ANNALES

DE

L'INSTITUT FOURIER

Kelly JABBUSCH & Stefan KEBEKUS

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Tome 61, nº 6 (2011), p. 2277-2290.

<http://aif.cedram.org/item?id=AIF_2011__61_6_2277_0>

Ann. Inst. Fourier, Grenoble
61, 6 (2011) 2277-2290

POSITIVE SHEAVES OF DIFFERENTIALS COMING FROM COARSE MODULI SPACES

by Kelly JABBUSCH & Stefan KEBEKUS (*)

ABSTRACT. — Consider a smooth projective family of canonically polarized complex manifolds over a smooth quasi-projective complex base Y°, and suppose the family is non-isotrivial. If Y is a smooth compactification of Y°, such that $D := Y \setminus Y^\circ$ is a simple normal crossing divisor, then we can consider the sheaf of differentials with logarithmic poles along D. Viehweg and Zuo have shown that for some $m > 0$, the m^{th} symmetric power of this sheaf admits many sections. More precisely, the m^{th} symmetric power contains an invertible sheaf whose Kodaira-Iitaka dimension is at least the variation of the family. We refine this result and show that this "Viehweg-Zuo sheaf" comes from the coarse moduli space associated to the given family, at least generically.

As an immediate corollary, if Y° is a surface, we see that the non-isotriviality assumption implies that Y° cannot be special in the sense of Campana.

RÉSUMÉ. — On considère une famille projective lisse de variétés canoniquement polarisées sur une base quasi-projective lisse Y. Si la famille n'est pas iso-triviale, Viehweg et Zuo ont montré que toute bonne compactification de Y admet des formes pluricanoniques avec au plus des pôles logarithmiques le long du bord. Plus précisément leur résultat montre qu'une puissance symétrique suffisamment grande du faisceau des différentielles logarithmiques contient un sous-faisceau inversible dont la dimension de Kodaira-Iitaka est au moins égale à la variation de la famille. En suivant la construction de Viehweg-Zuo on montre que le faisceau inversible de Viehweg-Zuo provient, au moins génériquement, de l'espace de module "grossier" associé à la famille.

Comme corollaire immédiat on obtient que la base d'une famille non-isotriviale ne peut pas être spéciale au sens de Campana.

Keywords: Moduli space, positivity of differentials.
Math. classification: 14D07, 14D22.
(*) Kelly Jabbusch and Stefan Kebekus were supported by the DFG-Forschergruppe "Classification of Algebraic Surfaces and Compact Complex Manifolds" in full and in part, respectively. The work on this paper was finished while the authors visited the 2009 Special Year in Algebraic Geometry at the Mathematical Sciences Research Institute, Berkeley. Both authors would like to thank the MSRI for support.

1. Introduction and statement of main result

1.1. Introduction

Throughout this paper, we consider a smooth projective family $f^\circ : X^\circ \to Y^\circ$ of canonically polarized complex manifolds, of relative dimension n, over a smooth complex quasi-projective base. We assume that the family is not isotrivial, and let $\mu : Y^\circ \to \mathfrak{M}$ be the associated map to the coarse moduli space, whose existence is shown, e.g. in [10, Thm. 1.11]. We fix a smooth projective compactification Y of Y° such that $D := Y \setminus Y^\circ$ is a divisor with simple normal crossings.

In this setup, Viehweg and Zuo have shown the following fundamental result concerning the existence of logarithmic pluri-differentials on Y°.

Theorem 1.1 (Existence of pluri-differentials on Y, [12, Thm. 1.4(i)]). *There exists a number $m > 0$ and an invertible sheaf $\mathscr{A} \subseteq \operatorname{Sym}^m \Omega^1_Y(\log D)$ whose Kodaira-Iitaka dimension is at least the variation of the family, $\kappa(\mathscr{A}) \geqslant \operatorname{Var}(f^\circ)$.* □

The "Viehweg-Zuo" sheaf $\mathscr{A}$ was crucial in the study of hyperbolicity properties of manifolds that appear as bases of families of maximal variation and has been used to show that any minimal model program of the pair (Y, D) factors the moduli map, [4, 5, 6], see also the survey [7]. In spite of its importance, little is known about further properties of the sheaf $\mathscr{A}$. For example, it is unclear to us how the Viehweg-Zuo construction behaves under base change. The goal of this short note is to refine the result of Viehweg and Zuo somewhat, and show that the Viehweg-Zuo sheaf $\mathscr{A}$ comes from the coarse moduli space $\mathfrak{M}$, at least generically. A precise statement is given in Theorem 1.4 below.

