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THE p-RANK STRATIFICATION OF ARTIN-SCHREIER CURVES

by Rachel PRIES & Hui June ZHU (*)

Abstract. — We study a moduli space $\mathcal{AS}_g$ for Artin-Schreier curves of genus g over an algebraically closed field k of characteristic p. We study the stratification of $\mathcal{AS}_g$ by p-rank into strata $\mathcal{AS}_{g.s}$ of Artin-Schreier curves of genus g with p-rank exactly s. We enumerate the irreducible components of $\mathcal{AS}_{g,s}$ and find their dimensions. As an application, when $p = 2$, we prove that every irreducible component of the moduli space of hyperelliptic k-curves with genus g and 2-rank s has dimension $g - 1 + s$. We also determine all pairs (p, g) for which $\mathcal{AS}_g$ is irreducible. Finally, we study deformations of Artin-Schreier curves with varying p-rank.

Résumé. — Nous étudions un espace de modules $\mathcal{AS}_g$ des courbes d'Artin Schreier de genre g sur k, un corps algébriquement clos de caractéristique p. Nous étudions la stratification de $\mathcal{AS}_g$ par le p-rang, dont la strate $\mathcal{AS}_{g,s}$ décrit les courbes de genre g et de p-rang s. On énumère les composantes irréductibles de $\mathcal{AS}_{g,s}$ et on donne leurs dimensions. Une application, dans le cas $p = 2$, est que chaque composante irréductible de l'espace de modules des courbes hyperelliptiques sur k de genre g et de 2-rang s est de dimension $g - 1 + s$. Nous déterminons toutes les paires (p, g) pour lesquelles $\mathcal{AS}_g$ est irréductible. Finalement, nous étudions les déformations des courbes d'Artin-Schreier dont le p-rang varie.

1. Introduction

Let k be an algebraically closed field of characteristic $p > 0$. An *Artin-Schreier k-curve* is a smooth projective connected k-curve Y which is a $(\mathbb{Z}/p)$-cover of the projective line. The Riemann-Hurwitz formula implies that the genus g of Y is of the form $g = d(p - 1)/2$ for some integer $d \geqslant 0$.

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The *p-rank* of Y is the integer s such that the cardinality of $\mathrm{Jac}(Y)[p](k)$ is p^s. It is well known that $0 \leqslant s \leqslant g$. By the Deuring-Shafarevich formula, $s = r(p-1)$ for some integer $r \geqslant 0$.

In this paper, we study a moduli space $\mathcal{AS}_g$ for Artin-Schreier k-curves of genus g. We study its stratification by p-rank into strata $\mathcal{AS}_{g,s}$ whose points correspond to Artin-Schreier curves of genus g with p-rank exactly s. Throughout, we assume $g = d(p-1)/2$ and $s = r(p-1)$ for some integers $d \geqslant 1$ and $r \geqslant 0$ since the problem is trivial otherwise. We denote by $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ the floor and ceiling of a real number, respectively, and use the notation $\{\cdots\}$ to denote a multi-set. We prove:

THEOREM 1.1. — *Let $g = d(p-1)/2$ with $d \geqslant 1$ and $s = r(p-1)$ with $r \geqslant 0$.*

1. *The set of irreducible components of $\mathcal{AS}_{g,s}$ is in bijection with the set of partitions $\{e_1, \dots e_{r+1}\}$ of $d+2$ into $r+1$ positive integers such that each $e_j \not\equiv 1 \bmod p$.*
2. *The irreducible component of $\mathcal{AS}_{g,s}$ for the partition $\{e_1, \dots e_{r+1}\}$ has dimension*

$$d - 1 - \sum_{j=1}^{r+1} \lfloor (e_j - 1)/p \rfloor .$$

The proof uses ideas from [3, Section 5.1], [11], and [18]. As an application of Theorem 1.1, we determine all cases when $\mathcal{AS}_g$ is irreducible, using the fact that every irreducible component of $\mathcal{AS}_g$ has dimension $d-1$, [15, Cor. 3.16].

COROLLARY 1.2. — *The moduli space $\mathcal{AS}_g$ is irreducible in exactly the following cases: (i) $p = 2$; or (ii) $g = 0$ or $g = (p-1)/2$; or (iii) $p = 3$ and $g = 2, 3, 5$.*

When $p = 2$, the moduli space $\mathcal{AS}_g$ is the same as $\mathcal{H}_g$, the moduli space of hyperelliptic k-curves of genus g. By [13, Thm. 4.1], $\mathcal{H}_g$ is irreducible of dimension $2g - 1$ when $p = 2$. Let $\mathcal{H}_{g,s} \subset \mathcal{H}_g$ denote the stratum whose points correspond to hyperelliptic k-curves of genus g with 2-rank s. Theorem 1.1 yields the following description of $\mathcal{H}_{g,s}$. This also generalizes the result $\dim(\mathcal{H}_{g,0}) = g - 1$ when $p = 2$ from [19, Prop. 4.1].

COROLLARY 1.3. — *Let $p = 2$ and $g \geqslant 1$. The irreducible components of $\mathcal{H}_{g,s}$ are in bijection with partitions of $g+1$ into $s+1$ positive integers. Every component has dimension $g - 1 + s$.*

The geometry of $\mathcal{AS}_g$ is more complicated when $p \geqslant 3$. For example, Theorem 1.1 shows that, for fixed g and s, the irreducible components of

$\mathcal{AS}_{g,s}$ can have different dimensions and thus $\mathcal{AS}_{g,s}$ is not pure in general when $p \geqslant 3$, Corollary 3.13.

Here is some motivation for these results, which also gives another illustration how the geometry of $\mathcal{AS}_g$ is more complicated when $p \geqslant 3$. Recall that the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties over k of dimension g can be stratified by p-rank. Let $V_{g,s} \subset \mathcal{A}_g$ denote the stratum of abelian varieties with p-rank s. By [17, 1.6], every component of $V_{g,s}$ has codimension $g - s$ in $\mathcal{A}_g$. Suppose M is a subspace of $\mathcal{M}_g$, the moduli space of k-curves of genus g. One can ask whether the image $T(M)$ of M under the Torelli morphism is in general position relative to the p-rank stratification. A necessary condition for an affirmative answer is that $\operatorname{codim}(T(M) \cap V_{g,s}, T(M)) = g - s$. This has been verified when $M = \mathcal{M}_g$ in [8, Thm. 2.3] and when $M = \mathcal{H}_g$ for $p \geqslant 3$ in [9, Thm. 1]. Corollary 1.3 shows that this necessary condition is satisfied for $M = \mathcal{H}_g$ when $p = 2$. Corollary 3.13 shows that it is not satisfied for $M = \mathcal{AS}_g$ when $p \geqslant 3$.

Finally, we study how the components of $\mathcal{AS}_{g,s}$ (with varying s) fit together inside $\mathcal{AS}_g$. This is related to the study of deformations of wildly ramified degree p covers with non-constant branch locus. Under the obvious necessary conditions, we prove that the p-rank of an Artin-Schreier curve can be increased by exactly $p - 1$ in a flat deformation. This yields the following result.

Theorem 1.4. — *Suppose $0 \leqslant s \leqslant g-(p-1)$. If η is an irreducible component of $\mathcal{AS}_{g,s}$ which is not open and dense in an irreducible component of $\mathcal{AS}_g$, then η is in the closure of $\mathcal{AS}_{g,s+(p-1)}$ in $\mathcal{AS}_g$.*

When $p = 2$, we are further able to give a complete combinatorial description of how the irreducible components of $\mathcal{H}_{g,s}$ (with varying s) fit together in $\mathcal{H}_g$, Corollary 4.8.

Here is an outline of the paper. In Section 2, we describe the p-ranks of Artin-Schreier curves and the relationship between irreducible components and partitions. Section 3 contains the proof of the main results. One finds Theorem 1.1 in Section 3.4, Corollary 1.2 in Section 3.5, and Corollary 1.3 in Section 3.6. The deformation results, including Theorem 1.4, are in Section 4. We conclude with some open questions.

