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# CHARACTERIZATION OF JACOBIAN NEWTON POLYGONS OF PLANE BRANCHES AND NEW CRITERIA OF IRREDUCIBILITY

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ABSTRACT. — In this paper we characterize, in two different ways, the Newton polygons which are jacobian Newton polygons of a plane branch. These characterizations give in particular combinatorial criteria of irreducibility for complex series in two variables and necessary conditions which a complex curve has to satisfy in order to be the discriminant of a complex plane branch.

RÉSUMÉ. — Nous caractérisons de deux manières différentes les polygones de Newton jacobiens des branches planes. Ces caractérisations donnent, en particulier, des critères combinatoires d'irréductibilité des séries complexes en deux variables et des conditions nécessaires pour qu'une courbe dans le plan complexe soit le discriminant d'une branche plane.

*Dedicated to Professor Arkadiusz Płoski on his 60<sup>th</sup> birthday*

## 1. Introduction

Teissier in [23] introduced the notion of *jacobian Newton polygon* which is the Newton polygon *in the coordinates*  $(u, v)$  of the discriminant, *i.e.*, the image of the critical locus, of a map

$$(\ell, f): (\mathbf{C}^{n+1}, 0) \longrightarrow (\mathbf{C}^2, 0)$$

given by  $(u, v) = (l(x_0, \dots, x_n), f(x_0, \dots, x_n))$ , where  $l$  is a sufficiently general linear form and  $f(x_0, \dots, x_n)$  is a convergent power series and he proved that these jacobian Newton polygons are constant for the members

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of a family of equisingular germs of complex analytic isolated hypersurface singularities.

The inclinations of the compact edges of these jacobian Newton polygons are rational numbers called the *polar invariants* of the germ.

For germs of plane complex analytic curves, with the usual definitions of equisingularity, and in the case of a germ of plane irreducible curve (*i.e.*, a branch), Merle shows in [19] that the datum of the jacobian Newton polygon determines and is determined by the equisingularity class of the curve (or equivalently its embedded topological type). The formulae of Merle have been generalized to the case of *reduced* plane curve germs by Casas, Eggers, García Barroso, Gwoździewicz-Płoski, Maugendre and Wall among others (see [4], [5], [6], [10], [17], [18] and [26]) and they depend only on the equisingularity class of the curve. In contrast to the usual Newton polygon, the jacobian Newton polygon is independent of the choice of coordinates, and encodes a lot of information about the local geometry of a plane curve (see 4.3 of [23] for the irreducible case), for example the Łojasiewicz exponents for the inequalities  $|\text{grad } f(z)| \geq C_1|z|^\theta$  and  $|\text{grad } f(z)| \geq C_2|f(z)|^\theta$ , with  $z$  near  $0 \in \mathbb{C}^2$  and  $C_1, C_2$  constants (see [23], Corollaire 2, page 270). The behavior of the curvature of the Milnor fibers is also determined by the jacobian Newton polygon (see [7]).

In 1982, P. Maisonobe in [16], gave necessary and sufficient conditions on Puiseux exponents of an irreducible plane curve germ of  $\mathbb{C}^2$  so that it is the discriminant curve of an analytic map germ  $(l, f) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  where  $f$  has an irreducible critical locus. In this paper we characterize, in two different ways, the special convenient Newton polygons which are jacobian Newton polygons of a branch, answering a question of A. Lenarcik and A. Płoski. These characterizations give in particular combinatorial criteria of irreducibility for complex series in two variables and a necessary condition which a complex curve has to satisfy in order to be the discriminant of a complex plane branch.

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## 2. Plane branches, Puiseux expansions and semigroup

For us, a branch is an irreducible germ of a complex analytic curve. A plane branch is given by a convergent power series  $f(x, y) \in \mathbb{C}\{x, y\}$  which

is not a unit and is irreducible in that ring. The branch is the germ at 0 of the set of solutions of  $f(x, y) = 0$ .

By the theorem of Newton, after possibly a change of coordinates to achieve that  $x = 0$  is transversal to it at 0, the branch  $C$  can be parametrized near 0 as follows

$$\begin{aligned} x(t) &= t^d \\ y(t) &= a_e t^e + a_{e+1} t^{e+1} + \dots + a_{e+j} t^{e+j} + \dots \quad \text{with } e \geq d \end{aligned}$$

or equivalently by

$$y(x^{1/d}) = a_e x^{e/d} + a_{e+1} x^{(e+1)/d} + \dots + a_{e+j} x^{(e+j)/d} + \dots \quad \text{with } e \geq d$$

where  $d$  is the order of the series  $f(x, y)$ .

According to Puiseux the branch  $C$  admits  $d$  different Newton expansions  $\{y_i(x^{1/d})\}_{i=1}^d$ , with  $y_i(x^{1/d}) = y(\omega_i x^{1/d})$  where  $\omega_i$  are the  $d$ -roots of the unity in  $\mathbf{C}$ . Moreover we may write  $f$  as the product

$$f(x, y) = \text{unit} \prod_{i=1}^d (y - y_i(x^{1/d})).$$

The above expansions are called *Puiseux roots of the branch C*.

It is well-known that if  $d > 1$  then there exist  $g \in \mathbf{N} \setminus \{0\}$  and integer numbers  $\beta_1 < \beta_2 < \dots < \beta_g$  such that  $\text{g. c. d.}(d, \beta_1, \dots, \beta_g) = 1$  and the orders of the series  $y_i(x^{1/d}) - y_j(x^{1/d})$  with  $i \neq j$  are exactly the rational numbers  $\left\{ \frac{\beta_i}{d} \right\}_{i=1}^g$ . The sequence  $(\beta_0 = d, \beta_1, \dots, \beta_g)$  is called the *Puiseux characteristic* or *Puiseux exponents of the branch C* defined by the equation  $f(x, y) = 0$ .

Recall that an increasing sequence  $\delta_0 < \dots < \delta_h$  of positive integer numbers is the Puiseux characteristic of a branch if and only if  $\text{g. c. d.}(\delta_0, \dots, \delta_h) = 1$  and  $\text{g. c. d.}(\delta_0, \dots, \delta_k) < \text{g. c. d.}(\delta_0, \dots, \delta_{k-1})$  for all  $k \in \{1, \dots, h\}$ .

The *semigroup* associated to the branch  $f(x, y) = 0$  is by definition  $\Gamma := \{\text{ord } g(t^d, y(t)) : g \not\equiv 0 \pmod{f}\} \subseteq \mathbf{N}$ .

Zariski (see [28]) proved that the semigroup  $\Gamma$  admits a minimal set of generators  $\bar{\beta}_0 < \bar{\beta}_1 < \dots < \bar{\beta}_g$ , that is

$$\Gamma = \langle \bar{\beta}_0, \bar{\beta}_1, \dots, \bar{\beta}_g \rangle := \bar{\beta}_0 \mathbf{N} + \bar{\beta}_1 \mathbf{N} + \dots + \bar{\beta}_g \mathbf{N}.$$

This set of generators is uniquely determined by the semigroup  $\Gamma$ , and determines it.

On the other hand, according to Bresinsky (see Theorem 2, p. 383 of [3]), a semigroup  $\Gamma$  of positive integer numbers generated by  $\gamma_0 < \dots < \gamma_r$ , is the semigroup of a plane branch iff it verifies the following three properties:

- (1)  $\text{g. c. d.}(\gamma_0, \dots, \gamma_r) = 1$ ,

- (2)  $\text{g. c. d.}(\gamma_0, \dots, \gamma_i) < \text{g. c. d.}(\gamma_0, \dots, \gamma_{i-1})$  for all  $i \in \{1, \dots, r\}$ , and
- (3)  $\frac{\text{g.c.d.}(\gamma_0, \dots, \gamma_{i-1})}{\text{g.c.d.}(\gamma_0, \dots, \gamma_i)} \gamma_i < \gamma_{i+1}$  for  $1 \leq i \leq r - 1$ .

