J'} = τω_J$, then we choose $φ_{J'} = τφ_J$ as representative for $ω_{J'}$. Note that $φ_{J'}$ is well-defined (because it does not depend on the choice of $τ \in Ω$ with $τω_J = ω_{J'}$) and is $A_{J'}$-stable. This choice is compatible with Convention 3.12.

Now, for $z \in H^1(F', Z(L_J))$, we choose $t_z \in \mathfrak{T}$ such that $t_z^{-1}F'(t_z) \in z$ and define

$$φ_{J,z} = t_zφ_J.$$  (4.4)

Then the family $(φ_{J,z})_{J \subset Δ, z \in H^1(F', Z(L_J))}$ is a system of representatives of the $\tilde{T}^{F'}$-orbits of $\text{Irr}_l(U^{F'})$. Moreover, if we write $Z_J = Z(L_J)$, then for every $z \in H^1(F', Z(L_J))$, we have $\text{Stab}_{\tilde{T}^{F'}}(φ_{J,z}) = Z_J^{F'}$. Now, for $J \subset Δ, z \in H^1(F', Z(L_J))$ and $ψ \in \text{Irr}(Z_J^{F'})$, we define

$$χ_{J,z,ψ} = \text{Ind}_{U^{F'} \rtimes Z_J^{F'}}(φ_{J,z} ⊗ ψ),$$  (4.5)

where $\hat{φ}_{J,z}$ is the extension of $φ_{J,z}$ to $U^{F'} \rtimes Z_J^{F'}$ defined by $\hat{φ}_{J,z}(ut) = φ_{J,z}(u)$ for all $u \in U^{F'}$ and $t \in Z_J^{F'}$.

LEMMA 4.4. — Assume that Convention 4.3 holds. For $τ \in A, J \subset Δ, z \in H^1(F', Z(L_J))$ and $ψ \in \text{Irr}(Z_J^{F'})$, we have

$$τχ_{J,z,ψ} = χ_{τ(J),τ(z),τ^*ψ}.$$
Proof. — Using the induction formula [19, 5.1], we have

\[ \tau \chi_{J,z,\psi} = \text{Ind}_{B^F}^{B^{F'}} (\tau \phi_{J,z} \otimes \tau \psi). \]

Since \( \tau \) and \( F' \) commute, we have \( \tau (Z_{J,\tau}) = \tau (Z_J)^{F'} = Z_{\tau(J)}^{F'} \), because \( \tau (L_J) = L_{\tau(J)} \). Moreover, the choices in Convention 4.3 imply that \( \tau \phi_{J,z} \) is \( T^{F'} \)-conjugate to \( \phi_{\tau(J),\tau(z)} \), and the result follows.

\[ \square \]

4.2. Equivariant bijections

We define \( B = D \rtimes A \), where \( D \) is the group of outer diagonal automorphism of \( G^{F'} \) induced by the inner automorphisms of \( T^{F'}/T^{F'} \). We denote by \( \mathcal{O}_D \) and \( \mathcal{O}'_D \) the set of \( D \)-orbits of \( \text{Irr}_s(G^{F'}) \) and \( \text{Irr}_s(B^{F'}) \). The group \( D \) has order 1 or 3. For \( i \in \{1,3\} \), we write \( \mathcal{O}_{D,i} \) and \( \mathcal{O}'_{D,i} \) for the subset of elements of \( \mathcal{O}_D \) and \( \mathcal{O}'_D \) of size \( i \). For \( \nu \in \text{Irr}(Z(G^{F'})) \), we denote by \( \text{Irr}(G^{F'}|\nu) \) the set of irreducible characters of \( G^{F'} \) lying over \( \nu \), that is \( \chi \in \text{Irr}(G^{F'}|\nu) \) if and only if \( \chi|_{Z(G^{F'})} = \chi(1) \cdot \nu \). We recall that \( Z(B) = Z(G) \), and for \( \nu \in \text{Irr}(Z(G^{F'})) \) we set

\[ \text{Irr}_s(G^{F'}|\nu) = \text{Irr}(G^{F'}|\nu) \cap \text{Irr}_s(G^{F'}) \]

and

\[ \text{Irr}_s(B^{F'}|\nu) = \text{Irr}(B^{F'}|\nu) \cap \text{Irr}_s(B^{F'}), \]

and denote by \( \mathcal{O}_D,\nu \) and \( \mathcal{O}'_{D,\nu} \) the set of \( D \)-orbits of \( \text{Irr}(G^{F'}|\nu) \) and \( \text{Irr}(B^{F'}|\nu) \), respectively. For \( i \in \{1,3\} \), we write \( \mathcal{O}_{D,\nu,i} \) (resp. \( \mathcal{O}'_{D,\nu,i} \)) for the subset of elements of \( \mathcal{O}_{D,\nu} \) (resp. of \( \mathcal{O}'_{D,\nu} \)) of size \( i \).

Remark 4.5. — By Theorem 3.1(2), every \( D \)-orbit of \( \text{Irr}_s(G^{F'}) \) is the set of constituents of some \( \rho_s \) with \( s \in \mathcal{S} \). We denote by \( \delta_s \) the \( D \)-orbit corresponding to \( s \in \mathcal{S} \). Note that \( |\delta_s| = |A_{G^*}(s)^F'| \). For \( J \subseteq \Delta \) and \( \psi \in \text{Irr}(Z_{J'}) \), we define

\[ \delta_{J,\psi} = \{ \chi_{J,z,\psi} \mid z \in H^1(F', Z(L_J)) \}. \]

Then the \( D \)-orbits of \( \text{Irr}_s(B^{F'}) \) are the sets \( \delta_{J,\psi} \) with \( J \subseteq \Delta \) and \( \psi \in \text{Irr}(Z_{J'}) \). Moreover, we have \( |\delta_{J,\psi}| = |H^1(F', Z(L_J))| \).

Lemma 4.6. — Let \( \nu \in \text{Irr}(Z(G^{F'})) \). Write \( A_\nu = \text{Stab}_A(\nu) \) and suppose that \( \mathcal{O}_{D,\nu,k} \) and \( \mathcal{O}'_{D,\nu,k} \) for \( k \in \{1,3\} \) are \( A_\nu \)-equivalent. Then \( \text{Irr}_s(G^{F'}|\nu) \) and \( \text{Irr}_s(B^{F'}|\nu) \) are \( D \rtimes A_\nu \)-equivalent.
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Proof. — We choose \( A_{\nu} \)-equivariant bijections \( f_1 : \mathcal{O}_{D,\nu,1} \to \mathcal{O}'_{D,\nu,1} \) and \( f_3 : \mathcal{O}_{D,\nu,3} \to \mathcal{O}'_{D,\nu,3} \). We define \( \Psi_\nu : \text{Irr}_s(G^{F'})|\nu) \to \text{Irr}_s(B^{F'})|\nu) \) as follows. Let \( \delta_s \in \mathcal{O}_{D,\nu} \). If \( \delta_s \in \mathcal{O}_{D,\nu,k} \) for \( k \in \{1,3\} \), then by Remark 4.5 there is \( J \subseteq \Delta \) and \( \psi \in \text{Irr}(Z_j^{F'}) \) such that \( f_k(\delta_s) = \delta_J,\psi \). Then we set

\[
\Psi_\nu(\rho_{s,z}) = \chi_{J,z,\psi}.
\]

Note that, if \( H^1(F',Z(L_J)) \) is not trivial, then \( H^1(F',Z(L_J)) \) and \( H^1(F',Z(G)) \) are identified by the map \( h^1_J : H^1(F',Z(G)) \to H^1(F',Z(L_J)) \) defined in [14, 14.31] (which is an isomorphism in this case). Hence, the map \( \Psi_\nu \) is well-defined and is an \( D \times \text{A}_\nu \)-equivariant bijection by Theorem 3.1, Lemma 4.4, Proposition 3.3 and [5, 3.6].

