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ALMOST-EINSTEIN MANIFOLDS WITH NONNEGATIVE ISOTROPIC CURVATURE

by Harish SESHADRI (*)

ABSTRACT. — Let (M, g), $n \geqslant 4$, be a compact simply-connected Riemannian n-manifold with nonnegative isotropic curvature. Given $0 < l \leqslant L$, we prove that there exists $\varepsilon = \varepsilon(l, L, n)$ satisfying the following: If the scalar curvature s of g satisfies

$$l \leqslant s \leqslant L$$

and the Einstein tensor satisfies

$$\left|\mathrm{Ric} - \frac{s}{n} g\right| \leqslant \varepsilon$$

then M is diffeomorphic to a symmetric space of compact type.

This is related to the result of S. Brendle on the metric rigidity of Einstein manifolds with nonnegative isotropic curvature.

RÉSUMÉ. — Soit (M, g), une variété riemannienne compacte simplement connexe de dimension $n \geqslant 4$, à courbure isotrope positive ou nulle. Nous montrons que pour tout $0 < l < L$, il existe un $\varepsilon = \varepsilon(l, L, n)$ qui satisfait la propriété suivante : si la courbure scalaire s de g satisfait

$$l \leqslant s \leqslant L$$

et que le tenseur d'Einstein satisfait

$$\left|\mathrm{Ric} - \frac{s}{n} g\right| \leqslant \varepsilon$$

alors M est difféomorphe à un espace symétrique de type compact.

Ceci est lié au résultat de S. Brendle sur la rigidité métrique des variétés d'Einstein à courbure isotrope positive ou nulle.

1. Introduction

A Riemannian manifold (M, g) is said to have *nonnegative isotropic curvature* if

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \geqslant 0$$

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for every orthonormal 4-frame $\{e_1, e_2, e_3, e_4\}$.

In the case of strict inequality above we say that the manifold has positive isotropic curvature. Recently S. Brendle [1] proved that a compact Einstein manifold with nonnegative isotropic curvature has to be a locally symmetric space of compact type. In this note we relax the restriction that the metric is Einstein to the condition that the Einstein tensor is small in norm and obtain the following smooth rigidity result:

Theorem 1.1. — *Let (M^n, g), $n \geqslant 4$, be a compact simply-connected Riemannian manifold with nonnegative isotropic curvature. Given $0 < l \leqslant L$, there exists $\varepsilon = \varepsilon(l, L, n)$ satisfying the following: If the scalar curvature s of g satisfies*

$$l \leqslant s \leqslant L$$

and the Einstein tensor satisfies

$$\left|\mathrm{Ric} - \frac{s}{n} g\right| \leqslant \varepsilon$$

then M is diffeomorphic to a symmetric space of compact type.

This result was inspired by the paper of P. Petersen and T. Tao [7] where it is proved that "almost" quarter-pinching of sectional curvatures again leads to smooth rigidity as above. The main difference between their conclusion and ours is that symmetric spaces of rank $\geqslant 2$ are allowed in our case, while almost quarter-pinching gives only rank-1 spaces.

We remark that for any L, ε the conditions $s \leqslant L$ and $\left|\mathrm{Ric} - \frac{s}{n} g\right| \leqslant \varepsilon$ can be achieved just by rescaling the metric by a large constant. In particular, consider the connected sum $S^{n-1} \times S^1 \# S^{n-1} \times S^1$ which admits a metric with positive isotropic curvature by [5]. Rescaling this metric gives the two bounds above. However this manifold does not support a locally symmetric metric (irreducible or reducible) of compact type. This is seen by observing that the fundamental group of the latter space has to contain an abelian subgroup of finite index. Hence the lower bound on scalar curvature is necessary. On the other hand it is not known if just positive Ricci curvature and nonnegative isotropic curvature already imply that the underlying compact manifold is diffeomorphic to a locally symmetric space, even without the assumption of simple-connectivity.

We make a few remarks about the proofs. The proof of Theorem 1.1 proceeds as follows: Let (M, g) be a n-manifold satisfying the hypotheses of Theorem 1.1. By choosing $\varepsilon \leqslant \frac{l}{2n}$, we can ensure that the Ricci curvature is uniformly (*i.e.* depending only on n, l and L) positive. An elementary argument (Lemma 3.1) shows that we have an uniform upper bound on the norm of the curvature tensor. One can improve this by applying the Ricci flow for a short time: This gives a nearby metric with uniform bounds on the higher covariant derivatives of the curvature tensor (Lemma 2.1). One also has an uniform upper bound on diameter since the Ricci curvature is uniformly positive. A theorem of Petrunin-Tuschmann also guarantees an uniform lower bound on injectivity radius. To apply their result one needs finite second homotopy group which is the case if (M, g) has *positive* isotropic curvature. All these ingredients enable one to prove Theorem 1.1 by contradiction when (M, g) has positive isotropic curvature.