Zariski ([28]) proves that the Puiseux characteristic and the minimal set of the generators of the semigroup of a branch are equivalent. More precisely if the Puiseux characteristic of the branch  $C$  is  $(\beta_0, \dots, \beta_g)$  and the generators of its semigroup are  $\bar{\beta}_0, \dots, \bar{\beta}_g$  then  $\bar{\beta}_i = \beta_i$  for  $i \in \{0, 1\}$  and  $\bar{\beta}_{i+1} = n_i \bar{\beta}_i + \beta_{i+1} - \beta_i$  for  $i \in \{1, \dots, g-1\}$  where  $n_i = \frac{\text{g.c.d.}(\beta_0, \dots, \beta_{i-1})}{\text{g.c.d.}(\beta_0, \dots, \beta_i)}$ . Further, in the proof of Corollary 9.9 we will need an arithmetic property of the semigroup  $\Gamma$ :

PROPERTY 2.1.

$$(2.1) \quad \text{g. c. d.} (\bar{\beta}_0, (n_1 - 1)\bar{\beta}_1, \dots, (n_g - 1)\bar{\beta}_g) = 1$$

*Proof.* — We prove by induction on  $k$  that

$$\text{g. c. d.} (\bar{\beta}_0, (n_1 - 1)\bar{\beta}_1, \dots, (n_k - 1)\bar{\beta}_k) = \text{g. c. d.} (\bar{\beta}_0, \bar{\beta}_1, \dots, \bar{\beta}_k).$$

By inductive hypothesis

$$\begin{aligned} & \text{g. c. d.} (\bar{\beta}_0, (n_1 - 1)\bar{\beta}_1, \dots, (n_k - 1)\bar{\beta}_k, (n_{k+1} - 1)\bar{\beta}_{k+1}) \\ &= \text{g. c. d.} (\text{g. c. d.} (\bar{\beta}_0, (n_1 - 1)\bar{\beta}_1, \dots, (n_k - 1)\bar{\beta}_k), (n_{k+1} - 1)\bar{\beta}_{k+1}) \\ &= \text{g. c. d.} (\text{g. c. d.} (\bar{\beta}_0, \bar{\beta}_1, \dots, \bar{\beta}_k), (n_{k+1} - 1)\bar{\beta}_{k+1}) \\ &= \text{g. c. d.} (\text{g. c. d.} (\bar{\beta}_0, \bar{\beta}_1, \dots, \bar{\beta}_k), -\bar{\beta}_{k+1}) = \text{g. c. d.} (\bar{\beta}_0, \bar{\beta}_1, \dots, \bar{\beta}_{k+1}). \end{aligned}$$

In above computation we used elementary properties of the greatest common divisor and the relation

$$n_{k+1}\bar{\beta}_{k+1} = \frac{\bar{\beta}_{k+1}}{\text{g. c. d.}(\bar{\beta}_0, \bar{\beta}_1, \dots, \bar{\beta}_{k+1})} \text{g. c. d.}(\bar{\beta}_0, \bar{\beta}_1, \dots, \bar{\beta}_k).$$

□

Both Puiseux characteristic and semigroup form complete sets of topological invariants of a branch. It means that they characterize the equisingularity class of the branch (see 4.3 of [23]).

### 3. Newton polygons

Let  $\mathbf{R}_+ = \{x \in \mathbf{R} : x \geq 0\}$ . Any two subsets  $A, B$  of the first quadrant  $\mathbf{R}_+^2$  can be added coordinate-wise, to give the Minkowski sum  $A + B = \{a + b : a \in A \text{ and } b \in B\}$  of  $A$  and  $B$ . The subset  $\mathcal{N}$  of  $\mathbf{R}_+^2$  is a *Newton polygon* if  $\mathcal{N}$  is the convex hull of  $S + \mathbf{R}_+^2$  for some  $S \subset \mathbf{N}^2$ , and in this case we denote it by  $\mathcal{N}(S)$ . The boundary of a Newton polygon is a broken

line with infinite horizontal and vertical sides, possibly different from the coordinate axis and a finite number of compact edges.

According to Teissier (see [22] and [24]) we put  $\{\frac{a}{b}\} = \mathcal{N}(\{(a, 0), (0, b)\})$ ,  $\{\frac{a}{\infty}\} = \mathcal{N}(\{(a, 0)\})$  and  $\{\frac{\infty}{b}\} = \mathcal{N}(\{(0, b)\})$  for any  $a, b > 0$  and call such polygons *elementary Newton polygons*. If  $\{\frac{a}{b}\}$  is an elementary Newton polygon, with  $a \neq \infty, b \neq \infty$ , then its *inclination* is by definition the rational number  $\frac{a}{b}$ .

The Newton polygons form a semigroup (see [22], Section 3.6, page 616) with the Minkowski sum, and the elementary Newton polygons generate it. Every Newton polygon  $\mathcal{N}$  can be written, in a unique way, called *canonical form*, as a finite sum

$$\mathcal{N} = \sum_{i=1}^r \left\{ \frac{a_i}{b_i} \right\}$$

where the inclinations of the terms form an strictly increasing sequence.

The *height* of the Newton polygon  $\mathcal{N}$  is by definition the length of the projection of compact edges of  $\mathcal{N}$  on the vertical coordinate axis which we will denote  $\text{ht}(\mathcal{N})$ .

A Newton polygon is *convenient* if intersects both coordinate axis (see [11]) and it is *special* if the inclinations of its compact faces are greater than 1 and intersects the vertical coordinate axis.

Note that any convenient Newton polygon  $\mathcal{N} = \sum_{i=1}^r \left\{ \frac{a_i}{b_i} \right\}$  is determined by the inclinations  $\left\{ \frac{a_i}{b_i} \right\}_{i=1}^r$  of its compact faces and their respective heights  $\{b_i\}_{i=1}^r$ .

Fix a complex nonsingular surface *i.e.*, a complex holomorphic variety of dimension 2. Let  $(x, y)$  be a chart centered at  $O$ . In all this paper we consider *reduced* plane curve germs, that is curves with local equations  $f(x, y) = \sum a_{ij}x^i y^j \in \mathbf{C}\{x, y\}$  without multiple factors. We put  $\mathcal{N}_{x,y}(C) = \mathcal{N}(S)$  where  $S = \{(i, j) \in \mathbf{N}^2 : a_{ij} \neq 0\}$  which is called *Newton polygon of the curve  $C \equiv f(x, y) = 0$  in the coordinates  $(x, y)$* . Clearly  $\mathcal{N}_{x,y}(C)$  depends on  $(x, y)$ .

We can extend the definitions of this section to any subset  $S \subseteq \mathbf{Q}^2$  such that there exists a positive integer  $m$  with  $m.S \subseteq \mathbf{N}^2$ . The Newton polygons obtained by this generalization will be called *rational Newton polygons*. The usual Newton polygon with integral vertices will be called *integral Newton polygons*.

### 4. Main result

Let  $f = f(x, y) \in \mathbf{C}\{x, y\}$  be a power series of order  $d$  without multiple factors such that the vertical axis is transverse to the curve  $f(x, y) = 0$ . Using Puiseux factorization we may write  $f$  and its partial derivative  $f'_y$  as the products

$$(4.1) \quad \begin{aligned} f(x, y) &= \text{unit} \prod_{i=1}^d (y - \alpha_i(x)), \\ f'_y(x, y) &= \text{unit} \prod_{j=1}^{d-1} (y - \gamma_j(x)), \end{aligned}$$

where  $\alpha_i(x), \gamma_j(x)$  are fractional power series, that is, elements of the ring  $\mathbf{C}\{x\}^* = \bigcup_{n \in \mathbf{N}} \mathbf{C}\{x^{1/n}\}$ .

If  $f'_y = g_1 \cdots g_s$  is the factorization of  $f'_y$  in irreducible factors, Teissier proves ([23]) that the jacobian Newton polygon  $\mathcal{N}_J(f)$  of  $f(x, y) = 0$  equals

$$\sum_{j=1}^s \left\{ \frac{(f, g_j)_0}{\text{ord } g_j} \right\},$$

where  $(f, g_j)_0$  denotes the intersection number of  $f$  and  $g_j$ , where different elementary Newton polygons may have the same inclinations.

We can rewrite the above sum in terms of Puiseux roots of  $f'_y$ . By Zeuthen's rule (cf. e.g. [20] Proposition 2.1) for every  $j \in \{1 \dots s\}$  we have  $\left\{ \frac{(f, g_j)_0}{\text{ord } g_j} \right\} = \sum_i \left\{ \frac{\text{ord } f(x, \gamma_i(x))}{1} \right\}$  where the sum runs over  $\gamma_i(x)$  which are Puiseux roots of  $g_j$ .

So we can write

$$\mathcal{N}_J(f) = \sum_{j=1}^{d-1} \left\{ \frac{\text{ord } f(x, \gamma_j(x))}{1} \right\}$$

as a rational Newton polygon.

The polar invariants of  $f$ , that is, the inclinations of  $\mathcal{N}_J(f)$  are

$$\text{ord } f(x, \gamma_j(x)).$$

If  $q_j$  is the polar invariant of  $f$  associated to the edge  $\Gamma$  of  $\mathcal{N}_J(f)$ , we call the height of  $\Gamma$  the *multiplicity* of  $q_j$  and we denote it by  $m_j$ , consequently  $\mathcal{N}_J(f)$  is determined by the unordered sequence  $Q(f) = \langle q_0 : m_0, q_1 : m_1, \dots, q_r : m_r \rangle$ . So the canonical form of  $\mathcal{N}_J(f)$  is

$$\mathcal{N}_J(f) = \sum_{j=0}^r \left\{ \frac{m_j q_j}{m_j} \right\}.$$

It is well-known (see [23]) that  $\mathcal{N}_J(f)$  is determined by the equisingularity class of  $\{f = 0\}$ . If  $\{f = 0\}$  is a branch having semigroup  $\langle \bar{\beta}_0, \bar{\beta}_1, \dots, \bar{\beta}_g \rangle$  then by Merle (see [19]) the canonical form of  $\mathcal{N}_J(f)$  is

$$(4.2) \quad \mathcal{N}_J(f) = \sum_{k=1}^g \left\{ \frac{(n_k - 1)\bar{\beta}_k}{(n_k - 1)n_1 \dots n_{k-1}} \right\}.$$

Using (4.2) we can compute  $\langle \bar{\beta}_0, \bar{\beta}_1, \dots, \bar{\beta}_g \rangle$  from  $\mathcal{N}_J(f)$ . Hence for  $f$  irreducible  $\mathcal{N}_J(f)$  determines the equisingularity class of  $\{f = 0\}$ .

