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THE C^1 INVARIANCE OF THE ALGEBRAIC MULTIPLICITY OF A HOLOMORPHIC VECTOR FIELD

by Rudy ROSAS

Abstract. — We prove that the algebraic multiplicity of a holomorphic vector field at an isolated singularity is invariant by C^1 equivalences.

Résumé. — On démontre que la multiplicité algébrique d'une singularité d'un champ de vecteurs holomorphe est invariante par C^1-equivalences.

1. Introduction

Given a curve $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, singular at $0 \in \mathbb{C}^2$, we define its *algebraic multiplicity* as the degree of the first nonzero jet of f, that is, $\nu(f) = \nu$ where

$$f = f_\nu + f_{\nu+1} + \cdots$$

is the Taylor development of f and $f_\nu \neq 0$. A well known result by Burau [2] and Zariski [15] states that ν is a *topological invariant*, that is, given $\widetilde{f} : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ and a homeomorphism $\mathrm{h} : U \to \widetilde{U}$ between neighborhoods of $0 \in \mathbb{C}^2$ such that $\mathrm{h}(f^{-1}(0) \cap U) = \widetilde{f}^{-1}(0) \cap V$ then $\nu(f) = \nu(\widetilde{f})$. Consider now a holomorphic vector field Z in $\mathbb{C}^2$ with a singularity at $0 \in \mathbb{C}^2$. If

$$Z = Z_\nu + Z_{\nu+1} + \cdots, Z_\nu \neq 0$$

we define $\nu = \nu(Z)$ as the *algebraic multiplicity* of Z. The vector field Z defines a holomorphic foliation by curves $\mathcal{F}$ with isolated singularity in a neighborhood of $0 \in \mathbb{C}^2$ and the algebraic multiplicity $\nu(Z)$ depends only on the foliation $\mathcal{F}$. A natural question, posed by J.F.Mattei is: is $\nu(\mathcal{F})$ a

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topological invariant of $\mathcal{F}$?. In [4], the authors give a positive answer if $\mathcal{F}$ is a *generalized curve*, that is, if the desingularization of $\mathcal{F}$ does not contain complex saddle-nodes. In this work, we consider the problem in dimension $n \geqslant 2$ and impose conditions on the topological equivalence. Let $\mathcal{F}$ be a holomorphic foliation by curves of a neighborhood U of $0 \in \mathbb{C}^n$ with a unique singularity at $0 \in \mathbb{C}^n (n \geqslant 2)$. We assume that $\mathcal{F}$ is generated by the holomorphic vector field

$$V = \sum_{i=1}^{n} a_i \frac{\partial}{\partial z_i},\ a_i \in \mathcal{O}_U,\ g.c.d.(a_1, a_2, \ldots, a_n) = 1.$$

The algebraic multiplicity of $\mathcal{F}$ (at $0 \in \mathbb{C}^n$) is the minimum vanishing order at $0 \in \mathbb{C}^n$ of the functions a_i. Let $\widetilde{\mathcal{F}}$ be another holomorphic foliation by curves of a neighborhood $\widetilde{U}$ of $0 \in \mathbb{C}^n$ and let $\mathrm{h} : U \to \widetilde{U}$ be a topological equivalence between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$, that is, a homeomorphism taking leaves of $\mathcal{F}$ to leaves of $\widetilde{\mathcal{F}}$. Let $\pi : \widehat{\mathbb{C}^n} \to \mathbb{C}^n$ be the quadratic blow up with center at $0 \in \mathbb{C}^n$. Clearly the map $h := \pi^{-1} \mathrm{h}\, \pi$ is a homeomorphism between $\pi^{-1}(U \backslash \{0\})$ and $\pi^{-1}(\widetilde{U} \backslash \{0\})$. Then we prove the following:

Theorem 1.1. — *Suppose that h extends to the divisor $\pi^{-1}(0)$ as a homeomorphism between $\pi^{-1}(U)$ and $\pi^{-1}(\widetilde{U})$. Then the algebraic multiplicities of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are the same.*

If h is a C^1 diffeomorphism, we prove that h extends to the divisor. Thus, we obtain that the algebraic multiplicity is invariant by C^1 equivalences:

Theorem 1.2. — *Let $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ be two foliations by curves of neighborhoods U and $\widetilde{U}$ of $0 \in \mathbb{C}^n$, $n \geqslant 2$. Let $\mathrm{h} : U \to \widetilde{U}$ be a C^1 equivalence between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$, that is, a C^1 diffeomorphism taking leaves of $\mathcal{F}$ to leaves of $\widetilde{\mathcal{F}}$. Then the algebraic multiplicities of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are equal.*

It is known that there exists a unique way of extending the pull back foliation $\pi^*(\mathcal{F}|_{U\backslash\{0\}})$ to a singular analytic foliation $\mathcal{F}_0$ on $\pi^{-1}(U)$ with singular set of codimension $\geqslant 2$. We say that $\mathcal{F}_0$ is the strict transform of $\mathcal{F}$ by π. Let $\widetilde{\mathcal{F}}_0$ be the strict transform of $\widetilde{\mathcal{F}}$ by π. In order to prove Theorem 1.1 we show that the algebraic multiplicity of $\mathcal{F}$ depends on the Chern class of the tangent bundle of $\mathcal{F}_0$. To relate the Chern classes of the tangent bundles of $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$ we use the following theorem (see [7]).

Theorem 1.3. — *Let $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ be foliations by curves on the complex manifolds M and $\widetilde{M}$ respectively. Let $c(T\mathcal{F})$ denote the Chern class of the tangent bundle $T\mathcal{F}$ of $\mathcal{F}$. Let $h : M \to \widetilde{M}$ be a topological equivalence between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ and consider the map $h^* : H^2(M, \mathbb{Z}) \to H^2(\widetilde{M}, \mathbb{Z})$ induced in the cohomology. Then $h^*(c(T\mathcal{F})) = c(T\widetilde{\mathcal{F}})$.*

Clearly the homeomorphism $h : \pi^{-1}(U\backslash\{0\}) \to \pi^{-1}(\widetilde{U}\backslash\{0\})$ is a topological equivalence between $\mathcal{F}_0|_{\pi^{-1}(U\backslash\{0\})}$ and $\widetilde{\mathcal{F}}_0|_{\pi^{-1}(\widetilde{U}\backslash\{0\})}$. To be able to apply Theorem 1.3 we show that h extends as a topological equivalence between $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$. This is the non trivial part of the proof. Thus, we prove the following.

Theorem 1.4. — *Let V and $\widetilde{V}$ be complex manifolds, let $Y \subset V$ and $\widetilde{Y} \subset \widetilde{V}$ be analytic subvarieties of codimension $\geqslant 1$ and, let $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ be holomorphic foliations by curves on V and $\widetilde{V}$ respectively. Suppose there is a homeomorphism h between V and $\widetilde{V}$ with $h(Y) = \widetilde{Y}$ and such that $h|_{V\backslash Y}$ is a topological equivalence between $\mathcal{F}|_{V\backslash Y}$ and $\widetilde{\mathcal{F}}|_{\widetilde{V}\backslash\widetilde{Y}}$. Then h is a topological equivalence between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$.*

This paper is organized as follows. In section 2 we prove Theorem 1.4. In section 3 we relate the algebraic multiplicity of the foliation and the Chern class of its strict transform, and prove Theorem 1.1. Finally, section 4 discusses the C^1 case.

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2. An extension theorem

This section is devoted to prove Theorem 1.4. We start with some definitions. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{B} = \{z \in \mathbb{C}^{n-1} : ||z|| < 1\}$ where $n \geqslant 2$. Let M be a complex manifold of complex dimension n and let D be a subset of M homeomorphic to a disc. We say that D is a *singular disc* if for all $x \in D$ there exist a neighborhood $\mathcal{D}$ of x in D, and an injective holomorphic function $f : \mathbb{D} \to M$ such that $f(\mathbb{D}) = \mathcal{D}$, $f(0) = x$. If $f'(0) = 0$ we say that x is a *singularity* of D, otherwise x is a *regular point* of D (this does not depend on f). The set S of singularities of D is discrete and closed in D and we have that $D\backslash S$ is a complex submanifold of M. Thus, if x is a regular point of D, there is a neighborhood U of x in M and holomorphic coordinates (w, z), $w \in \mathbb{B}$, $z \in \mathbb{D}$ on U such that $D \cap U$ is represented by $(w = 0)$. If D does not have singularities we say that it is a *regular disc*. In this case, by uniformization, there is a holomorphic map

$f : E \to M$, where $E = \mathbb{D}$ or $\mathbb{C}$, such that f is a biholomorphism between E and D.

Example. — Let $\mathcal{F}$ be a holomorphic foliation by curves on the complex manifold M and let $D \subset M$ be a topological disc contained in a leaf of $\mathcal{F}$. Then D is a regular disc.

The following Lemma will be fundamental in the proof of Theorem 1.4.

