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Jet schemes of complex plane branches and equisingularity

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JET SCHEMES OF COMPLEX PLANE BRANCHES
AND EQUISINGULARITY

by Hussein MOURTADA (*)

Abstract. — For $m \in \mathbb{N}$, we determine the irreducible components of the $m$–th Jet Scheme of a complex branch $C$ and we give formulas for their number $N(m)$ and for their codimensions, in terms of $m$ and the generators of the semigroup of $C$. This structure of the Jet Schemes determines and is determined by the topological type of $C$.

Résumé. — Pour $m \in \mathbb{N}$, nous déterminons les composantes irréductibles des $m$–èmes espaces des jets d’une branche plane complexe $C$ et nous donnons des formules pour leur nombre $N(m)$ et leurs codimensions, en fonction de $m$ et des générateurs du semigroupe de $C$. Cette structure des espaces des jets détermine et elle est déterminée par le type topologique de $C$.

1. Introduction

Let $\mathbb{K}$ be an algebraically closed field. The space of arcs $X_\infty$ of an algebraic $\mathbb{K}$–variety $X$ is a non-noetherian scheme in general. It has been introduced by Nash in [10]. Nash has initiated its study by looking at its image by the truncation maps $X_\infty \rightarrow X_m$ in the jet schemes of $X$. The $m$th–jet scheme $X_m$ of $X$ is a $\mathbb{K}$– scheme of finite type which parametizes morphisms $\text{Spec } \mathbb{K}[t]/(t)^{m+1} \rightarrow X$. From now on, we assume $\text{char } \mathbb{K} = 0$.

In [10], Nash has derived from the existence of a resolution of singularities of $X$, that the number of irreducible components of the Zariski closure of the set of the $m$–truncations of arcs on $X$ that send 0 into the singular locus of $X$ is constant for $m$ large enough. Besides a theorem of Kolchin asserts that if $X$ is irreducible, then $X_\infty$ is also irreducible. More recently,

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the jet schemes have attracted attention from various viewpoints. In [9], Mustata has characterized the locally complete intersection varieties having irreducible $X_m$ for $m \geq 0$. In [2], a formula comparing the codimensions of $Y_m$ in $X_m$ with the log canonical threshold of a pair $(X, Y)$ is given. In this work, we consider a curve $C$ in the complex plane $\mathbb{C}^2$ with a singularity at $0$ at which it is analytically irreducible (i.e. the formal neighborhood $(C, 0)$ of $C$ at $0$ is a branch). We determine the irreducible components of the space $C_m^0 := \pi_m^{-1}(0)$ where $\pi_m : C_m \rightarrow C$ is the canonical projection, and we show that their number is not bounded as $m$ grows. More precisely, let $x$ be a transversal parameter in the local ring $\mathcal{O}_{\mathbb{C}^2, 0}$, i.e. the line $x = 0$ is transversal to $C$ at $0$ and following [2], for $e \in \mathbb{N}$, let

$$\text{Cont}_e(x)_m (\text{resp.} \text{Cont}_{> e}(x)_m) := \{ \gamma \in C_m \mid \text{ord}_t x \circ \gamma = e (\text{resp.} > e) \},$$

where $\text{Cont}$ stands for contact locus. Let $\Gamma(C) = \langle \beta_0, \ldots, \beta_g \rangle$ be the semigroup of the branch $(C, 0)$ and let $e_i = \text{gcd}(\beta_0, \ldots, \beta_i)$, $0 \leq i \leq g$. Recall that $\Gamma(C)$ and the topological type of $C$ near $0$ are equivalent data and characterize the equisingularity class of $(C, 0)$ as defined by Zariski in [13]. We show in theorem 4.9 that the irreducible components of $C_m^0$ are

$$C_{m\kappa \ell} = \text{Cont}_{\kappa \beta_0}(x)_m,$$

for $1 \leq \kappa$ and $\kappa \beta_0 \beta_1 + e_1 \leq m$,

$$C_{m\kappa \nu}^j = \text{Cont}_{e_j^{-1}}^{\kappa \beta_0}(x)_m$$

for $2 \leq j \leq g, 1 \leq \kappa, \kappa \neq 0 \mod \frac{e_{j-1}}{e_j}$ and $\kappa \beta_0 \beta_1 + e_1 \leq m < \kappa \beta_j$,

$$B_m = \text{Cont}_{e_1^{-1} q}^{\beta_0 \beta_1}(x)_m,$$

if $q \beta_0 \beta_1 + e_1 \leq m < (q + 1)n_1 \beta_1 + e_1$.

These irreducible components give rise to infinite and finite inverse systems represented by a tree. We recover $\langle \beta_0, \ldots, \beta_g \rangle$ from the tree and the multiplicity $\beta_0$ in corollary 4.13, and we give formulas for the number of irreducible components of $C_m^0$ and their codimensions in terms of $m$ and $(\beta_0, \ldots, \beta_g)$ in proposition 4.7 and corollary 4.10. We recover the fact coming from [2] and [6] that

$$\min_m \frac{\text{codim}(C_m^0, \mathbb{C}^2_m)}{m + 1} = \frac{1}{\beta_0} + \frac{1}{\beta_1}.$$

The structure of the paper is as follows: The basics about Jet schemes and the results that we will need are presented in section 2. In section 3
we present the definitions and the results we will need about branches. The last section is devoted to the proof of the main result and corollaries.

2. Jet schemes

Let \( K \) be an algebraically closed field of arbitrary characteristic. Let \( X \) be a \( K \)-scheme of finite type over \( k \) and let \( m \in \mathbb{N} \). The functor \( F_m : K \text{-Schemes} \to \text{Sets} \) which to an affine scheme defined by a \( K \)-algebra \( A \) associates

\[
F_m(\text{Spec}(A)) = \text{Hom}_K(\text{Spec}A[t]/(t^{m+1}), X)
\]

is representable by a \( K \)-scheme \( X_m \) [12]. \( X_m \) is the \( m \)-th jet scheme of \( X \), and \( F_m \) is isomorphic to its functor of points. In particular the closed points of \( X_m \) are in bijection with the \( K[t]/(t^{m+1}) \) points of \( X \).

For \( m, p \in \mathbb{N}, m > p \), the truncation homomorphism \( A[t]/(t^{m+1}) \to A[t]/(t^{p+1}) \) induces a canonical projection \( \pi_{m,p} : X_m \to X_p \). These morphisms clearly verify \( \pi_{m,p} \circ \pi_{q,m} = \pi_{q,p} \) for \( p < m < q \).

Note that \( X_0 = X \). We denote the canonical projection \( \pi_{m,0} : X_m \to X_0 \) by \( \pi_m \).

**Example 2.1.** — Let \( X = \text{Spec} \frac{K[x_0,\ldots,x_n]}{(f_1,\ldots,f_r)} \) be an affine \( K \)-scheme. For a \( K \)-algebra \( A \), to give a \( A \)-point of \( X_m \) is equivalent to give a \( K \)-algebra homomorphism

\[
\varphi : \frac{K[x_0,\ldots,x_n]}{(f_1,\ldots,f_r)} \to A[t]/(t^{m+1}).
\]

The map \( \varphi \) is completely determined by the image of \( x_i, i = 0, \ldots, n \)

\[
x_i \mapsto \varphi(x_i) = x_i^{(0)} + x_i^{(1)} t + \cdots + x_i^{(m)} t^m
\]

such that \( f_l(\varphi(x_0),\cdots,\varphi(x_n)) \in (t^{m+1}), l = 1, \ldots, r \).

If we write

\[
f_l(\varphi(x_0),\cdots,\varphi(x_n)) = \sum_{j=0}^{m} F_l^{(j)}(x_0^{(0)},\cdots,x_n^{(0)}) t^j \mod (t^{m+1})
\]

where \( x^{(j)} = (x_0^{(j)},\cdots,x_n^{(j)}) \), then

\[
X_m = \text{Spec} \frac{K[x_0^{(0)},\cdots,x_n^{(m)}]}{(F_l^{(j)})_{j=0,\ldots,m, l=1,\ldots,r}}
\]
Example 2.2. — From the above example, we see that the m-th jet scheme of the affine space $\mathbb{A}^n_K$ is isomorphic to $\mathbb{A}^{(m+1)n}_K$ and that the projection $\pi_{m,m-1} : (\mathbb{A}^n_K)_m \rightarrow (\mathbb{A}^n_K)_{m-1}$ is the map that forgets the last $n$ coordinates.

Let $\text{char}(\mathbb{K}) = 0$, $S = \mathbb{K}[x_0, \cdots, x_n]$ and $S_m = \mathbb{K}[x^{(0)}, \cdots, x^{(m)}]$. Let $D$ be the $\mathbb{K}$-derivation on $S_m$ defined by $D(x_i^{(j)}) = x_i^{(j+1)}$ if $0 \leq j < m$, and $D(x_i^{(m)}) = 0$. For $f \in S$ let $f^{(1)} := D(f)$ and we recursively define $f^{(m)} = D(f^{(m-1)})$.

**Proposition 2.3.** — Let $X = \text{Spec}(S/(f_1, \cdots, f_r)) = \text{Spec}(R)$ and $R_m = \Gamma(X_m)$. Then

$$R_m = \text{Spec}\left(\frac{\mathbb{K}[x^{(0)}, \cdots, x^{(m)}]}{(f_i^{(j)})_{i=1, \cdots, r}}\right).$$

**Proof.** — For a $\mathbb{K}$-algebra $A$, to give an $A$-point of $X_m$ is equivalent to give an homomorphism

$$\phi : \mathbb{K}[x_0, \cdots, x_n] \rightarrow A[t]/(t^{m+1})$$

which can be given by

$$x_i \mapsto \frac{x_i^{(0)}}{0!} + \frac{x_i^{(1)}}{1!}t + \cdots + \frac{x_i^{(m)}}{m!}t^m.$$

Then for a polynomial $f \in S$, we have

$$\phi(f) = \sum_{j=0}^{m} \frac{f^{(j)}(x^{(0)}, \cdots, x^{(j)})}{j!} t^j.$$ 

To see this, it is sufficient to remark that it is true for $f = x_i$, and that both sides of the equality are additive and multiplicative in $f$, and the proposition follows. \hfill $\square$

**Remark 2.4.** — Note that the proposition shows the linearity of the equations $F_i^{(j)}(x^{(0)}, \cdots, x^{(j)})$ defining $X_m$ with respect to the new variables i.e $x^{(j)}$. We can deduce from this that if $X$ is a nonsingular $\mathbb{K}$-variety of dimension $n$, then the projections $\pi_{m,m-1} : X_m \rightarrow X_{m-1}$ are locally trivial fibrations with fiber $\mathbb{A}^n_K$. In particular, $X_m$ is a non singular variety of dimension $(m+1)n$.