Theorem 1.4 directly relates to a conjecture of Campana. In [1, Conj. 12.19] Campana conjectured that the assumption "f° not isotrivial" immediately implies that the base manifold Y° is not *special*. In other words, any family of canonically polarized varieties over a special base manifold is necessarily isotrivial. In the case where Y° is a surface, the conjecture is claimed in [1, Thm. 12.20]. However, we had difficulties following the proof. We will show in Section 4 that Campana's conjecture in dimension two is an immediate corollary to Theorem 1.4. Using a more advanced line of argumentation, Campana's conjecture in dimension three can also be deduced. Details will appear in a forthcoming paper(1).

(1) Note added in proof: the paper has appeared as "Families over special base manifolds and a conjecture of Campana", Mathematische Zeitschrift, DOI: 10.1007/s00209-010-0758-6

Throughout the present paper we work over the field of complex numbers.

1.2. Statement of the main result

Roughly speaking, the main result of this paper is that the Viehweg-Zuo sheaf comes from the coarse moduli space $\mathfrak{M}$. To formulate the statement precisely, we use the following notation.

Notation 1.2. — Consider the subsheaf $\mathscr{B} \subseteq \Omega^1_Y(\log D)$, defined on presheaf level as follows: if $U \subset Y$ is any open set and $\sigma \in \Gamma\big(U, \Omega^1_Y(\log D)\big)$ any section, then $\sigma \in \Gamma\big(U, \mathscr{B}\big)$ if and only if the restriction $\sigma|_{U'}$ is in the image of the differential map

$$d\mu|_{U'} : \mu^*\big(\Omega^1_{\mathfrak{M}}\big)|_{U'} \longrightarrow \Omega^1_{U'},$$

where $U' \subseteq U \cap Y^\circ$ is the open subset where the moduli map μ has maximal rank.

Remark 1.3. — By construction, it is clear that the sheaf $\mathscr{B}$ is a saturated subsheaf of $\Omega^1_Y(\log D)$. We say that $\mathscr{B}$ is the saturation of $\operatorname{Image}(d\mu)$ in $\Omega^1_Y(\log D)$.

With this notation, the main result of the paper is then formulated as follows.

THEOREM 1.4 (Refinement of the Viehweg-Zuo Theorem 1.1). — *There exists a number $m > 0$ and an invertible subsheaf $\mathscr{A} \subseteq \operatorname{Sym}^m \mathscr{B}$ whose Kodaira-Iitaka dimension is at least the variation of the family, $\kappa(\mathscr{A}) \geqslant \operatorname{Var}(f^\circ)$.*

1.3. Outline of the paper

We begin the proof of Theorem 1.4 in Section 2 with a summary of Viehweg-Zuo's proof of Theorem 1.1. Using the notation and results of Section 2, a proof of Theorem 1.4 is given in Section 3. We end this paper with Section 4, where we briefly recall Campana's notion of a special logarithmic pair, give the precise statement of his conjecture and give an extremely short proof for families over surfaces.

Acknowledgments

Campana's Conjecture 4.4 was brought to our attention during the 2007 Levico conference in Algebraic Geometry. We would like to thank Frédéric Campana and Eckart Viehweg for a number of discussions on the subject, and the anonymous referee for careful reading, and for finding a mistake in the first version of the paper.

2. Review of Viehweg-Zuo's proof of Theorem 1.1

To prepare for the proof of Theorem 1.4, we give a very brief synopsis of Viehweg-Zuo's proof of Theorem 1.1, covering only the material used in the proof of Theorem 1.4. The reader who is interested in a detailed understanding is referred to the original paper [12] and to the survey [11]. The overview contained in this section and the facts outlined in Section 2.4 can perhaps serve as a guideline to the original references.

2.1. Setup of notation

Throughout the present Section 2, we choose a smooth projective compactification X of X° such that the following holds:

(1) The difference $\Delta := X \backslash X^\circ$ is a divisor with simple normal crossings.
(2) The morphism f° extends to a projective morphism $f : X \to Y$.

It is then clear that $\Delta = f^{-1}(D)$ set-theoretically. Removing a suitable subset $S \subset Y$ of codimension $\operatorname{codim}_Y S \geqslant 2$, the following will then hold automatically on $Y' := Y \setminus S$ and $X' := X \setminus f^{-1}(S)$, respectively.

(3) The restricted morphism $f' := f|_{X'}$ is flat.
(4) The divisor $D' := D \cap Y'$ is smooth.
(5) The divisor $\Delta' := \Delta \cap X'$ is a relative normal crossing divisor, i.e. a normal crossing divisor whose components and all their intersections are smooth over the components of D'.

In the language of Viehweg-Zuo, [12, Def 2.1(c)], the restricted morphism $f' : X' \to Y'$ is a "good partial compactification of f°".