Lemma 2.1. — *Let $F : \mathbb{D} \times [0,1] \to \mathbb{C}^n$ be a continuous map such that for all $t \in [0,1]$, the map $F(*,t) : \mathbb{D} \to \mathbb{C}^n$ is a homeomorphism onto its image. Thus, we have a continuous family of discs $D_t := F(\mathbb{D} \times \{t\})$. Suppose D_t is a regular disc for each $t > 0$. Then D_0 is a singular disc.*

Proof. — We give a sketch of the proof. Let $p = F(x_0, 0)$ be any point in D_0. Let $\mathrm{U} \subset \mathbb{D}$ be a disc centered at x_0 and such that $\overline{\mathrm{U}} \subset \mathbb{D}$. Let $t_k > 0$ be such that $t_k \to 0$ as $k \to \infty$ an define $\mathcal{D}_k = F(\mathrm{U} \times \{t_k\})$. By uniformization there is a holomorphic map $f_k : \mathbb{D} \to \mathbb{C}^n$ which is a biholomorphism between $\mathbb{D}$ and $\mathcal{D}_k$. We may assume that $f_k(0) = F(x_0, t_k)$ for all k and it is well known that f_k extends as a homeomorphism $f_k : \overline{\mathbb{D}} \to \overline{\mathcal{D}}_k$. By Montel's theorem we can assume that f_k converges uniformly on compact sets to a holomorphic function $f : \mathbb{D} \to \mathbb{C}^n$, $f(0) = p$. Clearly it is sufficient to show that f is not a constant function ($f \not\equiv p$). Let $\mathbb{S}^1 := \partial\mathbb{D}$ and consider for each k the homeomorphism

$$\varphi_k := f_k|_{\mathbb{S}^1} : \mathbb{S}^1 \to \partial\mathcal{D}_k.$$

By taking a subsequence if necessary, it is not difficult to see that we may assume that φ_k converges a.e. to a function

$$\varphi : \mathbb{S}^1 \to \partial\mathcal{D}_0.$$

Fix $x \in \mathbb{D}$. Since $\{\varphi_k\}$ is uniformly bounded, by the dominated convergence theorem we have that

$$\frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{\varphi_k(w)}{w-x}dw \to \frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{\varphi(w)}{w-x}dw \tag{2.1}$$

as $k \to \infty$. By Cauchy's Integral Formula the left part of (2.1) is equal to $f_k(x)$ and, since $f_k(x) \to f(x)$, we conclude that

$$f(x) = \frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{\varphi(w)}{w-x}dw. \tag{2.2}$$

Finally, it is not difficult to prove from this equation that $f \equiv p$ implies $\varphi = p$ a.e., which is a contradiction because $\varphi(\mathbb{S}^1) \subset \partial\mathcal{D}_0$ and $p \notin \partial\mathcal{D}_0$. □

We now show that Theorem 1.4 is a consequence of the following theorem.

THEOREM 2.2. — *Let $\mathcal{F}$ be a foliation by curves on the complex manifold M. Let $X \subset M$ be an analytic subvariety of codimension $\geqslant 1$. Suppose that:*

(*i*) *$\mathcal{F}$ is generated by a holomorphic vector field.*
(*ii*) *There exists a homeomorphism $h : \Sigma \times D \to M$, where Σ is a ball in $\mathbb{C}^{n-1}$ and D is a disc in $\mathbb{C}$.*
(*iii*) *If $D_z := h(\{z\} \times D)$ then for all z: either D_z is contained in X, or $D_z \cap X$ is discrete and $D_z \backslash X$ is contained in a leaf of $\mathcal{F}$.*

Then $\mathcal{F}$ is regular and the sets D_z are the leaves of $\mathcal{F}$.

Proof of Theorem 1.4. — Let p be a point in Y which is regular for $\mathcal{F}$. Let Σ denote a ball in $\mathbb{C}^{n-1}$ and D a disc in $\mathbb{C}$. Consider a neighborhood W of p on which $\mathcal{F}$ is a product foliation, that is, $W \simeq \Sigma \times D$ and the sets $\{z\} \times D$ are the leaves of $\mathcal{F}|_W$. We take W small enough such that $\widetilde{\mathcal{F}}$ restricted to $M := h(W)$ is generated by a holomorphic vector field. Let X be the intersection between M and $\widetilde{Y}$. We will show that the hypothesis of Theorem 2.2 hold for $\widetilde{\mathcal{F}}$ restricted to M. Hypothesis (i) and (ii) of 2.2 evidently hold. Let $D_z = h(\{z\} \times D)$. Then it is easy to see that

ASSERTION 1. — *For all $z \in \Sigma$, either $\{z\} \times D$ is contained in Y, or $S'_z := (\{z\} \times D) \cap Y$ is discrete and closed in $\{z\} \times D$.*

Suppose that D_z is not contained in X. Let $S_z = h(S'_z)$, where S'_z is given by Assertion 1. Then S_z is discrete in D_z. Observe that $(\{z\} \times D) \backslash S'_z$ is contained in a leaf of $\mathcal{F}|_{M\backslash Y}$. Then, since $h|_{M\backslash Y}$ is a topological equivalence between $\mathcal{F}|_{V\backslash Y}$ and $\widetilde{\mathcal{F}}|_{\widetilde{V}\backslash\widetilde{Y}}$, it follows that

$$D_z \backslash S_z = h((\{z\} \times D) \backslash S'_z)$$

is contained in a leaf of $\widetilde{\mathcal{F}}$. Thus, hypothesis (iii) of 2.2 holds. Then $\widetilde{\mathcal{F}}$ is regular on $M = h(W)$ and every D_z is contained in a leaf of $\widetilde{\mathcal{F}}$. Therefore we conclude:

ASSERTION 2. — *If p is a point in Y which is regular for $\mathcal{F}$, then p is mapped by h to a regular point of $\widetilde{\mathcal{F}}$. Moreover, there exists a neighborhood Ω of p in its leaf which is mapped by h onto a neighborhood of $h(p)$ in its leaf.*

Now, by using Assertion 2 for h and h^{-1}, we deduce that p is regular for $\mathcal{F}$ if and only if $h(p)$ is regular for $\widetilde{\mathcal{F}}$. Hence

$$h(\mathrm{Sing}(\mathcal{F})) = \mathrm{Sing}(\widetilde{\mathcal{F}}).$$

It remains to prove that h maps any leaf of $\mathcal{F}$ onto a leaf of $\widetilde{\mathcal{F}}$. Let p be a regular point of $\mathcal{F}$. Let L be the leaf of $\mathcal{F}$ passing through p and let

$\widetilde{L}$ be the leaf of $\widetilde{\mathcal{F}}$ passing through $h(p)$. Let A be the set of points in L which are mapped by h into $\widetilde{L}$. By Assertion 2, if $x \in A$ there exists a neighborhood of x in L_p contained in A. Therefore A is open. Now, let $x \notin A$. Then $h(x) \notin \widetilde{L}$. Thus, if $L' \neq L$ is the leaf of $\widetilde{\mathcal{F}}$ passing through $h(x)$ it follows by Assertion 2 that there exists a neighborhood Ω of x in L which is mapped by h into $L' \neq \widetilde{L}$, hence Ω is contained in $L \backslash A$. Then A is also closed and it follows by conectedness that $A = L$, that is, $h(L) \subset \widetilde{L}$. Analogously, we prove that $h^{-1}(\widetilde{L}) \subset L$. Therefore $h(L) = \widetilde{L}$. □

We proceed now to prove Theorem 2.2.

Proposition 2.3. — *Let $\mathcal{F}$ be a foliation by curves on the complex manifold M. Let $X \subset M$ be an analytic subvariety of codimension $\geqslant 1$. Suppose that:*

(*i*) *There exists a homeomorphism $h : \Sigma \times D \to M$, where Σ is a ball in $\mathbb{C}^{n-1}$ and D is a disc in $\mathbb{C}$.*

(*ii*) *If $D_z := h(\{z\} \times D)$ then for all z: either D_z is contained in X, or D_z is contained in a leaf of $\mathcal{F}$.*

Consider $z' \in \Sigma$ and suppose that $D_{z'}$ is a singular disc. Let $S_{z'}$ the set of singularities of $D_{z'}$. Then $D_{z'} \backslash S_{z'}$ is contained in a leaf of $\mathcal{F}$.

Proof. — It is sufficient to prove the following.

Assertion. — *If $p \in D_{z'} \backslash S_{z'}$ then p has a neighborhood in $D_{z'} \backslash S_{z'}$ contained in a leaf of $\mathcal{F}$.*

Suppose Assertion holds. Let L be a leaf of $\mathcal{F}$ and let $x \in (D_{z'} \backslash S_{z'}) \cap L$. By Assertion, there is a neighborhood Δ of x in $D_{z'} \backslash S_{z'}$ such that $\Delta \subset L$. Then $\Delta \subset (D_{z'} \backslash S_{z'}) \cap L$ and it follows that the intersection of $D_{z'} \backslash S_{z'}$ with any leaf is open in $D_{z'} \backslash S_{z'}$. Then, since $D_{z'} \backslash S_{z'}$ is connected, we have that it is contained in a unique leaf.

