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Pham Hoang HIEP

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HÖLDER CONTINUITY OF SOLUTIONS TO THE MONGE-AMPÈRE EQUATIONS ON COMPACT KÄHLER MANIFOLDS

by Pham Hoang HIEP

Abstract. — We study Hölder continuity of solutions to the Monge-Ampère equations on compact Kähler manifolds. T. C. Dinh, V.A. Nguyen and N. Sibony have shown that the measure ω_u^n is moderate if u is Hölder continuous. We prove a theorem which is a partial converse to this result.

Résumé. — Nous étudions la continuité de Hölder des solutions des équations de Monge-Ampère sur des variétés Kählériennes compactes. T. C. Dinh, V.A. Nguyen et N. Sibony ont prouvé que ω_u^n est modéré si u est Hölder-continue. Nous démontrons dans quelques cas la réciproque de ce résultat.

1. Introduction

Let X be a compact n-dimensional Kähler manifold equipped with a fundamental form ω satisfying $\int_X \omega^n = 1$. An upper semicontinuous function $\varphi\colon X \to [-\infty, +\infty)$ is called ω-plurisubharmonic (ω-psh) if $\varphi \in L^1(X)$ and $\omega_\varphi := \omega + dd^c\varphi \geqslant 0$. By $\mathrm{PSH}(X,\omega)$ (resp. $\mathrm{PSH}^-(X,\omega)$) we denote the set of ω-psh (resp. negative ω-psh) functions on X. The complex Monge-Ampère equation $\omega_u^n = f\omega^n$ was solved for smooth positive f in the fundamental work of S. T. Yau (see [31]). Later S. Kołodziej showed that there exists a continuous solution if $f \in L^p(\omega^n)$, $f \geqslant 0$, $p > 1$ (see [24]). Recently in [27] he proved that this solution is Hölder continuous in this case (see also [18] for the case $X = \mathbf{C}P^n$). In Corollary 1.2 in [16] the authors have shown that the measure ω_u^n is moderate if u is Hölder continuous. The main result is the following theorem which give a partial answer to the converse problem:

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Theorem A. — *Let μ be a non-negative Radon measure on X such that*

$$\mu(B(z,r)) \leqslant Ar^{2n-2+\alpha},$$

for all $B(z,r) \subset X$ ($A, \alpha > 0$ are constants). Then for every $f \in L^p(d\mu)$ with $p > 1$, $\int_X f d\mu = 1$, there exists a Hölder continuous ω-psh function u such that $\omega_u^n = f d\mu$.

The following results are simple applications of Theorem A:

Corollary B. — *Let $\varphi \in \mathrm{PSH}(X, \omega)$ be a Hölder continuous function. Then for every $f \in L^p(\omega_\varphi \wedge \omega^{n-1})$ with $p > 1$, $\int_X f\omega_\varphi \wedge \omega^{n-1} = 1$, there exists a Hölder continuous ω-psh function u such that $\omega_u^n = f\omega_\varphi \wedge \omega^{n-1}$.*

Corollary C. — *Let S be a C^1 smooth real hypersurface in X and V_S be the volume measure on S. Then for every $f \in L^p(dV_S)$ with $p > 1$, $\int_X f dV_S = 1$, there exists a Hölder continuous ω-psh function u such that $\omega_u^n = f dV_S$.*

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2. Preliminaries

First we recall some elements of pluripotential theory that will be used throughout the paper. Details can be found in [2]–[3], [5]–[6], [4], [7], [9]–[8], [13]–[15], [19]–[20], [21], [23]–[27], [28], [29]–[30], [32]–[33].

2.1. In [24] Kołodziej introduced the capacity C_X on X by

$$C_X(E) := \sup\Big\{\int_E \omega_\varphi^n \colon \varphi \in \mathrm{PSH}(X,\omega),\ -1 \leqslant \varphi \leqslant 0\Big\}$$

for all Borel sets $E \subset X$.

2.2. In [19] Guedj and Zeriahi introduced the Alexander capacity T_X on X by

$$T_X(E) = e^{-\sup_X V_{E,X}^*}$$

for all Borel sets $E \subset X$. Here $V_{E,X}^*$ is the global extremal ω-psh function for E defined as the smallest upper semicontinuous majorant of $V_{E,X}$ i.e,

$$V_{E,X}(z) = \sup\Big\{\varphi(z) \colon \varphi \in \mathrm{PSH}(X,\omega),\ \varphi \leqslant 0 \text{ on } E\Big\}.$$

2.3. The following definition was introduced in [18]: A probability measure μ on X is said to satisfy the condition $\mathcal{H}(\alpha, A)$ $(\alpha, A > 0)$ if

$$\mu(K) \leqslant A C_X(K)^{1+\alpha},$$

for any Borel subset K of X.