**Theorem 4.7.** — Suppose that \( G \) is a simple simply-connected group of type \( E_6 \) defined over \( \mathbb{F}_p \) with corresponding Frobenius \( F' \). We suppose that \( F' \) is split. With the above notation, if \( \nu \in \text{Irr}(Z(G)^{F'}) \), then the sets \( \text{Irr}_s(G^{F'})|\nu) \) and \( \text{Irr}_s(B^{F'})|\nu) \) are \( D \times \text{A}_\nu \)-equivalent.

Proof. — Write \( q = p^n \), \( F \) the split Frobenius map of \( G \) over \( \mathbb{F}_p \) which stabilizes \( T \) and \( B \), and \( \sigma \) the graph automorphism of \( G \) with respect to \( T \) and \( B \), as above. Recall that \( F' = F^n \). We will prove that \( \text{Irr}_s(G^{F'})|\nu) \) and \( \text{Irr}_s(B^{F'})|\nu) \) are \( D \times \text{A}_\nu \)-equivalent, using Lemma 4.6. In order to prove that \( \mathcal{O}_{D,\nu,k} \) and \( \mathcal{O}'_{D,\nu,k} \) for \( k \in \{1,3\} \) are \( \text{A}_\nu \)-equivalent, we use [19, 13.23].

First, we suppose that \( Z(G)^{F} = 1 \) and \( Z(G)^{F'} = Z(G) \), that is \( p \neq 1 \) mod 3 and \( q \equiv 1 \) mod 3. We write \( \text{Irr}(Z(G)^{F'}) = \{1_Z, \varepsilon, \varepsilon^2\} \). Note that \( n \) is even. Moreover, we have

\[
A_{1Z} = A \quad \text{and} \quad A_\varepsilon = A_{\varepsilon^2} = \langle \sigma F \rangle.
\]

Suppose that \( k = 3 \) and \( \nu = 1_Z \). Let \( H \subseteq A \) (the subgroups of \( A \) are described in Lemma 4.2). By Lemma [5, 5.7] and Equation (3.8) (which is valid for any element of \( A \)), we deduce that \( |\mathcal{O}_{D,1Z,3}^{H}| \) is equal to the number of \( H^* \)-stable classes of \( s(G^*) \) with disconnected centralizer, where \( H^* \) denotes the subgroup of automorphisms of \( G^* \) induced by elements of \( H \). If \( H = \langle \sigma \rangle \times \langle F_j \rangle \) or \( H = \langle \sigma^IF_j \rangle \), then we write \( d = \text{Ord}(F_j) \). We claim that \( |\mathcal{O}_{D,1Z,3}^{H}| = p^{2n/d} \). Indeed, Theorem 2.2(3) implies that every semisimple class of \( G \) with disconnected centralizer is \( \sigma \)-stable, and we conclude with Table 2.3.

Now, in the proof of [5, 5.9], it is shown that \( |\delta_{J,\psi}| = 3 \), if and only if \( Z_j^{F'} = T_j^{F'} \times H_j^{F'} \), where \( T_J \) is a torus of rank \( |\Delta| - |J| \) and \( H_j^{F'} \) is isomorphic to \( H^1(F',Z(L_J)) \). Moreover, the elements of \( \delta_{J,\psi} \) lie over \( \varepsilon^m \). So, \( \mathcal{O}_{D,\varepsilon^m,3} \) consists of the orbits \( \delta_{J,\psi} \) with \( |H^1(F',Z(L_J))| = 3 \). By [1, Lemme 2.16, Table 2.17], a subset \( J \subseteq \Delta \) parametrizes such an orbit if and
only if it contains \(\{\alpha_1, \alpha_3, \alpha_5, \alpha_6\}\). Furthermore, by [17, Lemma 4.4.7], the group \(G^\sigma\) is a simple group of type \(F_4\) with root system \(\Phi_\sigma = \{\tilde{\alpha} \mid \alpha \in \Phi\}\) where \(\tilde{\alpha} = \frac{1}{2}(\alpha + \rho(\alpha))\), and the set of simple roots of \(G^\sigma\) with respect to \(T^\sigma\) and \(B^\sigma\) is \(\Delta_\sigma = \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_4\}\). Note that the labelling is as in [4, Planche VIII]. In particular, \(\tilde{\alpha}_1 = \alpha_2\) and \(\tilde{\alpha}_2 = \alpha_4\) are the long roots of \(\Delta_\sigma\). Moreover, the root subgroup corresponding to \(\tilde{\alpha} \in \Phi_\sigma\) is \(\tilde{X}_\alpha\), where \(\tilde{X}_\alpha = X_\alpha\) if \(\alpha = \rho(\alpha)\) and \(\tilde{X}_\alpha = X_\alpha \cdot X_{\rho(\alpha)}\) if \(\alpha \neq \rho(\alpha)\); see the proof of [17, Lemma 4.4.7]. We associate to \(J \subseteq \Delta\) the subset \(\tilde{J} \subseteq \Delta_\sigma\) such that, if a \(\sigma\)-orbit of \(\Delta\) lies in \(J\), the corresponding root of \(\Delta_\sigma\) lies in \(\tilde{J}\). Write \(\Phi_{\tilde{J}} = \Phi_\sigma \cap \tilde{Z}_{\tilde{J}}\). Then we have

\[
L_{\tilde{J}} = \langle T, \tilde{X}_\tilde{\alpha}, \tilde{\alpha} \in \Phi_{\tilde{J}} \rangle \quad \text{and} \quad L_{\tilde{J}}^\sigma = \langle T^\sigma, \tilde{X}_\alpha^\sigma, \tilde{\alpha} \in \Phi_{\tilde{J}} \rangle,
\]