Proof of Assertion. — Let p in $D_{z'} \backslash S_{z'}$. Since p is a regular point of the singular disc $D_{z'}$, on a neighborhood $U \subset M$ of p we may consider coordinates (w, y), $w \in \mathbb{B}$, $y \in \mathbb{D}$ with $p = (0,0)$ and such that $D_{z'} \cap U$ is represented by $(w = 0)$. Suppose that $p = h(z', t')$. Let Σ' be a ball in Σ containing z' and let D' be a disc in D containing t'. Then $W = \Sigma' \times D'$ is a neighborhood of (z', t') and, by taking W small enough, we assume $h(\overline{W}) \subset U$. Let $D'_z = h(\{z\} \times D')$. Note that $D'_{z'} \subset D_{z'} \cap U$, hence $D'_{z'}$ is contained in $(w = 0)$. Let $g : U \to \mathbb{D}$ be the projection $g(w, y) = y$. Consider $z \in \Sigma'$ and suppose $D_z \backslash X \neq \emptyset$. By hypothesis (ii), D_z is contained in a leaf of $\mathcal{F}$. Therefore D'_z is contained in leaf of $\mathcal{F}$ and we have that $g|_{D'_z} : D'_z \to \mathbb{D}$ is a holomorphic map. Remember that $D'_{z'} \subset (w = 0)$.

Then $g|_{D'_{z'}} : D'_{z'} \to \mathbb{D}$ is given by $(0, y) \to y$ and is therefore a one to one map. Then $g(D'_{z'})$ is a disc in $\mathbb{D}$ with $g(\partial D'_{z'})$ as boundary. Note that $p = (0,0) \in D'_{z'}$, hence 0 is contained in the disc $g(D'_{z'})$. Therefore the curve $g(\partial D'_{z'})$ winds once around 0. By the continuity of h we assume Σ' small enough such that $g(\partial D'_z)$ is homotopic to $g(\partial D'_{z'})$ in $\mathbb{D}\backslash\{0\}$ for all $z \in \Sigma'$. Then $g(\partial D'_z)$ winds once around 0 and $g|_{D'_z}$ has therefore a unique zero. In other words, the plaque D'_z intersects $Y = \mathbb{B} \times \{0\} \subset U$ at a unique point. Thus, we can define the map $f : h(W)\backslash X \to Y$ by $f(D'_z\backslash X) = D'_z \cap Y$ whenever $D'_z\backslash X \neq \emptyset$. We have that f is holomorphic because it is constant along the leaves and, restricted to any transversal, is a holonomy map. Since f is bounded and X has codimension $\geqslant 1$, by the generalized Riemann's extension theorem, f extends to a holomorphic function on $h(W)$. Observe that f restricted to Y is the identity map, then f is a submersion in a neighborhood V of Y. Hence f defines a regular foliation $\mathcal{N}$ on V. It is easy to see that $\mathcal{N}$ coincides with $\mathcal{F}$ on $V\backslash X$, thus $\mathcal{N} = \mathcal{F}$. Therefore $p \in Y$ is a regular point of $\mathcal{F}$.

Now, by reducing the neighborhood $W = \Sigma' \times D'$ of (z', t'), we may assume that $h(W)$ is contained in a neighborhood of p where $\mathcal{F}$ is given by a submersion f. Obviously $D'_{z'}$ is a neighborhood of p in D_z. We shall prove that $D'_{z'}$ is contained in a leaf of $\mathcal{F}$ (the leaf passing through p). If $D'_{z'}$ is not contained in X, so is $D_{z'}$ and, by hypothesis (ii), we have that $D'_{z'}$ is contained in a leaf of $\mathcal{F}$. On the other hand, suppose that $D'_{z'}$ is contained in X. Then there exists a sequence of points $z_k \to z'$ such that $h(\{z_k\} \times D)$ is not contained in X, otherwise $h(\Sigma'' \times D) \subset X$ for some neighborhood $\Sigma'' \subset \Sigma$ of z', which is a contradiction because X has codimension $\geqslant 1$. Thus, by (ii), we have that D'_{z_k} is contained in a leaf of $\mathcal{F}$ for all k. Recall $D'_{z_k} \subset h(W)$ is contained in a domain where $\mathcal{F}$ is given by the submersion f. Then f is constant over $D'_{z_k} = h(\{z_k\} \times D')$ and in particular, for all $t \in D'$ we have $f(h(z_k, t)) = f(h(z_k, t'))$. Then:

$$\begin{aligned} f(h(z',t)) &= f(h(\lim_{k\to\infty} z_k, t)) = \lim_{k\to\infty} f(h(z_k,t)) \\ &= \lim_{k\to\infty} f(h(z_k,t')) = f(h(\lim_{k\to\infty} z_k, t')) \\ &= f(h(z',t')). \end{aligned}$$

Therefore, for all $t \in D'$ we have that $h(z', t)$ and $h(z', t')$ are contained in the same leaf. It follows that $D'_{z'}$ is contained in the leaf passing trough $h(z', t')$. Thus, Assertion is proved. □

Proposition 2.4. — *Let $\mathcal{F}$ be a foliation by curves on the complex manifold M such that:*

(*i*) $\mathcal{F}$ *is generated by a holomorphic vector field.*

(*ii*) *There exists a homeomorphism* $h : \Sigma \times D \to M$*, where* Σ *is a ball in* $\mathbb{C}^{n-1}$ *and* D *is a disc in* $\mathbb{C}$*.*

(*iii*) *For all* z*, there is a discrete closed set* $S_z \subset D_z := h(\{z\} \times D)$ *such that* $D_z \backslash S_z$ *is contained in a leaf of* $\mathcal{F}$*.*

Then $\mathcal{F}$ *is regular and the sets* D_z *are the leaves of* $\mathcal{F}$*.*

The following lemmas are easy consequences of well known facts and we left the proofs to the reader.

Lemma 2.5. — *Let* $f : \overline{\mathbb{D}} \to \mathbb{C}$ *be smooth, and holomorphic on* $\mathbb{D}$*. Suppose that* f *is regular on* $\mathbb{S}^1 := \overline{\mathbb{D}}$*. Then* f *is a regular map if and only if the curve* $f|_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{C}$ *has degree 1*[(1)]*.*

Lemma 2.6. — *Let* M *be a complex manifold and* $D \subset M$ *a singular disc. Then there exists a holomorphic injective map* $g : E \to M$*, where* $E = \mathbb{D}$ *or* $\mathbb{C}$*, such that* $g(E) = D$*.*

Lemma 2.7. — *Let* $\mathcal{D} \subset \mathbb{C}^n$ *be a set homeomorphic to a disc such that for some point* $p \in \mathcal{D}$ *the annulus* $\mathcal{D}\backslash\{p\}$ *is a complex submanifold. Then* $\mathcal{D}$ *is a singular disc.*

Proof of Proposition 2.4. —

Assertion 1. — *For all* z*, we have that* D_z *is a singular disc and the sets* $D_z\backslash \operatorname{Sing}(\mathcal{F})$ *are the nonsingular leaves of* $\mathcal{F}$*.*

Proof . — Let $x \in D_z$. Since S_z is a discrete closed subset of D_z, there is a disc $\mathcal{D} \subset D_z$ with $x \in \mathcal{D}$ such that $\mathcal{D}\backslash\{x\} \subset D_z\backslash S_z$. Then, from hypothesis (*iii*), $\mathcal{D}\backslash\{x\}$ is contained in a leaf of $\mathcal{F}$. If $\mathcal{D}$ is small enough, we may think that $\mathcal{D}$ is contained in $\mathbb{C}^n$. Hence, by applying Lemma 2.7, there exists a holomorphic injective map $g : \mathbb{D} \to M$ with $g(\mathbb{D}) = \mathcal{D}$. Since that $x \in D_z$ was arbitrary, it follows that D_z is a singular disc.

Let L be a leaf of $\mathcal{F}$ and suppose that $x \in L \cap (D_z\backslash \operatorname{Sing}(\mathcal{F}))$ for some z. Take $\mathcal{D} \subset D_z$ as above. We assume $\mathcal{D}$ small enough such that it is contained in a neighborhood U of x where $\mathcal{F}$ is trivial and given by the submersion f. Then $\mathcal{D}\backslash\{x\}$ is contained in a leaf of $\mathcal{F}|_U$ and f is therefore constant over $\mathcal{D}\backslash\{x\}$. Hence, by continuity, f is constant over $\mathcal{D}$. Then $\mathcal{D}$ is contained in a leaf of $\mathcal{F}|_U$ and we have therefore $\mathcal{D} \subset L$. Thus we have $\mathcal{D} \subset L \cap (D_z\backslash \operatorname{Sing}(\mathcal{F}))$. It follows that $L \cap (D_z\backslash \operatorname{Sing}(\mathcal{F}))$ is an open subset of both L and $D_z\backslash \operatorname{Sing}(\mathcal{F})$ for all L and z. Now, fix a leaf L.

(1) The degree of a parameterized regular curve in the plane is defined as the winding number around 0 of its velocity vector.