A probability measure μ on X is said to satisfy the condition $\mathcal{H}(\infty)$ if for any $\alpha > 0$ there exist $A(\alpha) > 0$ dependent on α such that

$$\mu(K) \leqslant A(\alpha) C_X(K)^{1+\alpha},$$

for any Borel subset K of X.

2.4. The following definition was introduced in [17]: A measure μ is said to be moderate if for any open set $U \subset X$, any compact set $K \subset\subset U$ and any compact family $\mathcal{F}$ of plurisubharmonic functions on U, there are constants $\alpha > 0$ such that

$$\sup\Big\{\int_K e^{-\alpha\varphi} d\mu \colon \varphi \in \mathcal{F}\Big\} < +\infty.$$

2.5. The following class of ω-psh functions was investigated by Guedj and Zeriahi in [20]:

$$\mathcal{E}(X, \omega) = \Big\{\varphi \in \mathrm{PSH}(X, \omega) \colon \lim_{j \to \infty} \int_{\{\varphi > -j\}} \omega^n_{\max(\varphi, -j)} = \int_X \omega^n = 1\Big\}.$$

Let us also define

$$\mathcal{E}^-(X, \omega) = \mathcal{E}(X, \omega) \cap \mathrm{PSH}^-(X, \omega).$$

We refer to [20] for the properties of the class $\mathcal{E}(X, \omega)$.

2.6. S is called a C^1 smooth real hypersurface in X if for all $z \in X$ there exists a neighborhood U of z and $\chi \in C^1(U)$ such that $S \cap U = \{z \in U \colon \chi(z) = 0\}$ and $D\chi(z) \neq 0$ for all $z \in S \cap U$.

Next we state a well-known result needed for our work.

2.7. Proposition. — *Let μ be a non-negative Radon measure on X such that $\mu(B(z, r)) \leqslant A r^{2n-2+\alpha}$ for all $B(z, r) \subset X$ (A, $\alpha > 0$ are constants). Then $\mu \in \mathcal{H}(\infty)$.*

Proof. — By Theorem 7.2 in [33] and Proposition 7.1 in [19] we can find $\epsilon, C > 0$ such that

$$\mu(K) \leqslant A h^{2n-2+\alpha}(K) \leqslant \frac{AC}{\alpha} T_X(K)^{\epsilon\alpha} \leqslant \frac{ACe}{\alpha} e^{-\frac{\epsilon\alpha}{C_X(K)^{\frac{1}{n}}}},$$

for all Borel subsets K of X, where $h^{2n-2+\alpha}$ is the Hausdorff content of dimension $2n-2+\alpha$. This implies that $\mu \in \mathcal{H}(\infty)$. □

3. Stability of the solutions

The stability estimate of solutions to the Monge-Ampère equations on compact Kähler manifolds was obtained by Kołodziej ([24]). Recently, in [12] S. Dinew and Z. Zhang proved a stronger version of this estimate. We will show a generalization of the stability theorem by S. Kołodziej. As a first step we have the following proposition. This proof follows ideas of the proof of Theorem 2.5 in [11]. We include a proof for the reader's convenience.

3.1. Proposition. — *Let $\varphi, \psi \in \mathcal{E}^-(X,\omega)$ be such that $\omega_\varphi^n \in \mathcal{H}(\alpha, A)$. Then there exist constants $t \in \mathbf{R}$ and $C(\alpha, A) \geqslant 0$ such that*

$$\int_{\{|\varphi-\psi-t|>a\}} (\omega_\varphi^n + \omega_\psi^n) \leqslant C(\alpha, A) a^{n+1},$$

here $a = [\int_X \|\omega_\varphi^n - \omega_\psi^n\|]^{\frac{1}{2n+3+\frac{n+1}{1+\alpha}}}$.