2. Partitions and Artin-Schreier curves

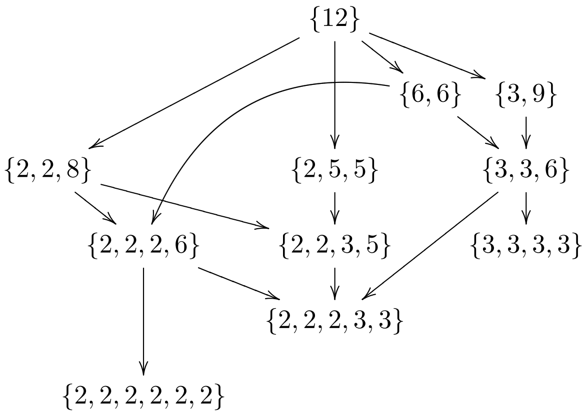
2.1. Partitions

Fix a prime $p > 0$ and an integer $d \geqslant 1$, with d even if $p = 2$. Let Ω_d be the set of partitions of $d + 2$ into positive integers $e_1, e_2, \ldots$ with each $e_j \not\equiv 1 \bmod p$. Let $\Omega_{d,r}$ be the subset of Ω_d consisting of partitions of length $r + 1$. If $\vec{E} \in \Omega_d$, let $r := r(\vec{E})$ be the integer so that $\vec{E} \in \Omega_{d,r}$. Write $\vec{E}$ as a multi-set $\{e_1, \ldots, e_{r+1}\}$ with $e_1 \leqslant \cdots \leqslant e_{r+1}$.

There is a natural partial ordering $\prec$ on Ω_d so that $\vec{E} \prec \vec{E}'$ if $\vec{E}'$ is a refinement of $\vec{E}$, in other words, if the entries of $\vec{E}'$ can be divided into disjoint subsets whose sums are in bijection with the entries of $\vec{E}$. Using this partial ordering, one can construct a directed graph G_d. The vertices of the graph correspond to the partitions $\vec{E}$ in Ω_d. There is an edge from $\vec{E}$ to $\vec{E}'$ if and only if $\vec{E} \prec \vec{E}'$, and $\vec{E} \neq \vec{E}'$, and there is no partition lying strictly in between them (*i.e.*, if $\vec{E} \prec \vec{E}'' \prec \vec{E}'$ for some $\vec{E}''$ in Ω_d then $\vec{E}'' = \vec{E}$ or $\vec{E}'' = \vec{E}'$).

An edge $\vec{E} \prec \vec{E}'$ in the directed graph G_d can be of two types. The first type has $r(\vec{E}) = r(\vec{E}') - 1$. In this case, one entry e of $\vec{E}$ splits into two entries e_1 and e_2 of $\vec{E}'$ such that $e = e_1 + e_2$ and none of the three is congruent to 1 modulo p. One can summarize this by writing $\{e\} \mapsto \{e_1, e_2\}$. The second type has $r(\vec{E}) = r(\vec{E}') - 2$. In this case, one entry e of $\vec{E}$ splits into three entries e_1, e_2, e_3 of $\vec{E}'$ such that $e = e_1 + e_2 + e_3$ and each $e_j \equiv (p + 1)/2 \bmod p$. It follows that none of the four is congruent to 1 modulo p. One can summarize this by writing $\{e\} \mapsto \{e_1, e_2, e_3\}$.

Example 2.1. — Let $p = 3$ and $d = 10$. Here is the graph G_{10} for Ω_{10}.



We skip the proofs of some of the following straightforward results. Lemma 2.2 is used in [2], while Lemmas 2.3 and 2.4 are used in Section 3.5.

LEMMA 2.2. — *The set* $\Omega_{d,0}$ *is nonempty if and only if* $p \nmid (d+1)$. *If* $p \nmid (d+1)$, *then* $\Omega_{d,0}$ *contains one partition* $\{d+2\}$ *which is an initial vertex of* G_d. *If* $p \mid (d+1)$, *then* $\Omega_{d,1}$ *consists of* $\lceil (d+1)(p-2)/2p \rceil$ *partitions, and every vertex of* G_d *is larger than one of these.*

LEMMA 2.3. — *If* $p = 2$, *there is a unique maximal partition* $\{2, \ldots, 2\}$ *in* Ω_d *with length* $d/2 + 1$.

LEMMA 2.4. — *Let* $p \geqslant 3$. *A partition is maximal if and only if its entries all equal two or three. Every integer* $r + 1$ *with* $(d-1)/3 \leqslant r \leqslant d/2$ *occurs exactly once as the length of a maximal partition. There are* $\lfloor d/2 \rfloor - \lceil (d-4)/3 \rceil$ *maximal partitions. There is a unique maximal partition if and only if* $d \in \{1, 2, 3, 5\}$.

Proof. — The first statement is true since if $e \geqslant 4$ then there are $e_1, e_2 \in \mathbb{Z}_{>0}$ so that $e_j \not\equiv 1 \bmod p$ and $e_1 + e_2 = e$. For the other statements, let $\vec{E}$ be a maximal partition of $d + 2$. Let b denote the number of the entries of $\vec{E}$ which equal 3. Note that $0 \leqslant b \leqslant (d+2)/3$. Let $r + 1$ be the length of $\vec{E}$. Then $d + 2 = 2(r+1) + b$ and $(d+2)/3 \leqslant r + 1 \leqslant (d+2)/2$. Any choice of $r + 1$ in this range yields a unique choice of b which determines a unique partition $\vec{E}$. □

Remark 2.5. — When $p = 2$, every path in G_d from the partition $\{d+2\}$ to the partition $\{2, \ldots, 2\}$ has the same length, which is $d/2$. When $p = 3$, every path in G_d from a minimal to a maximal vertex has the same length, which is $\lfloor d/3 \rfloor$. This property does not hold in general for $p \geqslant 5$.

2.2. Artin-Schreier curves

Here is a review of some basic Artin-Schreier theory. Let Y be an Artin-Schreier k-curve. Then there is a $\mathbb{Z}/p$-cover $\phi : Y \to \mathbb{P}^1_k$ with an affine equation of the form $y^p - y = f(x)$ for some non-constant rational function $f(x) \in k(x)$. At each ramification point, there is a filtration of the inertia group $\mathbb{Z}/p$, called the filtration of higher ramification groups in the lower numbering [21, IV].

Let $\{P_1, \ldots, P_{r+1}\}$ be the set of poles of $f(x)$ on the projective line $\mathbb{P}^1_k$. Let d_j be the order of the pole of $f(x)$ at P_j. One may assume that $p \nmid d_j$ by Artin-Schreier theory. Then d_j is the *lower jump* at P_j, *i.e.*, the last index for which the higher ramification group above P_j is nontrivial. Let $e_j = d_j + 1$. Then $e_j \geqslant 2$ and $e_j \not\equiv 1 \bmod p$. The ramification divisor of ϕ is $D := \sum_{j=1}^{r+1} e_j P_j$.

Lemma 2.6. — *The genus of* Y *is* $g_Y = ((\sum_{j=1}^{r+1} e_j) - 2)(p-1)/2$. *The* p*-rank of* Y *is* $s_Y = r(p-1)$.

Proof. — The first statement follows from the Riemann-Hurwitz formula using [21, IV, Prop. 4] and the second from the Deuring-Shafarevich formula [6, Cor. 1.8]. See [23, Remark 1.4] or [24, Section 2] for details. ☐

2.3. The p-rank of Artin-Schreier curves and partitions

The Artin-Schreier curves of genus $g = d(p-1)/2$ with p-rank $r(p-1)$ are intimately related to the partition sets $\Omega_{d,r}$ as defined in Section 2.1.