### 3. Semigroup of complex branches

The main references for this section are [14],[8],[1],[11],[5],[4],[7]. Let $f \in \mathbb{C}[[x,y]]$ be an irreducible power series, which is $y$-regular (i.e $f(0, y) =$
$y^\beta_0 u(y)$ where $u$ is invertible in $\mathbb{C}[[y]]$ and such that $\text{mult}_0 f = \beta_0$ and let $C$ be the analytically irreducible plane curve (branch for short) defined by $f$ in $\text{Spec} \mathbb{C}[[x, y]]$. By the Newton-Puiseux theorem, the roots of $f$ are

$$y = \sum_{i=0}^{\infty} a_i w^i x^{\frac{1}{\beta_0}}$$

where $w$ runs over the $\beta_0$-th roots of unity in $\mathbb{C}$. This is equivalent to the existence of a parametrization of $C$ of the form

$$x(t) = t^{\beta_0}, \quad y(t) = \sum_{i \geq \beta_0} a_it^i.$$ 

We recursively define $\beta_i = \min\{i, a_i \neq 0, \gcd(\beta_0, \ldots, \beta_{i-1}) \text{ is not a divisor of } i\}$.

Let $e_0 = \beta_0$ and $e_i = \gcd(e_{i-1}, \beta_i), i \geq 1$. Since the sequence of positive integers

$$e_0 > e_1 > \cdots > e_i > \cdots$$

is strictly decreasing, there exists $g \in \mathbb{N}$, such that $e_g = 1$. The sequence $(\beta_1, \ldots, \beta_g)$ is the sequence of Puiseux exponents of $C$. We set

$$n_i := \frac{e_{i-1}}{e_i}, m_i := \frac{\beta_i}{e_i}, i = 1, \ldots, g$$

and by convention, we set $\beta_{g+1} = +\infty$ and $n_{g+1} = 1$.

On the other hand, for $h \in \mathbb{C}[[x, y]]$, we define the intersection number

$$(f, h)_0 = (C, C_h)_0 := \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, h) = \text{ord}_t h(x(t), y(t))$$

where $C_h$ is the Cartier divisor defined by $h$ and $\{x(t), y(t)\}$ is as above.

The mapping $v_f : \mathbb{C}[[x, y]]/(f) \to \mathbb{N}, h \mapsto (f, h)_0$ defines a divisorial valuation.

We define the semigroup of $C$ to be the semigroup of $v_f$ i.e $\Gamma(C) = \Gamma(v_f) = \{(f, h)_0 \in \mathbb{N}, h \neq 0 \mod(f)\}$.

The following propositions and theorem from [14] characterize the structure of $\Gamma(C)$.

**Proposition 3.1.** — There exists a unique sequence of $g + 1$ positive integers $(\bar{\beta}_0, \ldots, \bar{\beta}_g)$ such that:

i) $\bar{\beta}_0 = \beta_0$,

ii) $\bar{\beta}_i = \min\{\Gamma(C) \setminus < \bar{\beta}_0, \ldots, \bar{\beta}_{i-1} >, 1 \leq i \leq g,$

iii) $\Gamma(C) = < \bar{\beta}_0, \ldots, \bar{\beta}_g >,$

where for $i = 1, \ldots, g + 1, < \bar{\beta}_0, \ldots, \bar{\beta}_{i-1} >$ is the semigroup generated by $\bar{\beta}_0, \ldots, \bar{\beta}_{i-1}$. By convention, we set $\bar{\beta}_{g+1} = +\infty.$
Proposition 3.2. — The sequence \((\tilde{\beta}_0, \cdots, \tilde{\beta}_g)\) verifies:

i) \(e_i = \gcd(\tilde{\beta}_0, \cdots, \tilde{\beta}_i), 0 \leq i \leq g\),

ii) \(\tilde{\beta}_0 = \beta_0, \tilde{\beta}_1 = \beta_1\) and \(\tilde{\beta}_i = n_{i-1}\tilde{\beta}_{i-1} + \beta_i - \beta_{i-1}\). In particular \(n_i\tilde{\beta}_i < \tilde{\beta}_{i+1}\), for \(i = 2, \cdots, g\).

Theorem 3.3. — The sequence \((\tilde{\beta}_0, \cdots, \tilde{\beta}_g)\) and the sequence \((\beta_0, \cdots, \beta_g)\) are equivalent data. They determine and are determined by the topological type of \(C\).

Then from the appendix of [14], [1] or [11], we can choose a system of approximate roots (or a minimal generating sequence) \(\{x_0, \cdots, x_{g+1}\}\) of the divisorial valuation \(v_f\). We set \(x = x_0, y = x_1\); for \(i = 2, \cdots, g+1, x_i \in \mathbb{C}[[x, y]]\) is irreducible; for \(1 \leq i \leq g\), the analytically irreducible curve \(C_i = \{x_i = 0\}\) has \(i-1\) Puiseux exponents and \(C_{g+1} = C\). This sequence also verifies:

i) \(v_f(x_i) = \tilde{\beta}_i, 0 \leq i \leq g\),

ii) \(\Gamma(C_i) := \langle \frac{\tilde{\beta}_0}{e_i-1}, \cdots, \frac{\tilde{\beta}_{i-1}}{e_i-1} \rangle\), and the Puiseux sequence of \(C_i\) is \((\frac{\beta_0}{e_i-1}, \cdots, \frac{\beta_{i-1}}{e_i-1}), 2 \leq i \leq g+1\).

iii) for \(1 \leq i \leq g\), there exists a unique system of nonnegative integers \(b_{ij}, 0 \leq j < i\) such that for \(1 \leq j < i, b_{ij} < n_j\) and \(n_i\tilde{\beta}_i = \sum_{0 \leq j < i} b_{ij}\beta_j\). Furthermore, for \(1 \leq i \leq g\), one can choose \(x_i\) such that they satisfy identities of the form

\[x_{i+1} = x_i^{n_i} - c_i x_0^{b_i} \cdots x_{i-1}^{b_{i-1}} - \sum_{\gamma = (\gamma_0, \cdots, \gamma_i)} c_\gamma x_0^{\gamma_0} \cdots x_i^{\gamma_i},(*)\]

with, \(0 \leq \gamma_j < n_j\), for \(1 \leq j < i\), and \(\sum_j \gamma_j \beta_j > n_i\beta_i\) and with \(c_\gamma, c_i \in \mathbb{C}\) and \(c_i \neq 0\). These last equations (\(\ast\)) let us realize \(C\) as a complete intersection in \(\mathbb{C}^{g+1} = \text{Spec} \mathbb{C}[[x_0, \cdots, x_g]]\) defined by the equations

\[f_i = x_{i+1} - (x_i^{n_i} - c_i x_0^{b_i} \cdots x_{i-1}^{b_{i-1}} - \sum_{\gamma = (\gamma_0, \cdots, \gamma_i)} c_\gamma x_0^{\gamma_0} \cdots x_i^{\gamma_i})\]

for \(1 \leq i \leq g\), with \(x_{g+1} = 0\) by convention.

Let \(h \in \mathbb{C}[[x, y]]\) be a \(y\)-regular irreducible power series with multiplicity \(p = \text{ord}_y h(0, y)\). Let \(y(x^{\frac{1}{\beta_0}})\) and \(z(x^{\frac{1}{\beta}})\) be respectively roots of \(f\) and \(h\) as in (1). We call contact order of \(f\) and \(h\) in their Puiseux series the following rational number

\[o_f(h) := \max\{\text{ord}_x (y(wx^{\frac{1}{\beta_0}}) - z(x^{\frac{1}{\beta}})); w^{\beta_0} = 1, \lambda^p = 1\} = \max\{\text{ord}_x (y(wx^{\frac{1}{\beta_0}}) - z(x^{\frac{1}{\beta}})); w^{\beta_0} = 1\} = \max\{\text{ord}_x (y(x^{\frac{1}{\beta_0}}) - z(x^{\frac{1}{\beta}})); \lambda^p = 1\} = o_h(f)\]

The following formula is from [8], see also [5].
In the sequel, we will denote the integral part of a rational number and the ideal defining coordinates we can write strictly positive integer such that \( o_f(h) \leq \frac{\beta_i}{\beta_0} \). Then
\[
\frac{(f,h)_0}{p} = \sum_{k=1}^{i-1} \frac{e_{k-1} - e_k}{\beta_0} \beta_k + e_{i-1} o_f(h) = (\frac{\beta_i}{\beta_0} e_{i-2} + (\beta_0 o_f(h) - \beta_{i-1}) e_{i-1}) \frac{1}{\beta_0}.
\]

**Corollary 3.5.** — \([1][5]\) Let \( i > 0 \) be an integer. Then \( o_f(h) \leq \frac{\beta_i}{\beta_0} \) iff \( \frac{(f,h)_0}{p} \leq e_{i-1} \frac{\beta_i}{\beta_0} \). Moreover \( o_f(h) = \frac{\beta_i}{\beta_0} \) iff \( \frac{(f,h)_0}{p} = e_{i-1} \frac{\beta_i}{\beta_0} \). In particular \( o_f(x_i) = \frac{\beta_i}{\beta_0}, 1 \leq i \leq g \). We say that \( C_i x_i = 0 \) has maximal contact with \( C \).

### 4. Jet schemes of complex branches

We keep the notations of sections 2 and 3. We consider a curve \( C \subset \mathbb{C}^2 \) with a branch of multiplicity \( \beta_0 > 1 \) at 0, defined by \( f \). Note that in suitable coordinates we can write
\[
f(x_0, x_1) = (x_1^{m_1} - c x_0^{m_1}) e_1 + \sum_{a \beta_0 + b \beta_1 > 0} c_{ab} x_0^a x_1^b; c \in \mathbb{C}^* \text{ and } c_{ab} \in \mathbb{C}. \ (\diamond)
\]
We look for the irreducible components of \( C_m := (\pi^{-1}_m(0)) \) for every \( m \in \mathbb{N} \), where \( \pi_m : C_m \to C \) is the canonical projection. Let \( J^m_0 \) be the radical of the ideal defining \( (\pi^{-1}_m(0)) \) in \( \mathbb{C}^2 \).

In the sequel, we will denote the integral part of a rational number \( r \) by \([r]\).