Proof. — Since $\int_{\{|\varphi-\psi-t|>a\}} (\omega_\varphi^n + \omega_\psi^n) \leqslant 2$, it suffices to consider the case when a is small. Set

$$\epsilon = \frac{1}{2} \inf\Big\{ \int_{\{|\varphi-\psi-t|>a\}} \omega_\varphi^n \colon t \in \mathbf{R} \Big\}$$

Hence

$$\int_{\{|\varphi-\psi-t|\leqslant a\}} \omega_\varphi^n \leqslant 1 - 2\epsilon$$

for all $t \in \mathbf{R}$. Set

$$t_0 = \sup\Big\{ t \in \mathbf{R} \colon \int_{\{\varphi<\psi+t+a\}} \omega_\varphi^n \leqslant 1 - \epsilon \Big\}$$

Replacing ψ by $\psi + t_0$ we can assume that $t_0 = 0$. Then $\int_{\{\varphi<\psi+a\}} \omega_\varphi^n \leqslant 1-\epsilon$ and $\int_{\{\varphi\leqslant\psi+a\}} \omega_\varphi^n \geqslant 1-\epsilon$. Hence

$$\begin{aligned}\int_{\{\psi<\varphi+a\}} \omega_\varphi^n &= 1 - \int_{\{\varphi+a\leqslant\psi\}} \omega_\varphi^n \\ &= 1 - \int_{\{\varphi\leqslant\psi+a\}} \omega_\varphi^n + \int_{\{\psi-a<\varphi\leqslant\psi+a\}} \omega_\varphi^n \leqslant 1 - \epsilon.\end{aligned}$$

Since $\int_{\{|\varphi-\psi|\leqslant a\}} \omega_\varphi^n \leqslant 1$ we can choose $s \in [-a + a^{n+2}, a - a^{n+2}]$ satisfying

$$\int_{\{|\varphi-\psi-s|<a^{n+2}\}} \omega_\varphi^n \leqslant 2a^{n+1}.$$

Replacing ψ by $\psi + s$ we can assume that $s = 0$. One easily obtains the following inequalities

$$\int_{\{\varphi<\psi+a^{n+2}\}} \omega_\varphi^n \leqslant 1-\epsilon, \qquad \int_{\{\psi<\varphi+a^{n+2}\}} \omega_\varphi^n \leqslant 1-\epsilon, \qquad \int_{\{|\varphi-\psi|<a^{n+2}\}} \omega_\varphi^n \leqslant 2a^{n+1}. \tag{1}$$

By [20] we can find $\rho \in \mathcal{E}(X,\omega)$, such that

$$\omega_\rho^n = \frac{1}{1-\epsilon} 1_{\{\varphi<\psi\}}\omega_\varphi^n + c1_{\{\varphi\geqslant\psi\}}\omega_\varphi^n \text{ and } \sup_X \rho = 0, \tag{2}$$

($c \geqslant 0$ is chosen so that the measure has total mass 1). For simplicity of notation we set $\beta = \frac{n+1}{1+\alpha}$. Set

$$U = \Big\{(1-a^{n+2+\beta})\varphi < (1-a^{n+2+\beta})\psi + a^{n+2+\beta}\rho\} \subset \{\varphi<\psi\Big\}.$$

From Theorem 2.1 in [15] and (2) we get

$$\omega_\varphi^{n-1} \wedge \omega_{(1-a^{n+2+\beta})\psi+a^{n+2+\beta}\rho} \geqslant (1-a^{n+2+\beta})\omega_\varphi^{n-1}\wedge\omega_\psi + \frac{a^{n+2+\beta}}{(1-\epsilon)^{\frac{1}{n}}}\omega_\varphi^n, \tag{3}$$

on U. From Theorem 2.3 in [15], Lemma 2.6 in [11] and (3) we obtain

$$\begin{aligned}
&(1-a^{n+2+\beta})\int_U \omega_\varphi^{n-1}\wedge\omega_\psi + \frac{a^{n+2+\beta}}{(1-\epsilon)^{\frac{1}{n}}}\int_U \omega_\varphi^n \\
&\quad\leqslant \int_U \omega_{(1-a^{n+2+\beta})\psi+a^{n+2+\beta}\rho}\wedge\omega_\varphi^{n-1} \\
&\quad\leqslant \int_U \omega_{(1-a^{n+2+\beta})\varphi}\wedge\omega_\varphi^{n-1} \\
&\quad= (1-a^{n+2+\beta})\int_U \omega_\varphi^n + a^{n+2+\beta}\int_U \omega\wedge\omega_\varphi^{n-1} \\
&\quad\leqslant (1-a^{n+2+\beta})\Big(\int_U \omega_\varphi^{n-1}\wedge\omega_\psi + 2a^{2n+3+\beta}\Big) + a^{n+2+\beta}\int_U \omega\wedge\omega_\varphi^{n-1}.
\end{aligned}$$

Hence

$$\frac{1}{(1-\epsilon)^{\frac{1}{n}}}\int_U \omega_\varphi^n \leqslant 2a^{n+1} + \int_U \omega\wedge\omega_\varphi^{n-1}. \tag{4}$$