Remark 2.1 (Restriction to a partial compactification). — Let $\mathscr{G}$ be a locally free sheaf on Y, and let $\mathscr{F}' \subseteq \mathscr{G}|_{Y'}$ be an invertible subsheaf. Since $\operatorname{codim}_Y S \geqslant 2$, there exists a unique extension of the sheaf $\mathscr{F}'$ to an invertible subsheaf $\mathscr{F} \subseteq \mathscr{G}$ on Y. Furthermore, the restriction map $\Gamma\big(Y, \mathscr{F}\big) \to \Gamma\big(Y', \mathscr{F}'\big)$ is an isomorphism. In particular, the notion of Kodaira-Iitaka dimension makes sense for the sheaf $\mathscr{F}'$, and $\kappa(\mathscr{F}') = \kappa(\mathscr{F})$.

2.2. Construction of the $\tau^0_{p,q}$

The starting point of the Viehweg-Zuo construction is the standard sequence of logarithmic differentials associated to the flat morphism f',

$$0 \to (f')^*\Omega^1_{Y'}(\log D') \to \Omega^1_{X'}(\log \Delta') \to \Omega^1_{X'/Y'}(\log \Delta') \to 0, \tag{2.1}$$

where $\Omega^1_{X'/Y'}(\log \Delta')$ is locally free. It is a standard fact, [3, II, Ex. 5.16], that Sequence (2.1) defines a filtration of the p^{th} exterior power,

$$\Omega^p_{X'}(\log \Delta') = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^p \supseteq F^{p+1} = \{0\},$$

with $F^r/F^{r+1} \cong (f')^*\big(\Omega^r_{Y'}(\log D')\big) \otimes \Omega^{p-r}_{X'/Y'}(\log \Delta')$. Taking the sequence

$$0 \longrightarrow F^1 \longrightarrow F^0 \longrightarrow F^0\big/F^1 \longrightarrow 0$$

modulo F^2, we obtain

$$0 \to (f')^*\big(\Omega^1_{Y'}(\log D')\big) \otimes \Omega^{p-1}_{X'/Y'}(\log \Delta') \to F^0\big/F^2 \to \Omega^p_{X'/Y'}(\log \Delta') \to 0. \tag{2.2}$$

Setting $\mathscr{L} := \Omega^n_{X'/Y'}(\log \Delta')$, twisting Sequence (2.2) with $\mathscr{L}^{-1}$ and pushing down, the connecting morphisms of the associated long exact sequence give maps

$$\tau^0_{p,q} : F^{p,q} \longrightarrow F^{p-1,q+1} \otimes \Omega^1_{Y'}(\log D'),$$

where

$$F^{p,q} := R^q f'_*(\Omega^p_{X'/Y'}(\log \Delta') \otimes \mathscr{L}^{-1})\big/\text{torsion}.$$

Set $\mathscr{N}_0^{p,q} := \ker(\tau^0_{p,q})$.

2.3. Alignment of the $\tau^0_{p,q}$

The morphisms $\tau^0_{p,q}$ and $\tau^0_{p-1,q+1}$ can be composed if we tensor the latter with the identity morphism on $\Omega^1_{Y'}(\log D')$. More specifically, we consider the following morphisms,

$$\underbrace{\tau^0_{p,q} \otimes \mathrm{Id}_{\Omega^1_{Y'}(\log D')^{\otimes q}}}_{=:\tau_{p,q}} : F^{p,q} \otimes \big(\Omega^1_{Y'}(\log D')\big)^{\otimes q} \to F^{p-1,q+1} \otimes \big(\Omega^1_{Y'}(\log D')\big)^{\otimes q+1},$$

and their compositions

$$\underbrace{\tau_{n-k+1,k-1} \circ \cdots \circ \tau_{n-1,1} \circ \tau_{n,0}}_{=:\tau^k} : F^{n,0} \to F^{n-k,k} \otimes \big(\Omega^1_{Y'}(\log D')\big)^{\otimes k}. \tag{2.3}$$

2.4. Fundamental facts about $\tau^{\mathbf{k}}$ and $\mathscr{N}_{\mathbf{0}}^{\mathbf{p},\mathbf{q}}$

Theorem 1.1 is shown by relating the morphism $\tau^0_{p,q}$ with the structure morphism of a Higgs-bundle coming from the variation of Hodge structures associated with the family f°. Viehweg's positivity results of push-forward sheaves of relative differentials, as well as Zuo's results on the curvature of kernels of generalized Kodaira-Spencer maps are the main input here. Rather than recalling the complicated line of argumentation, we simply state two central results from the argumentation of [12].

Fact 2.2 (Factorization via symmetric differentials, [12, Lem. 4.6]). — *For any k, the morphism τ^k factors via the symmetric differentials*

$$\operatorname{Sym}^k \Omega^1_{Y'}(\log D') \subseteq \big(\Omega^1_{Y'}(\log D')\big)^{\otimes k}.$$

More precisely, the morphism τ^k takes its image in

$$F^{n-k,k} \otimes \operatorname{Sym}^k \Omega^1_{Y'}(\log D').$$

□

Consequence 2.3. — Using Fact 2.2 and the observation that $F^{n,0} \cong \mathscr{O}_{Y'}$, we can therefore view τ^k as a morphism

$$\tau^k : \mathscr{O}_{Y'} \longrightarrow F^{n-k,k} \otimes \operatorname{Sym}^k \Omega^1_{Y'}(\log D').$$

While the proof of Fact 2.2 is rather elementary, the following deep result is at the core of Viehweg-Zuo's argument.