where \(L_{\tilde{J}}^\sigma\) is the standard Levi subgroup of \(G^\sigma\) with respect to \(T^\sigma\) corresponding to \(\tilde{J}\), because \(T^\sigma\) is connected. Note that \(L_{\tilde{J}}^\sigma = L_{\tilde{J}}^\sigma\) and \(D(L_{\tilde{J}}) = D(L_{\tilde{J}})^{\sigma}\), where \(D(L_{\tilde{J}})\) denotes the derived subgroup of \(L_{\tilde{J}}\). Furthermore, we have \(T_{\tilde{J}} = \text{Rad}(L_{\tilde{J}})\), which implies that \(T_{\tilde{J}}\) is \(\sigma\)-stable. Since \(L_{\tilde{J}}^\sigma\) is connected (as a Levi subgroup), we deduce from [15, 1.31] and [25, 2.2.1] that \(L_{\tilde{J}}^\sigma = T_{\tilde{J}}^\sigma D(L_{\tilde{J}})^{\sigma}\). This product is direct, because \(\sigma\) fixes non-negative elements of the center of \(D(L_{\tilde{J}})\). It follows that \(T_{\tilde{J}}^\sigma\) is connected (as group isomorphic to the connected quotient \(L_{\tilde{J}}^\sigma / D(L_{\tilde{J}})^{\sigma}\)). In particular, \(T_{\tilde{J}}^\sigma\) is a subgroup of \(\text{Rad}(L_{\tilde{J}})\). Moreover, since \(D(L_{\tilde{J}})^{\sigma} = D(L_{\tilde{J}}^\sigma)\), we deduce from [25, 2.3.3] that \(\dim(T_{\tilde{J}}^\sigma) = \dim(\text{Rad}(L_{\tilde{J}}^\sigma))\), and [25, 1.8.2] implies that

\[
(4.8) \quad T_{\tilde{J}}^\sigma = \text{Rad}(L_{\tilde{J}}^\sigma).
\]

Let \(\delta_{J,\psi}^{1,1} \in O_{D,1,3}^{(\sigma \times \langle F^{\rho} \rangle)}\). Then \(J\) is \(\sigma\)-stable and contains \(\{\alpha_1, \alpha_3, \alpha_5, \alpha_6\}\), and \(\sigma \psi = \psi\). Moreover, we have \((T_{\tilde{J}}^\sigma)^{\langle \sigma \cdot F^{\rho} \rangle} = (T_{\tilde{J}}^\sigma)^{F^{\rho}}\) and Equation (4.8) implies that the set \(O_{D,1,3}^{(\sigma \times \langle F^{\rho} \rangle)}\) is in bijection with the set of characters \(\tilde{X}_{\tilde{J},\tilde{\psi}}\) of \(\text{Irr}_s((B^\sigma)^{F^{\rho}})\) such that \(\tilde{J}\) contains \(\{\tilde{\alpha}_3, \tilde{\alpha}_4\}\). Therefore, [7, Lemma 5.4] implies that

\[
(4.9) \quad \left| O_{D,1,3}^{(\sigma \times \langle F^{\rho} \rangle)} \right| = p^{2n/d}.
\]

Similarly, since \((\sigma \cdot F^{\rho})^n = F'\) (because \(n\) is even), for a \(\sigma^i\)-stable \(J \subseteq \Delta\), we have \((T_{\tilde{J}}^\sigma)^{\sigma^i F^{\rho}} = T_{\tilde{J}}^{\sigma^i F^{\rho}}\), and we deduce that \(O_{D,1,3}^{(\sigma^i \cdot F^{\rho})}\) is in bijection with the \(D\)-orbits of size 3 of \(\text{Irr}_s((B^{\sigma^i} F^{\rho})^{1,1})\). As above, [1, 2.17] and [7, Lemma 5.4] implies that \(\left| O_{D,1,3}^{(\sigma^i \cdot F^{\rho})} \right| = p^{2n/d}\). This discussion proves that, if \(\nu = 1,\) then for every \(H \subseteq A\) we have

\[
(4.10) \quad \left| O_{D,1,3}^{H} \right| = \left| O_{D,1,3}^{H} \right|.
\]
Now, if \( \nu = \varepsilon^m \) with \( m = \pm 1 \), then \( A_\nu = \langle \sigma F \rangle \) and for every \( H = \langle \sigma^t F^{n/d} \rangle \leq A_\nu \), the same argument shows that
\[
|O_{D,\varepsilon^m,3}^H| = p^{2n/d} = |O_{D,\varepsilon^m,3}'^H|.
\]

So, this proves that for every \( \nu \in \text{Irr}(Z(G)^F) \), the sets \( O_{D,\nu,3} \) and \( O_{D,\nu,3}' \) are \( A_\nu \)-equivalent.

Suppose now that \( k = 1 \). Note that \( s(G^{F^*}) \) and \( O_D \) are \( A \)-equivalent. Let \( d \) be a divisor of \( n \). Then, we have \( |s(G^{F^*})^{\langle \sigma^t F^{n/d} \rangle}| = |s(G^{F^{n/d}})| \), because the set of representatives \( T \) of \( F' \)-stable semisimple classes of \( G^* \) can be chosen such that if the class of \( t \in T \) is \( \sigma^{n/d} F^{n/d} \)-stable, then \( \sigma^{n/d} F^{n/d}(t) = t \). We then conclude using the fact that a power of \( \sigma^{n/d} F^{n/d} \) equals \( F' \) and with Equation (4.14). So, by [7, 1.1], we deduce that
\[
|s(G^{F^*})^{\langle \sigma^{n/d} F^{n/d} \rangle}| = \begin{cases} 
p^n|\Delta/d|/d + 2p^{2n/d} & \text{if } Z(G)^{\sigma^{n/d}} = Z(G) \\
p^n|\Delta/d|/d & \text{otherwise} \end{cases}
\]

Furthermore, if \( t \in T \) is chosen \( F^{n/d} \)-stable when the class of \( t \) in \( G^* \) is \( F^{n/d} \)-stable, then we have \( |s(G^{F^*})^{\langle \sigma^* \times (F^{n/d}) \rangle}| = |s(G^{F^{n/d}})^{\sigma^*}| \).

Thanks to Theorem 3.8, we deduce that
\[
|s(G^{F^*})^{\langle \sigma^* \times (F^{n/d}) \rangle}| = p^n|\Delta^o|/d.
\]

Now, using the case \( k = 3 \), the fact that \( O_D = O_{D,1} \cup O_{D,3} \) and Equations (4.12) and (4.13), we deduce that
\[
|O^{\langle \sigma^{n/d} F^{n/d} \rangle}_{D,1}| = p^n|\Delta/d|/d - p^{2n/d} \quad \text{and} \quad |O^{\langle \sigma^* \times (F^{n/d}) \rangle}_{D,1}| = p^n|\Delta^o|/d - p^{2n/d}.
\]