To deal with nonnegative isotropic curvature we use the results of the author [8] and S. Brendle [1] which allow us to reduce the nonnegative case to the positive case.

2. A smoothing lemma

LEMMA 2.1. — *Let (M, g) be a compact Riemannian manifold of dimension n. Let $A > 0$ be such that $|R| \leqslant A$, where R is the curvature tensor of (M, g).*

(i) *There exists $\delta = \delta(A, n) > 0$ and $E = E(A, n)$ such that the Ricci flow starting at g is defined on the time interval $[0, \delta]$. On this interval*
$$|R(t)| \leqslant E.$$
(ii) *For each $k = 1, 2, \ldots$ there exists $C(n, k) > 0$ such that*
$$|\nabla^k R(t)| \leqslant C(n, k)$$
for $t \in [0, \delta]$.

(iii) *If the injectivity radius $\mathrm{inj}(M, g)$ is bounded below by a constant $\alpha > 0$, we can assume that there is a $\beta = \beta(\alpha, A, n) > 0$ such that $\mathrm{inj}(M, g(\delta)) > \beta$.*

(iv) *If the scalar curvature of (M, g) is bounded below by $a > 0$ then the scalar curvature of $(M, g(\delta))$ is still bounded below by a.*

(v) *If g has positive isotropic curvature so does $g(\delta)$.*

(vi) *There is a $D = D(A, n)$ such that if the norm of the Einstein tensor Z of (M, g) satisfies $|Z| \leqslant \varepsilon$, then $|Z(\delta)| \leqslant \varepsilon D$*

Proof. — We refer the reader to the notes of P. Topping [9] for details in the proof below.

(i) and (ii) follow from the evolution equation for the curvature tensor and its covariant derivatives under Ricci flow. Note that one just needs an uniform bound on $|R|$ for the validity of (i) and (ii).

Since $|R(t)|$ is uniformly bounded for $[0,\delta]$ by (i), it follows directly from the Ricci flow equation that the metrics $g(t)$ are uniformly bi-Lipshitz equivalent to g. More precisely, one knows that if the Ricci curvature satisfies $|\operatorname{Ric}(t)| \leqslant c$ along a Ricci flow then

$$e^{-2ct}g(0) \leqslant g(t) \leqslant e^{2ct}g(0).$$

Hence we have (iii).

(iv) is a consequence of the fact that the infimum of scalar curvature increases along Ricci flow.

Since Ricci flow preserves positive isotropic curvature, by the work of Brendle and Schoen [2], we have (v).

Finally we prove (vi): Let $Z := \operatorname{Ric} - \frac{s}{n} g$ denote the Einstein tensor. We have, using $\frac{\partial g^{ij}}{\partial t} = 2\operatorname{Ric}^{ij}$,

$$\begin{aligned}\frac{\partial |Z|^2}{\partial t} &= \frac{\partial}{\partial r}\Big(g^{ij}g^{kl}Z_{ik}Z_{jl}\Big) \\ &= 2\Big(\operatorname{Ric}^{ij}g^{kl} + g^{ij}\operatorname{Ric}^{kl}\Big)Z_{ik}Z_{jl} + 2\Big\langle Z, \frac{\partial Z}{\partial t}\Big\rangle.\end{aligned} \tag{2.1}$$

The evolution of Z is given by

$$\begin{aligned}\frac{\partial Z}{\partial t}(X,W) = \triangle Z(X,W) - 2Z(W,\operatorname{Ric}(X)) + 2\Big\langle R(X,.,W,.),\operatorname{Ric}\Big\rangle \\ - \frac{2}{n}|\operatorname{Ric}|^2\langle X,W\rangle,\end{aligned}$$

where $\operatorname{Ric}(X)$ (in the second term on the right hand side) denotes Ricci curvature regarded as an operator. Note that the bilinear form

$$T(X,Y) := -2Z(W,\operatorname{Ric}(X)) + 2\Big\langle R(X,.,W,.),\operatorname{Ric}\Big\rangle - \frac{2}{n}|\operatorname{Ric}|^2\langle X,W\rangle$$