Since the intersection of L with any $D_z \setminus \operatorname{Sing}(\mathcal{F})$ is open in L, it follows by connectedness that L is contained in a unique $D_z \setminus \operatorname{Sing}(\mathcal{F})$. For this $D_z \setminus \operatorname{Sing}(\mathcal{F})$, we also have that its intersection with any leaf is open in $D_z \setminus \operatorname{Sing}(\mathcal{F})$. Again by connectedness $D_z \setminus \operatorname{Sing}(\mathcal{F})$ is contained in a unique leaf, thus we necessarily have that $D_z \setminus \operatorname{Sing}(\mathcal{F}) = L$. Therefore Assertion 1 is proved.

Fix $p \in M$. We have $p \in D_{z'}$ for some $z' \in \Sigma$. Take p' in $D_{z'} \setminus S_{z'}$. From hypothesis (iii), p' is a regular point of $\mathcal{F}$. We have $p' = h(z', t')$ with $t' \in D$. If $B \subset \Sigma$ is a ball containing z', then $\Sigma_0 := B \times \{t'\}$ is a $(n-1)$ ball passing through (z', t'). We assume B small enough such that $\overline{\Sigma}_0$ is mapped by h into a neighborhood W of p' where $\mathcal{F}$ is equivalent to a product foliation. Let $\widetilde{\Sigma}$ (submanifold of W) be a global transversal to $\mathcal{F}|_W$. If w is a point contained in $\overline{h(\Sigma_0)}$, the leaf of $\mathcal{F}|_W$ passing through it intersects $\widetilde{\Sigma}$ in a unique point $\psi(w)$. We claim that ψ is a homeomorphism of $\overline{h(\Sigma_0)}$ onto its image. Since $\overline{h(\Sigma_0)}$ is compact, it suffices to prove that ψ is injective on $\overline{h(\Sigma_0)}$. Suppose that w_1 and w_2 are two points in $\overline{h(\Sigma_0)}$ contained in the same leaf L of $\mathcal{F}|_W$. From Assertion 1, we have that $L \subset D_z$ for some z. Then $h^{-1}(L) \subset \{z\} \times D$, hence $h^{-1}(w_1)$ and $h^{-1}(w_2)$ are two different points in the intersection of $(z \times D)$ with $\overline{\Sigma}_0$, which is a contradiction because $\overline{\Sigma}_0 \subset \Sigma \times \{t'\}$ intersects $(z \times D)$ only at (z, t').

If we redefine $\widetilde{\Sigma}$ as $\widetilde{\Sigma} = \psi(h(\Sigma_0))$, it follows from above that for all $z \in B$, D_z intersects $\widetilde{\Sigma}$ at the unique point $\psi(h(z, t_0))$. Thus we may define the map

$$g : V = h(B \times \mathbb{D}) \to \widetilde{\Sigma},$$

$$g(D_z) = D_z \cap \widetilde{\Sigma}.$$

By Assertion 1, each leaf of $\mathcal{F}$ is contained in some D_z. Then g is constant along the leaves. Therefore, since the restriction of g to any transversal is a holonomy map, we have that g is holomorphic on $V \setminus \operatorname{Sing}(\mathcal{F})$. Actually, since $\operatorname{Sing}(\mathcal{F})$ has codimension $\geqslant 2$, g is holomorphic on V.

Consider $x \in \widetilde{\Sigma} \setminus g(\operatorname{Sing}(\mathcal{F}))$. Then $D = g^{-1}(x)$ does not intersect $\operatorname{Sing}(\mathcal{F})$. Clearly D is equal to some D_z. Then, by Assertion 1, $D \setminus \operatorname{Sing}(\mathcal{F}) = D$ is a leaf of $\mathcal{F}$. Thus, we conclude that for all $x \in \widetilde{\Sigma} \setminus g(\operatorname{Sing}(\mathcal{F}))$, the leaf passing through x is simply connected. Moreover, since $\operatorname{Sing}(\mathcal{F})$ has codimension $\geqslant 2$, we have that $g(\operatorname{Sing}(\mathcal{F}))$ has codimension $\geqslant 1$ in $\widetilde{\Sigma}$ and we have therefore that:

Assertion 2. — *For all x in a dense subset of $\widetilde{\Sigma}$, the leaf passing through x is simply connected.*

Let Z be a holomorphic vector field which generates $\mathcal{F}$ on V and φ the local complex flow of Z. Let L be a leaf of $\mathcal{F}|_V$ and let x_L be its intersection with $\widetilde{\Sigma}$ ($g(L) = \{x_L\}$). There exists $\varepsilon_L > 0$ such that $\varphi(x_L, *)$ maps the disc $|t| < \varepsilon_L$ biholomorphically onto a neighborhood D_L of x_L in L. Thus, given any x in D_L there exists a unique $\tau_L(x)$ with $|\tau_L(x)| < \varepsilon_L$ such that $\varphi(x_L, \tau_L(x)) = x$. The function $\tau_L : D_L \to \mathbb{C}$ is the complex time between x_L and x. Clearly τ_L is holomorphic on D_L.

Assertion 3. — *The function τ_L can be analytically continued on L along any path $\gamma : [0,1] \to L$ with $\gamma(0) = x_L$.*

Proof . — Since γ does not intersect $\mathrm{Sing}(\mathcal{F})$ there exists $\delta > 0$ such that for all x in $\gamma([0,1])$, the map $\varphi(x, *)$ is a biholomorphism between $\mathbb{D}_{2\delta}$ and its image. Denote x_L by x_0 and let $0 = s_0 < s_1 < \cdots < s_r = 1$ and $x_1 = \gamma(s_1), \ldots, x_r = \gamma(s_r)$ be such that:

(*i*) The open sets $\varphi(x_i, \mathbb{D}_\delta)$ for $i = 0, \ldots, r$ cover $\gamma([0,1])$.
(*ii*) x_i is contained in $\varphi(x_{i-1}, \mathbb{D}_\delta)$ for $i = 1, \ldots, r$.

For each $i = 0, \ldots, r$ let $\tau_i' : \varphi(x_i, \mathbb{D}_{2\delta}) \to \mathbb{D}_{2\delta}$ be defined by $\varphi(x_i, \tau_i'(x)) = x$. Let $x \in \varphi(x_{i-1}, \mathbb{D}_\delta) \cap \varphi(x_i, \mathbb{D}_\delta)$. Let $t_i = \tau_{i-1}'(x_i)$ for $i = 1, \ldots, r$ and define $t_0 = 0$. Clearly, $|t_i|$ and $|\tau_i'(x)|$ are less than δ, hence $|t_i + \tau_i'(x)| < 2\delta$ and we have that

$$\begin{aligned}\varphi(x_{i-1}, t_i + \tau_i'(x)) &= \varphi(\varphi(x_{i-1}, t_i), \tau_i'(x))\\ &= \varphi(\varphi(x_{i-1}, \tau_{i-1}'(x_i)), \tau_i'(x))\\ &= \varphi(x_i, \tau_i'(x))\\ &= x.\end{aligned}$$

Then, by definition of τ_{i-1}' we obtain:

$$t_i + \tau_i'(x) = \tau_{i-1}'(x). \tag{2.3}$$

For each $i = 1, \ldots, r$ let τ_i be the holomorphic function on $\varphi(x_i, \mathbb{D}_\delta)$ defined by

$$\tau_i = \tau_i' + t_0 + \cdots + t_i.$$

By using (2.3) we deduce that $\tau_{i-1} = \tau_i$ on $\varphi(x_{i-1}, \mathbb{D}_\delta) \cap \varphi(x_i, \mathbb{D}_\delta)$. Moreover, it follows from the definition that τ_0 is equal to τ_L in a neighborhood of $x_0 = x_L$. Therefore, τ_0,...,τ_r give an analytic continuation of τ_L along γ.

Assertion 4. — *Let L be any leaf of $\mathcal{F}|_V$ and let $\gamma', \gamma'' : [0,1] \to L$ be paths such that $\gamma'(0) = \gamma''(0) = x_L$ and $\gamma'(1) = \gamma''(1) = x \in L$. Let τ_L' be the analytic continuation of τ_L along γ' and let τ_L'' be the analytic continuation of τ_L along γ''. Then $\tau_L'(x) = \tau_L''(x)$. Thus, τ_L extends as a* *holomorphic function on L. Therefore we may define* $\tau : V\setminus \mathrm{Sing}(\mathcal{F}) \to \mathbb{C}$ *by* $\tau = \tau_L$ *on L. Then* τ *is holomorphic on* $U\setminus \mathrm{Sing}(\mathcal{F})$ *and extends to U because* $\mathrm{Sing}(\mathcal{F})$ *has codimension* $\geqslant 2$*. Moreover, if restricted to a leaf,* τ *is a regular map. In particular,* τ *is a submersion on* $U\setminus \mathrm{Sing}(\mathcal{F})$*.*

Proof . — Fix L and denote x_L by x_0. Let $0 = s_0 < \cdots < s_r = 1$, let $\Sigma_0, \ldots, \Sigma_r$ be transversals to the foliation at the points $x_0, x_1 = \gamma(s_1), \ldots, x_r = \gamma(s_r)$ respectively, and let $\delta > 0$ with the following properties:

- (i) $\Sigma_0 \subset \widetilde{\Sigma}$.
- (ii) The flow φ maps $\Sigma_i \times \mathbb{D}_{2\delta}$ biholomorphically onto its image, for all $i = 0, \ldots, r$.
- (iii) The transversal Σ_i is contained in $\varphi(\Sigma_{i-1} \times \mathbb{D}_\delta)$, for all $i = 1, \ldots, r$.
- (iv) For all $i = 1, \ldots, r$ we have that $\Sigma_i = h_i(\Sigma_0)$, where h_i is the holonomy map along γ.