Lemma 2.7. — *There exists an Artin-Schreier* k*-curve of genus* g *with* p*-rank* $r(p-1)$ *if and only if* $d := 2g/(p-1)$ *is a nonnegative integer and* $\Omega_{d,r}$ *is nonempty.*

Proof. — By Lemma 2.6, the existence of an Artin-Schreier k-curve with genus $g = d(p-1)/2$ and p-rank $r(p-1)$ is equivalent to the existence of $f(x) \in k(x)$ whose poles have orders $\{e_1 - 1, \ldots, e_{r+1} - 1\}$ where each $e_j \not\equiv 1 \bmod p$ and $\sum_{j=1}^{r+1} e_j = d + 2$. This is equivalent to $\Omega_{d,r}$ being nonempty. ☐

Example 2.8. — Let $p = 2$. Let $g \geqslant 0$ and $0 \leqslant s \leqslant g$. Then $\Omega_{2g,s}$ is non-empty since $2g + 2$ can be partitioned into $s + 1$ even integers. Therefore, there exists an Artin-Schreier k-curve of genus g and p-rank s in characteristic 2.

Example 2.9. — Let $p \geqslant 3$. There exists an Artin-Schreier k-curve of genus $g = d(p-1)/2$ with p-rank 0 if and only if $p \nmid (d+1)$ by Lemma 2.2. There exists an ordinary Artin-Schreier k-curve (*i.e.*, with p-rank g) if and only if $2 \mid d$. If $2 \nmid d$, the largest p-rank which occurs for an Artin-Schreier k-curve of genus g is $s = g - (p-1)/2$ by Lemma 2.4.

3. Moduli spaces of Artin-Schreier curves

Consider fixed parameters p, $g = d(p-1)/2$ with $d \geqslant 1$, and $s = r(p-1)$ with $0 \leqslant s \leqslant g$. In this section, we study the p-rank s strata $\mathcal{AS}_{g,s}$ of the moduli space $\mathcal{AS}_g$ of Artin-Schreier curves of genus g. We show the irreducible components of $\mathcal{AS}_{g,s}$ are in bijection with the elements of $\Omega_{d,r}$ and find the dimensions of these components.

3.1. Artin-Schreier covers

Let S be a k-scheme. An *S-curve* is a proper flat morphism $Y \to S$ whose geometric fibres are smooth connected curves. An *Artin-Schreier curve* Y over S is an S-curve for which there exists an (unspecified) inclusion $\iota : \mathbb{Z}/p \hookrightarrow \mathrm{Aut}_S(Y)$ such that the quotient $Y/\iota(\mathbb{Z}/p)$ is a ruled scheme. This means that there is an (unspecified) isomorphism between each geometric fibre of $Y/\iota(\mathbb{Z}/p)$ and $\mathbb{P}^1$. An *Artin-Schreier cover* over S is a $\mathbb{Z}/p$-cover $\phi : Y \to \mathbb{P}^1_S$. In other words, it is an Artin-Schreier curve Y over S along with the data of a specified inclusion $\iota : \mathbb{Z}/p \hookrightarrow \mathrm{Aut}_S(Y)$ and a specified isomorphism $Y/\iota(\mathbb{Z}/p) \simeq \mathbb{P}^1_S$.

Consider the following contravariant functors from the category of k-schemes to sets: $\mathcal{AS}_g$ (resp. $\mathcal{AS}\mathrm{cov}_g$) which associates to S the set of isomorphism classes of Artin-Schreier curves (resp. covers) over S with genus g. As in [15, Prop. 2.7], one can show that there is an algebraic stack representing $\mathcal{AS}_g$ which we denote again by the symbol $\mathcal{AS}_g$. Similarly, *e.g.*, [14, pg. 1], there is an algebraic stack representing $\mathcal{AS}\mathrm{cov}_g$ which we denote again by the symbol $\mathcal{AS}\mathrm{cov}_g$. The next lemma is about a natural map from $\mathcal{AS}\mathrm{cov}_g$ to $\mathcal{AS}_g$.

Lemma 3.1. — *Let $g \geqslant 2$. There is a morphism $F : \mathcal{AS}\mathrm{cov}_g \to \mathcal{AS}_g$ and the fibre of F over every geometric point of $\mathcal{AS}_g$ has dimension* 3.

Proof. — There is a functorial transformation $\mathcal{AS}\mathrm{cov}_g(S) \to \mathcal{AS}_g(S)$ that takes the isomorphism class of a given Artin-Schreier cover $\phi : Y \to \mathbb{P}^1_S$ over S to the isomorphism class of the Artin-Schreier curve Y over S. In other words, the transformation is defined by forgetting the inclusion ι and the isomorphism $Y/\iota(\mathbb{Z}/p) \simeq \mathbb{P}^1$ (and taking the quotient of the set of inclusions ι by the action of $\mathrm{Aut}(\mathbb{Z}/p)$). This transformation yields a morphism $F : \mathcal{AS}\mathrm{cov}_g \to \mathcal{AS}_g$ by Yoneda's lemma.

To prove the second claim, it suffices to work locally in the étale topology. Given an Artin-Schreier S-curve Y, by [3, pg. 232], after an étale extension of S, there exists an inclusion $\iota : \mathbb{Z}/p \hookrightarrow \mathrm{Aut}_S(Y)$ and an isomorphism $I : Y/\iota(\mathbb{Z}/p) \to \mathbb{P}^1_S$. Thus Y is in the image of F. There are only finitely many choices for ι since $\mathrm{Aut}_S(Y)$ is finite for $g \geqslant 2$ (*e.g.*, [7, Theorem 1.11]). There is a three-dimensional choice for the isomorphism I since $\dim(\mathrm{Aut}(\mathbb{P}^1_S)) = 3$. Thus the fibre of F over Y has dimension three. □

3.2. The ramification divisor

This section is about the ramification divisor of a given Artin-Schreier cover.

Let S be a k-scheme and let $n \in \mathbb{Z}_{>0}$. Consider the contravariant functor $\mathcal{C}\mathrm{div}_n$, from the category of k-schemes to sets, which associates to S the set of isomorphism classes of relative effective Cartier divisors of $\mathbb{P}^1_S$ of constant degree n. This functor is represented by a (Hilbert) scheme which we denote also by $\mathcal{C}\mathrm{div}_n$.

There is a discrete invariant $\vec{E}$ which induces a natural stratification $\mathcal{C}\mathrm{div}_{n,\vec{E}}$ of $\mathcal{C}\mathrm{div}_n$. To see this, suppose $S = \mathrm{Spec}(K)$ where K is a field with $\mathrm{char}(K) = p$. Given $D \in \mathcal{C}\mathrm{div}_n(S)$, one can associate to D a locally principal effective Weil divisor of $\mathbb{P}^1_S$ with degree n by [12, II, Prop. 6.11, Remark 6.11.2]. After a finite flat extension $S' \to S$, one can write $D' = D \times_S S' = \sum_{j=1}^{r+1} e_j P_j$ where $e_j \geqslant 1$ and $\sum_{j=1}^{r+1} e_j = n$ and where $\{P_1, \dots, P_{r+1}\}$ is a set of distinct horizontal sections of $\mathbb{P}^1_{S'}$. Let $\vec{E}(D) = \{e_1, \dots, e_{r+1}\}$ with $e_1 \leqslant \cdots \leqslant e_{r+1}$. The partition $\vec{E}(D)$ of n induces a natural stratification $\mathcal{C}\mathrm{div}_{n,\vec{E}}$ of $\mathcal{C}\mathrm{div}_n$ (where the sections $\{P_1, \dots, P_{r+1}\}$ associated to D can vary). Let $\mathcal{C}\mathrm{div}^1_n = \cup_{\vec{E} \in \Omega_{n-2}} \mathcal{C}\mathrm{div}_{n,\vec{E}}$ (*i.e.,* all $\vec{E}$ for which each $e_j \not\equiv 1 \bmod p$). For fixed $\vec{E}$, let $H_{\vec{E}} \subset S_{r+1}$ be the subgroup of the symmetric group generated by all transpositions (j_1, j_2) for which $e_{j_1} = e_{j_2}$.