is identically zero if the metric is Einstein. In general, we claim that there is a constant $C_1 = C_1(A,n)$ such that

$$|T(X,Y)| \leqslant C_1|Z||X||W|.$$

This can be seen as follows: Since $\operatorname{Ric} = Z + \frac{s}{n}g$, we have

$$\begin{aligned}T(X,Y) = -2Z(W,\operatorname{Ric}(X)) + 2\Big\langle R(X,.,W,.), Z + \frac{s}{n}g\Big\rangle \\ - \frac{2}{n}\Big\langle Z + \frac{s}{n}g, Z + \frac{s}{n}g\Big\rangle\langle X,W\rangle \\ = -2Z(W,\operatorname{Ric}(X)) + 2\Big\langle R(X,.,W,.), Z\Big\rangle - \frac{2}{n}|Z|^2\langle X,W\rangle \\ + 2\Big\langle R(X,.,W,.), \frac{s}{n}g\Big\rangle - \frac{2s^2}{n^2}\langle X,W\rangle\end{aligned}$$

It is clear there is a constant $C' = C'(A,n)$ such that the absolute value of the first three terms in the last expression above is bounded by $C'|Z||X||W|$ for all X, W. As for the last two terms, note that $\langle R(X,.,W,.), g\rangle = \mathrm{Ric}(X,W)$ and hence

$$2\Big\langle R(X,.,W,.), \frac{s}{n} g\Big\rangle - \frac{2s^2}{n^2}\langle X, W\rangle = \frac{2s}{n} Z(X,W).$$

This completes the proof of the claim.

Therefore, for some $C_2 = C_2(A,n)$, we have

$$\begin{aligned} 2\Big\langle Z, \frac{\partial Z}{\partial t}\Big\rangle &\leqslant\ 2\big\langle \triangle Z, Z\big\rangle + C_2|Z|^2 \\ &=\ \triangle|Z|^2 - 2|\nabla Z|^2 + C_2|Z|^2 \\ &\leqslant\ \triangle|Z|^2 + C_2|Z|^2 \end{aligned} \tag{2.2}$$

Plugging (2.2) in (2.1) and again using the upper bound on the Riemann curvature tensor, we get, for some $C_3 = C_3(A,n)$,

$$\frac{\partial |Z|^2}{\partial t} \leqslant \triangle|Z|^2 + C_3|Z|^2.$$

It follows from the maximum principle that $|Z|^2(t) \leqslant |Z|^2(0)e^{C_3 t}$, yielding the required estimate. $\square$

3. Proof of Theorem 1.1

We begin with a simple but useful lemma. Let $c \in \mathbb{R}$. By

$$K^{\mathrm{iso}} \geqslant c$$

we mean that

$$K^{\mathrm{iso}}(e_i, e_j, e_k, e_l) :=\ R_{ikik} + R_{ilil} + R_{jkjk} + R_{jljl} - 2R_{ijkl} \geqslant c$$

for every orthonormal 4-frame $\{e_i, e_j, e_k, e_l\}$.

Lemma 3.1. — *Given $c, C \in \mathbb{R}$, there exists $b = b(c, C, n)$ such that if (M^n, g) is a Riemannian manifold with*

$$K^{\mathrm{iso}} \geqslant c, \qquad s \leqslant C,$$

then the norm of the Weyl tensor W is bounded by b:

$$|W| \leqslant b.$$

Proof. — The proof is similar to that of Proposition 2.5 of [5]. Note that

$$K^{\text{iso}}(e_i,e_j,e_k,e_l)+K^{\text{iso}}(e_i,e_j,e_l,e_k):=2(R_{ikik}+R_{ilil}+R_{jkjk}+R_{jljl}).$$