Our aim is to show that jacobian Newton polygons of irreducible series are distinguished among all jacobian Newton polygons.

**THEOREM 4.1.** — *Let  $f, g \in \mathbf{C}\{x, y\}$  be such that  $\mathcal{N}_J(f) = \mathcal{N}_J(g)$  and assume that  $f$  is irreducible. Then  $g$  is also irreducible.*

The aim of the next three sections is to prove Theorem 4.1.

### 5. The Kuo-Lu Lemma

Let  $\phi, \psi$  be fractional power series of variable  $x$ . We will denote the order of difference  $\phi(x) - \psi(x)$  by  $O(\phi, \psi)$  and call it the *contact order*.

For every  $\phi_1, \phi_2, \phi_3 \in \mathbf{C}\{x\}^*$  we have (see for example page 69 of [27]):

$$(5.1) \quad \text{if } O(\phi_1, \phi_2) \leq O(\phi_2, \phi_3) \leq O(\phi_1, \phi_3) \text{ then } O(\phi_1, \phi_2) = O(\phi_2, \phi_3).$$

Let  $\phi \in \mathbf{C}\{x\}^*$  and  $r \in \mathbf{R}^+ \cup \{+\infty\}$ . We will call the set

$$B(\phi, r) = \{ \psi \in \mathbf{C}\{x\}^* : O(\phi, \psi) \geq r \}$$

a *pseudo-ball with center at  $\phi$  and radius  $r$* .

It follows from (5.1) that pseudo-balls have the following metric properties:

- (i) every element of a pseudo-ball is its center,
- (ii) if  $B_1 = B(\phi_1, r_1), B_2 = B(\phi_2, r_2)$  are two disjoint pseudo-balls and  $\psi_1 \in B_1, \psi_2 \in B_2$  then  $O(\phi_1, \phi_2) = O(\psi_1, \psi_2)$  i.e., contact order between elements of two disjoint pseudo-balls does not depend on the choice of these elements,
- (iii) for all pseudo-balls  $B_1 = B(\phi_1, r_1)$  and  $B_2 = B(\phi_2, r_2)$  one of three possibilities holds:

$$\begin{aligned} B_1 \cap B_2 &= \emptyset && \text{if } O(\phi_1, \phi_2) < \min(r_1, r_2), \\ B_1 \subset B_2 &&& \text{if } O(\phi_1, \phi_2) \geq \min(r_1, r_2) \text{ and } r_1 \geq r_2, \\ B_2 \subset B_1 &&& \text{if } O(\phi_1, \phi_2) \geq \min(r_1, r_2) \text{ and } r_2 \geq r_1. \end{aligned}$$



Take a power series  $f(x, y)$  without multiple factors of the form (4.1), let  $\text{Zer } f = \{\alpha_1, \dots, \alpha_d\}$  be the set of its Puiseux roots and put

$$\mathcal{B} := \{ B(\alpha, O(\alpha, \alpha')) : \alpha, \alpha' \in \text{Zer } f \},$$

which is a partially ordered set with the inclusion operation.

We will denote by  $h(B)$  the radius of  $B \in \mathcal{B}$  and call this number the *height* of  $B$ .

Inclusion relation gives  $\mathcal{B}$  a structure of a tree called *the Kuo-Lu tree-model*  $T(f)$  (see [13]). The root of  $T(f)$  is the pseudo-ball of the minimal height  $\min O(\alpha_i, \alpha_j)$  which contains all Puiseux roots of  $f$ . Leaves of  $T(f)$  are pseudo-balls  $B(\alpha_i, +\infty) = \{\alpha_i\}$  of infinite heights which can be identified with Puiseux roots of  $f$ . A path from the root to the leaf  $\{\alpha_i\}$  connects successive  $B \in \mathcal{B}$  of increasing heights for which  $\alpha_i \in B$ . Finally a pseudo-ball  $B_1$  is a *child* of the pseudo-ball  $B_2$  if  $B_1 \subsetneq B_2$  and there is not a pseudo-ball  $B \in \mathcal{B}$  such that  $B_1 \subsetneq B \subsetneq B_2$ .

Let  $B$  be an element of  $\mathcal{B}$  and let  $\gamma$  be any fractional power series. We will say that  $\gamma$  *grows from*  $B$  if and only if  $\gamma \in B$  and  $O(\gamma, \alpha) = h(B)$  for all  $\alpha \in \text{Zer } f \cap B$ . In this case contact orders between  $\gamma$  and Puiseux roots of  $f$  are given by  $O(\gamma, \alpha) = O(B, \alpha)$  where

$$O(B, \alpha) := \begin{cases} h(B) & \text{if } \alpha \in B \\ O(\alpha', \alpha) & \text{otherwise} \end{cases}$$

and  $\alpha'$  is any element of  $B$ . Put  $q(B) := \sum_{\alpha \in \text{Zer } f} O(B, \alpha)$ . Then in the case when  $\gamma$  grows from  $B$  we have

$$(5.2) \quad \text{ord } f(x, \gamma(x)) = \sum_{\alpha \in \text{Zer } f} O(\gamma, \alpha) = \sum_{\alpha \in \text{Zer } f} O(B, \alpha) = q(B).$$

The next lemma is a reformulation of Lemma 3.3 in [13] (see also Lemma 2.2 in [8]).

LEMMA 5.1 (The Kuo-Lu Lemma).

- (i) For every  $\gamma \in \text{Zer } f'_y$  there exists  $B \in T(f)$  of finite height such that  $\gamma$  grows from  $B$ .
- (ii) For given  $B \in T(f)$  of finite height the number of Puiseux roots of  $f'_y$  which grow from  $B$  counted with multiplicities is equal to  $t(B) - 1$  where  $t(B)$  is the number of children of  $B$  in  $T(f)$ .

*Proof.* — Recall that  $\text{Zer } f = \{\alpha_1, \dots, \alpha_d\}$  and let  $\text{Zer } f'_y := \{\gamma_1, \dots, \gamma_{d-1}\}$ . According to Lemma 3.3 in [13], for given  $\alpha \in \text{Zer } f$  and a positive rational number  $r$ ,

$$(5.3) \quad \#\{j : O(\alpha, \gamma_j) = r\} = \#\{k : O(\alpha, \alpha_k) = r\}.$$

For the proof of the first statement fix  $\gamma \in \text{Zer } f'_y$  and let  $\alpha \in \text{Zer } f$  be such that  $O(\alpha, \gamma) = \max_k O(\alpha_k, \gamma)$ . Then from (5.3) there exists  $\alpha' \in \text{Zer } f$  such that  $O(\alpha, \gamma) = O(\alpha, \alpha')$  and consequently  $\gamma$  grows from the pseudo-ball  $B(\alpha, O(\alpha, \alpha'))$ .

For the proof of the second statement suppose that  $B_1, \dots, B_k$  are the children of  $B_0$ . By (5.3) for every  $B \in T(f)$  we have  $\sharp(B \cap \text{Zer } f) = \sharp\{j : \gamma_j \in B\} + 1$ . Hence

$$\begin{aligned} \sharp\{j : \gamma_j \in \text{Zer } f'_y \text{ grows from } B_0\} &= \sharp\{j : \gamma_j \in B_0\} - \sum_{i=1}^k \sharp\{j : \gamma_j \in B_i\} \\ &= (\sharp(B_0 \cap \text{Zer } f) - 1) - \sum_{i=1}^k (\sharp(B_i \cap \text{Zer } f) - 1) \\ &= k - 1, \end{aligned}$$

since  $B_0 \cap \text{Zer } f = \bigsqcup_{i=1}^k (B_i \cap \text{Zer } f)$ . □

The Kuo-Lu lemma together with (5.2) gives a complete information on polar invariants and their multiplicities in terms of  $T(f)$ . More precisely if  $\tilde{\mathcal{B}} := \{B \in T(f) : h(B) < +\infty\}$  then the jacobian Newton polygon of  $f(x, y) = 0$  equals

$$\sum_{B \in \tilde{\mathcal{B}}} \left\{ \frac{(t(B) - 1)q(B)}{t(B) - 1} \right\}.$$

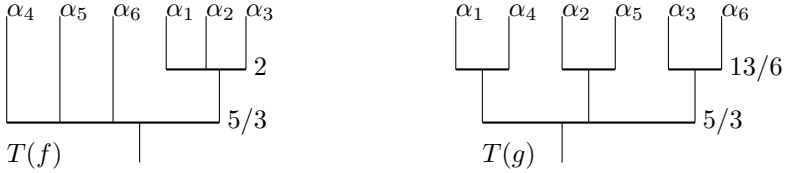
*Example 5.2.* — Let  $f(x, y) = (y^3 - x^5) \prod_{i=1}^3 (y - a_i x^2)$ ,  $a_i \neq a_j$  for  $i \neq j$ . The Puiseux roots of  $f$  are:  $\alpha_i = a_i x^2$  for  $i = 1, 2, 3$  and  $\alpha_i = \epsilon^i x^{5/3}$  for  $i = 4, 5, 6$  and  $\epsilon = e^{2\pi i/3}$ .