Fact 2.4 (Negativity of $\mathscr{N}_0^{p,q}$, [12, Claim 4.8]). — *Given any numbers p and q, there exists a number k and an invertible sheaf $\mathscr{A}' \in \operatorname{Pic}(Y')$ of Kodaira-Iitaka dimension $\kappa(\mathscr{A}') \geqslant \operatorname{Var}(f^0)$ such that*

$$(\mathscr{A}')^* \otimes \operatorname{Sym}^k\big((\mathscr{N}_0^{p,q})^*\big)$$

is generically generated. □

2.5. End of proof

To end the sketch of proof, we follow [12, p. 315] almost verbatim. By Fact 2.4, the trivial sheaf $F^{n,0} \cong \mathscr{O}_{Y'}$ cannot lie in the kernel $\mathscr{N}_0^{n,0}$ of $\tau^1 = \tau^0_{n,0}$. We can therefore set $1 \leqslant m$ to be the largest number with $\tau^m(F^{n,0}) \neq \{0\}$. Since m is maximal, $\tau^{m+1} = \tau_{n-m,m} \circ \tau^m \equiv 0$ and

$$\operatorname{Image}(\tau^m) \subseteq \ker(\tau_{n-m,m}) = \mathscr{N}_0^{n-m,m} \otimes \operatorname{Sym}^m \Omega^1_{Y'}(\log D').$$

In other words, τ^m gives a non-trivial map

$$\tau^m : \mathscr{O}_{Y'} \cong F^{n,0} \longrightarrow \mathscr{N}_0^{n-m,m} \otimes \operatorname{Sym}^m \Omega^1_{Y'}(\log D').$$

Equivalently, we can view τ^m as a non-trivial map

$$\tau^m : (\mathscr{N}_0^{n-m,m})^* \longrightarrow \operatorname{Sym}^m \Omega^1_{Y'}(\log D'). \tag{2.4}$$

By Fact 2.4, there are many morphisms $\mathscr{A}' \to \operatorname{Sym}^k\big((\mathscr{N}_0^{n-m,m})^*\big)$, for k large enough. Together with (2.4), this gives a non-zero morphism $\mathscr{A}' \to \operatorname{Sym}^{k\cdot m} \Omega^1_{Y'}(\log D')$.

We have seen in Remark 2.1 that the sheaf $\mathscr{A}' \subseteq \operatorname{Sym}^{k\cdot m} \Omega^1_{Y'}(\log D')$ extends to a sheaf $\mathscr{A} \subseteq \operatorname{Sym}^{k\cdot m} \Omega^1_{Y}(\log D)$ with $\kappa(\mathscr{A}) = \kappa(\mathscr{A}') \geqslant \operatorname{Var}(f^\circ)$. This ends the proof of Theorem 1.1. □

3. Proof of Theorem 1.4

3.1. Setup and assumptions

The proof of Theorem 1.4 makes use of essentially all results explained in Section 2. Since the assumptions of Theorems 1.1 and 1.4 agree, we maintain the full setup and all notation introduced in Section 2.

3.2. Reduction to a study of the $\tau^0_{p,q}$

The construction outlined in Section 2 essentially says that the sheaf $\mathscr{A}$ constructed by Viehweg-Zuo is a symmetric product of the image sheaves of the $\tau^0_{p,q}$. The precise statement is the following.

Proposition 3.1. — *To prove Theorem 1.4, it suffices to show that*

$$\operatorname{Image}(\tau^0_{p,q}) \subseteq F^{p-1,q+1} \otimes \mathscr{B}' \tag{3.1}$$

for all p and q, where $\mathscr{B}' := \mathscr{B}|_{Y'}$ and $\mathscr{B} \subseteq \Omega^1_Y(\log D)$ is the sheaf defined in Notation 1.2.

Remark 3.2. — *Since $\mathscr{B}$ is saturated, it is enough to check inclusion* (3.1) *on an open set.*

Proof of Proposition 3.1. — If (3.1) holds, the image of the morphisms τ^k defined in (2.3) lies in $F^{n-k,k} \otimes \big(\mathscr{B}'\big)^{\otimes k}$. Furthermore, by Fact 2.2,

$$\operatorname{Image}(\tau^k) \subseteq F^{n-k,k} \otimes \operatorname{Sym}^k \mathscr{B}'.$$

If we chose the number m as in Section 2.5 above, the image of τ^m is then contained in $\mathscr{N}_0^{n-m,m} \otimes \operatorname{Sym}^m \mathscr{B}'$, and τ^m can be seen as a non-trivial map

$$\tau^m : (\mathscr{N}_0^{n-m,m})^* \longrightarrow \operatorname{Sym}^m \mathscr{B}'.$$