From this it follows that $4s$ can be expressed as a sum of $n(n-1)$ isotropic curvatures as in [5]. Since we have an upper bound on s and a lower bound on K^{iso}, we get an upper bound $b_1=b_1(c,C,n)$ for K^{iso}. Since

$$4W_{ijkl}=4R_{ijkl}=K^{\text{iso}}(e_i,e_j,e_l,e_k)-K^{\text{iso}}(e_i,e_j,e_k,e_l)$$

the two-sided bounds on K^{iso} give an upper bound $b_2=b_2(c,C,n)$ on $|W_{ijkl}|=|R_{ijkl}|$. Since this holds for an arbitrary orthonormal 4-frame, we can apply the above bound to the 4-frame

$$\left\{e_i,\ \frac{1}{\sqrt{2}}(e_j-e_l),\ e_k,\ \frac{1}{\sqrt{2}}(e_j+e_l)\right\}$$

to get

$$\begin{aligned}2b_2 &\geqslant \left|W(e_i,\ e_j-e_l,\ e_k,\ e_j+e_l)\right|\\ &=\left|W_{ijkj}-W_{ilkl}+W_{ijkl}-W_{ilkj}\right|\\ &\geqslant \left|W_{ijkj}-W_{ilkl}\right|-\left|W_{ijkl}-W_{ilkj}\right|\end{aligned}$$

Hence

$$\left|W_{ijkj}-W_{ilkl}\right|\leqslant 2b_2+\left|W_{ijkl}-W_{ilkj}\right|\leqslant 4b_2.$$

Since $\sum_p W_{ipkp}=0$, this implies that $|W_{ipkp}|\leqslant b_3(c,C,n)$. Finally, to get a bound on terms of the form $|W_{ipip}|$, we repeat the above argument *i.e.* observe that the bound on $|W_{ipkp}|$ holds for any orthonormal triple $\{e_p,e_i,e_k\}$. In particular it applies to $\left\{e_p,\frac{1}{\sqrt{2}}(e_i-e_k),\ \frac{1}{\sqrt{2}}(e_i+e_k)\right\}$. Repeating the above steps and using $\sum_i W_{ipip}=0$, we get an upper bound $b_4(c,C,n)$ on any term of the form W_{ipip}. Hence $|W|\leqslant b_5(c,C,n)$. □

An immediate corollary of Lemma 3.1 is that *an upper bound on scalar curvature, a lower bound on isotropic curvature and an upper bound on the norm of the Ricci tensor gives a bound on the norm of the Riemann curvature tensor*. This applies, in particular, to a metric satisfying the hypotheses of Theorem 1.1.

Turning to the proof of Theorem 1.1, the first restriction we impose on ε is that $\varepsilon\leqslant\frac{l}{2n}$. This implies that the Ricci curvature is uniformly positive:

$$\mathrm{Ric}\geqslant\frac{l}{2\sqrt{n}}g. \tag{3.1}$$

We first claim that Theorem 1.1 is true if we assume that (M,g) has *positive* isotropic curvature. Then, by Micallef-Moore [4], $\pi_2(M)=0$.

Theorem 0.4 of [7] states that the injectivity radius of a compact simply-connected Riemannian n-manifold with finite second homotopy group, bounded sectional curvature $|K| \leqslant a$ and positive Ricci curvature $\mathrm{Ric} > bg$ has a positive lower bound on injectivity radius dependent only on a, b and n. The comment following Lemma 3.1 and (3.1) give us the required bounds on curvature.

Remark. — If M is even-dimensional, then one has the following alternative proof for a lower bound on injectivity radius inj. If inj $\to 0$, then all the characteristic numbers of M, in particular the Euler characteristic , would have to vanish. On the other hand, a simply-connected Riemannian n-manifold with positive isotropic curvature has to be homeomorphic to the n-sphere [4]. This contradiction shows that collapse cannot occur in even-dimensions.

Now suppose that there is no ε for which the conclusion of Theorem 1.1 holds. Then we get a sequence of (M_i, g_i) of Riemannian n-manifolds, none of which is diffeomorphic to a symmetric space of compact type, with uniformly bounded sectional curvatures and diameter (by Myers-Bonnet, since (3.1) holds) and injectivity radius bounded below.

We now apply the smoothing Lemma 2.1 of the previous section to each of the manifolds (M_i, g_i). We obtain the "smoothed" Riemannian manifolds $(M_i, g_i(\delta))$. By (ii) and (iii) of Lemma 2.1we can assume that a subsequence of $(M_i, g_i(\delta))$ converges in the C^∞ topology to a smooth complete Riemannian n-manifold (M, g). This manifold will have to be positive Einstein (hence compact) by (vi) and (iv) of Lemma 2.1, and of nonnegative isotropic curvature by (v). By [1], (M, g) is isometric to a symmetric space of compact type. Hence M_i is diffeomorphic to a symmetric space of compact type for large i, which is a contradiction.

Hence we have established the existence of

$$\varepsilon_p = \varepsilon_p(l, L, n) \tag{3.2}$$

which yields the conclusion in the presence of positive isotropic curvature.