Let us draw the Kuo-Lu tree of  $f$ . Following [13] we draw pseudo-balls of finite height as horizontal bars and we do not draw pseudo-balls of infinite height. Three Puiseux roots of  $f'_y$  grow from a bar  $B_1$  of height  $5/3$  and two Puiseux roots of a partial derivative grow from a bar  $B_2$  of height 2. Since  $q(B_1) = 6 \cdot (5/3) = 10$  and  $q(B_2) = 3 \cdot 2 + 3 \cdot (5/3) = 11$  we have  $Q(f) = \langle 10 : 3, 11 : 2 \rangle$ .

Now take an irreducible series  $g(x, y) = (y^3 - x^5)^2 - 9x^{11}$ . Its Puiseux roots are  $\alpha_i(x) = \epsilon^{10i} x^{5/3} + \epsilon^{13i} x^{13/6} + \dots$  for  $i = 1, \dots, 6$  where  $\epsilon$  is 6-th primitive root of unity and dots mean terms of higher degrees.

The tree  $T(g)$  has one bar  $B_1$  of height  $5/3$  and three bars  $B_i$  of height  $13/6$ . Two Puiseux roots of  $f'_y$  grow from  $B_1$  and  $q(B_1) = 6 \cdot (5/3) = 10$ . From every  $B_i$  ( $i = 2, 3, 4$ ) grows exactly one Puiseux root of  $f'_y$  and  $q(B_i) =$

$2 \cdot (13/6) + 4 \cdot (5/3) = 11$  for  $i = 2, 3, 4$ . Hence  $Q(g) = \langle 10 : 2, 11 : 3 \rangle$ .



**COROLLARY 5.3.** — *Let  $f = f(x, y) \in \mathbf{C}\{x, y\}$  be a power series such that the vertical axis is transverse to  $\{f = 0\}$  and let  $w = w(x, y) \in \mathbf{C}\{x\}[y]$  be the Weierstrass polynomial of  $f$ . Then  $\mathcal{N}_J(f) = \mathcal{N}_J(w)$ .*

*Proof.* — Write  $f(x, y)$  in the form (4.1). Then  $w(x, y) = \prod_{i=1}^d (y - \alpha_i(x))$  and clearly  $\text{Zer } f = \text{Zer } w$ . Since polar invariants and their multiplicities depend only on Kuo-Lu tree we have  $\mathcal{N}_J(f) = \mathcal{N}_J(w)$ . □

### 6. Similarity lemma

Let  $f$  be an irreducible power series in two variables of order greater than 1. Then after an analytic change of coordinates we may assume that

$$(6.1) \quad f(x, y) = (y^n - x^m)^a + \sum_{ni+mj > anm} f_{i,j} x^i y^j$$

where  $1 < n < m$  are coprime integers. By Merle’s formula (see [19],[8]) the smallest polar invariant of  $f$  is  $am$  with multiplicity  $n - 1$ . We will show that the converse is also true.

**LEMMA 6.1.** — *Let  $f \in \mathbf{C}\{x, y\}$  be a power series such that  $\text{ord } f = an$  and the smallest polar invariant of  $f$  is  $am$  with multiplicity  $n - 1$ . If  $1 < n < m$  are coprime integers, then after an analytic change of coordinates*

$$f(x, y) = (y^n - x^m)^a + \sum_{ni+mj > anm} f_{i,j} x^i y^j.$$

*Proof.* — Choose a system of coordinates such that the  $y$ -axis is transverse to  $\{f = 0\}$  and write a Puiseux factorization of  $f$

$$f(x, y) = \text{unit} \prod_{i=1}^{an} (y - \alpha_i(x)).$$

Let  $B$  be the root of  $T(f)$ . Since contact orders  $O(\alpha_i, \alpha_j)$  are greater than or equal to  $h(B)$  all Puiseux roots of  $f$  have a form  $\alpha_i(x) = \lambda(x) + c_i x^{h(B)} + \dots$  where  $\lambda(x) = a_1 x^{\delta_1} + a_2 x^{\delta_2} + \dots + a_k x^{\delta_k}$  ( $1 \leq \delta_1 < \delta_2 < \dots < \delta_k$ ) is a finite sum of terms of degrees smaller than  $h(B)$  and at least one  $c_i$  is non

zero. We will show that all exponents in  $\lambda(x)$  are integers. Suppose to the contrary that  $\delta_j$  are integers for  $1 \leq j < s$  and  $\delta_s = p/q$ , for  $p, q$  coprime integers. Then by [25], p. 107 Theorem 4.1, if  $\omega$  is a  $q$ -th primitive root of unity, a series  $\bar{\alpha}$  of the form

$$\bar{\alpha}(x) = a_1x^{\delta_1} + \dots + a_{s-1}x^{\delta_{s-1}} + \omega^p a_s x^{p/q} + \dots,$$

which is a conjugate of

$$\alpha(x) = a_1x^{\delta_1} + \dots + a_{s-1}x^{\delta_{s-1}} + a_s x^{p/q} + \dots$$

is a Puiseux root of  $f$  and we get a contradiction because  $O(\alpha, \bar{\alpha}) = p/q < h(B)$ .

We checked that  $\lambda(x)$  is a polynomial. After an analytic substitution  $y := y - \lambda(x)$  we may assume that

$$\alpha_i(x) = c_i x^{h(B)} + \dots \quad \text{for } i = 1, \dots, an.$$

Computing the smallest polar invariant of  $f$  by formula (5.2) we get

$$am = q(B) = \sum_{\alpha \in \text{Zer } f} O(B, \alpha) = \sum_{\alpha \in \text{Zer } f} h(B) = an h(B).$$

Hence  $h(B) = m/n$ .

Fix a weight  $w$  such that  $w(x) = n$ ,  $w(y) = m$  and let in  $f$  denotes the initial quasi-homogeneous part of  $f$  with respect to  $w$ . Since all factors  $y - c_i x^{m/n}$  are quasi homogeneous we have

$$(6.2) \quad \text{in } f(x, y) = \text{const} \prod_{i=1}^{an} (y - c_i x^{m/n}).$$

On the other hand because in  $f(x, y)$  is a quasi-homogeneous polynomial it can be written as

$$\text{const} \prod_{j=1}^s (y^n - C_j x^m)^{k_j}$$

where  $C_i \neq C_j$  for  $i \neq j$  and  $k_1 + \dots + k_s = a$ . We want to show that  $s = 1$ . Suppose to the contrary that  $s \geq 2$ . Assume for simplicity that  $C_1 \neq 0$ . Then  $(y^n - C_1 x^m)(y^n - C_2 x^m)$  equals

$$\prod_{j=1}^n (y - \omega^j \sqrt[n]{C_1} x^{m/n})(y - \omega^j \sqrt[n]{C_2} x^{m/n}) \quad \text{if } C_2 \neq 0$$

$$y^n \prod_{j=1}^n (y - \omega^j \sqrt[n]{C_1} x^{m/n}) \quad \text{if } C_2 = 0$$

where  $\omega$  is an  $n$ -th primitive root of unity. Hence (6.2) has at least  $n + 1$  different factors. Because different  $c_i$  yield different edges of  $T(f)$  growing

from the root of  $T(f)$ , the multiplicity of the smallest polar invariant is at least  $n$  and we arrive at contradiction.

We checked that in  $f(x, y) = \text{const}(y^n - C_1 x^m)^a$  and finally a substitution  $x := C_1^{1/m} x$  gives  $C_1 = 1$ . □

### 7. Reduction technique

Take a distinguished Weierstrass polynomial  $f(x, y) \in \mathbf{C}\{x\}[y]$  of the form (6.1) not necessarily irreducible. Let  $F(x, y) = f(x^n, y)$  and let  $\omega$  be an  $n$ -th primitive root of unity. By equality

$$(y^n - x^{nm})^a = (y - x^m)^a (y - \omega x^m)^a \dots (y - \omega^{n-1} x^m)^a$$

and Hensel's lemma we get a factorization  $F = f_0 f_1 \dots f_{n-1}$  where  $f_i$  are Weierstrass polynomials with quasi-homogeneous initial forms  $(y - \omega^i x^m)^a$  for  $i = 0, \dots, n - 1$ . Denote by  $\tilde{f}$  the factor  $f_0$ . This transformation has nice properties.