From Proposition 3.6 in [19] and (4) we get

$$(5)\quad \frac{1}{(1-\epsilon)^{\frac{1}{n}}}\Big[\int_{\{\varphi\leqslant\psi-a^{n+2}\}}\omega_\varphi^n - C_1(\alpha,A)a^{n+1}\Big]$$
$$\leqslant \frac{1}{(1-\epsilon)^{\frac{1}{n}}}\Big[\int_{\{\varphi\leqslant\psi-a^{n+2}\}}\omega_\varphi^n - A[C_X(\{\rho\leqslant -\frac{1}{2a^\beta}\})]^{1+\alpha}\Big]$$
$$\leqslant \frac{1}{(1-\epsilon)^{\frac{1}{n}}}\Big[\int_{\{\varphi\leqslant\psi-a^{n+2}\}}\omega_\varphi^n - \int_{\{\rho\leqslant -\frac{1}{2a^\beta}\}}\omega_\varphi^n\Big]$$
$$\leqslant \frac{1}{(1-\epsilon)^{\frac{1}{n}}}\int_U \omega_\varphi^n$$
$$\leqslant 2a^{n+1} + \int_U \omega\wedge\omega_\varphi^{n-1}$$
$$\leqslant 2a^{n+1} + \int_{\{\varphi<\psi\}} \omega\wedge\omega_\varphi^{n-1},$$

Similarly to ρ we define $\vartheta\in\mathcal{E}(X,\omega)$, such that

$$\omega_\vartheta^n = \frac{1}{1-\epsilon}1_{\{\varphi<\psi\}}\omega_\varphi^n + l1_{\{\psi\geqslant\varphi\}}\omega_\varphi^n \text{ and } \sup_X \vartheta = 0,$$

(l plays the same role as c above). Set

$$V = \Big\{(1-a^{n+2+\beta})\psi < (1-a^{n+2+\beta})\varphi + a^{n+2+\beta}\vartheta\Big\} \subset \{\psi<\varphi\}.$$

We get

$$(6)\quad \frac{1}{(1-\epsilon)^{\frac{1}{n}}}\Big[\int_{\{\psi\leqslant\varphi-a^{n+2}\}}\omega_\varphi^n - C_1(\alpha,A)a^{n+1}\Big] \leqslant 2a^{n+1} + \int_{\{\psi<\varphi\}}\omega\wedge\omega_\varphi^{n-1}.$$

From (1), (5) and (6) we obtain

$$\frac{1}{(1-\epsilon)^{\frac{1}{n}}}\Big[1-2a^{n+1}-2C_1(\alpha,A)a^{n+1}\Big]$$
$$\leqslant \frac{1}{(1-\epsilon)^{\frac{1}{n}}}\Big[\int_{\{|\varphi-\psi|\geqslant a^{n+1}\}}\omega_\varphi^n - 2C_1(\alpha,A)a^{1+\alpha}\Big]$$
$$\leqslant 4a^{n+1}+1.$$

Hence

$$\epsilon \leqslant 1 - \Big[\frac{1-2(C_1(\alpha,A)+1)a^{n+1}}{4a^{n+1}+1}\Big]^n \leqslant C_2(\alpha,A)a^{n+1}.$$

This implies that there exists $t\in\mathbf{R}$ satisfying

$$\int_{\{|\varphi-\psi-t|>a\}}\omega_\varphi^n \leqslant 2C_2(\alpha,A)a^{n+1}.$$

Finally we have

$$\int_{\{|\varphi-\psi-t|>a\}}(\omega_\varphi^n+\omega_\psi^n) = 2\int_{\{|\varphi-\psi-t|>a\}}\omega_\varphi^n + \int_{\{|\varphi-\psi-t|>a\}}(\omega_\psi^n-\omega_\varphi^n)$$
$$\leqslant 2C_2(\alpha,A)a^{n+1} + a^{2n+3+\beta} \leqslant C(\alpha,A)a^{n+1}.$$

□

The second step in proving our stability theorem is the following

3.2. Proposition. — *Let $\varphi,\psi \in \mathcal{E}^-(X,\omega)$ be such that $\omega_\varphi^n,\omega_\psi^n \in \mathcal{H}(\alpha,A)$. Then there exist constants $t\in\mathbf{R}$ and $C(\alpha,A)\geqslant 0$ such that*

$$C_X(\{|\varphi-\psi-t|>a\}) \leqslant C(\alpha,A)a,$$

here $a = [\int_X \|\omega_\varphi^n - \omega_\psi^n\|]^{\frac{1}{2n+3+\frac{n+1}{1+\alpha}}}$.