As in Section 2.5, we obtain a map $\mathscr{A}' \to \operatorname{Sym}^{k\cdot m} \mathscr{B}'$, with $\kappa(\mathscr{A}') \geqslant \operatorname{Var}(f^\circ)$, and Remark 2.1 gives the extension to a sheaf $\mathscr{A} \subset \operatorname{Sym}^{k\cdot m} \mathscr{B}$, with $\kappa(\mathscr{A}) = \kappa(\mathscr{A}')$. □

3.3. Proof of Inclusion (3.1) in a simple case

It remains to check Inclusion (3.1). Before tackling the problem in general, we consider a trivial case first.

Proposition 3.2. — *If the variation of f° is maximal, i.e.* $\operatorname{Var}(f^\circ) = \dim Y^\circ$, *then Inclusion* (3.1) *holds.*

Proof. — If the variation of f° is maximal, then the moduli map $Y^\circ \to \mathfrak{M}$ is generically finite onto the closure of its image. In particular, the sheaf $\mathscr{B}$ introduced in Notation 1.2 equals $\Omega^1_Y(\log D)$. Inclusion (3.1) is therefore trivially satisfied. □

3.4. Comparing families with respect to Inclusion (3.1)

Given two families, one the pull-back of the other via a dominant morphism, an elementary comparison of the morphisms $\tau^0_{p,q}$ associated with the families shows that Inclusion (3.1) holds for one of the families if and only if it also holds for the other. We will later use the following Comparison Proposition to show that the Viehweg-Zuo sheaf of a family essentially only depends on the image of the base in the coarse moduli space, and not so much on the family itself.

Proposition 3.3 (Comparison Proposition). — *Consider a Cartesian diagram of smooth projective families of n-dimensional canonically polarized manifolds over smooth quasi-projective base manifolds, as follows*

$$\begin{tikzcd} \hat{X}^\circ \arrow[r, "\Gamma"] \arrow[d, "\hat{f}^\circ"'] & \tilde{X}^\circ \arrow[d, "\tilde{f}^\circ"] \\ \hat{Y}^\circ \arrow[r, "\gamma", "\text{dominant}"'] & \tilde{Y}^\circ. \end{tikzcd}$$

Let $\hat{f}' : \hat{X}' \to \hat{Y}'$ and $\tilde{f}' : \tilde{X}' \to \tilde{Y}'$ be two good partial compactifications, in the sense introduced in Section 2.1. Then Inclusion (3.1) *holds for $\hat{f}'$ if and only if it holds for $\tilde{f}'$.*

Proof. — We have noted in Remark 3.2 that it suffices to check Inclusion (3.1) on an open subset. In particular, it suffices to consider the restrictions of the morphisms $\tau^0_{p,q}$ and of all relevant sheaves to $\hat{Y}^\circ$ and $\tilde{Y}^\circ$. This greatly simplifies notation because the logarithmic boundary terms do not appear in the restrictions, and we can write, e.g., $\Omega^p_{\tilde{Y}^\circ}$ instead of the more complicated $\Omega^p_{\tilde{Y}'}(\log \tilde{D}')$.

Shrinking $\hat{Y}^\circ$ and $\tilde{Y}^\circ$ further, if necessary, we may assume without loss of generality that γ is surjective and smooth. We may also assume that the moduli map $\tilde{\mu} : \tilde{Y}^\circ \to \mathfrak{M}$ has maximal rank. By assumption, the moduli map $\hat{\mu} : \hat{Y}^\circ \to \mathfrak{M}$ is the composition $\hat{\mu} = \tilde{\mu} \circ \gamma$.

As in Section 2, we need to discuss the connecting morphisms $\tau^0_{p,q}$ on $\hat{Y}^\circ$ and on $\tilde{Y}^\circ$, respectively. For clarity of notation we indicate the relevant space by indexing all morphisms and sheaves with either a hat or a tilde. That way, we write

$$\hat{\tau}^0_{p,q} : \hat{F}^{p,q} \to \hat{F}^{p-1,q+1} \otimes \Omega^1_{\hat{Y}^\circ} \quad \text{and} \quad \tilde{\tau}^0_{p,q} : \tilde{F}^{p,q} \to \tilde{F}^{p-1,q+1} \otimes \Omega^1_{\tilde{Y}^\circ},$$

where $\hat{F}^{p,q} := R^q \hat{f}^\circ_* \big(\Omega^p_{\hat{X}^\circ/\hat{Y}^\circ} \otimes (\Omega^n_{\hat{X}^\circ/\hat{Y}^\circ})^{-1}\big)$ and the sheaf $\tilde{F}^{p,q}$ on $\tilde{Y}^\circ$ is defined analogously. Finally, set

$$\widehat{\mathscr{B}}^\circ := \operatorname{Image}\big(d\hat{\mu} : \hat{\mu}^*(\Omega^1_{\mathfrak{M}}) \to \Omega^1_{\hat{Y}^\circ}\big) \quad \text{and}$$
$$\widetilde{\mathscr{B}}^\circ := \operatorname{Image}\big(d\tilde{\mu} : \tilde{\mu}^*(\Omega^1_{\mathfrak{M}}) \to \Omega^1_{\tilde{Y}^\circ}\big).$$