LEMMA 7.1.

- (i)  $f$  is irreducible if and only if  $\tilde{f}$  is irreducible.
- (ii) Let  $Q(f) = \langle q_0 : m_0, q_1 : m_1, \dots, q_r : m_r \rangle$  be the system of polar invariants of  $f$  with  $q_{i-1} < q_i$  for  $i = 1, \dots, r$ . Then  $Q(\tilde{f}) = \langle q'_1 : m'_1, \dots, q'_r : m'_r \rangle$  where

$$m'_i = m_i / (m_0 + 1)$$

$$q'_i = q_i (m_0 + 1) - q_0 m_0$$

for  $i = 1, \dots, r$ .

- (iii) If  $f$  is irreducible with Puiseux characteristic  $(\beta_0, \dots, \beta_r)$  then the Puiseux characteristic of  $\tilde{f}$  is  $(\frac{\beta_0}{n}, \beta_2, \dots, \beta_r)$ .

*Proof.* — Assume that  $f$  is irreducible. Then all Puiseux roots  $\alpha_i$  of  $f$  are fractional power series and the exponents of those series have a least common denominator  $na$ . Since Puiseux roots of  $F(x, y) = f(x^n, y)$  are  $\alpha_i(x^n)$  the exponents of these series have a least common denominator  $a$ . Thus the order of irreducible factors of  $F$  is exactly  $a$  and we see that  $F = f_0 f_1 \dots f_{n-1}$  is a decomposition into irreducible factors.

If  $f = h_1 h_2$  is a product then clearly  $\tilde{f} = \tilde{h}_1 \tilde{h}_2$ . Thus we ended proof of (i).

To prove (ii) we will compute the system of polar invariants  $Q(F)$  in two ways. Write Puiseux factorizations of  $f$  and  $f'_y$ :

$$f(x, y) = \prod_{i=1}^{an} (y - \alpha_i(x)),$$

$$f'_y(x, y) = an \prod_{j=1}^{an-1} (y - \gamma_j(x)).$$

The Puiseux roots of  $F$  are  $\bar{\alpha}_i(x) = \alpha_i(x^n)$  for  $i = 1, \dots, an$  and by equality  $\frac{\partial F}{\partial y}(x, y) = \frac{\partial f}{\partial y}(x^n, y)$ , the Puiseux roots of  $F'_y$  are  $\bar{\gamma}_i(x) = \gamma_i(x^n)$  for  $i = 1, \dots, an - 1$ . We have

$$(7.1) \quad \text{ord } F(x, \bar{\gamma}_i(x)) = \text{ord } f(x^n, \gamma_i(x^n)) = n \text{ ord } f(x, \gamma_i(x))$$

for  $i = 1, \dots, an - 1$ . Hence

$$(7.2) \quad Q(F) = \langle nq_0 : m_0, \dots, nq_r : m_r \rangle.$$

In the second part of computation we will show that the Kuo-Lu tree-model of  $F = f_0 \cdots f_{n-1}$  has a special structure. It separates above the root to  $n$  sub-trees and each of them is isomorphic to  $T(\tilde{f})$ . This together with Kuo-Lu Lemma will allow to express the system of polar invariants of  $F$  by the system of polar invariants of  $\tilde{f}$ .

By [25] if  $\bar{\alpha} \in \text{Zer } F$  is a Puiseux root of  $f_i$  then the initial term of  $\bar{\alpha}$  is  $\omega^i x^m$ . Hence for every  $\bar{\alpha}, \bar{\alpha}' \in \text{Zer } F$

$$O(\bar{\alpha}, \bar{\alpha}') = m \quad \text{if } \bar{\alpha} \in \text{Zer } f_i, \bar{\alpha}' \in \text{Zer } f_j, i \neq j$$

$$O(\bar{\alpha}, \bar{\alpha}') > m \quad \text{if } \bar{\alpha}, \bar{\alpha}' \in \text{Zer } f_i.$$

It follows that the Kuo-Lu tree-model  $T(F)$  has a root  $B_0$  of height  $m$  and above the root it separates to  $n$  sub-tree-models  $T_i$  ( $i = 0, \dots, n - 1$ ); the leaves of  $i$ -th sub-tree-model are roots of  $f_i$ . Moreover if  $\phi$  grows from  $B \in T_i$  (in short  $\phi$  grows from  $T_i$ ) then  $\phi = \omega^i x^m + \dots$

We shall establish a one-to-one correspondence between the Puiseux roots of  $F'_y$  which grow from  $T_0$  and those which grow from  $T_i$  ( $i = 1, \dots, n - 1$ ). To do this we need some properties of action of complex roots of unity on fractional power series.

Let  $D$  be an integer such that every  $\phi \in \text{Zer } f \cup \text{Zer } f'_y$  can be written as fractional power series with common denominator  $D$ .

$$\phi(x) = a_1 x^{n_1/D} + a_2 x^{n_2/D} + \dots, \quad 1 \leq D < n_1 < n_2 < \dots$$

Clearly  $D$  is a multiple of  $n$  because  $m/n$  is the smallest exponent of  $\phi \in \text{Zer } f$ . For every  $\theta \in \mathbf{C}$  such that  $\theta^D = 1$  we define a conjugate  $\theta(\phi)$  by

formula

$$\theta(\phi)(x) = a_1\theta^{n_1}x^{n_1/D} + a_2\theta^{n_2}x^{n_2/D} + \dots$$

It is shown in [14] (page 293) that for every  $D$ -th root of unity  $\theta$  the action of  $\theta$  permutes the sets  $\text{Zer } f$  and  $\text{Zer } f'_y$  and preserves a contact order, that is  $O(\phi_1, \phi_2) = O(\theta(\phi_1), \theta(\phi_2))$ . □

CLAIM 7.2. —  $\text{ord } f(x, \gamma(x)) = \text{ord } f(x, \theta(\gamma)(x))$  for every  $\gamma \in \text{Zer } f'_y$ ,  $\theta^D = 1$ .

The proof is straightforward. From properties of conjugation mentioned above we get

$$\begin{aligned} \text{ord } f(x, \gamma(x)) &= \sum_i O(\gamma, \alpha_i) = \sum_i O(\theta(\gamma), \theta(\alpha_i)) = \sum_i O(\theta(\gamma), \alpha_i) \\ &= \text{ord } f(x, \theta(\gamma)(x)). \end{aligned}$$

Recall that  $\omega$  is a primitive  $n$ -th root of unity.

CLAIM 7.3. — *There exist  $\theta$  ( $\theta^D = 1$ ) such that for every  $\gamma \in \text{Zer } f'_y$  of the form  $\gamma(x) = x^{m/n} + \dots$  we have*

$$\theta^i(\gamma)(x) = \omega^i x^{m/n} + \dots \quad \text{for } i = 1, \dots, n - 1.$$

*Proof of Claim 7.3.* — Since  $m$  and  $n$  are coprime there is  $pm + qn = 1$  for some integers  $p, q$ . Let  $\theta$  be the complex number such that  $\theta^{D/n} = \omega^p$ . Clearly  $\theta^D = 1$ . We have  $(\theta^i)^{m(D/n)} = \omega^{imp} = \omega^{imp}\omega^{inq} = \omega^{imp+inq} = \omega^i$ . Writing  $\gamma$  as a power series with common denominator  $D$

$$\gamma(x) = x^{\frac{m(D/n)}{D}} + \dots$$

we see that  $\theta^i(\gamma)(x) = (\theta^i)^{m(D/n)}x^{\frac{m(D/n)}{D}} + \dots = \omega^i x^{m/n} + \dots$

Observe that for every  $\gamma \in \text{Zer } f'_y$  a series  $\gamma(x^n)$  grows from a sub-tree-model  $T_i$  if and only if  $\gamma$  has a form  $\gamma(x) = \omega^i x^{m/n} + \dots$  for  $i = 0, \dots, n - 1$ . Thus by Claim 7.3 the action of  $\theta^i$  yields a one-to-one correspondence between Puiseux roots of  $F'_y$  which grow from  $T_0$  and Puiseux roots of  $F'_y$  which grow from  $T_i$ . By (7.1) and Claim 7.2 the corresponding roots give the same polar invariants.

Take  $\bar{\gamma} \in \text{Zer } F'_y$ . If  $\bar{\gamma}$  grows from a root  $B_0$  then  $O(\bar{\gamma}, \bar{\alpha}) = m$  for all  $\bar{\alpha} \in \text{Zer } F$ . Hence  $\text{ord } F(x, \bar{\gamma}(x)) = \sum_{\bar{\alpha} \in \text{Zer } F} O(\bar{\gamma}, \bar{\alpha}) = anm$ . Because from  $B_0$  grow  $n$  sub-tree-models there are  $n - 1$  Puiseux roots of  $F'_y$  which give the smallest polar invariant  $anm$ .