Lemma 3.2. — *If $\vec{E} \in \Omega_{d,r}$, then $\mathcal{C}\mathrm{div}_{d+2,\vec{E}}$ is irreducible of dimension $r+1$.*

Proof. — Let Δ denote the weak diagonal of $(\mathbb{P}^1)^{r+1}$, consisting of $(r+1)$-tuples with at least two coordinates equal. The quotient of $(\mathbb{P}^1)^{r+1} - \Delta$ by the action of $H_{\vec{E}}$ is irreducible with dimension $r+1$. By the remarks preceding this lemma, the spaces $\mathcal{C}\mathrm{div}_{d+2,\vec{E}}$ and $[(\mathbb{P}^1)^{r+1} - \Delta]/H_{\vec{E}}$ are locally isomorphic for the finite flat topology, where the isomorphism identifies D with the equivalence class of $(P_1, \dots, P_{r+1})$. Thus $\mathcal{C}\mathrm{div}_{d+2,\vec{E}}$ is irreducible with dimension $r+1$. □

Proposition 3.3. — *Let $d = 2g/(p-1)$. There is a morphism $B : \mathcal{AS}\mathrm{cov}_g \to \mathcal{C}\mathrm{div}_{d+2}$ and the image of B is $\mathcal{C}\mathrm{div}^1_{d+2}$.*

Proof. — Given an Artin-Schreier cover ϕ over S, consider the closed subscheme D of the fixed points under $\iota(\mathbb{Z}/p)$; this is a relative Cartier divisor of $\mathbb{P}^1_S$ of constant degree $d+2$ by [3, Lemma 5.2.3, pg. 232]. The functorial transformation $\mathcal{AS}\mathrm{cov}_g(S) \to \mathcal{C}\mathrm{div}_{d+2,\vec{E}}(S)$ yields a morphism $B : \mathcal{AS}\mathrm{cov}_g \to \mathcal{C}\mathrm{div}_{d+2}$ by Yoneda's lemma.

If $D \in \mathcal{C}\mathrm{div}_{d+2}(S)$, then there is a restriction on $\vec{E}(D)$. As before, one can identify a pullback $D' = D \times_S S'$ with an effective Weil divisor $\sum_{j=1}^{r+1} e_j P_j$ where $e_j \geqslant 1$ are such that $\sum_{j=1}^{r+1} e_j = d+2$ and where $\{P_1, \dots, P_{r+1}\}$ is a set of distinct horizontal sections of $\mathbb{P}^1_{S'}$. If $D = B(\phi)$ is in the image of B, then $\{P_1, \dots, P_{r+1}\}$ constitutes the branch locus of the pullback $\phi' = \phi \times_S S'$ of ϕ and $d_j = e_j - 1$ is the lower jump of ϕ' above the geometric generic point of P_j by [21, IV, Prop. 4]. As seen in Section 2.2, $e_j \not\equiv 1 \bmod p$ for $1 \leqslant j \leqslant r+1$. Thus the image of B is contained in $\mathcal{C}\mathrm{div}^1_{d+2}$.

Suppose $\vec{E} \in \Omega_d$. To prove that $\mathcal{C}\mathrm{div}^1_{d+2,\vec{E}}$ is contained in the image of B, by descent, it suffices to work locally in the finite flat topology. Given $D = \sum_{j=1}^{r+1} e_j P_j$, consider the divisor $\tilde{D} = \sum_{j=1}^{r+1}(e_j - 1)P_j$ of $\mathbb{P}^1_S$. There is a non-constant function $f(x) \in \mathcal{O}(S)(x)$ with $\mathrm{div}_\infty(f(x)) = \tilde{D}$. Consider the cover $\phi : Y \to \mathbb{P}^1_S$ given by the affine equation $y^p - y = f(x)$. Then ϕ is an Artin-Schreier cover with ramification divisor D and the fibres of Y have genus g by Lemma 2.6. Thus $D \in \mathrm{Im}(B)$. □

Let $\mathcal{AS}_{g,\vec{E}}$ (resp. $\mathcal{AS}\mathrm{cov}_{g,\vec{E}}$) denote the locally closed reduced subspace of $\mathcal{AS}_g$ (resp. $\mathcal{AS}\mathrm{cov}_g$) whose geometric points correspond to Artin-Schreier covers whose ramification divisor has partition $\vec{E}$. The morphisms F and B respect the partition $\vec{E}$. Let $F_{\vec{E}} : \mathcal{AS}\mathrm{cov}_{g,\vec{E}} \to \mathcal{AS}_{g,\vec{E}}$ and $B_{\vec{E}} : \mathcal{AS}\mathrm{cov}_{g,\vec{E}} \to \mathcal{C}\mathrm{div}_{d+2,\vec{E}}$ denote the natural restrictions.

3.3. Artin-Schreier covers with fixed ramification divisor

In this section, we fix a partition $\vec{E} \in \Omega_{g,r}$ and a divisor $D \in \mathcal{C}\mathrm{div}_{d+2,\vec{E}}$ and study the fibre of B over D. Using [3, Section 5.1], we show that this fibre is irreducible and compute its dimension. We provide some intuition by describing the equations for an Artin-Schreier cover with ramification divisor D.

Notation 3.4. — Let $\vec{E} \in \Omega_{d,r}$ be a fixed partition $\{e_1, \dots, e_{r+1}\}$ of $d+2$. Consider a fixed divisor D corresponding to a point of $\mathcal{C}\mathrm{div}_{d+2,\vec{E}}$. Let $\mathcal{AS}\mathrm{cov}_{g,D}$ be the fibre of $B_{\vec{E}} : \mathcal{AS}\mathrm{cov}_{g,\vec{E}} \to \mathcal{C}\mathrm{div}_{d+2,\vec{E}}$ over D.

Notation 3.5. — For $1 \leqslant j \leqslant r+1$, let $t_j = d_j - \lfloor d_j/p \rfloor$ where $d_j = e_j - 1$. Let $N_{\vec{E}} = \sum_{j=1}^{r+1} t_j$. Let $M_j = (\mathbb{A}^1)^{t_j - 1} \times (\mathbb{A}^1 - \{0\})$. Let $M = \times_{j=1}^{r+1} M_j$. There is an action on M by the subgroup $H_{\vec{E}} \subset S_{r+1}$ generated by all transpositions (j_1, j_2) for which $d_{j_1} = d_{j_2}$. Define $M_D = M/H_{\vec{E}}$.

Proposition 3.6. — *With notation as in 3.4 and 3.5, the fibre $\mathcal{AS}\mathrm{cov}_{g,D}$ of $B_{\vec{E}}$ over D is locally isomorphic for the finite flat topology to M_D. Thus $\mathcal{AS}\mathrm{cov}_{g,D}$ is irreducible with dimension $N_{\vec{E}}$ over k.*

Proof. — By the definition of M_D, the first claim implies the second. For the first claim, let η denote a labeling of the $r+1$ points in the support of D. Let $\mathcal{AS}\mathrm{cov}^{\eta}_{g,D}$ be the contravariant functor which associates to S the set of covers ϕ in the fibre $\mathcal{AS}\mathrm{cov}_{g,D}(S)$ along with a labeling η of the branch locus. It suffices to show that the moduli space for $\mathcal{AS}\mathrm{cov}^{\eta}_{g,D}$ is locally isomorphic to M. This statement can be found in [3, pg. 229, pg. 233]. □

Remark 3.7. — In [11, Cor. 2.10], the author constructs an ind-scheme $\mathcal{M}$ which is a fine moduli space for covers $Y \to \mathbb{P}^1$ of k-schemes with group $\mathbb{Z}/p$ and branch locus $\{P_1, \dots, P_{r+1}\}$ (where Y has unbounded genus). The k-points of $\mathcal{AS}\mathrm{cov}_{g,D}$ are in bijection with the k-points of $\mathcal{M}$ such that Y has genus g. Recall from [11] that $\mathcal{M}$ is a direct limit of affine schemes. This direct limit arises because if $S = \mathrm{Spec}(K)$ where K is not perfect, then there are non-trivial Artin-Schreier covers over S which become trivial after a finite flat extension of S. In [18], the author addressed this issue using a *configuration space* whose k-points are in bijection with covers defined over k. In Proposition 3.6, we instead followed the approach of [3, Section 5.1].

Remark 3.8. — Proposition 3.6 implies that a flat base change of $\mathcal{AS}\mathrm{cov}_{g,\vec{E}}$ is a $\mathbb{G}_a^n \times \mathbb{G}_m$-bundle over $\mathcal{C}\mathrm{div}_{d+2,\vec{E}}$ for some n.