Proof. — Since $C_X(\{|\varphi-\psi-t|>a\}) \leqslant C_X(X) = 1$, it suffices to consider the case when a is small. Without loss of generality we can assume that $\sup_X\varphi = \sup_X\psi = 0$. By Remark 2.5 in [18] there exists $M(\alpha,A)>0$ such that $\|\varphi\|_{L^\infty(X)} < M(\alpha,A)$, $\|\psi\|_{L^\infty(X)} < M(\alpha,A)$. By Proposition 3.1 we can find $t>0$ such that

$$\int_{\{|\varphi-\psi-t|>a\}}(\omega_\varphi^n+\omega_\psi^n) \leqslant C_1(\alpha,A)a^{n+1}.$$

We consider the case $a<\min(1,\frac{1}{C_1(\alpha,A)})$. Since $\int_{\{|\varphi-\psi-t|>a\}}(\omega_\varphi^n+\omega_\psi^n)<1$ we get $\{|\varphi-\psi-t|>a\}\neq X$. This implies that $|t|\leqslant \sup_X|\varphi-\psi|+1\leqslant M(\alpha,A)+1$. Replacing ψ by $\psi+t$ we can assume that $t=0$ and $\|\psi\|_{L^\infty(X)} < 2M(\alpha,A)+1$. Using Lemma 2.3 in [18] for $s=\frac{a}{2}$, $t=\frac{a}{2(2M(\alpha,A)+1)}$ we get

$$C_X(\{\varphi-\psi<-a\}) \leqslant C_X\Big(\Big\{\varphi-\psi<-\frac{a}{2}-\frac{a}{2(2M(\alpha,A)+1)}\Big\}\Big)$$
$$\leqslant \frac{2^n(2M(\alpha,A)+1)^n}{a^n}\int_{\{\varphi-\psi<-a\}}\omega_\varphi^n$$
$$\leqslant 2^n(2M(\alpha,A)+1)^nC_1(\alpha,A)a.$$

Similarly we get

$$C_X(\{\psi-\varphi<-a\}) \leqslant 2^n(2M(\alpha,A)+1)^nC_1(\alpha,A)a.$$

Combination of these inequalities yields

$$C_X(\{|\varphi-\psi|>a\}) \leqslant C(\alpha,A)a.$$

Now we prove the promised generalization of Kołodziej stability theorem (Theorem 1.1 in [27]). □

3.3. THEOREM. — *Let $\varphi, \psi \in \mathcal{E}^-(X,\omega)$ be such that $\sup_X \varphi = \sup_X \psi = 0$ and $\omega_\varphi^n, \omega_\psi^n \in \mathcal{H}(\alpha, A)$. Then there exists $C(\alpha, A) > 0$ such that*

$$\sup_X |\varphi - \psi| \leqslant C(\alpha, A)\Big[\int_X \|\omega_\varphi^n - \omega_\psi^n\|\Big]^{\frac{\min(1,\frac{\alpha}{n})}{2n+3+\frac{n+1}{1+\alpha}}}.$$

Proof. — Set

$$a = \Big[\int_X \|\omega_\varphi^n - \omega_\psi^n\|\Big]^{\frac{1}{2n+3+\frac{n+1}{1+\alpha}}}.$$

By Proposition 3.2 there exists $C_1(\alpha, A) > 0$ and $t \in \mathbf{R}$ such that $|t| \leqslant M(\alpha, A) + 1$ and

$$C_X(\{|\varphi - \psi - t| > a\}) \leqslant C_1(\alpha, A)a.$$

Moreover, by Proposition 2.6 in [18] there exists $C_2(\alpha, A) > 0$ such that

$$\begin{aligned}\sup_X |\varphi - \psi - t| &\leqslant 2a + C_2(\alpha, A)[C_X(\{|\varphi - \psi - t| > a\})]^{\frac{\alpha}{n}}\\ &\leqslant 2a + C_2(\alpha, A)[C_1(\alpha, A)a]^{\frac{\alpha}{n}}\\ &\leqslant C_3(\alpha, A)a^{\min(1,\frac{\alpha}{n})}.\end{aligned}$$

Moreover, since $\sup_X \varphi = \sup_X \psi = 0$ we obtain $|t| \leqslant C_3(\alpha, A)a^{\min(1,\frac{\alpha}{n})}$. Combination of these inequalities yields

$$\begin{aligned}\sup_X |\varphi - \psi| &\leqslant \sup_X |\varphi - \psi - t| + |t| \leqslant 2C_3(\alpha, A)a^{\min(1,\frac{\alpha}{n})}\\ &= C(\alpha, A)\Big[\int_X \|\omega_\varphi^n - \omega_\psi^n\|\Big]^{\frac{\min(1,\frac{\alpha}{n})}{2n+3+\frac{n+1}{1+\alpha}}}.\end{aligned}$$