**Proposition 4.1.** — For \( 0 < m < n \beta_1 \), we have that
\[
(C_m^0)_{\text{red}} = (\pi^{-1}_m(0))_{\text{red}} = \text{Spec } \mathbb{C}\left[ x_0^{(0)}, \ldots, x_0^{(m)}, x_1^{(0)}, \ldots, x_1^{(m)} \right] / \left( x_0^{(0)}, \ldots, x_0^{(\left\lfloor \frac{m}{\beta_1} \right\rfloor)}, x_1^{(0)}, \ldots, x_1^{(\left\lfloor \frac{m}{\beta_1} \right\rfloor)} \right),
\]
and
\[
(C_{n \beta_1}^0)_{\text{red}} = (\pi^{-1}_{n \beta_1}(0))_{\text{red}} = \text{Spec } \mathbb{C}\left[ x_0^{(0)}, \ldots, x_0^{(n \beta_1)}, x_1^{(0)}, \ldots, x_1^{(n \beta_1)} \right] / \left( x_0^{(0)}, \ldots, x_0^{(n - 1)}, x_1^{(0)}, \ldots, x_1^{(n - 1)} - c x_0^{(n)} \right),
\]

**Proof.** — We write \( f = \sum_{(a,b) \in I} c_{ab} f_{ab} \) where \( (a, b) \in \mathbb{N}^2 \), \( f_{ab} = x_0^{a} x_1^{b} \), \( c_{ab} \in \mathbb{C} \) and \( a \beta_0 + b \beta_1 > \beta_0 \beta_1 \) (the segment \( [(0, \beta_0)(\beta_1, 0)] \) is the Newton Polygon of \( f \)). Let \( \text{supp}(f) = \{(a, b) \in \mathbb{N}^2; c_{ab} \neq 0 \} \).
For $0 < m < n_1 \beta_1$, the proof is by induction on $m$. For $m = 1$, we have that

$$F^{(1)} = \sum_{(a,b) \in supp(f)} c_{ab} F_{ab}^{(1)}$$

where $(F^{(0)}, \ldots, F^{(i)})$ (resp. $(F_{ab}^{(0)}, \ldots, F_{ab}^{(i)})$) is the ideal defining the $i$-th jet scheme $C_i$ of $C$ (resp. $C_{ab}^i$ the $i$-th jet scheme of $C_{ab} = \{f_{ab} = 0\}$) in $C_i^2$.

Then we have

$$F_{ab}^{(1)} = \sum_{i_k = 1} x_0^{(i_1)} \cdots x_0^{(i_a)} x_1^{(i_{a+b})}$$

where $\beta_1(a + b) \geq a \beta_0 + b \beta_1 \geq \beta_0 \beta_1$ so $a + b \geq \beta_0 > 1$. Then for every $(a, b) \in supp(f)$ and every $(i_1, \ldots, i_a, \ldots, i_{a+b}) \in \mathbb{N}^{a+b}$ such that $\sum_{k=1}^{a+b} i_k = 1$ there exists $1 \leq k \leq a + b$ such that $i_k \neq 0$, this means that $F_{ab}^{(1)} \in (x_0^{(0)}, x_1^{(0)})$ and since we are looking over the origin, we have that $(x_0^{(0)}, x_1^{(0)}) \subseteq J_1^0$ therefore $(\pi_1^{-1}(0))_{red} = Spec \mathbb{C}[x_0^{(0)}, x_1^{(0)}, \cdots, x_1^{(m-1)}]$. In fact this is nothing but the Zariski tangent space of $C$ at $0$.

Suppose that the lemma holds until $m - 1$ i.e.

$$(\pi_1^{-1}(0))_{red} = Spec \mathbb{C}[x_0^{(0)}, \cdots, x_0^{(m-1)} - x_1^{(0)}, \cdots, x_1^{(m-1)}] / (x_0^{(0)}, \cdots, x_0^{(m-1)}, x_1^{(m-1)}).$$

First case: If $[m - 1]_{\beta_1} = [m]_{\beta_1}$ and $[m - 1]_{\beta_0} = [m]_{\beta_0}$. We have

$$F^{(m)} = \sum_{(a,b) \in supp(f)} c_{ab} \sum_{i_k = m} x_0^{(i_1)} \cdots x_0^{(i_a)} x_1^{(i_{a+b})}$$

Let $(a, b) \in supp(f)$; if for every $k = 1, \ldots, a$, we had $i_k \geq [m]_{\beta_1} + 1$, and for every $k = a + 1, \ldots, a + b$, we had $i_k \geq [m]_{\beta_0} + 1$, then

$$m \geq a([m]_{\beta_1} + 1) + b([m]_{\beta_0} + 1) > \frac{m}{\beta_1} a + \frac{m}{\beta_0} b = m \frac{a \beta_0 + b \beta_1}{\beta_0 \beta_1} \geq m.$$  

The contradiction means that there exists $1 \leq k \leq a$ such that $i_k < [m]_{\beta_1}$ or there exists $a + 1 \leq k \leq a + b$ such that $i_k < [m]_{\beta_0}$. So $F^{(m)}$ lies in the ideal generated by $J_{m-1}^0$ in $\mathbb{C}[x_0^{(0)}, \cdots, x_0^{(m-1)}, x_1^{(0)}, \cdots, x_1^{(m-1)}]$ and $J_m^0 = J_{m-1}^0 \mathbb{C}[x_0^{(0)}, \cdots, x_0^{(m)}, x_1^{(0)}, \cdots, x_1^{(m)}]$.

Second case: If $[m - 1]_{\beta_1} = [m]_{\beta_1}$ and $[m - 1]_{\beta_0} + 1 = [m]_{\beta_0}$ (i.e. $\beta_0$ divides $m$). We have that

$$F^{(m)} = F^{(m)}_{0, \beta_0} + \sum_{(a,b) \in supp(f) : (a,b) \neq (0, \beta_0)} F^{(m)}_{ab}, \quad (**)$$
where
\[ F_{0\beta_0}^{(m)} = \sum_{i_k=m} x_1^{(i_1)} \cdots x_1^{(i_{\beta_0})} \]
\[ = x_1^{(\frac{m}{\beta_0})} + \sum_{i_k=m; (i_1, \cdots, i_{\beta_0}) \neq (\frac{m}{\beta_0}, \cdots, \frac{m}{\beta_0})} x_1^{(i_1)} \cdots x_1^{(i_{\beta_0})}; \]
but \( \sum i_k = m \) and \((i_1, \cdots, i_{\beta_0}) \neq (\frac{m}{\beta_0}, \cdots, \frac{m}{\beta_0})\) implies that there exists \(1 \leq k \leq \beta_0\) such that \(i_k < \frac{m}{\beta_0}\), so
\[ \sum_{i_k=m; (i_1, \cdots, i_{\beta_0}) \neq (\frac{m}{\beta_0}, \cdots, \frac{m}{\beta_0})} x_1^{(i_1)} \cdots x_1^{(i_{\beta_0})} \in J_{m-1}^0 \mathbb{C}[x_0^{(0)}, \cdots, x_0^{(m)}, x_1^{(0)}, \cdots, x_1^{(m)}]. \]

For the same reason as above, we have that
\[ \sum_{(a,b) \in \text{supp}(f); (a,b) \neq (0,\beta_0)} F_{ab}^{(m)} \in J_{m-1}^0 \mathbb{C}[x_0^{(0)}, \cdots, x_0^{(m)}, x_1^{(0)}, \cdots, x_1^{(m)}]. \]

From (**) we deduce that \(x_1^{(\frac{m}{\beta_0})} \in J_m^0\) and
\[ F^{(m)} \in (x_0^{(0)}, \cdots, x_0^{(\frac{m}{\beta_1})}, x_1^{(0)}, \cdots, x_1^{(\frac{m}{\beta_0})}). \]
Then \(J_m^0 = (x_0^{(0)}, \cdots, x_0^{(\frac{m}{\beta_1})}, x_1^{(0)}, \cdots, x_1^{(\frac{m}{\beta_0})})\).

The third case i.e. if \(\frac{m-1}{\beta_1} + 1 = \frac{m}{\beta_1}\) and \(\frac{m-1}{\beta_0} = \frac{m}{\beta_0}\) is discussed as the second one. Note that these are the only three possible cases since \(m < n_1\beta_1 = lcm(\beta_0, \beta_1)\)(here \(lcm\) stands for the least common multiple).

For \(m = n_1\beta_1\), we have that \(F^{(m)}\) is the coefficient of \(t^m\) in the expansion of
\[ f(x_0^{(0)} + x_0^{(1)}t + \cdots + x_0^{(m)}t^m, x_1^{(0)} + x_1^{(1)}t + \cdots + x_1^{(m)}t^m). \]

But since we are interested in the radical of the ideal defining the \(m\)-th jet scheme, and we have found that \(x_0^{(0)}, \cdots, x_0^{(n_1-1)}, x_0^{(0)}, \cdots, x_1^{(m_1-1)} \in J_{m-1}^0 \subseteq J_m^0\), we can annihilate \(x_0^{(0)}, \cdots, x_0^{(n_1-1)}, x_1^{(0)}, \cdots, x_1^{(m_1-1)}\) in the above expansion. Using (\(\phi\)), we see that the coefficient of \(t^m\) is \((x_1^{(m_1)})^{n_1} - cx_0^{(n_1)}m_1)\cdot e_1. \]

In the sequel if \(A\) is a ring, \(I \subseteq A\) an ideal and \(f \in A\), we denote by \(V(I)\) the subvariety of \(Spec\ A\) defined by \(I\) and by \(D(f)\) the open set in \(Spec\ A\), \(D(f) := Spec\ A_f\).

The proof of the following corollary is analogous to that of proposition 4.1.
Corollary 4.2. — Let $m \in \mathbb{N}$; let $k \geq 1$ be such that $m = kn_1 \tilde{\beta}_1 + i; 1 \leq i \leq n_1 \tilde{\beta}_1$. Then if $i < n_1 \tilde{\beta}_1$, we have that

$$\text{Cont}^{>kn_1}(x_0)_m = (\pi^{-1}_{m, kn_1 \tilde{\beta}_1}(V(x_0^{(0)}, \ldots, x_0^{(kn_1)})))_{\text{red}} = \frac{\mathbb{C}[x_0^{(0)}, \ldots, x_0^{(m)}, x_1^{(0)}, \ldots, x_1^{(m)}]}{(x_0^{(0)}, \ldots, x_0^{(kn_1)}, \ldots, x_0^{(kn_1 + [\frac{i}{k_1}])}, x_1^{(0)}, \ldots, x_1^{(kn_1)}, \ldots, x_1^{(kn_1 + [\frac{i}{k_1}])})}$$

and if $i = n_1 \tilde{\beta}_1$

$$\pi^{-1}_{m, kn_1 \tilde{\beta}_1}(V(x_0^{(0)}, \ldots, x_0^{(kn_1)}))_{\text{red}} = \frac{\mathbb{C}[x_0^{(0)}, \ldots, x_0^{((k+1)n_1-1)}, x_1^{(0)}, \ldots, x_1^{((k+1)m_1-1)}, x_1^{(k+1)m_1)} - cx_0^{(k+1)m_1})}{(x_0^{(0)}, \ldots, x_0^{(k+1n_1 - 1)}, x_1^{(0)}, \ldots, x_1^{(k+1m_1)} - x_0^{(k+1m_1)})}.$$

We now consider the case of a plane branch with one Puiseux exponent.