Now take  $\bar{\gamma}(x) = x^m + \dots$  which grows from  $T_0$ . Then for  $i = 1, \dots, n - 1$  we have  $\text{ord } f_i(x, \bar{\gamma}(x)) = \sum_{\bar{\alpha} \in \text{Zer } f_i} O(\bar{\gamma}, \bar{\alpha}) = am$ . By equality  $\text{ord } F(x, \bar{\gamma}(x))$

$= \sum_{i=0}^{n-1} \text{ord } f_i(x, \bar{\gamma}(x))$  we get

$$(7.3) \quad \text{ord } F(x, \bar{\gamma}(x)) = \text{ord } \tilde{f}(x, \bar{\gamma}(x)) + (n - 1)am.$$

The sub-tree-model  $T_0$  is equal to  $T(\tilde{f})$ . By the Kuo-Lu lemma there is one-to-one correspondence between Puiseux roots of  $\tilde{f}'_y$  and Puiseux roots of  $F'_y$  which grow from  $T_0$ . Thus if  $Q(\tilde{f}) = \langle q'_1 : m'_1 \dots, q'_s : m'_s \rangle$  then from (7.3)

$$(7.4) \quad Q(F) = \langle amn : n-1, q'_1+(n-1)am : nm'_1 \dots, q'_s+(n-1)am : nm'_s \rangle.$$

Comparing (7.2) and (7.4) we get (ii).

To prove (iii) recall that the Puiseux roots of  $\tilde{f}$  are exactly the Puiseux roots  $\bar{\alpha}_i$  of  $F(x, y) = 0$  for which  $\text{in}(\bar{\alpha}_i) = x^m$ . Put  $\text{Zer}(\tilde{f}) = \{\bar{\alpha}_{j_1}, \dots, \bar{\alpha}_{j_a}\}$  then

$$\begin{aligned} \{\text{ord}(\bar{\alpha}_{j_k} - \bar{\alpha}_{j_l})\}_{k \neq l} &= \{n \cdot \text{ord}(\alpha_{j_k} - \alpha_{j_l})\}_{k \neq l} = \left\{ n \frac{\beta_2}{\beta_0}, \dots, n \frac{\beta_g}{\beta_0} \right\} \\ &= \left\{ \frac{\beta_2}{a}, \dots, \frac{\beta_g}{a} \right\} \end{aligned}$$

so  $\text{char}(\tilde{f}) = (a, \beta_2, \dots, \beta_g)$ . □

*Proof of Theorem 4.1.* — We apply induction with respect to  $\text{ord } f$ . If  $\text{ord } f = 1$  then  $f$  has an empty system of polar invariants. Hence also  $g'_y$  has no Puiseux root and consequently  $\text{ord } g = 1$ . In this case both  $f$  and  $g$  are irreducible.

Now assume that  $\text{ord } f > 1$  and that Theorem is true for smaller orders. We may assume that  $f$  has form (6.1). From  $\mathcal{N}_J(f) = \mathcal{N}_J(g)$  it follows that  $g$  satisfies the assumptions of Lemma 6.1 so we may also assume that  $g$  has form (6.1). By Corollary 5.3 we may replace  $f$  and  $g$  by their distinguished Weierstrass polynomials. Take  $\tilde{f}$  and  $\tilde{g}$ . By Lemma 7.1  $\mathcal{N}_J(\tilde{f}) = \mathcal{N}_J(\tilde{g})$  and  $\tilde{f}$  is irreducible. Hence by inductive hypothesis  $\tilde{g}$  is irreducible so also  $g$  is. □

*Remark 7.4.* — If we put  $f = y^2(y - x^2)^2 + x^{11}$  and  $g = y^3(y - x^2) + x^{11}$ , then  $\mathcal{N}_J(f) = \mathcal{N}_J(g) = \left\{ \frac{8}{1} \right\} + \left\{ \frac{22}{2} \right\}$  but  $f = 0$  has two branches while  $g = 0$  has four. So we cannot generalize Theorem 1 to multi-branched curves.

*Remark 7.5.* — In [15] the author gives an example of two polynomials  $f(x, y) = (y - x^2)(y^4 - x^{12})$  and  $g(x, y) = (y - x^2)^2 y^3 + x^{14}$  such that the curves  $f = 0$  and  $g = 0$  are unitangent and have the same jacobian Newton polygon, but  $f = 0$  is non-degenerate with 3 irreducible components and  $g = 0$  is degenerate with 5 irreducible components.



*Remark 7.6.* — In Example 5.2 we take two curves  $f = 0$  and  $g = 0$ , where  $f = 0$  has 4 branches and  $g = 0$  is irreducible. These curves have the same set of polar invariants, so in Theorem 4.1 we cannot replace the jacobian Newton polygon by the collection of inclinations of its edges.

Merle in [19] proves that the set of polar invariants and the order of a branch determine its equisingularity class. Nevertheless this data is the same for the curves  $f = 0$  and  $g = 0$  of Example 5.2, so the knowledge of this data is not enough to decide if a curve is irreducible or not.

Theorem 4.1 allows us to decide if a curve is irreducible using its jacobian Newton polygon. In the sequel, following this approach, we propose new criteria of irreducibility of complex plane curves.

## 8. Characterization of jacobian Newton polygons of branches using the reduction operation

Note that if  $\tilde{f}$  is the reduction of  $f$ , then by (ii) of Lemma 7.1

$$\mathcal{N}_J(\tilde{f}) := \sum_{j=1}^r \left\{ \frac{m'_j q'_j}{m'_j} \right\}.$$

We will call the transition from  $\mathcal{N}_J(f)$  to  $\mathcal{N}_J(\tilde{f})$  a reduction operation. In this section we extend this operation to all rational Newton polygons and using the results of section 7 we will characterize the special convenient Newton polygons which are jacobian Newton polygons of irreducible plane curves.

### The reduction operation over rational Newton polygons

Let  $\mathcal{N} = \sum_{i=1}^r \left\{ \frac{L_i}{M_i} \right\}$  be a convenient rational Newton polygon with  $r \geq 2$  and  $\frac{L_1}{M_1} < \dots < \frac{L_r}{M_r}$ . The *reduction* of  $\mathcal{N}$  is by definition

$$(8.1) \quad \mathcal{R}(\mathcal{N}) := \sum_{i=1}^{r-1} \left\{ \frac{L'_i}{M'_i} \right\}$$

where

$$(8.2) \quad L'_i := L_{i+1} - \frac{L_1}{1 + M_1} M_{i+1},$$

and

$$(8.3) \quad M'_i := \frac{M_{i+1}}{1 + M_1}.$$

Since  $\frac{L_{i+1}}{M_{i+1}} < \frac{L_{i+2}}{M_{i+2}}$ ,  $\mathcal{R}(\mathcal{N})$  is again a convenient rational Newton polygon written in its canonical form because  $\frac{L'_i}{M'_i} < \frac{L'_{i+1}}{M'_{i+1}}$ .

The reduction operation transforms a Newton polygon of  $r > 1$  compact faces to a Newton polygon of  $r - 1$  compact faces.

We denote by  $\mathcal{R}^i(\mathcal{N})$  the Newton polygon obtained after applying  $i$ -times the reduction operation to  $\mathcal{N}$ . By convention we put  $\mathcal{R}^0(\mathcal{N}) = \mathcal{N}$ .

If  $\mathcal{N} = \sum_{k=1}^r \left\{ \frac{L_k}{M_k} \right\}$  then we denote  $\mathcal{R}^i(\mathcal{N})$  by  $\sum_{k=1}^{r-i} \left\{ \frac{L_k^{(i)}}{M_k^{(i)}} \right\}$ .

PROPERTIES 8.1. —

- (1)  $\mathcal{R}(\mathcal{N}_J(f)) = \mathcal{N}_J(\tilde{f})$ .
- (2) If  $\mathcal{N} = \sum_{i=1}^r \left\{ \frac{L_i}{M_i} \right\}$  and  $\mathcal{N}^* = \sum_{i=1}^r \left\{ \frac{L_i^*}{M_i^*} \right\}$  satisfy  $\mathcal{R}(\mathcal{N}) = \mathcal{R}(\mathcal{N}^*)$  and  $\left\{ \frac{L_1}{M_1} \right\} = \left\{ \frac{L_1^*}{M_1^*} \right\}$  then  $\mathcal{N} = \mathcal{N}^*$ .