□

3.4. COROLLARY. — *Let μ be a non-negative Radon measure on X such that $\mu(B(z,r)) \leqslant Ar^{2n-2+\alpha}$ for all $B(z,r) \subset X$ (A, $\alpha > 0$ are constants). Given $p > 1, M > 0, \epsilon > 0$ and $f, g \in L^p(d\mu)$ with $\|f\|_{L^p(d\mu)}, \|g\|_{L^p(d\mu)} \leqslant M$ and $\int_X f d\mu = \int_X g d\mu = 1$. Assume that $\varphi, \psi \in \mathcal{E}^-(X,\omega)$ satisfy $\omega_\varphi^n = f d\mu$, $\omega_\psi^n = g d\mu$ and $\sup_X \varphi = \sup_X \psi = 0$. Then there exists $C(\alpha, A, M, \epsilon) > 0$ such that*

$$\sup_X |\varphi - \psi| \leqslant C(\alpha, A, M, \epsilon)\Big[\int_X |f - g| d\mu\Big]^{\frac{1}{2n+3+\epsilon}}.$$

Proof. — By Hölder inequality we have

$$\int_K f d\mu \leqslant \|f\|_{L^p(d\mu)}[\mu(K)]^{1-\frac{1}{p}} \leqslant M[\mu(K)]^{1-\frac{1}{p}},$$

$$\int_K g d\mu \leqslant \|g\|_{L^p(d\mu)}[\mu(K)]^{1-\frac{1}{p}} \leqslant M[\mu(K)]^{1-\frac{1}{p}},$$

for any Borel subset K of X. By Proposition 2.7 we get $fd\mu, gd\mu \in \mathcal{H}(\infty)$. Using Theorem 3.3 we can find $C(\alpha, A, M, \epsilon) > 0$ such that

$$\sup_X |\varphi - \psi| \leqslant C(\alpha, A, M, \epsilon)\Big[\int_X |f-g|d\mu\Big]^{\frac{1}{2n+3+\epsilon}}.$$

□

4. Local estimates in Potential theory

Let Ω be a bounded domain in $\mathbf{R}^n$ $(n \geqslant 2)$. By $\mathrm{SH}(\Omega)$ (resp. $\mathrm{SH}^-(\Omega)$) we denote the set of subharmonic (resp. negative subharmonic) functions on Ω. For each $u \in SH(\Omega)$ and $\delta > 0$ we denote

$$\tilde{u}_\delta(x) = \frac{1}{c_n\delta^n}\int_{B_\delta} u(x+y)dV_n(y),$$
$$u_\delta(x) = \sup_{y\in B_\delta} u(x+y),$$

for $x \in \Omega_\delta = \{x \in \Omega\colon d(x, \partial\Omega) > \delta\}$. Here $B_\delta = \{x \in \mathbf{R}^n\colon |x| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}} < \delta\}$ and c_n is the volume of the unit ball B_1. We state some results which will be used in our main theorems.

4.1. Theorem. — *Let μ be a non-negative Radon measure on Ω such that $\mu(B(z,r)) \leqslant Ar^{n-2+\alpha}$ for all $B(z,r) \subset D \subset\subset \Omega$ ($A, \alpha > 0$ are constants). Then for $K \subset\subset D$ and $\epsilon > 0$ there exists $C(\alpha, A, K, \epsilon)$ such that*

$$\int_K [\tilde{u}_\delta - u]d\mu \leqslant C(\alpha, A, K, \epsilon)\int_{\bar{D}} \Delta u\ \delta^{\frac{\alpha-\epsilon}{1+\alpha}},$$

for all $u \in SH(\Omega)$, where Δ is the Laplace operator.