Lemma 4.3. — Let $C$ be a plane branch with one Puiseux exponent. Let $m, k \in \mathbb{N}$, such that $k \neq 0$ and $m \geq kn_1 \tilde{\beta}_1 + 1$, and let $\pi_{m, kn_1 \tilde{\beta}_1} : C_m \to C_{kn_1 \tilde{\beta}_1}$ be the canonical projection. Then

$$C_m^k := \pi_{m, kn_1 \tilde{\beta}_1}^{-1}(V(x_0^{(0)}, \ldots, x_0^{(kn_1-1)}) \cap D(x_0^{(kn_1)}))_{\text{red}}$$

is irreducible of codimension $k(m_1 + n_1) + 1 + (m - kn_1 \tilde{\beta}_1)$ in $\mathbb{C}_m^2$.

Proof. — First note that since $e_1 = 1$, we have $m_1 = \frac{\beta_1}{e_1} = \tilde{\beta}_1$. Let $I_m^0$ be the ideal defining $C_m^k$ in $\mathbb{C}_m^2 \cap D(x_0^{(kn_1)})$. Since $m \geq kn_1 \tilde{\beta}_1$, by corollary 4.2, $x_0^{(0)}, \ldots, x_0^{(kn_1-1)} \in I_m^0$. So $I_m^k$ is the radical of the ideal $I_m^{*0k} := (x_0^{(0)}, \ldots, x_0^{(kn_1-1)}, x_1^{(0)}, \ldots, x_1^{(kn_1-1)}, F(0), \ldots, F(m))$. Now it follows from \diamond and proposition 2.3 that

$$F(l) \in (x_0^{(0)}, \ldots, x_0^{(kn_1-1)}, x_1^{(0)}, \ldots, x_1^{(kn_1-1)}) \text{ for } 0 < l < km_1m_1,$$

$$F(km_1m_1) \equiv x_1^{(kn_1)m_1} - cx_0^{(kn_1)m_1} \mod (x_0^{(0)}, \ldots, x_0^{(kn_1-1)}, x_1^{(0)}, \ldots, x_1^{(kn_1-1)}),$$

$$F(km_1m_1 + l) \equiv n_1 x_1^{(kn_1)m_1 - 1} x_0^{(km_1 + l)} - m_1 cx_0^{(kn_1)m_1 - 1} x_0^{(km_1 + l) + H_l(x_0^{(0)}, \ldots, x_0^{(kn_1-1)}, x_1^{(0)}, \ldots, x_1^{(kn_1-1)}) \mod (x_0^{(0)}, \ldots, x_0^{(kn_1-1)}, x_1^{(0)}, \ldots, x_1^{(kn_1-1)}),}$$

for $1 \leq l \leq m - km_1m_1$.

This implies that

$$I_m^{*0k} = (x_0^{(0)}, \ldots, x_0^{(kn_1-1)}, x_1^{(0)}, \ldots, x_1^{(kn_1-1)}, F(km_1m_1), \ldots, F(m)).$$
Moreover the subscheme of $\mathbb{C}^2_m \cap D(x_0^{(kn)})$ defined by $I_m^{*0k}$ is isomorphic to the product of $\mathbb{C}^*(\mathbb{C}^*)$ isomorphic to the regular locus of $x_1^{(km)} - cx_0^{(kn)m_1}$ by an affine space and its codimension is $k(m_1 + n_1) + 1 + (m - kn_1m_1)$ so it is reduced and irreducible, and it is nothing but $C^k_m$, or equivalently $I_m^{0k} = I_m^{*0k}$.

**COROLLARY 4.4.** — Let $C$ be a plane branch with one Puiseux exponent. Let $m \in \mathbb{N}, m \neq 0$. Let $q \in \mathbb{N}$ be such that $m = qn_1\bar{\beta}_1 + i; 0 < i < n_1\bar{\beta}_1$. Then $C_m^0 = \pi_m^{-1}(0)$ has $q + 1$ irreducible components which are:

$$C_{mkI} = \overline{C_m^0}, 1 \leq k \leq q,$$

and $B_m = \text{Cont}^{>q_1}(x) = \pi_m^{-1}(V(x_0^{(0)}, \cdots, x_0^{(q_1)}))$.

We have that

$$\text{codim}(C_{mkI}, \mathbb{C}^2_m) = k(m_1 + n_1) + 1 + (m - kn_1m_1)$$

and

$$\text{codim}(B_m, \mathbb{C}^2_m) = q(m_1 + n_1) + \left[\frac{i}{\beta_0}\right] + \left[\frac{i}{\beta_1}\right] + 2 = \left[\frac{m}{\beta_0}\right] + \left[\frac{m}{\beta_1}\right] + 2 \text{ if } i < n_1\bar{\beta}_1$$

$$\text{codim}(B_m, \mathbb{C}^2_m) = (q + 1)(m_1 + n_1) + 1 \text{ if } i = n_1\bar{\beta}_1.$$

**Proof.** — The codimensions and the irreducibility of $B_m$ and $C_{mkI}$ follow from corollary 4.2 and lemma 4.3. This shows that if $1 \leq k < k' \leq q$, we have $\text{codim}(C_{mkI}, \mathbb{C}^2_m) < \text{codim}(C_{mk'I}, \mathbb{C}^2_m)$, then $C_{mkI} \not\subseteq C_{mk'I}$. On the other hand, since $C_{mkI} \subseteq V(x_0^{(kn)})$ and $C_{mkI} \not\subseteq V(x_0^{(kn)})$, we have that $C_{mkI} \not\subseteq C_{mk'I}$. This also shows that $\text{dim } B_m \geq \text{dim } C_{mkI}$ for $1 \leq k \leq q$, therefore $B_m \not\subseteq C_{mkI}, 1 \leq k \leq q$. But $C_{mkI} \not\subseteq B_m$ because $B_m \subseteq V(x_0^{(qn)})$ and $C_{mkI} \not\subseteq V(x_0^{(qn)})$ for $1 \leq k \leq q$. We thus have that $C_{mkI} \not\subseteq B_m$ and $B_m \not\subseteq C_{mkI}$. We conclude the corollary from the fact that by construction $C_m^0 = \bigcup_{k=1}^q C_{mkI} \cup B_m$.

To understand the general case, i.e. to find the irreducible components of $C_m^0$, where $C$ has a branch with $g$ Puiseux exponents at 0, since for $kn_1\bar{\beta}_1 < m \leq (k + 1)n_1\bar{\beta}_1$, $m, k \in \mathbb{N}$ we know by corollary 4.2 the structure of the $m$-jets that project to $V(x_0^{(0)}, \cdots, x_0^{(kn)}) \cap C^0_{kn_1\bar{\beta}_1}$, we have to understand for $m > kn_1\bar{\beta}_1$ the $m$-jets that projects to $V(x_0^{(0)}, \cdots, x_0^{(kn_1-1)}) \cap D(x_0^{(kn_1)})$, i.e. $C_m^k := \pi_m^{-1}(V(x_0^{(0)}, \cdots, x_0^{(kn_1-1)}) \cap D(x_0^{(kn_1)}))_{\text{red}}$.

Let $m, k \in \mathbb{N}$ be such that $m \geq kn_1\bar{\beta}_1$. Let $j = \max\{l, n_2 \cdots n_{l-1} \text{ divides } k\}$ (we set $j = 2$ if the greatest common divisor $(k, n_2) = 1$ or if $g = 1$). Set $\kappa$ such that $k = \kappa n_2 \cdots n_{j-1}$, then we have $kn_1 = \kappa n_{j} \cdots n_{g}$.
Indeed, we have that
\[ C_k^m = \bar{\pi}_{m,\left[n_i^{-m}u_{g}\right]}(C_{m,\left[n_i^{-m}u_{g}\right]^k}), \]
where \( \bar{\pi}_{m,\left[n_i^{-m}u_{g}\right]} : C_m^2 \rightarrow C_{m,\left[n_i^{-m}u_{g}\right]^2} \) is the canonical map. For \( j < g + 1 \) and \( m \geq \beta \beta_j \), we have that
\[ C_m^k = \emptyset \]

**Proposition 4.5.** — Let \( 2 \leq j \leq g + 1 \); for \( i = 2, \ldots, g \), and \( kn_1 \beta_1 < m < \kappa e^{-\beta_{j-1}} \), we have that
\[ o_f(\tilde{f}) = o_{x_i}(\hat{f}). \] (It was pointed by the referee that this follows from [1]. For the convenience of the reader we give a detailed proof below.) Let \( y(x^{\frac{1}{\beta_0}}), z(x^{\frac{1}{n_i^{-m}u_{g}}}) \) and \( u(x^{\frac{1}{m(\tilde{f})}}) \) be respectively Puiseux-roots of \( f, x_i \) and \( \tilde{f} \). There exist \( w, \lambda \in \mathbb{C} \) such that \( w^{\frac{1}{n_i^{-m}u_{g}}} = 1, \lambda^m(\tilde{f}) = 1 \) and
\[ o_f(\tilde{f}) = ord_x(u(\lambda x^{\frac{1}{m(\tilde{f})}}) - y(x^{\frac{1}{\beta_0}})) \]
and
\[ o_f(x_i) = ord_x(y(x^{\frac{1}{\beta_0}}) - z(wx^{\frac{1}{n_i^{-m}u_{g}} - 1})). \]
Since \( o_f(\tilde{f}) < o_f(x_i) \), we have that
\[ o_f(\tilde{f}) = ord_x(u(\lambda x^{\frac{1}{m(\tilde{f})}}) - y(x^{\frac{1}{\beta_0}}) + y(x^{\frac{1}{\beta_0}}) - z(wx^{\frac{1}{m(\tilde{f})} - 1})) \]
\[ = ord_x(u(\lambda x^{\frac{1}{m(\tilde{f})}}) - z(wx^{\frac{1}{n_i^{-m}u_{g} - 1}})) \leq o_{x_j}(\hat{f}). \]
On the other hand, there exist \( \lambda \) and \( \delta \in \mathbb{C} \), such that \( \lambda^m(\tilde{f}) = 1, \delta^{\beta_0} = 1 \) and such that
\[ o_{x_j}(\hat{f}) = ord_x(u(\lambda x^{\frac{1}{m(\tilde{f})}}) - z(x^{\frac{1}{n_i^{-m}u_{g} - 1}})) \]
and
\[ o_f(x_i) = ord_x(y(\delta x^{\frac{1}{\beta_0}}) - z(x^{\frac{1}{n_i^{-m}u_{g} - 1}})). \]
We have then that
\[ o_{x_i}(\tilde{f}) = \text{ord}_x(u(\lambda x^{\frac{1}{m(i)}}) - y(\delta x^{\frac{1}{n_0}}) + y(\delta x^{\frac{1}{n_0}}) - z(w x^{\frac{1}{n_1-n_i-1}})). \]