Moreover, note that the argument at the beginning of the proof shows that the set \( O^{\langle \sigma^* \rangle \times (F^{n/d})}_{D} \) is in bijection with \( \text{Irr}_s(B^F)^{F^{n/d}} \), which has \( p^{d|\Delta^o|} \) elements by [6, Proposition 3]. The set \( O^{\langle \sigma^{n/d} F^{n/d} \rangle}_{D} \) is in bijection with \( \text{Irr}_s(B^{\sigma^t F^{n/d}}) \), which has \( p^n|\Delta/d|/d + 2p^{2n/d} \) elements if \( \sigma^t F^{n/d} \) acts trivially on \( Z(G) \), and \( p^d|\Delta| \) elements otherwise. Since \( O'_D = O'_{D,1} \cup O'_{D,3} \), we deduce from Equations (4.9), (4.10) and (4.11) that
\[
|O^{\langle \sigma^* F^{n/d} \rangle}_{D,1}| = p^n|\Delta/d|/d - p^{2n/d} \quad \text{and} \quad |O^{\langle \sigma^* \rangle \times (F^{n/d})}_{D,1}| = p^n|\Delta^o|/d - p^{2n/d}.
\]

Equations (4.14) and (4.15) prove that the sets \( O_{D,1} \) and \( O'_{D,1} \) are \( A \)-equivalent. So, by [7, Theorem 1.1], the sets \( \text{Irr}_s(G^{F'}, \nu) \) and \( \text{Irr}_s(B^{F'}, \nu) \) are in bijection. Since \( O_{D,\nu,3} \) and \( O'_{D,\nu,3} \) are in bijection by Equations (4.10) and (4.11), we deduce that \( O_{D,\nu,1} \) and \( O'_{D,\nu,1} \) have the same cardinal. Moreover, if \( H \) is not a subgroup of \( \langle \sigma F \rangle \), then every \( H \)-stable character
of \( \text{Irr}_s(\mathbf{G}^{F'}) \) (resp. \( \text{Irr}_s(\mathbf{B}^{F'}) \)) lies over \( 1_Z \). Now, let \( H = \langle \sigma^i F^{n/d} \rangle \) be a subgroup of \( \langle \sigma F \rangle \). We consider the norm map \( N_{F'/\sigma^i F^{n/d}} : Z(\mathbf{G}) \rightarrow Z(\mathbf{G}) \), which is well-defined because \( (\sigma^i F^{n/d})^n = F^n \). By [5, Lemma 5.8, Lemma 5.9], if \( N_{F'/\sigma^i F^{n/d}} \) is surjective, then every set \( \mathcal{O}'_{D,\nu,1} \) and \( \mathcal{O}'_{D,\nu,1} \) contains \( \frac{1}{3} (p^{n|\Delta|/d} - p^{2n/d}) \) characters invariant under \( \sigma^i F^{n/d} \). Otherwise (i.e., when \( N_{F'/\sigma^i F^{n/d}} \) is trivial), the \( \sigma^i F^{n/d} \)-stable characters of \( \text{Irr}_s(\mathbf{G}^{F'}) \) and \( \text{Irr}_s(\mathbf{B}^{F'}) \) lie over \( 1_Z \). So, this proves that, for every subgroup \( H \) of \( A_\nu \), we have

\[
|\mathcal{O}'_{D,\nu,1}| = |\mathcal{O}'_{D,\nu,1}|.
\]

Therefore, the sets \( \mathcal{O}'_{D,\nu,1} \) and \( \mathcal{O}'_{D,\nu,1} \) are \( A_\nu \)-equivalent, as required.

Now, we suppose that \( Z(\mathbf{G})^{F} \) and \( Z(\mathbf{G})^{F'} \) are trivial, that is, \( p \not\equiv 1 \mod 3 \) and \( q \not\equiv 1 \mod 3 \). Then \( D \) is trivial and we conclude using the argument of Equations (4.12) and (4.13), and the discussion following Equation (4.14).

Finally, suppose that \( Z(\mathbf{G})^{F} = Z(\mathbf{G}) \), i.e. \( p \equiv q \equiv 1 \mod 3 \). Then \( Z(\mathbf{G})^{F'} = Z(\mathbf{G}) \). If \( n \) is odd, we remark that if \( X \) is a \( \langle \sigma F^{n/d} \rangle \)-set and an \( \langle F' \rangle \)-set, and if \( x \) is fixed by \( \sigma F^{n/d}(x) \) and \( F' \), then \( F^{n/d}(x) = x \) and \( \sigma(x) = x \). In particular, we can compare \( |\text{Irr}_s(\mathbf{G}^{F'})^{\sigma F^{n/d}}| \) with the number of semisimple classes of \( \mathbf{G}^{F^{n/d}} \) fixed by \( \sigma \) (there are \( p^{n|\Delta|/d} \) such classes by Lemma 3.7 and Theorem 3.8), and \( |\text{Irr}_s((\mathbf{B}^{F'})^{\sigma F^{n/d}})| \) with \( |\text{Irr}_s((\mathbf{B}^{\sigma})^{F^{d}})| = p^{n|\Delta|/d} \). If \( n \) is even, then \( (\sigma^i F^j) = F' \) and we are in the same situation as the first case of the proof. Using the same argument as above we deduce the numbers of Table 4.1. We set \( a_{F^{n/d}} = b_{F^{n/d}} = p^{n|\Delta|/d} - p^{2n/d} \) when the norm map \( N_{F'}/F^{n/d} \) is surjective, and \( a_{F^{n/d}} = p^{n|\Delta|/d} - 3p^{2n/d} \) and \( b_{F^{n/d}} = 0 \) otherwise.

It follows that \( \mathcal{O}_{D,\nu,k} \) and \( \mathcal{O}'_{D,\nu,k} \) are \( A_\nu \)-equivalent. This proves the claim. \( \square \)

### 4.3. Inductive McKay condition

**Lemma 4.8.** — Let \( H = GC \) be a finite central product with \( Z = G \cap C \). Suppose that \( C \) is abelian and that \( \tau \in \text{Aut}(H) \) acts on \( G \) and \( C \). For \( \chi \in \text{Irr}(G)^\tau \), write \( \nu = \text{Res}_Z^G(\chi) \). If \( \nu \) extends to a \( \tau \)-stable character of \( C \), then we have

\[
|\text{Irr}(H|\chi)^\tau| = |(C/Z)^\tau|.
\]
Hence, we have $\text{Irr}(H \cap \nu) = \{ \chi \cdot \theta \mid \theta \in \text{Irr}(C|\nu) \}$.

Since $\chi$ is $\tau$-stable, so is $\nu$ and $\text{Irr}(C|\nu)$ is $\tau$-stable. Furthermore, $\chi \cdot \theta$ is $\tau$-stable if and only if $\theta$ is $\tau$-stable. In particular, the set $\text{Irr}(H|\chi)^\tau$ is parametrized by $\text{Irr}(C|\nu)^\tau$. Moreover, $C$ is abelian, then $\nu$ extends to a linear character of $C$ (denoted by the same symbol) [19, 5.4] and by assumption, $\nu$ can be supposed to be $\tau$-stable. It follows that the map $g_\nu : \text{Irr}(C|1_Z) \to \text{Irr}(C|\nu)$, $\theta \mapsto \theta \nu$ is a bijection such that $\text{Irr}(C|\nu)^\tau = g_\nu(\text{Irr}(C|1_Z)^\tau)$ (because if $\nu$ is $\tau$-stable, so is $\nu^{-1}$). But we can identify $\text{Irr}(C|1_Z)$ with $\text{Irr}(C/Z)$ and the action of $\tau$ on these sets is compatible. Hence, we have $|\text{Irr}(C|1_Z)^\tau| = |\text{Irr}(C/Z)^\tau|$ and the result follows from [19, 6.32].