Denote by V' the union of the sets $\varphi(\Sigma_i \times \mathbb{D}_\delta)$ for $i = 0, \ldots, r$. Consider $x \in V'$ and let L_x be the leaf passing through x. Let $k \in \{0, \ldots, r\}$ be such that $x \in \varphi(\Sigma_k \times \mathbb{D}_\delta)$. Then L_x intersects Σ_k and it follows from hypothesis (iv) that L_x intersects each Σ_i. Since $\Sigma_0 \subset \widetilde{\Sigma}$ we have that L_x intersects Σ_0 in a unique point and, by (iv), the same holds for each Σ_i. Then we may define $\rho_i : V' \to \Sigma_i$ such that $\rho_i(x)$ is the point of intersection between L_x and Σ_i. Let $\tau_i'(x) \in \mathbb{D}_\delta$ be defined by $\varphi(\rho_i(x), \tau_i'(x)) = x$. Since $\rho_i(x) \in \Sigma_i$, by hypothesis (iii) we have that $\rho_i(x) \in \varphi(\Sigma_{i-1} \times \mathbb{D}_\delta)$ for $i = 1, \ldots, r$. Then for $i = 1, \ldots, r$ we may define $t_i : V' \to \mathbb{D}_\delta$ as $t_i = \tau_{i-1}' \circ \rho_i$. Define $t_0 : V' \to \mathbb{D}_\delta$ as the zero function. Clearly, ρ_i, τ_i and t_i are holomorphic functions. We proceed as in the proof of Assertion 3. Let $x \in \varphi(\Sigma_i \times \mathbb{D}_\delta) \cap \varphi(\Sigma_{i-1} \times \mathbb{D}_\delta)$. Since $|t_i(x)|$ and $|\tau_i'(x)|$ are less than δ, then $|t_i(x) + \tau_i'(x)| < 2\delta$. Thus, by hypothesis (ii), $\varphi(\rho_{i-1}(x), t_i(x) + \tau_i'(x))$ is well defined and:

$$\begin{aligned}
\varphi(\rho_{i-1}(x), t_i(x) + \tau_i'(x)) &= \varphi(\varphi(\rho_{i-1}(x), t_i(x)), \tau_i'(x)) \\
&= \varphi(\varphi(\rho_{i-1}(x), \tau_{i-1}' \circ \rho_i(x)), \tau_i'(x)) \\
&= \varphi(\rho_i(x), \tau_i'(x)) \\
&= x.
\end{aligned}$$

Then by definition of τ_{i-1}' we deduce that

$$t_i(x) + \tau_i'(x) = \tau_{i-1}'(x).$$

Thus, the holomorphic functions on $\varphi(\Sigma_i \times \mathbb{D}_\delta)$ defined as

$$\tau_i(x) = \tau_i'(x) + t_0(x) + \cdots + t_i(x) \tag{2.4}$$

for each $i = 0, \dots, r$ are such that

$$\tau_i = \tau_{i-1}$$

on $\varphi(\Sigma_i \times \mathbb{D}_\delta) \cap \varphi(\Sigma_{i-1} \times \mathbb{D}_\delta)$. Observe that for any leaf L', the restriction $\tau_0|_{L'}$ coincides with $\tau_{L'}$ on a neighborhood of $x_{L'}$. Then $\tau_0|_{L'}, \cdots, \tau_r|_{L'}$ give an analytic continuation of $\tau_{L'}$. Thus, $\tau_r|_L$ is the analytic continuation of τ_L along γ', hence $\tau_r(x) = \tau_L'(x)$. We denote τ_r by τ'. Analogously we construct τ'' for γ''. Then we have that $\tau''|_{L'}$ is an analytic continuation of $\tau_{L'}$ and, $\tau''|_L$ is the analytic continuation of τ_L along γ'', hence $\tau''(x) = \tau_L(x)$. By Assertion 2, we may take a sequence $\{x_k\}$ of points in Σ_0 with $x_k \to x$ as $k \to \infty$ and such that the leaf L_k passing through x_k is simply connected for all k. From above $\tau'|_{L_k}$ and $\tau''|_{L_k}$ are analytic continuations of τ_{L_k}. Since L_k is simply connected and, by Assertion 2, τ_{L_k} has an analytic continuation along any path, then $\tau'|_{L_k}$ and $\tau''|_{L_k}$ coincide on a neighborhood of x_k. In particular, $\tau'(x_k) = \tau''(x_k)$. Making $k \to \infty$ it follows by continuity that $\tau'(x) = \tau''(x)$, that is, $\tau_L'(x) = \tau_L''(x)$. Therefore, τ_L extends to L.

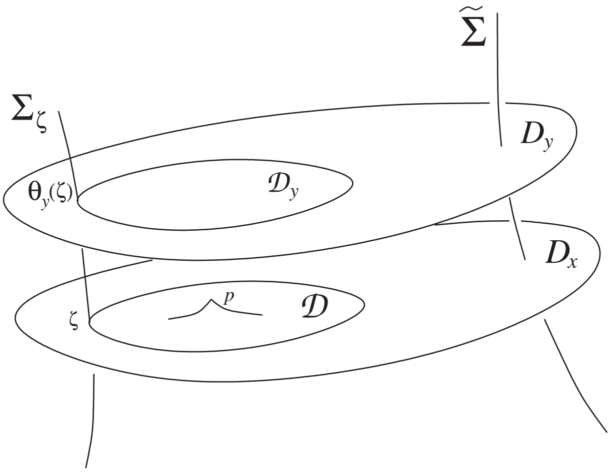
We define $\tau : V \backslash \operatorname{Sing}(\mathcal{F}) \to \mathbb{C}$ by $\tau|_L = \tau_L$. From above, τ coincides with the holomorphic function τ' on a neighborhood of the point x (arbitrary point). Then τ is holomorphic. Finally, remember (equation 2.4) that on a neighborhood of any non singular point, τ is expressed as

$$\tau_r(x) = \tau_r'(x) + t_0(x) + \cdots + t_r(x).$$

If we restrict x to a leaf, the first term of the sum above is a regular map and the other terms are constants. Hence τ is a regular map of any leaf. This finishes the proof of Assertion 4.

Given $x \in \widetilde{\Sigma}$, we know that $g^{-1}(x)$ is equal to D_z for some z. We denote $g^{-1}(x)$ by D_x. Thus, we have $p \in D_x$ for $x = g(p)$. It follows from hypothesis (iii) that there is a disc $\mathcal{D}' \subset D_x$ containing p such that $\mathcal{D}' \backslash \{p\}$ is contained in a leaf. Lemma 2.7 implies that there is a holomorphic bijective map $f : \Omega \to \mathcal{D}'$, $f(0) = p$, where $\Omega \subset \mathbb{C}$ is a disc containing $\mathbb{D}$. Thus if $\mathcal{D} = f(\mathbb{D})$, we have that $f : \overline{\mathbb{D}} \to \overline{\mathcal{D}}$ is holomorphic and regular on $\overline{\mathbb{D}} \backslash \{0\}$. Since $\overline{\mathcal{D}} \backslash \{p\}$ is contained in a leaf and by Assertion 3 we have that τ is a submersion on $U \backslash \operatorname{Sing}(\mathcal{F})$, then there exists a neighborhood V of $\partial\Delta$ on which τ defines a foliation by transversal balls along $\partial\Delta$. If we denote by Σ_ζ the transversal passing trough $\zeta \in \partial\Delta$ we have that τ is constant along Σ_ζ. Recall that $y \in \widetilde{\Sigma}$ is the unique point in the intersection of D_y and $\widetilde{\Sigma}$. It follows from the transversal uniformity of the foliation that if $y \in \widetilde{\Sigma}$ is close to x then D_y intersects only one time each transversal Σ_ζ. Let $\theta_y(\zeta)$ be the intersection of D_y with Σ_ζ. Since $\theta_y(\zeta)$ and ζ are both contained in

Σ_ζ, we have that $\tau(\theta_y(\zeta)) = \tau(\zeta)$ for all $\zeta \in \partial\Delta$. Note that $\theta_y := \theta_y(\partial\Delta)$ is a smooth Jordan curve in D_y. By Assertion 2, we may choose y such that D_y is a leaf. We consider $\mathcal{D}_y \subset D_y$, the regular disc bounded by θ_y.