Now
\[ \text{ord}_x(u(\lambda x^{\frac{1}{m(i)}}) - y(\delta x^{\frac{1}{n_0}})) \leq o_f(\tilde{f}) \]
\[ < o_f(x_i) = \text{ord}_x(y(\delta x^{\frac{1}{n_0}}) - z(w x^{\frac{1}{n_1-n_i-1}})). \]

So
\[ o_{x_i}(\tilde{f}) = \text{ord}_x(u(\lambda x^{\frac{1}{m(i)}}) - y(\delta x^{\frac{1}{n_0}})) \leq o_f(\tilde{f}). \]

We conclude that \( o_f(\tilde{f}) = o_{x_i}(\tilde{f}), \) and since the sequence of Puiseux exponents of \( C_i \) is \( (\frac{\beta_0}{n_1 \cdots n_g}, \ldots, \frac{\beta_{i-1}}{n_1 \cdots n_g}) \), applying proposition 3.4 to \( C \) and \( C_i \), we find that \( (f, \tilde{f})_0 = n_i \cdots n_g(x_i, \tilde{f})_0 \) and claim follows.

On the other hand by the corollary 3.5 applied to \( f \) and \( \tilde{f}, (f, \tilde{f})_0 \geq \kappa e_{i-1} \frac{\beta_i}{\beta_0} \) if and only if \( o_f(\tilde{f}) \geq \frac{\beta_i}{\beta_0} = o_{x_i}(f) = o_f(x_i) \) so \( o_f(\tilde{f}) \geq \frac{\beta_i}{\beta_0} \) if and only if \( o_{x_i}(\tilde{f}) \geq \frac{\beta_i}{\beta_0} \), therefore \( (x_i, \tilde{f})_0 \geq \kappa \frac{\beta_i}{e_{j-1}}. \) This proves the first assertion.

The second assertion is a direct consequence of lemma 5.1 in [5].

To further analyse the \( C_m^k \)'s, we realize, as in section 3, \( C \) as a complete intersection in \( \mathbb{C}^{g+1} = \text{Spec} \mathbb{C}[x_0, \ldots, x_g] \) defined by the ideal \( (f_1, \ldots, f_g) \) where
\[
 f_i = x_{i+1} - (x_i^{n_i} - c_i x_0^{b_{0i}} \cdots x_{i-1}^{b_{i-1}} - \sum_{\gamma=(\gamma_0, \ldots, \gamma_i)} c_i \gamma_0 x_0^{\gamma_0} \cdots x_i^{\gamma_i})
\]
for \( 1 \leq i \leq g \) and \( x_{g+1} = 0. \) This will let us see the \( C_m^k \)'s as fibrations over some reduced scheme that we understand well.

We keep the notations above and let \( I^{k}_m \) be the radical of the ideal defining \( C_m^k \) in \( \mathbb{C}^{g+1} \) and let \( I^{0k}_m \) be the ideal defining
\[
 C_m^k = (V(I^{0}_m, x_0^{(0)}, \ldots, x_0^{(kn_1)})) \cap D(x_0^{(kn_1)})_{\text{red}} \text{ in } D(x_0^{(kn_1)}).
\]

**Lemma 4.6.** — Let \( k \neq 0, j \) and \( \kappa \) as above. For \( 1 \leq i < j \leq g \) (resp. \( 1 \leq i < j-1 = g \)) and for \( \kappa n_i \cdots n_{j-1} \beta_i \leq m < \kappa n_i+1 \cdots n_{j-1} \beta_{i+1} \), we have
\[
 I^{0k}_m = (x_0^{(0)}, \ldots, x_0^{(\frac{n \beta_i}{n_j-n_g}-1)}),
\]
\[
 x_i^{(0)}, \ldots, x_i^{(\frac{n \beta_j}{n_j-n_g}-1)}, F_l^{(\frac{n \beta_i}{n_j-n_g})}, \ldots, F_l^{(m)}; 1 \leq l \leq i,
\]
\[
 x_{i+1}^{(0)}, \ldots, x_{i+1}^{(\frac{m}{n_i+1-n_g})},
\]
\[
 F_l^{(0)}, \ldots, F_l^{(m)}, i + 1 \leq l \leq g - 1.
\]
Moreover for $1 \leq l \leq i$,
\[
F_l^{(l)} = -(x_l^{(l)})_{0}^{n_l} - c_l x_0^{(l)} \cdots x_{l-1}^{(l)-1} \mod \left( \left( x_l^{(l)} \right)_{0}^{m} \right),
\]
for $1 \leq l < i$ and $\kappa \frac{n_l \beta_l}{n_j \cdots n_g} < n < \kappa \frac{n_{l+1} \beta_l}{n_j \cdots n_g}$ (resp. $l = i$ and $\kappa \frac{n_l \beta_l}{n_j \cdots n_g} < n \leq \frac{m}{n_{l+1} \cdots n_g}$)
\[
F_l^{(n)} \equiv -(n_l x_l^{(n-1)})_{0}^{n_l} - c_l \cdots (x_l^{(n-1)})_{0}^{n_l} \mod \left( \left( x_l^{(n-1)} \right)_{0}^{m} \right),
\]
for $1 \leq l < i$ and $\kappa \frac{n_l \beta_l}{n_j \cdots n_g} < n \leq m$ (resp. $l = i$ and $\frac{m}{n_{l+1} \cdots n_g} < n \leq m$), or $i + 1 \leq l \leq g - 1$ and $0 < n < m$,
\[
F_l^{(n)} = x_l^{(n)} + H_l(x_l^{(0)}, \cdots, x_l^{(n-1)}).
\]
For $i = j - 1 = g$ and $m \geq \kappa n_g \beta_g$,
\[
F_m^{(0)} = (x_0^{(0)}), \cdots, x_0^{(\beta_0 - 1)},
\]
\[
x_l^{(0)}, \cdots, x_l^{(\beta_l - 1)}, F_l^{(\beta_l)}, \cdots, F_l^{(m)}, 1 \leq l \leq g,
\]
where for $1 \leq l < g$ and $\kappa n_l \beta_l \leq n \leq m$, the above formula for $F_l^{(n)}$ remains valid,
\[
F_{g}^{(\kappa n_g \beta_g)} \equiv -(x_g^{(\kappa \beta_g)}_{0}^{n_g} - c_g x_0^{(\beta_0)} \cdots x_{g-1}^{(\beta_{g-1})})_{0}^{n_g} \mod \left( \left( x_g^{(\beta_g)} \right)_{0}^{m} \right),
\]
and for $\kappa n_g \beta_g < n \leq m$,
\[
F_{g}^{(n)} \equiv -(n_g x_g^{(\kappa \beta_g)}_{0}^{n_g} - c_g x_0^{(\beta_0)} \cdots x_{g-1}^{(\beta_{g-1})})_{0}^{n_g} \mod \left( \left( x_g^{(\beta_g)} \right)_{0}^{m} \right),
\]
\[
\sum_{0 \leq h \leq g-1} b_{g} x_0^{(\beta_0)} \cdots x_h^{(\beta_h)} x_h^{(\beta_h)_{0}^{n_g} - c_g x_0^{(\beta_0)} \cdots x_{g-1}^{(\beta_{g-1})}} + H_g(x_h^{(\beta_h)_{0}^{n_g}}), \cdots, x_l^{(\beta_l - 1)})_{0}^{n_g} \mod \left( \left( x_l^{(\beta_l - 1)} \right)_{0}^{m} \right).
\]
I

Assume that it holds for $0 < n < k_n \beta_1$. Now since $\frac{m}{n_2 \cdots n_g} \geq \left[ \frac{m}{n_2 \cdots n_g} \right] \geq k_n m_1$, we have

$$F_{(k_1 m_1)}^{(n)} = -(x_1^{(k_1 m_1)} - c_1 x_0^{(k_1 m_1)})$$

and

$$F_{(k_1 m_1)}^{(n)} = -(n_1 x_1^{(k_1 m_1 - 1)} x_1^{(k_1 m_1 - n - k_1 m_1)} - m_1 c_1 x_0^{(k_1 m_1 - 1)} x_0^{(k_1 m_1 - n - k_1 m_1)})$$

$$+ H_l(x_0^{(0)}, \ldots, x_0^{(k_1 m_1 - 1)} x_1^{(0)}, \ldots, x_1^{(k_1 m_1 - n - k_1 m_1 - 1)} x_2^{(0)}, \ldots, x_2^{(m - n_2 \cdots n_g - 1)})$$

for $k_n \beta_1 < n \leq \left[ \frac{m}{n_2 \cdots n_g} \right]$. Finally, for $l = 1$ and $[ \frac{m}{n_2 \cdots n_g} ] < n \leq m$, or $2 \leq l \leq g - 1$ and $0 \leq n \leq m$, we have

$$F_{l}^{(n)} = x_l^{(n)} + H_l(x_0^{(0)}, \ldots, x_0^{(m)}, \ldots, x_l^{(0)}, \ldots, x_l^{(n)}).$$