We recall that if $N$ is a normal subgroup of $G$, then we can associate to every $G$-invariant irreducible character of $N$ an element $[\chi]_{G/N}$ of the second cohomology group $H^2(G/N, \mathbb{C}^\times)$ of $G/N$.

\begin{table}[h!]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
$H$ & $n$ & $\langle \sigma \rangle \times \langle F^{n/d} \rangle$ & $\langle F^{n/d} \rangle$ & $\langle \sigma F^{n/d} \rangle$ \\
\hline
$|\mathcal{O}_{1,z,1}^{H}|$ & $p^{2n/d}$ & $p^{2n/d}$ & $p^{2n/d}$ \\
$|\mathcal{O}_{1,\varepsilon,1}^{H}|$ & $0$ & $p^{2n/d}$ & $0$ \\
$|\mathcal{O}_{1,\varepsilon,1}^{H}|$ & $p^{n|\Delta_\sigma|/d} - p^{2n/d}$ & $a_{F^{n/d}}$ & $p^{n|\Delta_\sigma|/d} - p^{2n/d}$ \\
$|\mathcal{O}_{1,\varepsilon,1}^{H}|$ & $0$ & $b_{F^{n/d}}$ & $0$ \\
$|\mathcal{O}_{1,\varepsilon,1}^{H}|$ & $p^{n|\Delta_\sigma|/d} - p^{2n/d}$ & $a_{F^{n/d}}$ & $p^{n|\Delta_\sigma|/d} - p^{2n/d}$ \\
$|\mathcal{O}_{1,\varepsilon,1}^{H}|$ & $0$ & $b_{F^{n/d}}$ & $0$ \\
\hline
\end{tabular}
\caption{Case when $Z(G)^F = Z(G)$}
\end{table}
Theorem 4.9. — Let $p > 3$ be a prime number and $q$ a $p$-power. Then the finite simple group $E_6(q)$ is “good” for the prime $p$.

Proof. — Let $X$ be a simple group of type $E_6$ with parameter $q = p^n$. In order to prove that $X$ is “good” for $p$, we will show that $X$ satisfies the properties (1)–(8) of [20, §10]. Let $G$ be a simple simply-connected group of type $E_6$ defined over $\mathbb{F}_q$ and with split Frobenius map $F' : G \to G$. Recall that $X = G^{F'}/Z(G)^{F'}$ and $G^{F'}$ is the universal cover of $X$. Moreover, the subgroup $U^F$ is a $p$-Sylow subgroup of $G^{F'}$ with normalizer $B^F$. We set $M = B^{F'}$ and we will show that $M$ has the required properties. By [6, Lemma 5], we have $\text{Irr}_s(G^{F'}) = \text{Irr}_{p'}(G^{F'})$ and $\text{Irr}_s(B^{F'}) = \text{Irr}_{p'}(B^{F'})$.

Let $\nu \in \text{Irr}(Z(G)^{F'})$ and $A_\nu = D \rtimes A_\nu$ as above. By Theorem 4.7, there is an $A_\nu$-equivariant bijection $\Phi_\nu : \text{Irr}_{p'}(G^{F'}|\nu) \to \text{Irr}_{p'}(B^{F'}|\nu)$. Thus, Properties (1)–(4) of [20, §10] are satisfied. Moreover, by Lemma 4.6, it is sufficient to prove the result for $\chi = \rho_{s,1}$ and $\Psi_\nu(\chi) = \chi_{J,1,\psi}$, where $(J, \psi)$ corresponds to $s$. For such a $\chi$, we define

$$G_\chi = G \rtimes \text{Stab}_{A_\nu}(\chi) \quad \text{and} \quad G'_\chi = G' \rtimes \text{Stab}_{A_\nu}(\Phi_\nu(\chi)),$$

where $G = G^{F'}$ (resp. $G' = B^{F'}$) if $\chi$ is not $D$-stable, and $G = \tilde{G}^{F'}$ (resp., $G' = \tilde{B}^{F'}$) otherwise. Recall that $\tilde{G}$ denotes the connected reductive group defined in Equation (3.3) with the convention of §4.1. We have $Z(G^{F'}) \leq Z(G_\chi)$ and $\text{Stab}_{D \rtimes A}(\chi)$ is induced by the conjugation action of $N_{G_\chi}(U^F)$. So, the property (5) of [20, §10] holds. We set $C = Z(G)$. Then $C = C_{G_\chi}(G^{F'})$ and the property (6) of [20, §10] is true.

In order to prove the properties (7) and (8) of [20, §10], we will first show that $\chi$ and $\Phi_\nu(\chi)$ extend to $G_\chi$ and $G'_\chi$, respectively. If $\chi$ is not $D$-stable, then $\text{Stab}_A(\chi) \leq \langle F^d, \sigma \rangle$. If $\text{Stab}_A(\chi)$ is cyclic, then $\chi$ extends to $G_\chi$ by [19, 11.22]. Otherwise, $\text{Stab}_A(\chi) = \langle \sigma, F^{n/d} \rangle$ for some divisor $d$ of $n$, and by Proposition 3.13, the extensions of $\chi$ to $G^{F'} \rtimes \sigma$ are $F^{n/d}$-stable, and thus extend to $G_\chi$ by [19, 11.22]. So, $\chi$ is extendible to $G_\chi$.