Since γ is smooth and the moduli map $\tilde{\mu}$ has maximal rank, $\hat{\mu}$ also has maximal rank, and both $\widetilde{\mathscr{B}}^\circ$ and $\widehat{\mathscr{B}}^\circ$ are saturated in $\Omega^1_{\tilde{Y}^\circ}$ and $\Omega^1_{\hat{Y}^\circ}$, respectively. Better still, the differential $d\gamma : \gamma^*(\Omega^1_{\tilde{Y}^\circ}) \to \Omega^1_{\hat{Y}^\circ}$ induces an isomorphism

$$d\gamma : \gamma^*(\widetilde{\mathscr{B}}^\circ) \xrightarrow{\cong} \widehat{\mathscr{B}}^\circ. \tag{3.3}$$

Since $\widetilde{\mathscr{B}}^\circ$ and $\widehat{\mathscr{B}}^\circ$ are saturated, to prove Proposition 3.3 it is equivalent to show that

$$\operatorname{Image}(\hat{\tau}^0_{p,q}) \subseteq \hat{F}^{p-1,q+1} \otimes \widehat{\mathscr{B}}^\circ \Longleftrightarrow \operatorname{Image}(\tilde{\tau}^0_{p,q}) \subseteq \tilde{F}^{p-1,q+1} \otimes \widetilde{\mathscr{B}}^\circ. \tag{3.4}$$

To prove (3.4), we aim to identify the sheaves $\hat{F}^{p,q}$ and $\gamma^*\big(\tilde{F}^{p,q}\big)$ and show that images of the $\hat{\tau}^0_{p,q}$ are naturally identified with the pull-backs of the images of $\tilde{\tau}^0_{p,q}$. For a precise statement, recall that there are isomorphisms

$\Gamma^*\big(\Omega^p_{\tilde{X}^\circ/\tilde{Y}^\circ}\big) \cong \Omega^p_{\hat{X}^\circ/\hat{Y}^\circ}$ for all p. Since taking cohomology commutes with flat base change, [3, III Prop. 9.3], we obtain isomorphisms

$$\iota^{p,q} : \gamma^*\big(\tilde{F}^{p,q}\big) \xrightarrow{\cong} \hat{F}^{p,q}$$

for all p and q. Tensoring $\iota^{p,q}$ with the differential $d\gamma : \gamma^*\big(\Omega^1_{\tilde{Y}^\circ}\big) \to \Omega^1_{\hat{Y}^\circ}$ gives a map

$$\iota^{p,q} \otimes d\gamma : \gamma^*\big(\tilde{F}^{p,q} \otimes \Omega^1_{\tilde{Y}^\circ}\big) \longrightarrow \hat{F}^{p,q} \otimes \Omega^1_{\hat{Y}^\circ}. \tag{3.5}$$

Equivalence (3.4), and hence Proposition 3.3, is an immediate consequence of the Isomorphism (3.3) and of the following claim.

Claim 3.6. — *Given any numbers p and q, the sheaf morphism* (3.5) *induces an isomorphism between the image of $\hat{\tau}^0_{p,q}$ and the pull-back of the image of $\tilde{\tau}^0_{p,q}$,*

$$\iota^{p,q} \otimes d\gamma : \gamma^*\big(\mathrm{Image}(\tilde{\tau}^0_{p,q})\big) \xrightarrow{\cong} \mathrm{Image}(\hat{\tau}^0_{p,q}).$$

It remains to prove Claim 3.6. Observe that Claim 3.6 follows trivially from the definitions of $\tilde{\tau}^0_{p,q}$ and $\hat{\tau}^0_{p,q}$ if we are in the simple case where $\hat{Y}^0$ is a product, say $\hat{Y}^\circ \cong \tilde{Y}^\circ \times \tilde{Z}^\circ$, and where γ is the projection to the first factor. Locally in the analytic topology, however, any smooth morphism looks like the projection morphism of a product. Since Claim 3.6 can be checked locally analytically, this proves the claim and ends the proof of Proposition 3.3. □

3.5. End of proof of Theorem 1.4

To complete the proof of Theorem 1.4, we compare our original family to one that is of maximal variation. The starting point is the existence of a universal family on a finite cover.