Proof. — Since the change of radii of the balls does not affect the statement we can assume that $\Omega = B_4$, $D = B_3$, $K = B_1$ and u is smooth on B_4. By [22] we have

$$u(x) = \int_{B_2} G(x,z)\Delta u(z) + h(x),$$

where $G(x,y)$ is the fundamental solution of Laplace equation and h is harmonic in B_2. By Fubini theorem we have

$$\begin{aligned}&\int_{B_1} [\tilde{u}_\delta(x) - u(x)]d\mu(x)\\ &\quad= \int_{B_1} \frac{1}{c_n\delta^n}\int_{B_\delta} [u(x+y) - u(x)]dV_n(y)d\mu(x)\\ &\qquad\cdot \frac{1}{c_n\delta^n}\int_{B_1}\int_{B_\delta}\int_{B_2} [G(x+y,z) - G(x,z)]\Delta u(z)dV_n(y)d\mu(x)\end{aligned}$$

$$= \int_{B_2} \Delta u(z) \frac{1}{c_n \delta^n} \int_{B_\delta} dV_n(y) \int_{B_1} [G(x+y,z) - G(x,z)] d\mu(x)$$

Set

$$F(y,z) = \int_{B_1} [G(x+y,z) - G(x,z)] d\mu(x).$$

It is enough to prove that $F(y,z) \leqslant C(\alpha, A, s) \delta^{\frac{\alpha-\epsilon}{1+\alpha}}$ for all $y \in B_\delta, z \in B_2$. We consider two cases:

Case 1: $n = 2$. For $y \in B_\delta, z \in B_2$, $\delta < \frac{1}{2}$, we have

$$\begin{aligned} F(y,z) &= \int_{B_1} [\ln|x+y-z| - \ln|x-z|] d\mu(x) \\ &= \int_{B_1 \cap \{|x-z| \geqslant |y|^{\frac{1}{1+\alpha}}\}} \ln\left|1 + \frac{y}{x-z}\right| d\mu(x) + \int_{B_1 \cap \{|x-z| < |y|^{\frac{1}{1+\alpha}}\}} \ln\Big|1 \\ &\qquad\qquad + \frac{y}{x-z}\Big| d\mu(x) \\ &\leqslant \int_{B_1 \cap \{|x-z| \geqslant |y|^{\frac{1}{1+\alpha}}\}} \ln(1 + |y|^{\frac{\alpha}{1+\alpha}}) d\mu(x) \\ &\quad + \ln 4 \int_{B_1 \cap \{|x-z| < |y|^{\frac{1}{1+\alpha}}\}} d\mu + \int_{B_1 \cap \{|x-z| < |y|^{\frac{1}{1+\alpha}}\}} \ln \frac{1}{|x-z|} d\mu(x) \\ &\leqslant |y|^{\frac{\alpha}{1+\alpha}} \mu(B_1) + A|y|^{\frac{\alpha}{1+\alpha}} \ln 4 \\ &\qquad + |y|^{\frac{\alpha-\epsilon}{1+\alpha}} \int_{\{|x-z| < |y|^{\frac{1}{1+\alpha}}\}} \frac{1}{|x-z|^{\alpha-\epsilon}} \ln \frac{1}{|x-z|} d\mu(x) \\ &\leqslant A(1+\ln 4)|y|^{\frac{\alpha}{1+\alpha}} + |y|^{\frac{\alpha-\epsilon}{1+\alpha}} C_1(\alpha,\epsilon) \int_{\{|x-z|<1\}} \frac{d\mu(x)}{|x-z|^{\alpha - \frac{\epsilon}{2}}} \\ &\leqslant A(1+\ln 4)|y|^{\frac{\alpha}{1+\alpha}} \\ &\qquad + C_1(\alpha,\epsilon)|y|^{\frac{\alpha-\epsilon}{1+\alpha}} \sum_{j=0}^{\infty} \int_{\{2^{-j-1} \leqslant |x-z| < 2^{-j}\}} \frac{d\mu(x)}{|x-z|^{\alpha-\frac{\epsilon}{2}}} \\ &\leqslant A(1+\ln 4)|y|^{\frac{\alpha}{1+\alpha}} + C_1(\alpha,\epsilon)|y|^{\frac{\alpha-\epsilon}{1+\alpha}} A \sum_{j=0}^{\infty} 2^{(j+1)(\alpha-\frac{\epsilon}{2}) - j\alpha} \\ &\leqslant C(\alpha, A, \epsilon)|y|^{\frac{\alpha-\epsilon}{1+\alpha}} \leqslant C(\alpha, A, \epsilon) \delta^{\frac{\alpha-\epsilon}{1+\alpha}}. \end{aligned}$$