As a consequence for $i = 1$, the subscheme of $C^{g+1} \cap D(x_0^{(k_1 m_1)})$ defined by $I_{m}^{0k}$ is isomorphic to the product of $C^*$ by an affine space, so it is reduced and irreducible and $I_{m}^{0k} = I_{m}^{0k}$ is a prime ideal in $C[x_0^{(0)}, \ldots, x_0^{(m)}, \ldots, x_g^{(0)}, \ldots, x_g^{(m)}]_{x_0^{(k_1 m_1)}}$, generated by a regular sequence, i.e. the proposition holds for $i = 1$. Assume that it holds for $i < j - 1 < g$ (resp. $i < j - 2 = g - 1$). For $k_n i + 1 \cdots n_{j - 1} \beta_{i + 1} \leq m < k_n i + 2 \cdots n_{j - 1} \beta_{i + 2}$, the ideal in $C[x_0^{(0)}, \ldots, x_0^{(m)}, \ldots, x_g^{(0)}, \ldots, x_g^{(m)}]_{x_0^{(k_1 m_1)}}$ generated by $I_{m}^{0k}$ is contained in $I_{m}^{0k}$. By the inductive hypothesis, $x_l^{(0)}, \ldots, x_l^{(n_2 \cdots n_g - 1)} \in$
$I_{\kappa n_{i+1} \cdots n_{j-1}}^{i+1}$ for $l = 1, \ldots, i + 1$. So $I_m^{0k}$ is the radical of

$$I_m^{0k} = (x_0^{(0)}, \ldots, x_l^{(\frac{n_i \beta_0}{n_j \cdots n_g} - 1)}, x_l^{(0)}, \ldots, x_l^{(\frac{n_i \beta_l}{n_j \cdots n_g} - 1)}, F_l^{(0)}, \ldots, F_l^{(m)}), 1 \leq l \leq i + 1,$$

$$x_{i+2}^{(0)}, \ldots, x_{i+2}^{(\frac{m}{n_{i+2} \cdots n_g})}, F_l^{(0)}, \ldots, F_l^{(m)}, i + 2 \leq l \leq g - 1).$$

Now for $0 \leq n < \frac{\kappa n_l \beta_l}{n_j \cdots n_g}$, we have

$$F_l^{(n)} \equiv x_{i+1}^{(n)} \mod (x_0^{(0)}, \ldots, x_l^{(\frac{n_i \beta_0}{n_j \cdots n_g} - 1)}, x_l^{(0)}, \ldots, x_l^{(\frac{n_i \beta_l}{n_j \cdots n_g} - 1)}, 1 \leq l \leq i + 1).$$

Here since $\overline{\beta}_{l+1} + n_l \overline{\beta}_l$, for $1 \leq l \leq i$ and $\frac{m}{n_{i+2} \cdots n_g} \geq \frac{1}{n_{i+2} \cdots n_g} \geq \frac{\kappa n_{i+1} \overline{\beta}_{i+1}}{n_j \cdots n_g},$ we can delete $F_l^{(n)}$, $1 \leq l \leq i + 1, 0 \leq n < \frac{\kappa n_l \beta_l}{n_j \cdots n_g}$ from the above generators of $I_m^{0k}$. The identities relative to the $F_l^{(n)}$ for $1 \leq l \leq i + 1$, $0 \leq n \leq m$ or $i + 2 \leq l \leq g - 1$ and $0 \leq n \leq m$ follow immediately from $(\circ)$. Hence the subscheme of $\mathbb{C}^{g+1} \cap D(x_0^{(kn_l)})$ defined by $I_m^{0k}$ is isomorphic to the product of $\mathbb{C}^*$ by an affine space, so it is reduced and irreducible and $I_m^{0k} = I_m^{0k}$ is a prime ideal in $\mathbb{C}[x_0^{(0)}, \ldots, x_0^{(m)}, \ldots, x_g^{(0)}, \ldots, x_g^{(m)}]_{x_0^{(kn_l)}},$ generated by a regular sequence, i.e the proposition holds for $i + 1$.

The case $i = j - 1 = g$ and $m \geq \kappa n_g \beta_g$ follows by similar arguments. □

As an immediate consequence we get

**Proposition 4.7.** — Let $C$ be a plane branch with $g$ Puiseux exponents. Let $k \neq 0, j$ and $\kappa$ as above. For $m \geq \kappa n_1 \beta_1$, let $\pi_{m,kn_1 \beta_1} : C_m \to C_{kn_1 \beta_1}$ be the canonical projection and let $C_m^k := \pi_{m,kn_1 \beta_1}^{-1}(D(x_0^{(kn_l)})) \cap V(x_0^{(0)}, \ldots, x_0^{(kn_l)})_{\text{red}}$. Then for $1 \leq i < j \leq g$ (resp.$1 \leq i < j - 1 = g$) and $\kappa n_i \cdots n_{j-1} \beta_i \leq m < \kappa n_{i+1} \cdots n_{j-1} \beta_{i+1}, C_m^k$ is irreducible of codimension

$$\frac{\kappa}{n_j \cdots n_g}(\beta_0 + \beta_1 + \sum_{l=1}^{i-1}(\beta_{l+1} - n_l \beta_l)) + ([\frac{m}{n_{i+1} \cdots n_g}] - \frac{\kappa n_i \beta_i}{n_j \cdots n_g}) + 1$$

in $\mathbb{C}_m$. (We suppose that the sum in the formula is equal to 0 when $i = 1$.) For $j \leq g$ and $m \geq \kappa \beta_j$ (resp.$j = g + 1$ and $m \geq \kappa n_g \beta_g$),

$$C_m^k = \emptyset$$
(resp. $C^k_m$ is of codimension
\[
\kappa(\tilde{\beta}_0 + \tilde{\beta}_1 + \sum_{l=1}^{g-1} (\tilde{\beta}_{l+1} - n_l\tilde{\beta}_l)) + m - \kappa n_g \tilde{\beta}_g + 1
\]
in $C^2_m$.

The referee kindly pointed out that for $m \in \mathbb{N}$ such that $\kappa n_i \cdots n_{j-1} \tilde{\beta}_i \leq m < \kappa n_{i+1} \cdots n_{j-1} \tilde{\beta}_{i+1}$, the codimension of $C^k_m$ can also be written as:
\[
\kappa \left( \frac{e_j - 1}{e_j} (\tilde{\beta}_0 + \beta_{i+1} - \tilde{\beta}_{i+1}) + \left( \frac{m}{e_i} - \frac{\kappa n_i \tilde{\beta}_i}{e_1} \right) + 1 \right.
\]

For $k' \geq k$ and $m \geq k'n_1 \tilde{\beta}_1$, we now compare $\text{codim}(C^k_m, C^2_m)$ and $\text{codim}(C^k_{m'}, C^2_m)$.

**Corollary 4.8.** — For $k' \geq k \geq 1$ and $m \geq k'n_1 \tilde{\beta}_1$, if $C^k_m$ and $C^k_{m'}$ are nonempty, we have
\[
\text{codim}(C^k_{m'}, C^2_m) \leq \text{codim}(C^k_m, C^2_m).
\]

**Proof.** — Let $\gamma^k : [kn_1 \tilde{\beta}_1, \infty[ \rightarrow [k(n_1 + m_1), \infty[ be the piecewise linear function given by
\[
\gamma^k(m) = \frac{k}{e_1} (\tilde{\beta}_0 + \tilde{\beta}_1 + \sum_{l=1}^{i-1} (\tilde{\beta}_{l+1} - n_l\tilde{\beta}_l)) + \left( \frac{m}{e_i} - \frac{\kappa n_i \tilde{\beta}_i}{e_1} \right) + 1
\]

for $1 \leq i \leq g$ and $\frac{k \tilde{\beta}_i}{n_2 \cdots n_{i-1}} \leq m < \frac{k \tilde{\beta}_{i+1}}{n_2 \cdots n_i}$. (Recall that by convention $\tilde{\beta}_{g+1} = \infty$)

In view of proposition 4.7, we have that $\text{codim}(C^k_m, C^2_m) = [\gamma^k(m)]$ for $k \equiv 0 \mod n_2 \cdots n_{j-1}$ and $k \not\equiv 0 \mod n_2 \cdots n_j$ with $2 \leq j \leq g$ and any integer $m \in [kn_1 \tilde{\beta}_1, \frac{k \tilde{\beta}_j}{n_2 \cdots n_{j-1}}[$ or for $k \equiv 0 \mod n_2 \cdots n_g$ and any integer $m \geq kn_1 \tilde{\beta}_1$. Similarly we define $\gamma^{k'} : [k'n_1 \tilde{\beta}_1, \infty[ \rightarrow [k'(n_1 + m_1), \infty[ by changing $k$ to $k'$.

Let $\Gamma^k$ (resp. $\Gamma^{k'}$) be the graph of $\gamma^k$ (resp. $\gamma^{k'}$) in $\mathbb{R}^2$. Now let $\tau : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $\tau(a, b) = (a, b - 1)$ and let $\lambda^{k'/k} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $\lambda^{k'/k}(a, b) = \frac{k'}{k} (a, b)$. We note that $\tau(\Gamma^{k'}) = \lambda^{k'/k}(\tau(\Gamma^k))$; we also note that the endpoints of $\tau(\Gamma^k)$ and $\tau(\Gamma^{k'})$ lie on the line through 0 with slope $\frac{\beta_0 + \beta_1}{e_1 n_1 \tilde{\beta}_1} = \frac{1}{e_1} \frac{n_1 + m_1}{n_1 m_1} < \frac{1}{e_1}$. Since $k' \geq 1$, the image of $\tau(\Gamma^k)$ by $\lambda^{k'/k}$ lies in the interior subset of $\mathbb{R}^2_{\geq 0}$ whith boundary the union of $\tau(\Gamma^k)$, of the segment joining its endpoint $(kn_1 \tilde{\beta}_1, \frac{k'}{k} (\beta_0 + \tilde{\beta}_1))$ to $(kn_1 \tilde{\beta}_1, 0)$ and of $[kn_1 \tilde{\beta}_1, \infty[ \times 0$. This implies that $\gamma^{k'}(m) \leq \gamma^k(m)$ for $m \geq k'n_1 \tilde{\beta}_1$, hence $[\gamma^{k'}(m)] \leq [\gamma^k(m)]$ and the claim. \qed
THEOREM 4.9. — Let $C$ be a plane branch with $g \geq 2$ Puiseux exponents. Let $m \in \mathbb{N}$. For $1 \leq m < n_1 \bar{\beta}_1 + e_1, C_m^0 = \text{Cont}^{>0}(x_0)_m$ is irreducible. For $q n_1 \bar{\beta}_1 + e_1 < m < (q + 1)n_1 \bar{\beta}_1 + e_1$, with $q \geq 1$ in $\mathbb{N}$, the irreducible components of $C_m^0$ are:

$$C_{m\kappa} = \text{Cont}^{\kappa \bar{\beta}_0}(x_0)_m$$

for $1 \leq \kappa$ and $\kappa \bar{\beta}_0 \bar{\beta}_1 + e_1 \leq m$,

$$C_{m\kappa V}^j = \text{Cont}^{\frac{\kappa \bar{\beta}_0}{n_j \cdots n_q}}(x_0)_m$$

for $j = 2, \ldots, g$, $1 \leq \kappa$ and $\kappa \not\equiv 0 \mod n_j$ and such that $\kappa n_1 \cdots n_{j-1} \bar{\beta}_1 + e_1 \leq m < \kappa \beta_j$,

$$B_m = \text{Cont}^{>n_1 q}(x_0)_m.$$

Proof. — We first observe that for any integer $k \neq 0$ and any $m \geq k n_1 \bar{\beta}_1$,

$$(C_m^0)_{\text{red}} = \cup_{1 \leq h \leq k} C_m^h \cup \text{Cont}^{>k n_1}(x_0)_m$$

where $C_m^h := \text{Cont}^{hn_1}(x_0)_m$. Indeed, for $k = 1$, we have that $(C_m^0)_{\text{red}} \subset V(x_0^{(0)}, \ldots, x_0^{(n_1-1)})$ by proposition 4.4. Arguing by induction on $k$, we may assume that the claim holds for $m \geq (k-1)n_1 \bar{\beta}_1$. Now by corollary 4.2, we know that for $m \geq k n_1 \bar{\beta}_1$, $\text{Cont}^{>(k-1)n_1}(x_0)_m \subset V(x_0^{(0)}, \ldots, x_0^{(k n_1-1)})$, hence the claim for $m \geq kn_1 \bar{\beta}_1$.