Let $f_y : \mathbb{D} \to \mathcal{D}_y$ be a uniformization map. Since θ_y is a smooth Jordan curve, f_y extends as a diffeomorphism $f_y : \overline{\mathbb{D}} \to \overline{\mathcal{D}}_y$ (see [14], p.323). By Assertion 3, we have that τ is regular on $\overline{\mathcal{D}}_y$. It follows that $\tau \circ f_y : \overline{\mathbb{D}} \to \mathbb{C}$ is a regular map. Therefore, by Lemma 2.5, the curve $\tau \circ f_y : \mathbb{S}^1 \to \mathbb{C}$ has degree 1. Remember that $\tau(\theta_y(\zeta)) = \tau(\zeta)$ for all $\zeta \in \partial\Delta$, thus $\tau(\partial\mathcal{D}_y) = \tau(\partial\mathcal{D})$. Then

$$\tau \circ f_y(\mathbb{S}^1) = \tau(\partial\mathcal{D}_y) = \tau(\partial\mathcal{D}) = \tau \circ f(\mathbb{S}^1).$$

Therefore $\tau \circ f : \mathbb{S}^1 \to \mathbb{C}$ is only a reparametrization of $\tau \circ f_y : \mathbb{S}^1 \to \mathbb{C}$, hence $\tau \circ f : \mathbb{S}^1 \to \mathbb{C}$ is regular and has degree 1. Again by Lemma 2.5, $\tau \circ f : \mathbb{D} \to \mathbb{C}$ is also a regular map and in particular, $\tau \circ f$ is locally injective. Therefore there exists a disc $\mathrm{U} \subset \mathbb{D}$, centered at 0, such that $\tau \circ f$ is injective on $\overline{\mathrm{U}}$. Then

$$\tau \circ f(\partial\,\mathrm{U})$$

is a Jordan curve in $\mathbb{C}$. We also denote $f(\mathrm{U})$ by $\mathcal{D}$. Again, let Σ_ζ be the transversal ball through $\zeta \in \partial\mathcal{D}$. Proceeding as above, if Σ' is a small enough ball in $\widetilde{\Sigma}$ containing $x = g(p)$, we obtain that for all $y \in \Sigma'$ the set D_y intersects each Σ_ζ at the unique point $\theta_y(\zeta)$. Thus we have the Jordan curve θ_y in D_y such that $\tau(\theta_y) = \tau(\partial D)$. Remember that $\tau(\partial D) = \tau \circ f(\partial\,\mathrm{U})$ is a Jordan curve in $\mathbb{C}$. It follows that $\tau(\theta_y)$ is Jordan curve in $\mathbb{C}$ for all y. Let $\mathcal{D}_y \subset D_y$ be the disc bounded by θ_y. Since D_y is a singular disc, by Lemma 2.6, there is an injective holomorphic map $f_y : E \to M$, where

$E = \mathbb{D}$ or $\mathbb{C}$, such that $f_y(E) = D_y$. Let $\Omega_y \subset E$ be such that $f_y(\Omega_y) = \mathcal{D}_y$. Clearly Ω_y is a disc and $f_y(\partial\Omega_y) = \partial\mathcal{D}_y$. Then

$$\tau \circ f_y(\partial\Omega_y) = \tau(\partial\mathcal{D}_y)$$

is, from above, a Jordan curve in $\mathbb{C}$. Hence we deduce that the holomorphic function $\tau \circ f_y : \Omega_y \to \mathbb{C}$ is injective on $\overline{\Omega}_y$. Thus, since f_y is injective, we conclude that

$$\tau : \overline{\mathcal{D}}_y \to \mathbb{C}$$

is injective for all $y \in \Sigma'$.

Denote by W the union of the discs $\mathcal{D}_y$ for all $y \in \Sigma'$. It is easy to see that W is a neighborhood of p. Define

$$F : \overline{W} \to \widetilde{\Sigma} \times \mathbb{C}$$

$$F(w) = (g(w), \tau(w))$$

Assertion 5. — *F is a biholomorphism between W and its image.*

Proof. — Clearly F is holomorphic on W. We shall prove that F injective on $\overline{W}$. Suppose $F(w) = F(w')$. Then $g(w) = g(w') = y$, hence $w, w' \in D_y$ and, since $\overline{W} \cap D_y = \overline{\mathcal{D}}_y$, we have $w, w' \in \overline{\mathcal{D}}_y$. On the other hand $\tau(w) = \tau(w')$ and since τ is injective on $\overline{\mathcal{D}}_y$ we conclude that $w = w'$. Now, since $\overline{W}$ is compact, F is a homeomorphism onto its image and it follows that F is a biholomorphism.

Now, we will prove that $p \in W$ is regular for $\mathcal{F}$. Let $\mathcal{N}$ be the regular foliation on $\widetilde{\Sigma} \times \mathbb{C}$ whose leaves are the sets $\{*\} \times \mathbb{C}$. Let $\mathcal{F}'$ be the pull-back foliation of $\mathcal{N}$ by the biholomorphism F. Then $\mathcal{F}'$ is regular and it is easy to see that $\mathcal{F}'$ coincides with $\mathcal{F}$ out on a open set of W (out of $\operatorname{Sing}(\mathcal{F})$). Then $\mathcal{F}' = \mathcal{F}$ on W and $\mathcal{F}$ is therefore regular at p. Since $p \in U$ was arbitrary, we have proved that $\operatorname{Sing}(\mathcal{F})$ is empty. Then, from Assertion 1, the sets D_z are the leaves of $\mathcal{F}$. The proof of Proposition 2.4 is complete. □

Proof of Theorem 2.2. —

Assertion 1. — *Let $z \in \Sigma$ such that D_z is not contained in X. Then D_z is contained in a leaf of $\mathcal{F}$.*

Proof. — Take $t_0 \in D_z$ such that $h(z, t_0) \notin X$. Since X is closed in M, if Σ' is a small enough neighborhood (ball) of z in Σ, we have that $h(z', t_0) \notin X$ for all $z' \in \Sigma'$. Hence, for all $z' \in \Sigma'$ we have that $D_{z'}$ is not contained in X. Then, by hypothesis (ii), $S_{z'} := D_{z'} \cap X$ is discrete and $D_{z'} \backslash S_{z'}$ is contained in a nonsingular leaf of $\mathcal{F}$. Therefore, $\mathcal{F}$ restricted to $M' := h(\Sigma' \times D)$ satisfies the hypothesis of Proposition 2.4 and we have therefore that D_z is contained in a leaf of $\mathcal{F}$.

ASSERTION 2. — *Let $z \in \Sigma$ such that D_z is contained in X. Then D_z is a singular disc.*

Proof . — Let $x \in D_z$, $x = h(z,t)$. Let $\Sigma' \subset \Sigma$ be a neighborhood (a ball) of z and $D' \subset D$ be a neighborhood (a disc) of t. If Σ' and D' are small enough, we may assume that $M' := h(\Sigma' \times D')$ is a domain in $\mathbb{C}^n$. Since X has codimension $\geqslant 1$, there is a path $x_s = h(z_t, t_s)$ in M' such that $x_0 = x$ and $x_s \notin X$ for all $s > 0$. Then $D_s := D_{z_s}$ is not contained in X for all $s > 0$ and it follows by Assertion 1 that D_s is contained in a leaf. Hence D_s is a regular disc for all $s > 0$. Then, we may apply Lemma 2.1 to the family of discs D_s and conclude that $D_z = D_0$ is a singular disc.

ASSERTION 3. — *Let z be such that $D_z \subset X$. Let S_z be the set of singularities of the singular disc D_z. Then $D_z \backslash S_z$ is contained in a leaf of $\mathcal{F}$.*

Proof . — By Assertion 2, if D_z is not contained in X we have that D_z is contained in a leaf of $\mathcal{F}$. Therefore, the hypothesis of Proposition 2.3 holds for $\mathcal{F}$ and Assertion 3 follows.

Let z be such that D_z is not contained in X. By hypothesis (iii) of 2.2, we have that $S_z := D_z \cap X$ is discrete and $D_z \backslash S_z$ is contained in a leaf of $\mathcal{F}$. From this and Assertion 3 we conclude: for all z there is a discrete set S_z such that $D_z \backslash S_z$ is contained in a leaf of $\mathcal{F}$. Therefore the hypothesis of Proposition 2.4 holds and Theorem 2.2 follows. □

3. The algebraic multiplicity and the Chern class of the tangent bundle of the strict transform

Let $\mathcal{F}_0$, $\widetilde{\mathcal{F}}_0$ and h as in §1.