Case 2: $n \geqslant 3$. Similarly for $y \in B_\delta, z \in B_2$, $\delta < \frac{1}{2}$, we have

$$F(y,z) = \int_{B_1} \Big[-\frac{1}{|x+y-z|^{n-2}} + \frac{1}{|x-z|^{n-2}} \Big] d\mu(x)$$

$$= \int_{B_1\cap\{|x-z|\geqslant|y|^{\frac{1}{1+\alpha}}\}} \frac{|x+y-z|^{n-2}-|x-z|^{n-2}}{|x+y-z|^{n-2}|x-z|^{n-2}} d\mu(x) + \int_{\{|x-z|<|y|^{\frac{1}{1+\alpha}}\}} \frac{d\mu(x)}{|x-z|^{n-2}}$$

$$\leqslant C_2(\alpha)|y|^{\frac{\alpha}{1+\alpha}} \int_{B_1\cap\{|x-z|\geqslant|y|^{\frac{1}{1+\alpha}}\}} d\mu(x) + |y|^{\frac{\alpha-\epsilon}{1+\alpha}} \int_{\{|x-z|<|y|^{\frac{1}{1+\alpha}}\}} \frac{d\mu(x)}{|x-z|^{n-2+\alpha-\epsilon}}$$

$$\leqslant AC_2(\alpha)|y|^{\frac{\alpha}{1+\alpha}} + |y|^{\frac{\alpha-\epsilon}{1+\alpha}} \int_{\{|x-z|<1\}} \frac{d\mu(x)}{|x-z|^{n-2+\alpha-\epsilon}}$$

$$\leqslant C(\alpha, A, \epsilon)|y|^{\frac{\alpha-\epsilon}{1+\alpha}}$$

$$\leqslant C(\alpha, A, \epsilon)\delta^{\frac{\alpha-\epsilon}{1+\alpha}},$$

□

4.2. Theorem. — *Let μ be a non-negative Radon measure on Ω such that $\mu(B(z,r)) \leqslant Ar^{n-2+\alpha}$ for all $B(z,r) \subset D \subset\subset \Omega$ ($A, \alpha > 0$ are constants). Then for $K \subset\subset D$ and $\epsilon > 0$ there exists $C(\alpha, A, K, \epsilon)$ such that*

$$\int_K [u_\delta - u] d\mu \leqslant C(\alpha, A, K, \epsilon)\|u\|_{L^\infty(\Omega)}\ \delta^{\frac{\alpha-\epsilon}{2(1+\alpha)}},$$

for all $u \in SH \cap L^\infty(\Omega)$.

We need a well-known lemma:

4.3. Lemma. — *Let $u \in SH \cap L^\infty(\Omega)$. Then*

$$|\tilde{u}_\delta(x) - \tilde{u}_\delta(y)| \leqslant \frac{\|u\|_{L^\infty(\Omega)}|x-y|}{\delta},$$

for all $x, y \in \Omega_\delta$.

Proof. — Proof of Theorem 4.2 By Lemma 4.3 we have

$$u_\delta(x) = \sup_{y\in B_\delta} u(x+y) \leqslant \sup_{y\in B_\delta} \tilde{u}_{\delta^{\frac{1}{2}}}(x+y) \leqslant \tilde{u}_{\delta^{\frac{1}{2}}}(x) + \delta^{\frac{1}{2}}\|u\|_{L^\infty(\Omega)}.$$

By Theorem 4.1 we get

$$\int_K [u_\delta - u] d\mu \leqslant \int_K [\tilde{u}_{\delta^{\frac{1}{2}}} - u] d\mu + \|u\|_{L^\infty(\Omega)}\mu(K)\delta^{\frac{1}{2}}$$
$$\leqslant C(\alpha, A, K, \epsilon)\|u\|_{L^\infty(\Omega)}\ \delta^{\frac{\alpha-\epsilon}{2(1+\alpha)}}.$$

Next we state a well-known result is a direct consequence of the Jensen formula (see [1]) □

4.4. Proposition. — *Let $u \in SH(B_2)$ be such that $|u(x) - u(y)| \leqslant A|x-y|^\alpha$ for all $x, y \in B_2$. Then there exists $C(\alpha, A) > 0$ such that*

$$\int_{B(x,r)} \Delta u \leqslant C(\alpha, A) r^{n-2+\alpha},$$

for all $B(x,r) \subset B_1$.

5. Main results

Proof of Theorem A. — We use the same scheme as the proof of Theorem 2.1 in [27]. From Corollary 3.4 and from Theorem 4.2 we can replace ω^n by $d\mu$. This implies that u is Hölder continuous with the Hölder exponent dependent on α, A, p, X and $\|f\|_{L^p(d\mu)}$.

Proof of Corollary B. — It follows from Proposition 4.4 and Theorem A.

Proof of Corollary C. — Direct application of Theorem A.

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Pham Hoang HIEP
University of Education (Dai hoc Su Pham Ha Noi)
Department of Mathematics
CauGiay, Hanoi (Vietnam)
phhiep_vn@yahoo.com