We thus get that for $q n_1 \bar{\beta}_1 + e_1 \leq m < (q + 1)n_1 \bar{\beta}_1 + e_1$,

$$(C_m^0)_{\text{red}} = \cup_{1 \leq k \leq q} C_m^k \cup \text{Cont}^{>q n_1}(x_0)_m.$$
by corollary 4.2. Let $\lambda^q : [qn_1 \bar{\beta}_1 + e_1, q(n_1 + m_1), \infty] \to [q(n_1 + m_1), \infty]$ be the function given by $\lambda^q(m) = q(n_1 + m_1) + \frac{m - qn_1 \bar{\beta}_1}{e_1} + 1$. For simplicity, set $i = m - qn_1 \bar{\beta}_1$. For any integer $i$ such that $e_1 \leq i < n_1 \bar{\beta}_1 = n_1 m_1 e_1$, we have $1 + \left[ \frac{i}{n_1 e_1} \right] + \left[ \frac{1}{m_1 e_1} \right] \leq \left[ \frac{i}{e_1} \right]$. Indeed this is true for $i = e_1$ and it follows by induction on $i$ from the fact that for any pair of integers $(b, a)$, we have $\left[ \frac{b+1}{a} \right] = \left[ \frac{b}{a} \right]$ if and only if $b + 1 \equiv 0 \mod a$ and $\left[ \frac{b+1}{a} \right] = \left[ \frac{b}{a} \right] + 1$ otherwise, since $i < n_1 m_1 e_1$. So $\delta^q(m) \leq [\lambda^q(m)]$.

But in the proof of corollary 4.8, we have checked that if $C^k_m \neq \emptyset$, then $\text{codim}(C^k_m, C^2_m) = [\gamma^k(m)]$. We have also checked that for $q \geq k$ and $m \geq q n_1 \bar{\beta}_1$, $\gamma^k(m) \geq \gamma^q(m)$. Finally in view of the definitions of $\gamma^q$ and $\lambda^q$, we have $\gamma^q(m) \geq \lambda^q(m)$, so $[\gamma^q(m)] \geq [\lambda^q(m)] \geq \delta^q(m)$.

For $m = (q + 1)n_1 \bar{\beta}_1$, we have $\delta^q(m) = (q + 1)(n_1 + m_1) + 1$ by corollary 4.2. For $m \in [(q + 1)n_1 \bar{\beta}_1, (q + 1)n_1 \bar{\beta}_1 + e_1]$, we have $\text{Cont}^{q n_1}(x_0)_m = C^{q+1}_m \cup \text{Cont}^{q+1}(x_0)_m$ and

$$\text{Cont}^{q+1}(x_0)_m = V(x_0^{(0)}, \ldots, x_0^{(q+1)n_1}, x_1^{(0)}, \ldots, x_1^{((q+1)m_1)})$$

again by corollary 4.2. If in addition we have $m < (q + 1)n_1 \bar{\beta}_2$, then by proposition 4.5 $C^{q+1}_m = V(x_0^{(0)}, \ldots, x_0^{((q+1)n_1-1)}, x_1^{(0)}, \ldots, x_1^{((q+1)m_1-1)}, x_1^{((q+1)m_1)} - c_1 x_0^{((q+1)n_1-1)}) \cap D(x_0^{((q+1)n_1)}),$ thus we have $\text{Cont}^{q n_1}(x_0)_m = C^{q+1}_m$ and $\delta^q(m) = (q + 1)(n_1 + m_1) + 1$. We have $(q + 1)n_1 \bar{\beta}_1 + e_1 \leq (q + 1)n_1 \bar{\beta}_2$ if $q + 1 \geq n_2$, because $\bar{\beta}_2 - n_1 \bar{\beta}_1 \equiv 0 \mod (e_2)$. If not, we may have $(q + 1)n_1 \bar{\beta}_2 < (q + 1)n_1 \bar{\beta}_1 + e_1$, so for $(q + 1)n_1 \bar{\beta}_2 \leq m < (q + 1)n_1 \bar{\beta}_1 + e_1$, we have $C^{q+1}_m = \emptyset, \text{Cont}^{q n_1}(x_0)_m = \text{Cont}^{q+1}(x_0)_m$ and $\delta^q(m) = (q + 1)(n_1 + m_1) + 2$.

In both cases, for $m \in [(q + 1)n_1 \bar{\beta}_1, (q + 1)n_1 \bar{\beta}_1 + e_1]$, we have $\delta^q(m) \leq (q + 1)(n_1 + m_1) + 2$. Since $[\lambda^q(m)] = (q + 1)(n_1 + m_1) + n_1 m_1 + 1$, we conclude that $[\lambda^q(m)] \geq \delta^q(m)$, so for $1 \leq k \leq q$, if $C^k_m \neq \emptyset$, we have $[\gamma^k(m)] \geq \delta^q(m)$. This proves that the irreducible components of $C^0_m$ are the $C^{m'}_m$ for $1 \leq k \leq q$ and $C^k_m \neq \emptyset$, and $\text{Cont}^{q n_1}(x_0)_m$, hence the claim in view of the characterization of the nonempty $C^k_m$'s given in proposition 4.5. $\square$

COROLLARY 4.10. — Under the assumption of theorem 4.9, let $q_0 + 1 = \min\{\alpha \in \mathbb{N}; \alpha(n_2 - n_1 \bar{\beta}_1) \geq e_1\}$. Then $0 \leq q_0 < n_2$. For $1 \leq m < (q_0 + 1)n_1 \bar{\beta}_1 + e_1$, $C^0_m$ is irreducible and we have $\text{codim}(C^0_m, C^2_m) = 2 + \left[ \frac{m}{\bar{\beta}_0} \right] + \left[ \frac{m}{\bar{\beta}_1} \right]$ for $0 \leq q < q_0$ and $qn_1 \bar{\beta}_1 + e_1 \leq m < (q + 1)n_1 \bar{\beta}_1$

or $0 \leq q \leq q_0$ and $(q + 1)n_1 \bar{\beta}_2 \leq m < (q + 1)n_1 \bar{\beta}_1 + e_1$. 

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that the empty $C$ with $1 \leq k \leq q$ and $(q+1)n_1 \beta_1 \leq m < (q+1) \beta_2$

or $(q_0 + 1)n_1 \beta_1 \leq m < (q_0 + 1)n_1 \beta_1 + e_1$.

For $q \geq q_0 + 1$ in $\mathbb{N}$ and $qn_1 \beta_1 + e_1 \leq m < (q+1)n_1 \beta_1 + e_1$, the number of irreducible components of $C^0_m$ is:

$$N(m) = q + 1 - \sum_{j=2}^{g} \left( \left\lfloor \frac{m}{\beta_j} \right\rfloor - \left\lfloor \frac{m}{n_j \beta_j} \right\rfloor \right)$$

and $\text{codim}(C^0_m, C^2_m) =$

$$2 + \left\lfloor \frac{m}{\beta_0} \right\rfloor + \left\lfloor \frac{m}{\beta_1} \right\rfloor \text{ for } qn_1 \beta_1 + e_1 \leq m < (q+1)n_1 \beta_1.$$

$$1 + \left\lfloor \frac{m}{\beta_0} \right\rfloor + \left\lfloor \frac{m}{\beta_1} \right\rfloor \text{ for } (q+1)n_1 \beta_1 \leq m < (q+1)n_1 \beta_1 + e_1.$$

**Proof.** — We have already observed that $n_2(\beta_2 - n_1 \beta_1) \geq e_1$ because $\beta_2 - n_1 \beta_1 \equiv 0 \mod (e_2)$, so $1 \leq q_0 + 1 \leq n_2$.

For $qn_1 \beta_1 + e_1 \leq m < (q+1)n_1 \beta_1 + e_1$, with $q \geq 1$, we have seen in the proof of theorem 4.9 that the irreducible components of $C^0_m$ are the $C^k_m$ for $1 \leq k \leq q$ and $C^k_m \neq \emptyset$, and $\text{Cont}^{\text{qm}}(x_0)_m$. We thus have to enumerate the empty $C^k_m$ for $1 \leq k \leq q$. By proposition 4.5, $C^k_m = \emptyset$ if and only if $j := \max\{l; l \geq 2 \text{ and } k \equiv 0 \mod n_2 \cdots n_{l-1}\} \leq g$ and $m \geq k - n_2 \cdots n_{j-1} \beta_j$.

Now recall that $\beta_{i+1} > n_i \beta_1$ for $1 \leq i \leq g - 1$ and that $\beta_2 - n_1 \beta_1 \geq e_2$. This implies that for $3 \leq j \leq g$, we have $\beta_j - n_1 \cdots n_{j-1} \beta_1 > n_2 \cdots n_{j-1} (\beta_2 - n_1 \beta_1) \geq n_2 \cdots n_{j-1} e_2 > e_1$. So if $j \geq 3$ and $\kappa$ is a positive integer such that $m \geq \kappa \beta_j$, we have $\frac{m-\kappa \beta_j}{n_1 \beta_1} > \kappa n_2 \cdots n_{j-1}$, hence $q = \left\lfloor \frac{m-\kappa \beta_j}{n_1 \beta_1} \right\rfloor \geq \kappa n_2 \cdots n_{j-1}$.

Therefore for $j \geq 3$, there are exactly $\left\lfloor \frac{m}{n_j \beta_j} \right\rfloor$ integers $\kappa \geq 1$ such that $m \geq \kappa \beta_j$ and $\kappa n_2 \cdots n_{j-1} \leq q$, among them $\left\lfloor \frac{m}{n_j \beta_j} \right\rfloor$ are $\equiv 0 \mod (n_j)$.