Theorem 3.4 (Existence of a universal family on a finite cover, [8, Prop. 2.7], see also [10, Thm. 9.25]). — *Let $\mathfrak{M}' \subseteq \mathfrak{M}$ be the reduced irreducible component that contains the image of Y°. Then there exists a reduced normal scheme $\widetilde{\mathfrak{M}}$, a finite and surjective morphism $\gamma : \widetilde{\mathfrak{M}} \to \mathfrak{M}'$ and a family of canonically polarized varieties $u : \tilde{U} \to \widetilde{\mathfrak{M}}$ such that γ is precisely the moduli map associated with the family u.* □

Let $Z' \subseteq Y^\circ \times_{\mathfrak{M}} \widetilde{\mathfrak{M}}$ be an irreducible component of the fiber product that dominates Y°, let $W' := \overline{\mathrm{Image}(Z')} \subseteq \widetilde{\mathfrak{M}}$ be the closure of the image of Z' in $\widetilde{\mathfrak{M}}$ and choose desingularizations $W \to W'$ and $Z \to Z'$ such that the natural map $Z' \to W'$ extends to give a dominant map $Z \to W$. Setting $X^\circ_Z := X^\circ \times_{Y^\circ} Z$, $\tilde{U}_Z := \tilde{U} \times_{\widetilde{\mathfrak{M}}} Z$ and $\tilde{U}_W := \tilde{U} \times_{\widetilde{\mathfrak{M}}} W$, we obtain linked Cartesian diagrams as follows

$$\begin{tikzcd}
X^\circ \arrow[d, "f^\circ"'] & X^\circ_Z \arrow[l] \arrow[d, "f^\circ_Z"'] & \tilde{U}_Z \arrow[r] \arrow[d, "u_Z"'] & \tilde{U}_W \arrow[r] \arrow[d, "u_W"', "\text{family of max. var.}"] & \tilde{U} \arrow[d, "u"', "\text{family of max. var.}"] & \\
Y^\circ & Z \arrow[l, "\text{dominant}"] \arrow[r, equal] & Z \arrow[r, "\text{dominant}"'] & W \arrow[r, "\text{gen. finite}"'] & \widetilde{\mathfrak{M}} \arrow[r, "\text{finite, surjective}", "\text{moduli map of } u"'] & \mathfrak{M}.
\end{tikzcd}$$

Recall from Proposition 3.2 that Inclusion (3.1) holds for the family u_W, which is of maximal variation. The Comparison Proposition 3.3 then asserts that Inclusion (3.1) also holds for the family u_Z.

The families f°_Z and u_Z are not necessarily isomorphic, but induce the same moduli map $Z \to \mathfrak{M}$. Since for any point $z \in Z$, the fibers $(f^\circ_Z)^{-1}(z)$ and $u_Z^{-1}(z)$ are isomorphic, the scheme of Z-isomorphisms,

$$I' := \mathrm{Isom}_Z\big(X^\circ_Z, \tilde{U}_Z\big) \subseteq \mathrm{Hom}_Z\big(X^\circ_Z, \tilde{U}_Z\big) \subseteq \mathrm{Hilb}_Z\big(X^\circ_Z \times_Z \tilde{U}_Z\big)$$

surjects onto Z. Since all fibers $(f^\circ_Z)^{-1}(z) \cong u_Z^{-1}(z)$ are canonically polarized manifolds and have only finitely many automorphisms, the natural map $I' \to Z$ is quasi-finite. Let I be a desingularization of a component of I' that dominates Z. Recall that taking Hilb, Hom and Isom commutes with base change. In particular, we have an isomorphism of I-schemes,

$$\mathrm{Isom}_I\big(X^\circ_Z \times_Z I, \tilde{U}_Z \times_Z I\big) \cong \mathrm{Isom}_Z\big(X^\circ_Z, \tilde{U}_Z\big) \times_Z I.$$

Looking at the right hand side, it is clear that there exists a section $I \to \mathrm{Isom}_I\big(X^\circ_Z \times_Z I, \tilde{U}_Z \times_Z I\big)$, i.e., an isomorphism of I-schemes, $X^\circ_Z \times_Z I \cong \tilde{U}_Z \times_Z I$. In summary, we obtain a diagram as follows,

$$\begin{tikzcd}
X^\circ \arrow[d, "f^\circ"'] & X^\circ_Z \times_Z I \arrow[l] \arrow[r, leftrightarrow, "\cong"] \arrow[d, "f^\circ_I"'] & \tilde{U}_Z \times_Z I \arrow[r] \arrow[d, "u_I"] & \tilde{U}_Z \arrow[d, "u_Z"] \\
Y^\circ & I \arrow[l, "\gamma_Y", "\text{dominant}"'] \arrow[r, equal] & I \arrow[r, "\gamma_Z", "\text{dominant}"'] & Z.
\end{tikzcd}$$

We have seen that Inclusion (3.1) holds for the family u_Z. Since γ_Z is dominant, the Comparison Proposition 3.3 applies to show that Inclusion (3.1) holds for the family u_I, or, equivalently, for the family f°_I. Another application of the Comparison Proposition 3.3 to the morphism γ_Y then shows that Inclusion (3.1) holds for the family f°. Theorem 1.4 then follows from Proposition 3.1. $\square$

4. Application of Theorem 1.4 to families over special surfaces

As an immediate corollary to Theorem 1.4, we see that any smooth projective family of canonically polarized manifolds over a special surface Y° is isotrivial, as conjectured by Campana. We first recall the precise definition of a special logarithmic pair below, taking the classical Bogomolov-Sommese vanishing theorem as our starting point.