PROPOSITION 3.1. — *If h extends to the divisor as a homeomorphism between $\pi^{-1}(U)$ and $\pi^{-1}(\widetilde{U})$, then the extension also denoted by h is a topological equivalence between $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$.*

Proof. — Is a direct application of Theorem 1.4. □

Proof of Theorem 1.1. — Suppose that $\mathcal{F}$ is generated on U by the holomorphic vector field

$$V = \sum_{i=1}^{n} a_i \frac{\partial}{\partial z_i},\ a_i \in \mathcal{O}_U,\ g.c.d.(a_1, a_2, \ldots, a_n) = 1.$$

For each $j = 1, 2, \ldots, n$, let $U_j = (x_j \neq 0)$ and $U'_j = \pi^{-1}(U_j)$. Let $V_j = \pi^*(V|_{U_j})$. If $(x_1^j, \ldots, x_n^j)$ are coordinates on U'_j such that

$$\pi(x_1^j, \ldots, x_n^j) = (x_j^j x_1^j, \ldots, x_j^j, \ldots, x_j^j x_n^j),$$

then

$$V_j = a_j \frac{\partial}{\partial x_j^j} + \sum_{i=1, i\neq j}^{n} \frac{a_i - x_i^j a_j}{x_j^j} \frac{\partial}{\partial x_i^j},$$

where $a_i = a_i \circ \pi$ for $i = 1, \ldots, n$. On U'_j, $\mathcal{F}_0$ is defined by the vector field

$$W_j = \frac{1}{(x_j^j)^{r-\xi}} V_j,$$

where r is the algebraic multiplicity of V at $0 \in \mathbb{C}^n$ and $\xi = 1$ or 0 depending on the divisor being invariant or not by $\mathcal{F}_0$. Evidently $V_i = V_j$ on $U'_i \cap U'_j$. Then

$$W_i = \left(x_j^j / x_i^i\right)^{r-\xi} W_j \quad \text{on} \quad U'_i \cap U'_j.$$

It follows from this equation that the tangent bundle $T\mathcal{F}_0$ of $\mathcal{F}_0$ is isomorphic to $L^{\xi - r}$, where L is the line bundle associated to the divisor $E = \pi^{-1}(0)$. Then the Chern class $c(T\mathcal{F}_0)$ of $T\mathcal{F}_0$ is equal to $(\xi - r)c(L)$. It is natural consider E as an element in $H_{n-2}(U', \mathbb{Z})$, where $U' = \pi^{-1}(U)$. We know that $c(L)$ is equal to $d(E) \in H^2(U', \mathbb{Z})$, the dual of E. Therefore

$$c(T\mathcal{F}_0) = (\xi - r)d(E).$$

On the other hand, make $\widetilde{U}' = \pi^{-1}(\widetilde{U})$ and observe that the divisor E is invariant by $\mathcal{F}_0$ if and only if it is by $\widetilde{\mathcal{F}}_0$. Then analogously we have

$$c(T\widetilde{\mathcal{F}}_0) = (\xi - \widetilde{r})\widetilde{d}(E),$$

where $\widetilde{r}$ is the algebraic multiplicity of $\widetilde{\mathcal{F}}$ and $\widetilde{d}(E) \in H^2(\widetilde{U}', \mathbb{Z})$ is the dual of E. By Proposition 3.1 we have that $h : U' \to \widetilde{U}'$ is a topological equivalence between $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$. Then Theorem 1.3 implies that

$$(\xi - r)h^*(d(E)) = (\xi - \widetilde{r})\widetilde{d}(E). \tag{3.1}$$

We may assume that U is a ball in $\mathbb{C}^n$. Thus, we have that U' is a tubular neighborhood of E and therefore $H^2(U', \mathbb{Z}) \simeq \mathbb{Z}$. Since the cohomology is invariant by homeomorphisms, we also have $H^2(\widetilde{U}', \mathbb{Z}) \simeq \mathbb{Z}$. Can be proved that $d(E)$ and $\widetilde{d}(E)$ are generators of $H^2(U', \mathbb{Z})$ and $H^2(\widetilde{U}', \mathbb{Z})$ respectively. Then we have that $h^*(d(E)) = \widetilde{d}(E)$ or $h^*(d(E)) = -\widetilde{d}(E)$. By using this in (3.1) we obtain $r = \widetilde{r}$ or $r + \widetilde{r} = 2\xi$. The second possibility implies $r = \widetilde{r} = \xi$, since $r \geqslant 1$, $\widetilde{r} \geqslant 1$ and $\xi = 1$ or 0. Therefore we conclude that $r = \widetilde{r}$. □

Remark. — Under the hypothesis of Theorem 1.1, we have another invariants. The restriction of $\mathcal{F}_0$ to the divisor is a foliation with $\operatorname{Sing}(\mathcal{F}_0)$ as singular set. It is well known that this foliation coincides out of the singular set with a unique foliation $\mathcal{N}$ of codimension $\geqslant 2$ in the divisor (the saturated foliation). We will say that $\mathcal{N}$ is the foliation induced by $\mathcal{F}_0$ in the divisor. Let $\widetilde{\mathcal{N}}$ be the foliation induced by $\widetilde{\mathcal{F}}_0$ in the divisor. It follows from Theorem 1.4 that $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ are topologically equivalent. Thus, since the divisor is isomorphic to $\mathbb{P}^{n-1}$, Theorem 1.3 implies that $d(\mathcal{N}) = d(\widetilde{\mathcal{N}})$. In other words, the degree of the foliation induced in the divisor is invariant.

From above, $\mathcal{F}_0$ is generated by the holomorphic vector fields W_i and

$$W_i = \left(x^j_j/x^i_i\right)^{r-\xi} W_j \quad \text{on} \quad U'_i \cap U'_j,$$

where $\xi = 1$ or 0. Let $x \in U'_i \cap U'_j$. Let $x^i = (x^i_1, \ldots, x^i_n)$ be the coordinates of x in U'_i and let $x^j = (x^j_1, \ldots, x^j_n)$ be the coordinates of x in U'_j. Since $\pi(x^i) = \pi(x^j)$, we have that

$$(x^i_i x^i_1, \ldots, x^i_i, \ldots, x^i_i x^i_n) = (x^j_j x^j_1, \ldots, x^j_j, \ldots, x^j_j x^j_n),$$

hence $x^j_j/x^i_i = x^i_j$. Replacing in last equation we obtain:

$$W_i = \left(x^i_j\right)^{r-\xi} W_j \quad \text{on} \quad U'_i \cap U'_j. \tag{3.2}$$

Observe that $\pi^{-1}(0) \cap U'_i$ is represented by $(x^i_i = 0)$. Recall that $\pi^{-1}(0)$ is canonically isomorphic to $\mathbb{P}^{n-1}$. A point p in $\pi^{-1}(0) \cap U'_i$ given by

$$(x^i_1(p), \ldots, 0_i, \ldots, x^i_n(p))$$

is represented in homogeneous coordinates by

$$[z_1 : \cdots : z_n](p) = [x^i_1(p) : \cdots : 1_i : \cdots : x^i_n(p)],$$

hence $x^i_j(p) = (z_j/z_i)(p)$. Thus, if $\mathcal{U}_i = U'_i \cap \pi^{-1}(0)$ and $J_i = W_i|_{\mathcal{U}_i}$, it follows from (3.2) that

$$J_i = (z_j/z_i)^{r-\xi} J_j \quad \text{on} \quad \mathcal{U}_i \cap \mathcal{U}_j. \tag{3.3}$$

Let S be the union of the components of codimension 2 of $\operatorname{Sing}(\mathcal{F}_0)$. Then S is the codimension 1 part (respect to the divisor) of the zero set of $\{J_i\}$. Each J_i may be expressed as $J_i = f_i Z_i$, where f_i is a holomorphic function on $\mathcal{U}_i$ and the vector field Z_i has singular set of codimension $\geqslant 2$. It follows from (3.3) that

$$Z_i = (f_j/f_i)\,(z_j/z_i)^{r-\xi} Z_j \ \text{ on } \ \mathcal{U}_i \cap \mathcal{U}_j.$$

From this equation, it is not difficult to conclude that

$$r = d(\mathcal{N}) - \deg(S) - 1 + \xi,$$

where $\deg(S)$ is the degree of S as a divisor of $\pi^{-1}(0)$. Then, since the algebraic multiplicity and the degree of the foliation induced in the divisor are invariants, we deduce that the degree of the codimension 1 part of the singular set of the strict transform is also an invariant. Moreover it is not difficult to see that $h(S) = \widetilde{S}$, where $\widetilde{S}$ is the union of the components of codimension 2 of $\mathrm{Sing}(\widetilde{\mathcal{F}}_0)$.

4. The case C^1

In this section we prove Theorem 1.2. In view of Theorem 1.1, it is sufficient to show the following.

Proposition 4.1. — *Let $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ be two foliations by curves of neighborhoods U and $\widetilde{U}$ of $0 \in \mathbb{C}^n$. Let $\mathrm{h} : U \to \widetilde{U}$ be a C^1 equivalence. Let $h : \pi^{-1}(U\backslash\{0\}) \to \pi^{-1}(\widetilde{U}\backslash\{0\})$ be as before. Then h can be extended to the divisor as a homeomorphism between $\pi^{-1}(U)$ and $\pi^{-1}(\widetilde{U})$.*

We start the proof.