Next consider the general case of nonnegative isotropic curvature.

Lemma 3.2. — *Let $(M^n,\ g)$, $n \geqslant 4$, be a compact simply-connected Riemannian manifold with nonnegative isotropic curvature. Suppose that*

$$0 < l \leqslant s_g \leqslant L, \quad \left|\mathrm{Ric}_g - \frac{s_g}{n} g\right|_g \leqslant \varepsilon$$

for some $0 < l \leqslant L$ and $\varepsilon \leqslant \frac{2l}{n}$.

Let (N^k, h) be an irreducible factor in the de Rham decomposition of N. If $k = 2$ or 3, N is diffeomorphic to S^2 or S^3. If $k \geqslant 4$, then (N, h) has nonnegative isotropic curvature and

$$0 < \frac{2l}{n} < s_h \leqslant L, \quad \Big|\mathrm{Ric}_h - \frac{s_h}{k} h\Big|_h \leqslant \varepsilon.$$

Proof. — The statement about $k = 2$ or 3 follows from the description of reducible manifolds with nonnegative isotropic curvature given by M. Micallef and M. Wang (Theorem 3.1, [5]). If $k \geqslant 4$, note that

$$\begin{aligned}\Big|\mathrm{Ric}_g - \frac{s_g}{n} g\Big|_g^2 &\geqslant \Big|\mathrm{Ric}_h - \frac{s_g}{n} h\Big|_h^2 \\ &= \Big|\mathrm{Ric}_h - \frac{s_h}{k}\Big|_h^2 + k\Big|\frac{s_h}{k} - \frac{s_g}{n}\Big|^2.\end{aligned}$$

Hence

$$\Big|\mathrm{Ric}_h - \frac{s_h}{k} h\Big|_h \leqslant \varepsilon.$$

and

$$s_h \geqslant \frac{k}{n} s_g - \sqrt{k}\varepsilon \geqslant \frac{l}{n}\sqrt{k}\Big(\sqrt{k} - \frac{1}{2}\Big) \geqslant \frac{2l}{n}.$$

Moreover, since (M, g) has positive Ricci curvature, so does each irreducible component and hence $s_h \leqslant s_g \leqslant L$. □

We can now complete the proof of the theorem by induction. The proof for the first nontrivial dimension $n = 4$ is the same as that for the inductive step, so we assume that the result is true in all dimensions less than n. Let (M^n, g) be a manifold as in Theorem 1.1 with the norm of the Einstein tensor being smaller than

$$\varepsilon_r(l, L, n) := \min\left\{\frac{2l}{n},\ \varepsilon\Big(\frac{2l}{n}, L, 4\Big), \dots, \varepsilon\Big(\frac{2l}{n}, L, n-1\Big)\right\}.$$

If (M, g) is reducible, it is enough to prove that each irreducible component of (M, g) is diffeomorphic to a symmetric space of compact type. Let $(N^k, h),\ 1 \leqslant k \leqslant n-1$ be such a component. By Lemma 3.2 and the inductive hypothesis we are done.

Suppose (M, g) is irreducible. We claim that if norm of the Einstein tensor of g is $\frac{1}{2}\varepsilon_p(\frac{l}{2}, 2L, n)$-small (where ε_p is defined by (3.2)) then we have the desired conclusion.

By the results of [8] and [1] the following holds: A compact orientable locally irreducible Riemannian manifold (X, h) with nonnegative isotropic curvature is either diffeomorphic to a locally symmetric space of compact type or we can find a metric $\bar{h}$ with *positive* isotropic curvature as close (in the C^∞ topology) to h as we want.

Hence if (M, g) is not diffeomorphic to a symmetric space of compact type, we can find $\bar{g}$ with positive isotropic curvature so close to g that

$$0 < \frac{l}{2} \leqslant s_{\bar{g}} \leqslant 2L, \quad \left|\mathrm{Ric}_{\bar{g}} - \frac{s_{\bar{g}}}{n}\bar{g}\right|_{\bar{g}} \leqslant \varepsilon_p.$$

Since $(M, \bar{g})$ has positive isotropic curvature and satisfies the above bounds, it is diffeomorphic to a symmetric space by our earlier result.

Finally we choose

$$\varepsilon(l, L, n) = \min\left\{\frac{1}{2}\ \varepsilon_p\left(\frac{l}{2}, 2L, n\right), \quad \varepsilon_r(l, L, n)\right\}.$$

□

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