Suppose now that $D$ is not trivial and $\chi$ is $D$-stable. In particular, $\chi$ is extendible to $\tilde{G}^{F'}$. Write $A_\chi = \text{Stab}_A(\chi)$. We will prove that $\chi$ extends to an $A_\chi$-stable character of $\tilde{G}^{F'}$. First, we suppose that $\sigma \in A_\chi$ and we write $H = GC$. Recall that $H$ is a central product and that $C = Z(G^{F'})$. By [3, §6.B], $H$ has index $|D|$ in $\tilde{G}^{F'}$. Note that $\chi$ is over $1_Z$. Since $1_C$ lies in $\text{Irr}(C|1_Z)$, we deduce from Lemma 4.8 that the character $\chi \cdot 1_C$ is $A_\chi$-stable. Moreover, Gallagher’s theorem implies that the elements of $E = \text{Irr}(\tilde{G}^{F'}|\chi \cdot 1_C)$ extend $\chi$ and $|E| = |D| = 3$. Moreover, $E$ is $A_\chi$-stable. Denote by $\rho : A_\chi \to \mathfrak{S}_E$ the homomorphism of groups induced by this operation. Suppose that $\sigma \in A_{\chi}$. Note that $\sigma$ does not act trivially.
on $E$ (because if $\sigma$ fixes a character of $E$, then the action of $\sigma$ on $E$ is equivalent to the action of $\sigma$ on $\tilde{G}^{F^i}/H$, which is not trivial). So, $\rho(\sigma) \neq 1$ and $\rho(\sigma)$ has order 2. Thus, $\sigma$ has to fix a character of $E$. Suppose now that there is $F^i \in A_\chi$. Since $\sigma$ and $F^i$ commute, we deduce that $\rho(F^i)$ centralizes $\rho(\sigma)$. But $\mathfrak{g}_E \simeq \mathfrak{g}_3$ and the centralizer of $\rho(\sigma)$ in $\mathfrak{g}_E$ has order 2. Thus, there is an $A_\chi$-stable character $\tilde{\chi}$ in $E$. By [15, 1.31], we have $\tilde{\chi}^2 = \text{Rad}(\tilde{G})^\sigma D(\mathfrak{g})^\sigma$, because $\sigma$ has order 2 and $|Z(D(\tilde{G}))| = 3$. Since $\text{Rad}(\tilde{G}) = T_0$ (cf. Equation (4.1)) has dimension 1, we deduce that $\sigma$ acts by inversion on $T_0$ and $|T_0^\sigma| = 2$. Hence, $\tilde{G}^\sigma$ is disconnected with connected component a simple group of type $F_4$ (by Lemma 4.1) of index 2. We deduce that $|Z(\tilde{G}^\sigma)| = 2$. Denote by $\tilde{\Psi}$ the character of $\tilde{G}$ constructed in Equation (3.12). The character $\tilde{\chi}$ extends to a character $E(\tilde{\chi})$ of $\tilde{G}^{F^i} \ltimes \langle \sigma \rangle$, and $(\tilde{\Psi}, E(\tilde{\chi}))_{\tilde{G}^{F^i}, \sigma} = \pm 1$. Moreover, by [23, Proposition 7.2] $\tilde{\Psi}$ has non-zero values only on $\tilde{U}_\sigma^{F^i}$ (see the proof of Proposition 3.13 for the definition), which is the union of the two classes $\tilde{U}_1$ and $\tilde{U}_{-1}$ of $\tilde{G}^{F^i} \ltimes \langle \sigma \rangle$ (by [23, Proposition 8.1], because $\sigma$ is semisimple). Then $E(\tilde{\chi})$ has a non-zero value on at least $\tilde{U}_1$ or $\tilde{U}_{-1}$. But $F^i$ fixes these two classes (because there is an $F^i$-stable element in $\tilde{U}_\sigma^{F^i}$. So, its $\tilde{G}^{F^i} \ltimes \langle \sigma \rangle$-class is $F^i$-stable and the other class has to be also $F^i$-stable). Then the argument of Proposition 3.13 shows that $E(\tilde{\chi})$ is $F^i$-stable. It follows that $E(\tilde{\chi})$ extends to $(\tilde{G}^{F^i} \ltimes \langle \sigma \rangle) \ltimes \langle F^i \rangle$ by [19, 11.22]. This proves that $\chi$ extends to $G_\chi$.

Now, we suppose that $\sigma \notin A_\chi$. Then $A_\chi = \langle \sigma^i F^{n/d} \rangle$ for some $d \neq n$. Moreover, as we have seen in the proof of Theorem 4.6, we have $F^i = (\sigma^i F^{n/d})^n$ (if not, $\sigma$ has to fix $\chi$). Then, there is a semisimple element $s$ of $G^{F^i}$ such that $\chi = \rho_{s,1}$. Since $A_{G^*}(s)^{F^i}$ is trivial, by the Lang-Steinberg theorem, we can suppose that $s$ is chosen to be $\sigma^i F^{n/d}$-stable. Let $\tilde{s}$ be an $\sigma^i F^{n/d}$-stable semisimple element of $G^*$ such that $i^*(\tilde{s}) = s$ (such elements exist by the Lang-Steinberg theorem, because $\ker(i^*)$ is connected and $\sigma^i F^{n/d}$-stable). Note that $\tilde{s} \in G^{F^i}$, and since $\tilde{s}$ is $\sigma^i F^{n/d}$-stable, the character $\rho_{\tilde{s}}$ is $\sigma^i F^{n/d}$-stable. Moreover, $\rho_{\tilde{s}}$ extends $\chi$. Since $G_\chi$ is a cyclic extension of $G^{F^i}$, we deduce from [19, 11.22] that $\chi$ extends to $G_\chi$, as required.

Write $\chi' = \Phi_\nu(\chi)$. Then there are $J \subseteq \Delta$, $z \in H^1(F', Z_J)$ and $\psi \in \text{Irr}(T_J)^{F'}$ with $\text{Res}_{Z(G)^{F^i}}(\psi) = \nu$, such that $\chi' = \chi_J, z, \psi$. Suppose that $\chi'$ is $D$-stable. Then $z = 1$ and $T_J$ is connected. Write $\tilde{T}_J = C_{\tilde{T}^{F^i}}(\phi_J)$, where $\phi_J$ is chosen as in Convention 4.3. Note that $\tilde{T}_J$ is a torus because the center of $\tilde{G}$ is connected. Then $T_J$ is a subtorus of $\tilde{T}_J$ and by [14, 0.5], there is an $F'$-stable subtorus $T'$ of $\tilde{T}_J$ such that $\tilde{T}_J = T_J \cdot T'$ (as direct
product). By [19, 6.17] and the construction of Equation (4.5) applied to $B$ and $\tilde{B}$, we deduce that

$$\text{Irr}(B^{F'} | \chi_{J,1,\psi}) = \{ \tilde{\chi}_{J,1,\psi} | \mu \in \text{Irr}(T'_{F'}) \},$$

where $\tilde{\chi}_{J,1,\psi} = \text{Res}_{B^{F'}}^{F'}(\psi) \otimes \mu$ denotes the character of $\tilde{B}^{F'}$ defined in Equation (4.5). Note that if $T_J$ is $\sigma^i F^j$-stable (for any $i, j$), then $T'$ is $\sigma^i F^j$-stable. Write $A_{\chi'} = \text{Stab}_A(\chi')$. It follows that the character $\tilde{\chi}_{J,1,\psi} = \text{Res}_{B^{F'}}^{F'}(\psi)$ is an $A_{\chi'}$-stable extension of $\chi'$. For $\tau \in A_{\chi'}$, $\phi_J$ and $\psi$ are $\tau$-stable. Write $\psi = \psi \otimes 1_{T'_{F'}}$. Then $T'_{F'}$ and $\phi_J \otimes \psi$ are $A_{\chi'}$-stable. Hence, as linear character, $\phi_J \otimes \psi$ extends to a linear character $\rho(\phi_J \otimes \psi)$ of $(U_{F'} \rtimes T'_{F'}) \rtimes A_{\chi'}$, and by [19, (5.6) p.74], we have

$$(4.16) \quad \tilde{\chi}_{J,1,\psi} = \text{Res}_{B^{F'}}^{F'}(\psi) \times A_{\chi'} \left( \text{Ind}_{B^{F'} \rtimes A_{\chi'}}^{F'}(U_{F'} \rtimes T'_{F'}) \times A_{\chi'}, E(\phi_J \otimes \psi) \right).$$

Hence, $\chi'$ extends to $G'_{\chi}$. Suppose that $\chi'$ is not $D$-stable. If $A_{\chi'}$ is cyclic, then $\chi'$ extends to $G'_{\chi}$ by [19, 11.22]. Otherwise, $z = 1$ and we can show that $\chi'$ extends to $A_{\chi'}$, because we can construct an extension of $\chi_{J,1,\psi}$ to $B^{F'} \rtimes A_{\chi'}$ as in Equation (4.16).