THEOREM 4.1 (Bogomolov-Sommese vanishing, [2, Sect. 6]). — *Let Y be a smooth projective variety and $D \subset Y$ a reduced, possibly empty divisor with simple normal crossings. If $p \leqslant \dim Y$ is any number and $\mathscr{A} \subseteq \Omega^p_Y(\log D)$ any invertible subsheaf, then the Kodaira-Iitaka dimension of $\mathscr{A}$ is at most p, i.e., $\kappa(\mathscr{A}) \leqslant p$.* □

In a nutshell, we say that a pair (Y, D) is special if the inequality in the Bogomolov-Sommese vanishing theorem is always strict.

DEFINITION 4.2 (Special logarithmic pair). — *In the setup of Theorem 4.1, a pair (Y, D) is called* special *if the strict inequality $\kappa(\mathscr{A}) < p$ holds for all p and all invertible sheaves $\mathscr{A} \subseteq \Omega^p_Y(\log D)$. A smooth, quasi-projective variety Y° is called special if there exists a smooth compactification Y such that $D := Y \setminus Y^\circ$ is a divisor with simple normal crossings and such that the pair (Y, D) is special.*

Remark 4.3. — If Y° is a smooth, quasi-projective variety and if (Y_1, D_1) and (Y_2, D_2) are two smooth compactifications with snc boundary divisors, as in Definition 4.2, then an elementary computation shows that the pair (Y_1, D_1) is special if and only if (Y_2, D_2) is special. Specialness can thus be checked on any snc compactification.

With this notation in place, Campana has conjectured the following.

Conjecture 4.4 (Generalization of Shafarevich Hyperbolicity, [1, Conj. 12.19]*).* — Let $f : X^\circ \to Y^\circ$ be a smooth family of canonically polarized varieties over a smooth quasi-projective base. If Y° is special, then the family f° is isotrivial.

As mentioned in the Introduction, in the case where Y° is a surface, Conjecture 4.4 is claimed in [1, Thm. 12.20]. However, we had difficulties following the proof, and offer a new proof, which is an immediate corollary to Theorem 1.4.

COROLLARY 4.5 (Campana's conjecture in dimension two). — *Conjecture 4.4 holds if* $\dim Y^\circ = 2$.

Proof. — We maintain the notation of Conjecture 4.4 and let $f : X^\circ \to Y^\circ$ be a smooth family of canonically polarized varieties over a smooth quasi-projective base, with Y° a special surface. Since Y° is special, it is not of log general type, and hence by [6, Thm. 1.1], $\operatorname{Var}(f^\circ) < 2$. Suppose $\operatorname{Var}(f^\circ) = 1$ and choose a compactification (Y, D) as in Definition 4.2, then by Theorem 1.4 there exists a number $m > 0$ and an invertible subsheaf $\mathscr{A} \subseteq \operatorname{Sym}^m \mathscr{B}$ such that $\kappa(\mathscr{A}) \geqslant 1$. However, since $\mathscr{B}$ is saturated in the locally free sheaf $\Omega^1_Y(\log D)$, it is reflexive, [9, Claim on p. 158], and since $\operatorname{Var}(f^\circ) = 1$, the sheaf $\mathscr{B}$ is of rank 1. Thus $\mathscr{B} \subseteq \Omega^1_Y(\log D)$ is an invertible subsheaf, [9, Lem. 1.1.15, on p. 154], and Definition 4.2 of a special pair gives that $\kappa(\mathscr{B}) < 1$, contradicting the fact that $\kappa(\mathscr{A}) \geqslant 1$. It follows that $\operatorname{Var}(f^\circ) = 0$ and that the family is hence isotrivial. □

A proof of Campana's Conjecture 4.4 in higher dimensions will appear in a forthcoming paper.

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Manuscrit reçu le 18 février 2010,
révisé le 3 mai 2010,
accepté le 15 octobre 2010.

Kelly JABBUSCH
KTH
Department of Mathematics
10044 Stockholm (Sweden)
jabbusch@math.kth.se

Stefan KEBEKUS
Albert-Ludwigs-Universität Freiburg
Mathematisches Institut
Eckerstraße 1, 79104 Freiburg (Germany)
stefan.kebekus@math.uni-freiburg.de