Remark 3.9. — For the convenience of the reader, we provide some intuition about Proposition 3.6. Let S be an irreducible affine k-scheme. Suppose $\phi \in \mathcal{AS}\mathrm{cov}_{g,D}(S)$ is an Artin-Schreier cover over S with ramification divisor D. Then ϕ has an affine equation $y^p - y = f(x)$ for some $f(x) \in \mathcal{O}(S)(x)$. The automorphism $\sigma = \iota(1)$ acts via $\sigma(y) = y + z$ for some $z \in (\mathbb{Z}/p)^*$. Two such covers $\phi_1 : y^p - y = f_1(x)$ and $\phi_2 : y^p - y = f_2(x)$ are isomorphic if and only if $f_2(x) = (z_2/z_1)f_1(x) + \delta^p - \delta$ for some $\delta \in \mathcal{O}(S)(x)$, see *e.g.*, [18, Lemma 2.1.5]. After possibly changing $f(x)$, one can suppose $z = 1$.

The cover ϕ is in *standard form* if $p \nmid i$ for any monomial $c_i x^i$ in $f(x)$ whose coefficient c_i is generically non-nilpotent. Given an Artin-Schreier cover ϕ, after a finite flat extension $S' \to S$, then $\phi \times_S S'$ has an affine equation in standard form. To prove this, one uses an étale cover $S'' \to S$ with equation $a^p - a = c_0$ to remove a constant coefficient $c_0 \in \mathcal{O}(S)$ from $f(x)$. If $f(x)$ contains a monomial cx^{pw} with $w \in \mathbb{Z}_{>0}$, one uses a purely inseparable cover $S' \to S''$ with equation $b^p = c$ to replace cx^{pw} with the monomial bx^w. These transformations are uniquely determined and do not change the isomorphism class of $\phi \times_S S'$.

Suppose $D = \sum_{j+1}^{r+1} e_j P_j$ where $\{P_1, \dots P_{r+1}\}$ is a fixed set of distinct horizontal sections of $\mathbb{P}^1_S$. If ϕ has ramification divisor D, then $f(x)$ has a partial fraction decomposition $f(x) = \sum_{j=1}^{r+1} g_j(x)$ where $g_j(x) \in (x - P_j)^{-1}\mathcal{O}(S)[(x-P_j)^{-1}]$ is a polynomial of degree d_j in the variable $(x - P_j)^{-1}$ with no constant term. (If $P_j = P_\infty$, let $(x-P_j)^{-1}$ denote x for consistency of notation.) If ϕ is in standard form, one can write $g_j(x) = \sum_{i=1}^{d_j} c_{i,j}(x-P_j)^{-i}$ where $c_{i,j} = 0$ if $p \mid i$ and $c_{d_j,j}$ is never zero. The isomorphism between M and $\mathcal{AS}\mathrm{cov}^{\eta}_{g,D}$ in the finite flat topology identifies $(\times_{j=1}^{r+1} \times_{i=1,\ p\nmid i}^{d_j} c_{i,j})$ with the isomorphism class of the Artin-Schreier cover $y^p - y = \sum_{j=1}^{r+1} g_j(x)$ (with the implicit labeling of $\{P_1, \dots, P_{r+1}\}$).

3.4. Irreducible components of the p-rank strata

Recall that $g = d(p-1)/2$ with $d \geqslant 1$ and d even if $p = 2$ and $s = r(p-1)$ with $0 \leqslant s \leqslant g$. The p-rank induces a stratification of $\mathcal{AS}_g$ (resp. $\mathcal{AS}\mathrm{cov}_g$). Let $\mathcal{AS}_{g,s}$ (resp. $\mathcal{AS}\mathrm{cov}_{g,s}$) denote the locally closed reduced subspace of $\mathcal{AS}_g$ (resp. $\mathcal{AS}\mathrm{cov}_g$) whose geometric points have p-rank s.

Theorem 3.10. — *The irreducible components of $\mathcal{AS}\mathrm{cov}_{g,s}$ are the strata $\mathcal{AS}\mathrm{cov}_{g,\vec{E}}$ with $\vec{E} \in \Omega_{d,r}$. If $\vec{E} = \{e_1, \dots, e_{r+1}\}$, then the dimension over k of the irreducible component $\mathcal{AS}\mathrm{cov}_{g,\vec{E}}$ is $d+2-\sum_{j=1}^{r+1}\lfloor (e_j-1)/p \rfloor$.*

Proof. — The image of $\mathcal{AS}\mathrm{cov}_{g,s}$ under B is the union of the strata $\mathcal{C}\mathrm{div}_{d+2,\vec{E}}$ of $\mathcal{C}\mathrm{div}^1_{d+2}$ with $r(\vec{E}) = r$ by Proposition 3.3. The stratum $\mathcal{C}\mathrm{div}_{d+2,\vec{E}}$ is irreducible of dimension $r+1$ by Lemma 3.2.

For $\vec{E} \in \Omega_{d,r}$, consider the morphism $B_{\vec{E}} : \mathcal{AS}\mathrm{cov}_{g,\vec{E}} \to \mathcal{C}\mathrm{div}_{d+2,\vec{E}}$. The fibre of $B_{\vec{E}}$ over a fixed divisor D is irreducible by Proposition 3.6. By Zariski's main theorem, $\mathcal{AS}\mathrm{cov}_{g,\vec{E}}$ is irreducible since $B_{\vec{E}}$ has irreducible fibres and image. Thus the irreducible components of $\mathcal{AS}\mathrm{cov}_{g,s}$ are the strata $\mathcal{AS}\mathrm{cov}_{g,\vec{E}}$ with $\vec{E} \in \Omega_{d,r}$.

The morphism $B_{\vec{E}}$ is flat since all its fibres are isomorphic. Thus the dimension of $\mathcal{AS}\mathrm{cov}_{g,\vec{E}}$ is the sum of the dimensions of $\mathcal{C}\mathrm{div}_{d+2,\vec{E}}$ and of the fibres of $B_{\vec{E}}$. This equals $r+1+\sum_{j=1}^{r+1}(d_j - \lfloor d_j/p \rfloor)$ by Lemma 3.2 and Proposition 3.6. This simplifies to $d+2-\sum_{j=1}^{r+1}\lfloor (e_j-1)/p \rfloor$. ☐

Theorem 1.1 in the introduction follows immediately from the next corollary.

Corollary 3.11. — *The irreducible components of* $\mathcal{AS}_{g,s}$ *are the strata* $\mathcal{AS}_{g,\vec{E}}$ *with* $\vec{E} \in \Omega_{d,r}$. *If* $\vec{E} = \{e_1, \dots, e_{r+1}\}$, *then the dimension* $d_{\vec{E}}$ *over* k *of the irreducible component* $\mathcal{AS}_{g,\vec{E}}$ *is* $d - 1 - \sum_{j=1}^{r+1} \lfloor (e_j - 1)/p \rfloor$.

Proof. — Let W be an irreducible component of $\mathcal{AS}_{g,s}$. By Lemma 3.1, $F^{-1}(W)$ is a union of irreducible components of $\mathcal{AS}\mathrm{cov}_{g,s}$. By Theorem 3.10, these are indexed by partitions $\vec{E} \in \Omega_{d,r}$. The morphism F respects the partition $\vec{E}$. In other words, given an Artin-Schreier curve Y, every Artin-Schreier cover $\phi : Y \to \mathbb{P}^1_k$ has the same partition. Thus there is a unique partition occurring for points in $F^{-1}(W)$, and so $F^{-1}(W)$ is irreducible. So the irreducible components of $\mathcal{AS}_{g,s}$ are the strata $\mathcal{AS}_{g,\vec{E}}$ with $\vec{E} \in \Omega_{d,r}$. The second statement follows by Lemma 3.1 for $g \geqslant 2$ since $\dim(W) = \dim(F^{-1}(W)) - 3$ and by direct computation for $g = 1$. ☐

Example 3.12. — Let $p = 3$ and $g = 10$. Here are the dimensions $d_{\vec{E}}$ of the irreducible components of $\mathcal{AS}_{10,s}$.

s	dimension
0	$d_{\{12\}} = 6$
2	$d_{\{3,9\}} = 7,\ d_{\{6,6\}} = 7$
4	$d_{\{2,2,8\}} = 7,\ d_{\{2,5,5\}} = 7,\ d_{\{3,3,6\}} = 8$
6	$d_{\{2,2,2,6\}} = 8,\ d_{\{2,2,3,5\}} = 8,\ d_{\{3,3,3,3\}} = 9$
8	$d_{\{2,2,2,3,3\}} = 9$
10	$d_{\{2,2,2,2,2,2\}} = 9$

The next corollary shows that the image of $\mathcal{AS}_g$ under the Torelli morphism is not in general position relative to the p-rank stratification of $\mathcal{A}_g$ when $p \geqslant 3$.