We now will prove the properties (7) and (8) of [20, §10]. If $\chi$ is not $D$-stable, then $C = Z(G)^{F'}$ and we choose $\gamma = \nu$. Then $\chi \cdot \gamma = \chi$ and $\Phi_\nu(\chi) \cdot \gamma = \Phi_\nu(\chi)$ and by the preceding discussion, these characters extend to $G_{\chi}$ and $G'_{\chi}$. Then by [19, 11.7], we have

$$[\chi]_{G_{\chi}/G^{F'}} = [\Phi_\nu(\chi)]_{G'_{\chi}/B^{F'}}.$$ and Properties (7) and (8) of [20, §10] are proven in this case.

Suppose now that $\nu$ is not trivial and that $\chi$ is $D$-stable. If $\nu = 1_Z$ then we set $\gamma = 1_C$, which is $G_{\chi}$ and $G'_{\chi}$-stable. Moreover, as we have seen, the characters $\chi \cdot 1_C$ and $\Phi_\nu(\chi) \cdot 1_C$ extend to $G_{\chi}$ and $G'_{\chi}$. If $\nu \neq 1$, then $G_{\chi} = \langle F'' \rangle$ with $F'' = \sigma^i F^{n/d}$ for some $d \neq 1$ and $F''^{n/d} = F'$. Moreover, since $\nu \neq 1$ is $F''$-stable, it follows that $F''$ acts trivially on $Z(G)^{F'}$. Note that $\Phi_\nu(\chi)$ is $D$-stable and extends to $G'_{\chi}$. Denote by $\bar{\chi}'$ such an extension and write

$$\gamma = \text{Res}_{Z(G)^{F'}}^{G'_{\chi}}(\bar{\chi}').$$

Then $\gamma$ is $G'_{\chi}$-stable and $G_{\chi}$-stable and lies over $\nu$. Furthermore, by [19, 6.17], we have

$$(4.17) \quad \left| \text{Irr}(\tilde{G}^{F'} | \chi)_{F''} \right| = \sum_{\theta \in \text{Irr}(\tilde{G}^{F'} | C_{\chi})_{F''}} \left| \text{Irr}(\tilde{G}^{F'} | \tilde{G}^{F'})_{F''} \right|.$$ As we have seen previously, $\chi = \rho_{s,1}$ for some $F''$-stable semisimple element $s \in G^{s,F''}$ and there is an $F''$-stable semisimple element $\tilde{s} \in \tilde{G}^{s,F''}$ such
that \(i^*(\widetilde{s}) = s\). Then by [3, (15.9)], we have \(\text{Irr}(\widetilde{G}^{F'})|\chi) = \{\rho_{z,1} | z \in C\}\) and [3, Lemma 8.3] implies that this set is in bijection with \(C\) because \(A_\text{G} \cdot (s)\) is trivial. Moreover, since \(\tilde{s}\) is \(F''\)-stable, these operations are \(F''\)-equivalent and \(|\text{Irr}(\widetilde{G}^{F'},|\chi)| = |C^{F''}|\) Now, Lemma 4.8 implies that \(|\text{Irr}(G^F|\chi)| = |C^F|/3\), because \(F''\) acts trivially on \(Z(G^F) = G^F \cap C\). Since \(|\text{Irr}(\widetilde{G}^{F'}|\theta)| |^{F''}| \leq 3\) (because \(G^F C\) has index 3 in \(G^{F'}\) and by [19, 6.17]), we deduce from Equation (4.17) that \(|\text{Irr}(\widetilde{G}^{F'})|\theta)|^{F''}|\) has to be equal to 3. But \(\chi \cdot \gamma \in \text{Irr}(G^F C|\chi)|^{F''}\). Thus, \(\chi \cdot \gamma\) extends to an \(F''\)-stable character of \(\widetilde{G}^{F'}\) and [19, 11.22] implies that \(\chi \cdot \gamma\) extends to \(G_\chi\). We conclude using [19, 11.7] that

\[
[X \cdot \gamma]|_{G_\chi/G^{F',C}} = [\Phi_\nu(\chi) \cdot \gamma]|_{\chi/G^{F',C}}.
\]

Hence, the properties (7) and (8) of [20, §10] hold, as required. \(\square\)

**Theorem 4.10.** — Let \(p > 3\) be a prime number and \(q\) a \(p\)-power. Then the finite simple group \(2E_6(q)\) is “good” for the prime \(p\).

**Proof.** — We set \(F' = \sigma F^n\). Then \(G^{F'}\) is the universal cover of \(2E_6(q)\), and the outer automorphism group of \(G^{F'}\) is \(D \times \langle F \rangle\), where \(F\) acts on \(G^{F'}\) as an automorphism of order \(2n\). For \(\omega \in \Delta/\rho\), we set \(X_\omega = \prod_{\alpha \in \omega} X_\alpha\).

Note that \(X_\omega\) is a subgroup of \(U\), because \(G\) is of type \(E_6\) and if the roots \(\alpha\) and \(\rho(\alpha)\) are distinct, then they are orthogonal. Moreover, by [9, §8.1] we have \(U^{F'}_1 \simeq \prod_{\omega \in \Delta/\rho} X_\omega\) (as direct product) and can construct the Gelfand-Graev characters, the regular characters and the semisimple characters of \(G^{F'}\) as in §3.1. Note that the analogue of Theorem 3.1 is valid (see [3, Proposition 15.3, Corollaire 15.14]) and the regular character \(\phi_1\) of \(\text{Irr}(U^{F'})\) can be chosen \(F\)-stable by [19, 6.32]. In particular, if \(s\) is an \(F^*\)-stable element of \(G^{* F'^*}\), then for every \(z \in H^1(F', Z(G))\), we have

\[
F^i|_{\rho_s, \omega_0(z)} = \rho_{s, \omega_0}(F^i(z)).
\]

For every \(J \subseteq \Delta/\rho\), we write \(\omega_J\) for the corresponding \(\widetilde{T}^{F'}\)-orbit in \(\text{Irr}(U^{F'})\) and we choose a representative \(\phi_{J} \in \omega_J\) as in Convention 4.3. Then we can define \(\chi_{J,z,\psi} \in \text{Irr}(B^{F'})\) as in Equation (4.5), which satisfies Lemma 4.4.