Similarly if $(q+1)n_1 \beta_1 + e_1 \leq (q+1) \beta_2$, or equivalently $q \geq q_0$, and if $\kappa$ is a positive integer such that $m \geq \kappa \beta_2$, we have $\kappa \leq \frac{m}{\beta_2} < q + 1$. Therefore if $q \geq q_0 + 1$, we conclude that there are $\sum_{j=2}^{g} \left( \left\lfloor \frac{m}{\beta_j} \right\rfloor - \left\lfloor \frac{m}{n_j \beta_j} \right\rfloor \right)$ empty $C^k_m$’s with $1 \leq k \leq q$. Moreover we have shown in the proof of theorem 4.9 that $\text{codim}(C^0_m, C^2_m) = \text{codim}(\text{Cont}^{\text{qm}}(x_0)_m, C^2_m) = 2 + \left\lfloor \frac{m}{\beta_0} \right\rfloor + \left\lfloor \frac{m}{\beta_1} \right\rfloor$ if $m < (q+1)n_1 \beta_1$ (resp. $1 + (q + 1)(n_1 + m_1) = 1 + \left\lfloor \frac{m}{\beta_0} \right\rfloor + \left\lfloor \frac{m}{\beta_1} \right\rfloor$ for $m \geq (q+1)n_1 \beta_1$). Also note that $q_0 \beta_2 < q_0 n_1 \beta_1 + e_1 < (q_0 + 1)n_1 \beta_1 + e_1 \leq (q_0 + 1) \beta_2 < n_2 \beta_2 < \beta_3 \cdots$. Therefore for $q_0 n_1 \beta_1 + e_1 \leq m < (q_0 + 1)n_1 \beta_1 + e_1$, we have $\left\lfloor \frac{m}{\beta_2} \right\rfloor = q_0, \left\lfloor \frac{m}{n_2 \beta_2} \right\rfloor = \left\lfloor \frac{m}{\beta_3} \right\rfloor = \cdots = 0$, so $N(m) = 1$, i.e. $C^0_m$ is irreducible.
Finally, assume that \( qn_1\beta_1 + e_1 \leq m < (q + 1)n_1\beta_1 + e_1 \) with \( q \geq 1 \) and \( q < q_0 \). Since \( q_0 < n_2 \), for \( 1 \leq k \leq q \) we have \( k \not\equiv 0 \mod(n_2) \) and \( m \geq qn_1\beta_1 + e_1 > q\beta_2 \), hence for \( 1 \leq k \leq q \), \( C_m^k = \emptyset \) and \( C_m^0 = \text{Cont}^{qn_1}(x_0)_m \) is irreducible. (The case \( q = q_0 \) was already known.) So for \( n_1\beta_1 \leq m < (q_0+1)n_1\beta_1 + e_1 \), \( C_m^0 = \text{Cont}^{qn_1}(x_0)_m \) is irreducible. (Recall that for \( 1 \leq m < q_0n_1\beta_1 + e_1 \), the irreducibility of \( C_m^0 \) is already known.) It only remains to check the codimensions of \( C_m^0 \) for \( 1 \leq m \leq q_0n_1\beta_1 + e_1 \). Here again we have seen in the proof of Theorem 4.9 that \( \text{codim}(C_m^0, C_m^2) = \text{codim}(\text{Cont}^{>qn_1}(x_0)_m, C_m^2) =: \delta^q(m) \) for any \( q \geq 1 \) and \( qn_1\beta_1 + e_1 \leq m < (q + 1)n_1\beta_1 + e_1 \) and that

\[
\delta^q(m) = 2 + \left[ \frac{m}{\beta_0} \right] + \left[ \frac{m}{\beta_1} \right] \text{ for any } q \geq 1 \text{ and } qn_1\beta_1 + e_1 \leq m < (q + 1)n_1\beta_1
\]

\[
(q + 1)(n_1 + m_1) + 1 = 1 + \left[ \frac{m}{\beta_0} \right] + \left[ \frac{m}{\beta_1} \right] \text{ for } q < q_0 \text{ and } (q + 1)n_1\beta_1 \leq m < (q + 1)\beta_2
\]

\[
(q + 1)(n_1 + m_1) + 2 = 2 + \left[ \frac{m}{\beta_0} \right] + \left[ \frac{m}{\beta_1} \right] \text{ for } q < q_0 \text{ and } (q + 1)\beta_2 \leq m < (q + 1)n_1\beta_1 + e_1.
\]

This completes the proof. \( \square \)

In [6], Igusa has shown that the log-canonical threshold of the pair \(((\mathbb{C}^2, 0), (C, 0))\) is \( \frac{1}{\beta_0} + \frac{1}{\beta_1} \). Here \((\mathbb{C}^2, 0)\)(resp.\((C, 0)\)) is the formal neighborhood of \( \mathbb{C}^2 \) (resp. \( C \)) at 0. Corollary 4.10 allows to recover corollary B of [2] in this special case.

**Corollary 4.11.** — If the plane curve \( C \) has a branch at 0, with multiplicity \( \beta_0 \), and first Puiseux exponent \( \beta_1 \), then

\[
\min_m \frac{\text{codim}(C_m^0, C_m^2)}{m + 1} = \frac{1}{\beta_0} + \frac{1}{\beta_1}.
\]

**Proof.** — For any \( m, p \neq 0 \) in \( \mathbb{N} \), we have \( m - p \left[ \frac{m}{p} \right] \leq p - 1 \) and \( m - p \left[ \frac{m}{p} \right] = p - 1 \) if and only if \( m + 1 \equiv 0 \mod(p) \); so for any \( m \in \mathbb{N}, 2 + \left[ \frac{m}{\beta_0} \right] + \left[ \frac{m}{\beta_1} \right] \geq (m + 1)(\frac{1}{\beta_0} + \frac{1}{\beta_1}) \) and we have equality if and only if \( m + 1 \equiv 0 \mod(\beta_0) \) and \( \beta_0 \) or equivalently \( m + 1 \equiv 0 \mod(n_1\beta_1) \) since \( n_1\beta_1 \) is the least common multiple of \( \beta_0 \) and \( \beta_1 \). If not we have \( 1 + \left[ \frac{m}{\beta_0} \right] + \left[ \frac{m}{\beta_1} \right] \geq (m + 1)(\frac{1}{\beta_0} + \frac{1}{\beta_1}) \). Now if \((q + 1)n_1\beta_1 \leq m < (q + 1)n_1\beta_1 + e_1 \) with \( q \in \mathbb{N} \), we have \((q + 1)n_1\beta_1 < m + 1 \leq (q + 1)n_1\beta_1 + e_1 < (q + 2)n_1\beta_1 \), so \( m + 1 \neq 0 \mod(n_1\beta_1) \). If \((q + 1)n_1\beta_1 \leq m < (q + 1)\beta_2 \) with \( q \in \mathbb{N} \) and \( q < q_0 \), then \((q + 1)n_1\beta_1 < m + 1 \leq (q + 1)n_1\beta_1 + e_1 < (q + 2)n_1\beta_1 \), so \( m + 1 \neq 0 \mod(n_1\beta_1) \).
(n_1 \bar{\beta}_1). So in both cases, we have $1 + \left[\frac{m}{\bar{\beta}_0}\right] + \left[\frac{m}{\bar{\beta}_1}\right] \geq (m + 1)(\frac{1}{\bar{\beta}_0} + \frac{1}{\bar{\beta}_1}).$ The claim follows from corollary 4.10. \hfill \Box

It also follows immediately from corollary 4.10.

**Corollary 4.12.** — Let $g_0 \in \mathbb{N}$ as in corollary 4.10. There exists $n_1 \bar{\beta}_1$ linear functions, $L_0, \cdots, L_{n_1 \bar{\beta}_1 - 1}$ such that $\text{dim}(C^0_m) = L_i(m)$ for any $m \equiv i \mod (n_1 \bar{\beta}_1)$ such that $m \geq g_0 n_1 \bar{\beta}_1 + e_1$.

The canonical projections $\pi_{m+1,m} : C^0_{m+1} \to C^0_m$, $m \geq 1$, induce infinite inverse systems

$$
\cdots B_{m+1} \to B_m \cdots \to B_1
$$

and finite inverse systems

$$
\cdots C_{(m+1)\kappa I} \to C_{m\kappa I} \cdots \to C_{(\kappa \beta_0 \bar{\beta}_1 + e_1)\kappa I} \to B_{\kappa \beta_0 \bar{\beta}_1 + e_1 - 1}
$$

for $2 \leq j \leq g$, and $\kappa \not\equiv 0 \mod (n_j)$.

We get a tree $T_{C,0}$ by representing each irreducible component of $C^0_m$, $m \geq 1$, by a vertex $v_{i,m}$, $1 \leq i \leq N(m)$, and by joining the vertices $v_{i,m+1}$ and $v_{i,m}$ if $\pi_{m+1,m}$ induces one of the above maps between the corresponding irreducible components.

This tree only depends on the semigroup $\Gamma$.

Conversely, we recover $\bar{\beta}_0, \cdots, \bar{\beta}_g$ from this tree and $\max\{m, \text{codim}(B_m, C^2_m) = 2\} = \bar{\beta}_0 - 1$. Indeed the number of edges joining two vertices from which an infinite branch of the tree starts is $\beta_0 \bar{\beta}_1$. We thus recover $\bar{\beta}_1$ and $e_1$. We recover $\bar{\beta}_2 - n_1 \bar{\beta}_1, \cdots, \bar{\beta}_j - n_1 \cdots n_{j-1} \bar{\beta}_1, \cdots, \bar{\beta}_g - n_1 \cdots n_{g-1} \bar{\beta}_1$, hence $\bar{\beta}_2, \cdots, \bar{\beta}_g$ from the number of edges in the finite branches.

**Corollary 4.13.** — Let $\mathcal{C}$ be a plane branch with $g \geq 1$ Puiseux exponents. The tree $T_{C,0}$ described above and $\max\{m, \text{dim} C^0_m = 2m\}$ determines the sequence $\bar{\beta}_0, \cdots, \bar{\beta}_g$ or equivalently the equisingularity class of $\mathcal{C}$ and conversely.

We represent below the tree for the branch defined by

$$f(x, y) = (y^2 - x^3)^2 - 4x^6 y - x^9 = 0,$$

whose semigroup is $< \bar{\beta}_0 = 4, \bar{\beta}_1 = 6, \bar{\beta}_2 = 15 >$, and for which we have $e_1 = 2$, $e_2 = 1$ and $n_1 = n_2 = 2$. 
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