Corollary 3.13. — *If* $p \geqslant 3$, *then* $\mathrm{codim}(\mathcal{AS}_{g,s}, \mathcal{AS}_g) < g - s$.

Proof. — Let $d = 2g/(p-1)$ and $r = s/(p-1)$. Let $\epsilon = \min \sum_{j=1}^{r+1} \lfloor (e_j - 1)/p \rfloor$ where the minimum ranges over all partitions $\{e_1, \dots, e_{r+1}\}$ with fixed sum $d + 2$. By Corollary 3.11, $\mathrm{codim}(\mathcal{AS}_{g,s}, \mathcal{AS}_g) = \epsilon$. Since $\lfloor (e_j - 1)/p \rfloor \leqslant (e_j - 2)/p$, one sees that $\epsilon \leqslant (d - 2r)/p = 2(g - s)/p(p-1)$. Thus $\epsilon < g - s$ if $p \geqslant 3$. ☐

3.5. Irreducibility of the Artin-Schreier locus

As an application of Theorem 1.1, we determine all pairs (p, g) for which $\mathcal{AS}_g$ is irreducible.

Corollary 1.2. *The moduli space $\mathcal{AS}_g$ is irreducible in exactly the following cases: (i) $p = 2$; or (ii) $g = 0$ or $g = (p-1)/2$; or (iii) $p = 3$ and $g = 2, 3, 5$.*

Proof. — Let $d = 2g/(p-1)$. Recall that $d_{\vec{E}}$ is the dimension of $\mathcal{AS}_{g,\vec{E}}$. The first claim is that there is a bijection between irreducible components of $\mathcal{AS}_g$ and partitions $\vec{E} \in \Omega_{d,r}$ so that $d_{\vec{E}} = d - 1$. To see this, note that [15, Cor. 3.16] implies that every irreducible component of $\mathcal{AS}_g$ has dimension $d-1$. If Γ is an irreducible component of $\mathcal{AS}_g$, then there is a partition $\vec{E} \in \Omega_d$ and an open subset $U \subset \Gamma$ so that $U \subset \mathcal{AS}_{g,\vec{E}}$. Then $d_{\vec{E}} = \dim(\Gamma) = d - 1$. Conversely, suppose $d_{\vec{E}} = d - 1$ for some $\vec{E} \in \Omega_d$. Then the irreducible space $\mathcal{AS}_{g,\vec{E}}$ is open in a unique irreducible component Γ of $\mathcal{AS}_g$.

Thus, $\mathcal{AS}_g$ is irreducible if and only if there is exactly one partition $\vec{E} \in \Omega_d$ with dimension $d_{\vec{E}} = d-1$. Write $\vec{E} = \{e_1, \ldots, e_{r+1}\}$. By Theorem 1.1, $d_{\vec{E}} = d - 1$ if and only if $e_j < p + 1$ for $1 \leqslant j \leqslant r + 1$.

If $p = 2$, only one partition satisfies the condition $e_j < 3$ for each j, namely the partition $\{2, \ldots, 2\}$, Lemma 2.3. Thus $\mathcal{A}_g$ is irreducible for all g when $p = 2$.

For arbitrary p, if $g = 0$ (resp. $g = (p-1)/2$) then $d = 0$ (resp. $d = 1$), and there is only one partition satisfying $e_j < p + 1$, namely the partition $\{2\}$ (resp. $\{3\}$). Thus $\mathcal{AS}_g$ is irreducible in these cases.

If $p = 3$ and $d = 2$ (resp. 3, 5), only one partition satisfies the condition $e_j < 4$, namely $\{2, 2\}$, (resp. $\{2, 3\}$, $\{2, 2, 3\}$). Thus $\mathcal{AS}_g$ is irreducible in these cases.

Suppose $p \geqslant 3$ and $d \geqslant 2$ and that $\mathcal{AS}_g$ is irreducible. If $\vec{E}$ is a maximal partition, then its entries satisfy $e_j \leqslant 3 < p + 1$. Thus Ω_d has a unique maximal partition. By Lemma 2.4, this implies $d \in \{2, 3, 5\}$. If $p \geqslant 5$, then there are at least two partitions satisfying $e_j < p+1$: for example, $\{4\}$ and $\{2, 2\}$ when $d = 2$; $\{5\}$ and $\{2, 3\}$ when $d = 3$; $\{2, 5\}$ and $\{2, 2, 3\}$ when $d = 5$. This is a contradiction and so cases (i)-(iii) are the only cases when $\mathcal{AS}_g$ is irreducible. □

3.6. Hyperelliptic curves in characteristic 2

Let $\mathcal{H}_g$ be the moduli space of hyperelliptic k-curves of genus g. Let $\mathcal{H}_{g,s}$ denote the locally closed reduced subspace of $\mathcal{H}_g$ parametrizing hyperelliptic k-curves of genus g with p-rank s. When $p = 2$, $\mathcal{H}_g$ is the same as $\mathcal{AS}_g$. This yields the following result.

Corollary 1.3. *Let $p = 2$. The irreducible components of $\mathcal{H}_{g,s}$ are in bijection with partitions of $g+1$ into $s+1$ positive integers. Every component has dimension $g - 1 + s$ over k.*

Proof. — By Corollary 3.11, the irreducible components of $\mathcal{H}_{g,s}$ are in bijection with the partitions of $d + 2 = 2g + 2$ into $s + 1$ even positive integers, which are in bijection with the partitions of $g + 1$ into $s + 1$ positive integers. The dimension of the irreducible component for $\vec{E} = \{e_1, \ldots, e_{s+1}\}$ is $(d - 1) - \sum_{j=1}^{s+1} \lfloor (e_j - 1)/2 \rfloor$. This simplifies to $g - 1 + s$ since e_j is even and $\lfloor (e_j - 1)/2 \rfloor = e_j/2 - 1$. ☐

4. Deformation results and open questions

In this section, we give some results on how the irreducible components of $\mathcal{AS}_{g,s}$ (with varying s) fit together within $\mathcal{AS}_g$. This involves deformations of wildly ramified covers with non-constant branch locus.

4.1. A deformation result for wildly ramified covers

The main result of this section is that, under the obvious necessary conditions, the p-rank of an Artin-Schreier curve can be increased by exactly $p - 1$ in a flat deformation. Let $S = \mathrm{Spec}(k[[t]])$ and let s be the closed point of S.

Proposition 4.1. — *Suppose $p \mid e_1$ or $p \mid e_2$. Suppose $\psi_\circ$ is an Artin-Schreier cover over k, branched at a point b with lower jump $e_1 + e_2 - 1$. Then there exists an Artin-Schreier cover ψ_S over S whose special fibre is isomorphic to $\psi_\circ$, whose generic fibre is branched at two points that specialize to b and which have lower jumps $e_1 - 1$ and $e_2 - 1$, and whose ramification divisor is otherwise constant.*

Proof. — Let $e = e_1 + e_2$. By hypothesis, $p \nmid (e - 1)$. Without loss of generality, suppose $p \mid e_1$.