Let \(E\) be an \(F\)- and \(F'\)-set, and let \(x \in E\) be such that \(F^k(x) = x\) and \(F'(x) = x\). Denote by \(d\) the order of \(F^k\) (viewed as an automorphism of \(G^{F'}\)). If \(d\) is even (resp. odd), then this is equivalent to \(\sigma(x) = x\) and \(F^k(x) = x\) (resp. \(\sigma F^n/d(x) = x\)). Moreover, note that if \(Z(G)^{F'} = Z(G)\), then \(Z(G)^F = \{1\}\) and \(n\) is odd. Using these facts, we can prove in a similar way as in the proof of Theorem 4.7, that for every \(\nu \in \text{Irr}(Z(G)^{F'})\) and \(A_\nu = \text{Stab}(F)\langle \nu \rangle\), there is an \(D \times A_\nu\)-equivariant bijection between

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\[\text{Irr}_\sigma(\mathbf{G}^{F'}, \nu) \text{ and } \text{Irr}_\sigma(\mathbf{B}^{F'}).\] Finally, we prove that the properties (5)-(8) of [20, §10] are satisfied as in the proof of Theorem 4.9. \qed

Remark 4.11. — This method is not sufficient to show the statement for \( p \in \{2, 3\}. \) Indeed, we need the assumption that \( p \) is a good prime for \( \mathbf{G} \) to apply the “relative” version of the McKay Conjecture proved in [7, Theorem 1.1], and [23, Proposition 8.1(ii)].

Proposition 4.12. — Suppose that \( X = \text{PSL}_\ell(p^n) \) or \( X = \text{PSU}_\ell(p^n) \) such that \( p \) is odd and \( \ell \) is an odd prime number that does not divide \( p. \) Then \( X \) is “good” for \( p. \)

Proof. — We set \( \tilde{\mathbf{G}} = \text{GL}_\ell(\mathbb{F}_p) \) and denote by \( F \) the standard Frobenius map of \( \tilde{\mathbf{G}} \) which acts by raising all entries of a matrix to the \( p \)-power. Write \( F' = F^n \) and \( \sigma \) for the non-trivial graph automorphism with respect to the \( F \)-stable Borel subgroup of lower triangular matrices and the \( F' \)-stable maximal torus \( \widetilde{T} \) of diagonal matrices of \( \tilde{\mathbf{G}}. \) We set \( \mathbf{G} = \text{SL}_\ell(\mathbb{F}_p), \) \( T = \widetilde{T} \cap \mathbf{G} \) and \( \mathbf{B} = \tilde{\mathbf{B}} \cap \mathbf{G}. \) Note that \( \tilde{\mathbf{G}}^\sigma = \text{GO}_\ell(\mathbb{F}_p) \) and \( \mathbf{G}^\sigma = \text{SO}_\ell(\mathbb{F}_p), \) which implies that \( Z(\mathbf{G}^\sigma) = \{1\} \) and \( Z(\tilde{\mathbf{G}}^\sigma) \) has order 2. Denote by \( \Delta = \{\alpha_1, \ldots, \alpha_{\ell-1}\} \) the set of simple roots of \( \mathbf{G} \) with respect to \( T \) and \( \mathbf{B}. \) We set \( \tilde{\alpha}_i = \frac{1}{2}(\alpha_i + \alpha_{\ell-i}) \) and \( \Delta_\sigma = \{\tilde{\alpha}_i \mid 1 \leq i \leq (\ell - 1)/2\}. \) Then it is proven in [17, Lemma 4.4.7] that \( \Delta_\sigma \) is the set of simple roots of \( \mathbf{G}^\sigma \) with respect to \( T^\sigma \) and \( \mathbf{B}^\sigma. \) Moreover, if we set \( \tilde{X}_{\tilde{\alpha}_i} = X_{\alpha_i}.X_{\alpha_{\ell-i}} \) for \( 1 \leq i \leq (\ell - 3)/2 \) and \( \tilde{X}_{\tilde{\alpha}_{(\ell-1)/2}} = (X_{\alpha_{(\ell-1)/2}}, X_{\alpha_{(\ell+1)/2}}), \) then the group \( \tilde{X}^\sigma_{\tilde{\alpha}_i} \) is the root subgroup corresponding to \( \tilde{\alpha}_i; \) see the proof of [17, Lemma 4.4.7]. Write \( \tilde{J} \) for the subset of \( \Delta_\sigma \) associated to a \( \sigma \)-stable subset \( J \) of \( \Delta. \) Since \( Z(\mathbf{G})^\sigma = \{1\}, \) it follows from the argument of the proof of Theorem 4.7 that \( T^\sigma_J \) (here, \( T_J \) denotes the radical of \( L_J \) as before) is the radical of the Levi subgroup \( L_{\tilde{J}} \) of \( \mathbf{G}^\sigma. \)

Therefore, by a similar argument to Lemma 4.6 and Theorem 4.7, we show that for \( \nu \in \text{Irr}(Z(\mathbf{G})^{F'}) \), there is an \( A_{\nu} \)-equivariant bijection between \( \text{Irr}_\nu(\mathbf{G}^{F'}|\nu) \) and \( \text{Irr}_\nu(\mathbf{B}^{F'}|\nu). \)

Now, we suppose that \( \chi \in \text{Irr}(\mathbf{G}^{F'}) \) is \( \tilde{T}^{F'} \)-stable and we denote by \( A_{\chi} \) its inertia subgroup in \( \text{Aut}(\mathbf{G}^{F'}). \) As above, we write \( E = \text{Irr}(\mathbf{G}^{F'}|\chi \cdot 1_C), \) where \( C = Z(\mathbf{G}^{F'}). \) Then \( |E| = \ell \) and \( \chi \cdot 1_C \) is \( A_{\chi} \)-stable. So, \( A_{\chi} \) acts on \( E. \) Suppose that \( \sigma \in A_{\chi}. \) Since \( \sigma \) has order 2 and that \( \ell \) is odd, \( \sigma \) fixes a character \( \tilde{\chi} \) of \( E. \) Thus, by Clifford theory and by [19, 6.32], the actions of \( \sigma \) on \( E \) and on \( \tilde{\mathbf{G}}^{F'}/\mathbf{G}^{F'}C \) are equivalent. Since \( \sigma \) acts by inversion on this group and has no non-trivial fixed point (because \( \ell \) is odd), we deduce that \( \tilde{\chi} \) is the unique \( \sigma \)-stable character of \( E. \) By \( \tau \in A_{\chi}. \) Then \( \tau \) and \( \sigma \) commute and \( \tau(\tilde{\chi}) \) is a \( \sigma \)-stable character of \( E. \) By unicity, \( \tau(\tilde{\chi}) = \tilde{\chi}, \)
which proves that $E$ has an $A_{\chi}$-stable element. Finally, we conclude with a similar argument to the proof of Theorem 4.9. The proof for a twisted Frobenius map is similar and the claim is proven. □

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BIBLIOGRAPHY


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Olivier BRUNAT
Université Denis Diderot - Paris 7
UFR de Mathématiques
175, rue du Chevaleret
F-75013 Paris.
email: brunat@math.jussieu.fr