*Proof.* — Using Lemma 7.1 we have (1). The property (2) follows from (8.2) and (8.3) since  $M_{i+1} = M_{i+1}^*$  and  $L_{i+1} = L_{i+1}^*$ . □

THEOREM 8.2. — Let  $\mathcal{N} = \sum_{k=1}^r \left\{ \frac{L_k}{M_k} \right\}$  be a special convenient integral Newton polygon satisfying

- (1)  $1 + \text{ht}(\mathcal{N}) < \frac{L_1}{M_1}$ ,
- (2)  $\mathcal{R}^i(\mathcal{N})$  is an integral Newton polygon and  $\frac{L_1^{(i)}}{M_1^{(i)}} \in \mathbf{N}$  for  $0 \leq i \leq r - 1$ ,
- (3)  $(1 + M_1^{(i)}) \text{ g. c. d. } \left( \frac{L_1^{(i)}}{M_1^{(i)}}, 1 + \text{ht}(\mathcal{R}^i(\mathcal{N})) \right) = 1 + \text{ht}(\mathcal{R}^i(\mathcal{N}))$  for  $0 \leq i \leq r - 1$ ,

then there exists  $f \in \mathbf{C}\{x, y\}$  irreducible such that  $\mathcal{N} = \mathcal{N}_J(f)$  and the Puiseux characteristic of  $f(x, y) = 0$  is  $\left( 1 + \text{ht}(\mathcal{N}), \frac{L_1^{(0)}}{M_1^{(0)}}, \frac{L_1^{(1)}}{M_1^{(1)}}, \dots, \frac{L_1^{(r-1)}}{M_1^{(r-1)}} \right)$ .

*Proof.* — We use induction on  $r$ . According to Merle ([19]) the Newton polygon  $\mathcal{N} = \left\{ \frac{L_1}{M_1} \right\}$  is the jacobian Newton polygon of a branch  $f(x, y) = 0$  if and only if  $\text{g. c. d.} \left( \frac{L_1}{M_1}, 1 + M_1 \right) = 1$  and  $1 + M_1 < \frac{L_1}{M_1}$ , moreover in this case the Puiseux characteristic of  $f(x, y) = 0$  is  $(1 + M_1, \frac{L_1}{M_1})$ .

We suppose now that Theorem 8.2 is true for any special convenient integral Newton polygon with  $r - 1$  compact edges and let  $\mathcal{N} = \sum_{k=1}^r \left\{ \frac{L_k}{M_k} \right\}$  be a special convenient integral Newton polygon satisfying the three conditions of Theorem 8.2. Then

CLAIM 8.3. — *The reduction  $\mathcal{R}(\mathcal{N})$  of  $\mathcal{N}$  also satisfies the hypothesis of Theorem 8.2.*

*The second and the third conditions are clearly satisfied. By hypothesis  $\frac{L_1}{M_1} > 1 + \text{ht}(\mathcal{N})$  so*

$$(8.4) \quad \frac{L_2}{M_2} > \frac{L_1}{M_1} > 1 + \text{ht}(\mathcal{N}) > \frac{1 + \text{ht}(\mathcal{N})}{1 + M_1} = 1 + \text{ht}(\mathcal{R}(\mathcal{N})).$$

Moreover

$$(8.5) \quad \frac{L'_1}{M'_1} > \frac{L_2}{M_2}$$

since  $M_1 L_2 - L_1 M_2 > 0$ , then (8.4) and (8.5) give  $\frac{L'_1}{M'_1} > 1 + \text{ht}(\mathcal{R}(\mathcal{N}))$ .

So there is an irreducible plane curve  $f'$  such that  $\mathcal{N}_J(f') = \mathcal{R}(\mathcal{N})$ . Put  $(\beta'_0, \beta'_1, \dots, \beta'_{r-1})$  the Puiseux characteristic of  $f' = 0$ . We define  $n$  and  $m$  from  $\mathcal{N}$  in the following way:

- (1)  $n := 1 + M_1,$
- (2)  $m := \frac{\frac{L_1}{M_1}}{\text{g.c.d.}(1 + \text{ht}(\mathcal{N}), \frac{L_1}{M_1})}.$

CLAIM 8.4. — *The numbers  $n$  and  $m$  are coprime integers and  $m > n$ .*

Put  $a := \text{g.c.d.}(1 + \text{ht}(\mathcal{N}), \frac{L_1}{M_1})$ . By definition  $\frac{L_1}{M_1} = ma$  and for  $i = 0$  the third condition of the theorem gives  $a = \text{g.c.d.}(na, ma)$ , then  $n$  and  $m$  are coprime. Moreover  $m > n$  since  $\frac{L_1}{M_1} > 1 + \text{ht}(\mathcal{N})$ .

CLAIM 8.5. — *The numbers  $(n\beta'_0, m\beta'_0, \beta'_1, \dots, \beta'_{r-1})$  form an increasing sequence of coprime integers.*

Observe that

$$\begin{aligned} \text{g.c.d.}(n\beta'_0, m\beta'_0, \beta'_1, \dots, \beta'_{r-1}) &= \text{g.c.d.}(\text{g.c.d.}(n\beta'_0, m\beta'_0), \beta'_1, \dots, \beta'_{r-1}) \\ &= \text{g.c.d.}(\beta'_0, \beta'_1, \dots, \beta'_{r-1}) = 1. \end{aligned}$$

In order to prove that  $(n\beta'_0, m\beta'_0, \beta'_1, \dots, \beta'_{r-1})$  is an increasing sequence, it is enough to prove that  $m\beta'_0 < \beta'_1$  that is equivalent to prove  $\frac{L_1}{M_1} < \frac{L'_1}{M'_1}$ , which is true by the inequality (8.5).

So there exists an irreducible plane curve  $f(x, y) = 0$  which Puiseux characteristic  $(n\beta'_0, m\beta'_0, \beta'_1, \dots, \beta'_{r-1})$ .

Consequently using Lemma 7.1 we obtain that  $\text{char}(\tilde{f}) = \text{char}(f')$  and then

$$\begin{aligned} \text{char}(f) &= (n \beta'_0, m \beta'_0, \beta'_1, \dots, \beta'_{r-1}) \\ &= \left( 1 + \text{ht}(\mathcal{N}), \frac{L_1^{(0)}}{M_1^{(0)}}, \frac{L_1^{(1)}}{M_1^{(1)}}, \dots, \frac{L_1^{(r-1)}}{M_1^{(r-1)}} \right). \end{aligned}$$

In order to finish we have to prove that  $\mathcal{N}_J(f) = \mathcal{N}$ . For that observe that  $f'$  and  $\tilde{f}$  are irreducible with  $\text{char}(f') = \text{char}(\tilde{f})$ , that is,  $f'$  and  $\tilde{f}$  are equisingular, so  $\mathcal{N}_J(f') = \mathcal{N}_J(\tilde{f})$  and by (1) of Properties 8.1 we can write  $\mathcal{N}_J(f') = \mathcal{R}(\mathcal{N}_J(f)) = \mathcal{R}(\mathcal{N})$ . By (4.2) the first segment of the jacobian Newton polygon  $\mathcal{N}_J(f)$  has height  $n - 1 = M_1$  and inclination  $m\beta'_0 = L_1^{(0)}/M_1^{(0)}$  hence equality  $\mathcal{N}_J(f) = \mathcal{N}$  follows from (2) of Properties 8.1.  $\square$

**COROLLARY 8.6.** — *A special convenient integral Newton polygon  $\mathcal{N} = \sum_{k=1}^r \left\{ \frac{L_k}{M_k} \right\}$  is the jacobian Newton polygon of a branch if and only if verifies the next three conditions:*

- (1)  $1 + \text{ht}(\mathcal{N}) < \frac{L_1}{M_1}$ ,
- (2)  $\mathcal{R}^i(\mathcal{N})$  is an integral Newton polygon and  $\frac{L_1^{(i)}}{M_1^{(i)}} \in \mathbf{N}$  for all  $0 \leq i \leq r - 1$ ,
- (3)  $(1 + M_1^{(i)}) \text{ g. c. d. } \left( \frac{L_1^{(i)}}{M_1^{(i)}}, 1 + \text{ht}(\mathcal{R}^i(\mathcal{N})) \right) = 1 + \text{ht}(\mathcal{R}^i(\mathcal{N}))$  for  $0 \leq i \leq r - 1$ .

*Proof.* — It is enough to show that if  $\mathcal{N} = \mathcal{N}_J(f)$  for an irreducible distinguished Weierstrass polynomial  $f(x, y) \in \mathbf{C}\{x\}[y]$  then  $\mathcal{N}$  satisfies conditions 1–3.

Let  $\langle \bar{\beta}_0, \bar{\beta}_1, \dots, \bar{\beta}_g \rangle$  be the semigroup of  $\{f = 0\}$ . Then by Merle formula (4.2)  $L_1 = (n_1 - 1)\bar{\beta}_1$ ,  $M_1 = n_1 - 1$  where  $n_1 = \bar{\beta}_0/\text{g. c. d.}(\bar{\beta}_0, \bar{\beta}_1)$  and  $\text{ht}(\mathcal{N}) + 1 = \bar{\beta}_0$ . Condition 1 reads as  $\bar{\beta}_0 < \bar{\beta}_1$ , condition 2 for  $i = 0$  is satisfied because  $\frac{L_1}{M_1} = \bar{\beta}_1$  and condition 3 for  $i = 0$  is an equation  $n_1 \text{ g. c. d.}(\bar{\beta}_1, \bar{\beta}_0) = \bar{\beta}_0$ . In order to check that conditions 2 and 3 are satisfied for  $1 \leq i \leq g - 1$  it is enough to observe that  $\mathcal{R}(\mathcal{N}) = \mathcal{N}_J(\tilde{f})$  and apply an induction because  $\tilde{f}$  is again an irreducible distinguished Weierstrass polynomial.  $\square$

*Remark 8.7.* — From Corollary 8.6 we have that if  $\mathcal{N} = \mathcal{N}_J(f)$ , with  $f$  irreducible and  $\text{char}(f) = (\beta_0, \dots, \beta_g)$  then  $\beta_0 = 1 + \text{ht}(\mathcal{N}_J(f))$  and  $\beta_{k+1} = \frac{L_1^{(k)}}{M_1^{(k)}}$ .