Consider the Artin-Schreier cover $\psi_\circ : Y_\circ \to Z_\circ$ which is wildly ramified at the point $y_\circ \in Y_\circ$ above b where it has lower jump $e - 1$. Let $\hat{\psi}_\circ : \hat{Y}_\circ \to \hat{Z}_\circ$ be the germ of $\psi_\circ$ at $y_\circ$. It is an Artin-Schreier cover of germs of curves. Using formal patching, see *e.g.*, [11, Prop. 2.7] or [3, Thm. 3.3.4], deformations of $\psi_\circ$ can be constructed locally via deformations of $\hat{\psi}_\circ$. With this technique, one can suppose that the deformation of $\psi_\circ$, and thus the ramification divisor, is constant away from b.

Now $\hat{Z}_\circ \simeq \mathrm{Spec}(k[[x^{-1}]])$. After a change of variables, one can suppose that the restriction of $\hat{\psi}_\circ$ to $\mathrm{Spec}(k((x^{-1})))$ has equation $y^p - y = x^{e-1}$.

Consider the deformation $\hat{\psi}_S$ of $\hat{\psi}_\circ$ over $S = \mathrm{Spec}(k[[t]])$ given by the normal extension of $\mathrm{Spec}(k[[x^{-1}, t]])$ with the following affine equation:

$$y^p - y = x^{e-1}/(1-xt)^{e_1}.$$

On the special fibre, when $t = 0$, then $\hat{\psi}_s$ is isomorphic to $\hat{\psi}_\circ$. On the generic fibre, when $t \neq 0$, then $\hat{\psi}_{S-s}$ is branched above $x^{-1} = 0$ and above $x^{-1} = t$. Let $F(x) = x^{e-1}/(1-xt)^{e_1}$. The order of the pole of $F(x)$ at $x^{-1} = 0$ is $e - 1 - e_1 = e_2 - 1$, which is prime-to-p by hypothesis. Thus the lower jump above $x^{-1} = 0$ is $e_2 - 1$.

To compute the lower jump above $x^{-1} = t$, one can expand $F(x)$ around $1/t$:

$$\begin{aligned} F(x) = (-1)^{e_1} t^{-(e+e_1-1)} (x - 1/t)^{-e_1} \\ + (e-1)(-1)^{e_1} t^{-(e+e_1-2)} (x-1/t)^{-e_1+1} + \dots. \end{aligned}$$

After a finite inseparable extension of $k((t))$ with equation $t_1^p = t$, the leading term of $F(x)$ is a pth power. The second term of $F(x)$ is non-zero since $p \nmid (e-1)$ and thus it becomes the leading term of the affine equation in standard form for $\hat{\psi}_{S-s}$. Thus the lower jump above $x^{-1} = t$ is $e_1 - 1$. Thus the cover $\hat{\psi}_{S-s}$ is branched at two points that specialize to b and which have lower jumps $e_1 - 1$ and $e_2 - 1$. By Lemma 2.6 and [20, Lemma IV.2.3], the deformation $\hat{\psi}_S$ of $\hat{\psi}_\circ$ over S is smooth. □

The next result shows that, under a mild necessary condition, the p-rank of an Artin-Schreier curve can be increased by exactly $p-1$ under a flat deformation. In particular, an Artin-Schreier curve of genus $g \geqslant p(p-1)/2$ and p-rank 0 can be deformed to an Artin-Schreier curve of genus g and p-rank $p-1$.

Proposition 4.2. — *Suppose that $Y_\circ$ is an Artin-Schreier k-curve of genus g and p-rank $r(p-1)$. Suppose there is a ramified point of $Y_\circ$ under the $\mathbb{Z}/p$-action whose lower jump d satisfies $d \geqslant p+1$. Then there exists an Artin-Schreier curve Y_S over S whose special fibre is isomorphic to $Y_\circ$ and whose generic fibre has genus g and p-rank $(r+1)(p-1)$.*

Proof. — Let $e_1 = p$ and $e_2 = d + 1 - p$. By hypothesis, there is an Artin-Schreier cover $\psi_\circ : Y_\circ \to \mathbb{P}^1_k$, branched at $r+1$ points, including one point b with lower jump $e_1 + e_2 - 1$. The result is then immediate from Proposition 4.1, because the generic fibre of ψ_S is branched at $r+2$ points. □

4.2. Preliminary closure results

In this section, we show that the combinatorial data in the graph G_d gives partial information about how the irreducible components of $\mathcal{AS}_{g,s}$ (with varying s) fit together in $\mathcal{AS}_g$. In fact, we will see in Section 4.3 that the graph G_d gives complete information about this question when $p = 2$.

For $i = 1, 2$, consider a partition $\vec{E}_i \in \Omega_{d,r_i}$. Let $s_i = r_i(p-1)$. Let $\Gamma_{\vec{E}_i} := \mathcal{AS}_{g,\vec{E}_i}$ be the irreducible component of $\mathcal{AS}_{g,s_i}$ corresponding to $\vec{E}_i$ as defined below Proposition 3.3. There is a partial ordering $\prec$ on Ω_d from Section 2.1.

Lemma 4.3. — *If $\Gamma_{\vec{E}_1}$ is in the closure of $\Gamma_{\vec{E}_2}$ in $\mathcal{AS}_g$, then $\vec{E}_1 \prec \vec{E}_2$.*

Proof. — Let $S = \mathrm{Spec}(k[[t]])$ and consider an Artin-Schreier cover ϕ_S so that the generic fibre yields a $k((t))$-point of $\Gamma_{\vec{E}_1}$ and the special fibre yields a k-point of $\Gamma_{\vec{E}_2}$. This is only possible if the branch points of ϕ_S coalesce when $t = 0$. Since $B(\phi_S)$ is a relative Cartier divisor of constant degree, the entries of the partition sum together under specialization and the partition decreases in size. ☐

The next example and lemma show that the condition $\vec{E}_1 \prec \vec{E}_2$ is frequently not sufficient for $\Gamma_{\vec{E}_1}$ to be in the closure of $\Gamma_{\vec{E}_2}$ in $\mathcal{AS}_g$ when $p \geqslant 5$.

Example 4.4. — Let $p = 5$ and $g = 4$ and consider $\vec{E}_1 = \{4\}$ and $\vec{E}_2 = \{2, 2\}$. Then $\Gamma_{\vec{E}_1}$ and $\Gamma_{\vec{E}_2}$ are both components of $\mathcal{AS}_4$ with dimension one. Although $\vec{E}_1 \prec \vec{E}_2$, at most a zero-dimensional subvariety of $\Gamma_{\vec{E}_1}$ can be in the closure of $\Gamma_{\vec{E}_2}$. In fact, $\Gamma_{\vec{E}_1}$ is the supersingular family parametrized by $y^5 - y = x^3 + cx^2$; while $\Gamma_{\vec{E}_2}$ is the ordinary family parametrized by $y^5 - y = x + c/x$.

For $a \in \mathbb{Z}_{>0}$, let $\overline{a}$ be the integer so that $\overline{a} \equiv a \bmod p$ and $0 \leqslant \overline{a} < p$.

Lemma 4.5. — *Suppose $\vec{E}_1 \prec \vec{E}_2$ with an edge from $\vec{E}_1$ to $\vec{E}_2$.*

1. *If the edge is of the form $\{e\} \mapsto \{e_1, e_2\}$ with $2 < \overline{e}_1 + \overline{e}_2 \leqslant p$, then $\dim_k(\Gamma_{\vec{E}_1}) = \dim_k(\Gamma_{\vec{E}_2})$ and $\Gamma_{\vec{E}_1}$ is not in the closure of $\Gamma_{\vec{E}_2}$ in $\mathcal{AS}_g$.*
2. *In all other cases, $\dim_k(\Gamma_{\vec{E}_1}) = \dim_k(\Gamma_{\vec{E}_2}) - 1$.*

Proof. — The dimension comparison follows from Theorem 1.1. If $\dim_k(\Gamma_{\vec{E}_1}) = \dim_k(\Gamma_{\vec{E}_2})$, then $\Gamma_{\vec{E}_1}$ is not in the closure of $\Gamma_{\vec{E}_2}$ since $\mathcal{AS}_g$ is separated. ☐

4.3. Closure of the p-rank strata

The main result of this section is Theorem 4.7 which states that every irreducible component of $\mathcal{AS}_{g,s}$ satisfying an obvious necessary condition is contained in the closure of $\mathcal{AS}_{g,s+(p-1)}$ in $\mathcal{AS}_g$. In the case that $p=2$, Corollary 4.8 strengthens this result.