Proposition 4.2. — *Under the conditions of Proposition 4.1, we have that $d\,\mathrm{h}(0) : \mathbb{C}^n \to \mathbb{C}^n$ maps complex lines onto complex lines. Furthermore, if $J : \mathbb{C}^n \to \mathbb{C}^n$ is the conjugation $J(z) = \bar{z}$, then either $d\,\mathrm{h}(0) : \mathbb{C}^n \to \mathbb{C}^n$ is a $\mathbb{C}$-linear isomorphism, or $d\,\mathrm{h}(0) = Q \circ J$, where $Q : \mathbb{C}^n \to \mathbb{C}^n$ is a $\mathbb{C}$-linear isomorphism. Thus, $d\,\mathrm{h}(0)$ induces a diffeomorphism of $\mathbb{P}^{n-1}$ onto itself.*

Proof. — let L be a complex line, $0 \in L \subset \mathbb{C}^n$. There exists $\mathbb{C}$-linear functions $A_i : \mathbb{C}^n \to \mathbb{C}$ for $i = 1, \ldots, (n-1)$, such that

$$L = \{z \in \mathbb{C}^n : \; A_i(z) = 0 \;\text{ for all }\; i = 1, \ldots, (n-1)\}.$$

Let $V : U \to \mathbb{C}^n$ be a holomorphic vector field which generates $\mathcal{F}$. The set:

$$B = \{z \in \mathbb{C}^n : \; A_i \circ V(z) = 0 \text{ for all } \quad i = 1, \ldots, (n-1)\}$$

is an analytic variety and it is easy to see that $0 \in B$. Then, there exists a complex curve contained in B and passing through 0. In particular there exists a sequence of points $z_k \in \mathbb{C}^n\backslash\{0\}$, $z_k \to 0$, such that $A_i \circ V(z_k) = 0$ for all $k \in \mathbb{N}$ and all $i = 1, 2, \ldots, (n-1)$. In other words, $T_{z_k}\mathcal{F} = L$ for all $k \in \mathbb{N}$. Now, since h is a C^1 equivalence, $d\,\mathrm{h}_{z_k}(T_{z_k}\mathcal{F}) = T_{\mathrm{h}(z_k)}\widetilde{\mathcal{F}}$, that is, $d\,\mathrm{h}_{z_k}(L) = T_{\mathrm{h}(z_k)}\widetilde{\mathcal{F}}$ is a complex line for all $k \in \mathbb{N}$. Making $k \to \infty$, since $\mathrm{h} \in C^1$ and the space of complex lines of $\mathbb{C}^n$ is compact, we obtain that $d\; h_0(L)$ is also a complex line. The second part of the proposition is an immediate consequence of the following lemma. □

LEMMA 4.3. — *Let $A : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ $(n \geqslant 2)$ be a $\mathbb{R}$-linear isomorphism. Identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ and assume that A maps complex lines onto complex lines. Then, either A is a $\mathbb{C}$-linear isomorphism, or $A = Q \circ J$ with $Q : \mathbb{C}^n \to \mathbb{C}^n$ a $\mathbb{C}$-linear isomorphism.*

Proof. — Since A maps any complex line onto a complex line, for all $v \in \mathbb{C}^n \backslash \{0\}$ there exists $\theta(v) \in \mathbb{C}\backslash\{0\}$ such that $A(iv) = \theta(v)A(v)$. Let v_1 and v_2 be two $\mathbb{C}$-linearly independent vectors. Then

$$A(iv_1 + iv_2) = A(iv_1) + A(iv_2) = \theta(v_1)A(v_1) + \theta(v_2)A(v_2).$$

Moreover:

$$\begin{aligned} A(iv_1 + iv_2) &= A(i(v_1 + v_2)) = \theta(v_1 + v_2)A(v_1 + v_2) \\ &= \theta(v_1 + v_2)A(v_1) + \theta(v_1 + v_2)A(v_2). \end{aligned}$$

From the equations above, we obtain:

$$(\theta(v_1) - \theta(v_1 + v_2))A(v_1) + ((\theta(v_2) - \theta(v_1 + v_2))A(v_2) = 0. \tag{4.1}$$

Let L_1 and L_2 be the complex lines generated by v_1 and v_2 respectively. Since v_1 and v_2 are $\mathbb{C}$-linearly independent, we have that L_1 and L_2 are different. This implies, since A is an isomorphism, that $A(L_1)$ and $A(L_2)$ are different complex lines. Then, since $A(L_1)$ and $A(L_2)$ are generated by $A(v_1)$ and $A(v_2)$ respectively, we have that $A(v_1)$ and $A(v_2)$ are $\mathbb{C}$-linearly independent. Thus, it follows from equation (4.1) that

$$\theta(v_1) = \theta(v_1 + v_2) = \theta(v_2).$$

It is now easy to see that $\theta(v) = \theta_0, \ \forall v \in \mathbb{C}^n\backslash\{0\}$. We know that there exists two $\mathbb{C}$-linear transformations $P : \mathbb{C}^n \to \mathbb{C}^n$ and $Q : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$A(z) = P(z) + Q(\bar{z}), \ \text{ for all } \ z \in \mathbb{C}^n.$$

Then

$$A(iz) = iP(z) - iQ(\bar{z}).$$

On the other hand

$$A(iz) = \theta_0 A(z) = \theta_0 P(z) + \theta_0 Q(\bar{z}), \ \text{ for all } \ z \in \mathbb{C}^n.$$

consequently

$$(\theta_0 - i)P(z) + (\theta_0 + i)Q(\bar{z}) = 0.$$

Since, as functions of z, $(\theta_0 - i)P$ and $(\theta_0 + i)Q \circ J$ are holomorphic and anti-holomorphic respectively, we have that

$$(\theta_0 - i)P \equiv 0, \ \ (\theta_0 + i)Q \circ J \equiv 0.$$

From this it is easy to see that either $P = 0$, or $Q = 0$. This proves the lemma. □

Definition 4.4. — *Let $\{z_k\}$ be a sequence of points in $\mathbb{C}^n \backslash \{0\}$. Let L be a complex line in $\mathbb{C}^n$. We say that $\{z_k\}$ is tangent to L at 0 if $z_k \to 0$ and every accumulation point of $\{z_k/||z_k||\}$ is contained in L.*

Let $\pi : \widehat{\mathbb{C}^n} \to \mathbb{C}^n$ be the blow up at $0 \in \mathbb{C}^n$. We know that $\pi^{-1}(0)$ is naturally isomorphic to $\mathbb{P}^{n-1}$. Thus, for each $p \in \pi^{-1}(0)$ we denote by L_p the respective complex line in $\mathbb{C}^n$. The following fact is well known and we left the proof to the reader:

Proposition 4.5. — *Let $\{p_k\}$ be a sequence of points in $\widehat{\mathbb{C}^n} \backslash \pi^{-1}(0)$. Then $p_k \to p \in \pi^{-1}(0)$ if and only if $\{\pi(p_k)\}$ is tangent to L_p at 0.*

Proof of Proposition 4.1. — Let $p \in \pi^{-1}(0)$ and $\{p_k\}$ any sequence of points in $\pi^{-1}(U) \backslash \pi^{-1}(0)$ such that $p_k \to p$.

Since $h \in C^1$, we have

$$\mathrm{h}(\pi(p_k)) = d\,\mathrm{h}_0(\pi(p_k)) + r(\pi(p_k)), \text{ where } \frac{r(\pi(p_k))}{||\pi(p_k)||} \to 0 \text{ as } k \to 0.$$

Then

$$\frac{\mathrm{h}(\pi(p_k))}{||\pi(p_k)||} = d\,\mathrm{h}_0\left(\frac{(\pi(p_k))}{||\pi(p_k)||}\right) + \frac{r(\pi(p_k))}{||\pi(p_k)||}. \tag{4.2}$$

By proposition 4.5, $\pi(p_k)$ is tangent to L_p at 0, hence any point of accumulation of the sequence $\{(\pi(p_k))/\|\pi(p_k)||\}$ is contained in L_p. Thus, it is easy to see from equation (4.2) that any point of accumulation of the sequence $\{\mathrm{h}(\pi(p_k))/||\pi(p_k)||\}$ is contained in $d\,\mathrm{h}_0(L_p)$ and the same holds for the sequence

$$\frac{\mathrm{h}(\pi(p_k))}{||\,\mathrm{h}(\pi(p_k))||} = \frac{\mathrm{h}(\pi(p_k))}{||\pi(p_k)||}\frac{||\pi(p_k)||}{||\,\mathrm{h}(\pi(p_k))||}.$$

From proposition 4.2 we have that $d\,\mathrm{h}_0(L_p)$ is a complex line. Then $\{\mathrm{h}(\pi(p_k)\})$ is tangent to $d\,\mathrm{h}_0(L_p)$ at 0. It follows by proposition 4.5 that $\pi^{-1} \circ \mathrm{h} \circ \pi(p_k) = h(p_k) \to q$, where $q \in \pi^{-1}(0)$ is such that $L_q = d\,\mathrm{h}_0(L_p)$. We extend h by making $h(p) = d\,\mathrm{h}_0(L_p)$ for all p in $\pi^{-1}(0)$. Finally, it is easy to prove that $h : \pi^{-1}(U) \to \pi^{-1}(\widetilde{U})$ is a homeomorphism. □

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