Recall, for $i=1,2$, that $\Gamma_{\vec{E}_i} := \mathcal{AS}_{g,\vec{E}_i}$ is the irreducible component of $\mathcal{AS}_{g,s_i}$ corresponding to $\vec{E}_i \in \Omega_{d,r_i}$, where $s_i = r_i(p-1)$. There are some earlier results about when $\Gamma_{\vec{E}_1}$ is in the closure of $\Gamma_{\vec{E}_2}$. For example, [16, Thm. 6.5.1] implies that $\Gamma_{\vec{E}_1}$ is in the closure of $\Gamma_{\vec{E}_2}$ for an edge of the form $\{2p-\ell+1\} \mapsto \{p, p-\ell+1\}$ as long as $\ell \mid (p-1)$. Here is another such result.

Proposition 4.6. — *Let $\vec{E}_1 \prec \vec{E}_2$ with an edge of the form $\{e\} \mapsto \{e_1, e_2\}$ from $\vec{E}_1$ to $\vec{E}_2$. If $p \mid e_1$ or $p \mid e_2$, then $\Gamma_{\vec{E}_1}$ is in the closure of $\Gamma_{\vec{E}_2}$ in $\mathcal{AS}_g$.*

In other words, under the hypothesis of Proposition 4.6, if $Y_\circ$ is an Artin-Schreier curve with partition $\vec{E}_1$ over k, then there exists an Artin-Schreier curve Y_S over $S = \operatorname{Spec}(k[[t]])$ whose special fibre is isomorphic to $Y_\circ$ and whose generic fibre has partition $\vec{E}_2$.

Proof. — For $i=1,2$, let $\Gamma_{\vec{E}_i} = \mathcal{AS}_{g,\vec{E}}$. Let $Y_\circ$ be the Artin-Schreier curve corresponding to a k-point of $\Gamma_{\vec{E}_1}$. There exists an Artin-Schreier cover $\phi_\circ : Y_\circ \to \mathbb{P}^1_k$ over k. The element e in the partition $\vec{E}_1$ determines a branch point $b \in \mathbb{P}^1_k$ so that the lower jump of $\phi_\circ$ above b is $e-1$.

Let $S = \operatorname{Spec}(k[[t]])$. By Proposition 4.1, there exists an Artin-Schreier cover ϕ_S over S whose special fibre is isomorphic to $\phi_\circ$ and whose generic fibre is branched at two points that specialize to b and that have lower jumps $e_1 - 1$ and $e_2 - 1$. Furthermore, the ramification divisor is otherwise constant. Thus the generic fibre of ϕ_S has partition $\vec{E}_2$. Thus $\Gamma_{\vec{E}_1}$ is in the closure of $\Gamma_{\vec{E}_2}$ in $\mathcal{AS}_g$. □

Theorem 4.7. — *Suppose $0 \leqslant s \leqslant g-(p-1)$. If η is an irreducible component of $\mathcal{AS}_{g,s}$ which is not open and dense in an irreducible component of $\mathcal{AS}_g$, then η is in the closure of $\mathcal{AS}_{g,s+(p-1)}$ in $\mathcal{AS}_g$.*

Proof. — The condition that η is not open and dense in an irreducible component of $\mathcal{AS}_g$ implies that $\dim(\eta) < d-1$ [15, Cor. 3.16]. By Theorem 3.11, $\eta = \mathcal{AS}_{g,\vec{E}}$ for some partition $\vec{E} \in \Omega_{d,r}$ containing an entry $e \geqslant p+2$. The result is then immediate from Proposition 4.6, letting $e_1 = p$ and $e_2 = e-p$. □

The next corollary shows that the graph G_d gives a complete combinatorial description of how the irreducible components of $\mathcal{AS}_{g,s}$ fit together in $\mathcal{AS}_g$ when $p = 2$. This result is used in [2].

Corollary 4.8. — *Suppose $p = 2$. Then $\Gamma_{\vec{E}_1}$ is in the closure of $\Gamma_{\vec{E}_2}$ in $\mathcal{AS}_g$ if and only if $\vec{E}_1 \prec \vec{E}_2$. Thus, if $0 \leqslant s < g$, then every component of $\mathcal{H}_{g,s}$ is in the closure of $\mathcal{H}_{g,s+1}$ in $\mathcal{H}_g$.*

Proof. — Lemma 4.3 implies the forward direction. For the converse, one reduces to the case that there is an edge from $\vec{E}_1$ to $\vec{E}_2$. Since $p = 2$, the edge has the form $\{e\} \mapsto \{e_1, e_2\}$ where e_1 and e_2 are even. Then Proposition 4.6 applies. ☐

4.4. Open questions

An answer to the following question would help determine whether $\mathcal{AS}_g$ is connected.

Question 1. What are necessary and sufficient conditions on the edge $\{e\} \mapsto \{e_1, e_2\}$ or the edge $\{e\} \mapsto \{e_1, e_2, e_3\}$ for $\Gamma_{\vec{E}_1}$ to be in the closure of $\Gamma_{\vec{E}_2}$ in AS_g?

Remark 4.9. — By Proposition 4.6, a sufficient condition for an affirmative answer to Question 4.4 for the edge $\{e\} \mapsto \{e_1, e_2\}$ is that $p \mid e_1 e_2$. Here is a heuristic why this condition may also be necessary. Suppose that K is a field of characteristic 0. If $\Phi : Y \to \mathbb{P}^1_K$ is a $\mathbb{Z}/p$-Galois cover and $y \in Y$ is a ramification point, then the identification of $\mathbb{Z}/p$ with $\mathrm{Gal}(\Phi)$ allows one to define a *canonical generator* $g_y \in \mathbb{Z}/p$ of the inertia group at y, see *e.g.*, [22, Section 2.2.1]. The inertia type of Φ is the multi-set $\{g_y\}$ for all ramification points y of Φ.

Now, if ϕ_1 is an Artin-Schreier cover (in characteristic p) with partition $\{e\}$, then ϕ_1 can be lifted to a $\mathbb{Z}/p$-cover of the projective line over a field of characteristic 0, and the inertia type of this lifting is the multi-set of length e of the form $\{1, \dots, 1, 1 - e\}$ [10, Ex. 3.3.1]. Similarly, if ϕ_2 is an Artin-Schreier cover with partition $\{e_1, e_2\}$, then the inertia type of the lifting is the multi-set of length $e_1 + e_2 = e$ of the form $\{1 - e_1, 1, \dots, 1, 1 - e_2\}$. So the inertia types of the liftings are the same if and only if either $1 - e_1 \equiv 1 \bmod p$ or $1 - e_2 \equiv 1 \bmod p$, in other words, if and only if $p \mid e_1 e_2$.

Question 2. Let $\vec{E} \in \Omega_{d,r}$. What Newton polygons occur for points of $\mathcal{AS}_{g,\vec{E}}$?

When $p \gg d$, the Newton polygon occurring for the generic point of $\mathcal{AS}_{g,\vec{E}}$ is found in [23]. Its limit as $p \to \infty$ has slopes 0 and 1 occurring with multiplicity $r(p-1)$ and slopes $\{\frac{1}{e_j-1}, \dots, \frac{e_j-2}{e_j-1}\}$ with multiplicity $p-1$ for each $1 \leqslant j \leqslant r+1$. See [5], [4] for recent results on this question.

Question 3. If $p \geqslant 3$ and $g > s \geqslant 0$, is every component of $\mathcal{H}_{g,s}$ in the closure of $\mathcal{H}_{g,s+1}$?

An answer to Question 3 would give more information about the geometry of the p-rank stratification of $\mathcal{H}_g$, thus generalizing Corollary 4.8. In [1, Cor. 3.15], the authors prove a related result: if $p \geqslant 3$ and $0 \leqslant s' < s \leqslant g$, then for each irreducible component S of $\mathcal{H}_{g,s}$, there exists an irreducible component $T_{s'}$ of $\mathcal{H}_{g,s'}$ such that $\bar{S}$ contains $T_{s'}$.

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