### 9. Characterization of jacobian Newton polygons of branches using the abrasion operation

The concept of approximate root was introduced and studied in [2]:

PROPOSITION 9.1. — *Let  $A$  be an integral domain. If  $f(y) \in \mathbf{A}[y]$  is monic of degree  $d$  and  $p$  is invertible in  $A$  and divides  $d$ , then there is a unique monic polynomial  $g(y) \in \mathbf{A}[y]$  such that the degree of  $f - g^p$  is less than  $d - \frac{d}{p}$ .*

This allows us to define:

DEFINITION 9.2. — *The unique monic polynomial of the preceding proposition is named the  $p$ -th approximate root of  $f$ .*

Suppose now that  $f \in \mathbf{C}\{x\}[y]$  is irreducible of Puiseux characteristic  $(\beta_0, \dots, \beta_g)$ . Put  $l_k := \text{g. c. d.}(\beta_0, \dots, \beta_k)$ . In particular  $l_k$  divides  $\deg f = \beta_0$  for all  $k \in \{1, \dots, g\}$ . We note in what follows  $f^{(k)}$  the  $l_k$ -th approximate root of  $f$  which we named *characteristic approximate roots* of  $f$ .

Next proposition is the main one in [2] (see also [9] and [21]):

PROPOSITION 9.3. — *Let  $f$  be an irreducible curve with characteristic Puiseux  $(\beta_0, \dots, \beta_g)$  and semigroup  $\bar{\beta}_0\mathbf{N} + \dots + \bar{\beta}_g\mathbf{N}$ . The approximate roots  $f^{(k)}$  of  $f$ , for  $0 \leq k \leq g$ , have the following properties:*

- (1) *The degree of  $f^{(k)}$  is equal to  $\frac{\beta_0}{l_k}$  and  $(f, f^{(k)})_0 = \bar{\beta}_{k+1}$ .*
- (2) *The polynomial  $f^{(k)}$  is irreducible and its Puiseux characteristic is  $(\frac{\beta_0}{l_k}, \dots, \frac{\beta_k}{l_k})$  and its semigroup is  $\frac{\bar{\beta}_0}{l_k}\mathbf{N} + \dots + \frac{\bar{\beta}_k}{l_k}\mathbf{N}$ .*

After [19] and from the above Proposition, if

$$Q(f) = \langle q_1 : m_1, \dots, q_g : m_g \rangle$$

then

$$Q(f^{(k)}) = \left\langle \frac{q_1}{l_k} : m_1, \dots, \frac{q_k}{l_k} : m_k \right\rangle.$$

This inspires the next operation between the rational Newton polygons.

#### The abrasion operation over rational Newton polygons

Let  $\mathcal{N} = \sum_{i=1}^r \left\{ \frac{L_i}{M_i} \right\}$  be a convenient rational Newton polygon with  $r \geq 2$  and  $\frac{L_1}{M_1} < \dots < \frac{L_r}{M_r}$ . The *abrasion* of  $\mathcal{N}$  is by definition

$$(9.1) \quad \mathcal{A}(\mathcal{N}) := \sum_{i=1}^{r-1} \left\{ \frac{\tilde{L}_i}{M_i} \right\}$$

where

$$(9.2) \quad \tilde{L}_i := \frac{1 + M_1 + \dots + M_{r-1}}{1 + M_1 + \dots + M_r} L_i,$$

for  $1 \leq i \leq r - 1$ .

Since  $\frac{L_i}{M_i} < \frac{L_{i+1}}{M_{i+1}}$ ,  $\mathcal{A}(\mathcal{N})$  is again a convenient rational Newton polygon written in its canonical form because  $\frac{\tilde{L}_i}{M_i} < \frac{\tilde{L}_{i+1}}{M_{i+1}}$ .

The abrasion operation transforms a Newton polygon of  $r > 1$  compact faces to a Newton polygon of  $r - 1$  compact faces.

We denote by  $\mathcal{A}^i(\mathcal{N})$  the Newton polygon obtained after applying  $i$ -times the abrasion operation to  $\mathcal{N}$ . Observe that  $\mathcal{A}^{i+1}(\mathcal{N}) = \mathcal{A}^i(\mathcal{A}(\mathcal{N})) = \mathcal{A}(\mathcal{A}^i(\mathcal{N}))$ . By convention we put  $\mathcal{A}^0(\mathcal{N}) = \mathcal{N}$ .

More precisely if  $\mathcal{N} = \sum_{k=1}^r \left\{ \frac{L_k}{M_k} \right\}$  we denote  $\mathcal{A}^i(\mathcal{N})$  by  $\sum_{k=1}^{r-i} \left\{ \frac{\tilde{L}_k^{(i)}}{\tilde{M}_k^{(i)}} \right\}$ .

PROPERTIES 9.4. — Let  $f$  be a branch with semigroup  $\langle \bar{\beta}_0, \bar{\beta}_1, \dots, \bar{\beta}_g \rangle$ . Then

- (1)  $\mathcal{A}(\mathcal{N}_J(f)) = \mathcal{N}_J(f^{(g-1)})$ ,
- (2)  $\mathcal{A}(\mathcal{N}_J(f^{(i+1)})) = \mathcal{N}_J(f^{(i)})$  for  $1 \leq i \leq g - 1$  and
- (3)  $\mathcal{A}^i(\mathcal{N}_J(f)) = \mathcal{N}_J(f^{(g-i)})$  for  $1 \leq i \leq g - 1$ .

Proof. — It is enough to prove that  $\mathcal{A}(\mathcal{N}_J(f)) = \mathcal{N}_J(f^{(g-1)})$ . After [19] we have that  $\mathcal{N}_J(f) = \sum_{i=1}^g \left\{ \frac{L_i}{M_i} \right\}$ , with  $L_i = (n_i - 1)\bar{\beta}_i$  and  $M_i = n_1 \cdots n_{i-1}(n_i - 1)$ , where  $n_i = \frac{\text{g.c.d.}(\bar{\beta}_0, \dots, \bar{\beta}_{i-1})}{\text{g.c.d.}(\bar{\beta}_0, \dots, \bar{\beta}_i)}$ . So  $\mathcal{A}(\mathcal{N}_J(f)) = \sum_{i=1}^{g-1} \left\{ \frac{\tilde{L}_i}{M_i} \right\}$  where  $\tilde{L}_i = \frac{1}{n_g} L_i$  since  $\frac{1+M_1+\dots+M_{g-1}}{1+M_1+\dots+M_g} = \frac{1}{n_g}$ . □

THEOREM 9.5. — Let  $\mathcal{N} = \sum_{k=1}^r \left\{ \frac{L_k}{M_k} \right\}$  be a special convenient integral Newton polygon satisfying

- (1)  $1 + \text{ht}(\mathcal{N}) < \frac{L_1}{M_1}$ ,
- (2)  $\mathcal{A}^i(\mathcal{N})$  is a special convenient integral Newton polygon,  $\frac{\tilde{L}_1^{(i)}}{\tilde{M}_1^{(i)}} \in \mathbf{N}$  for all  $i \in \{0, \dots, r-1\}$ , and  $(1 + \tilde{M}_1^{(i)} + \tilde{M}_2^{(i)} + \dots + \tilde{M}_{r-i-1}^{(i)}) \frac{\tilde{L}_{r-i}^{(i)}}{\tilde{M}_{r-i}^{(i)}} \in \mathbf{N}$  for all  $i \in \{0, \dots, r-2\}$ ,
- (3)  $\text{g.c.d.} \left( 1 + \text{ht}(\mathcal{A}^i(\mathcal{N})), \tilde{L}_1^{(i)}, \dots, \tilde{L}_{r-i}^{(i)} \right) = 1$  for  $0 \leq i \leq r - 1$ ,

then there exists an irreducible  $f \in \mathbf{C}\{x, y\}$  such that  $\mathcal{N} = \mathcal{N}_J(f)$ , and its semigroup is

$$(1 + \text{ht}(\mathcal{N}))\mathbf{N} + \frac{L_1}{M_1}\mathbf{N} + (1 + M_1)\frac{L_2}{M_2}\mathbf{N} + \dots + (1 + M_1 + M_2 + \dots + M_{r-1})\frac{L_r}{M_r}\mathbf{N}.$$

*Proof.* — We use induction on  $r$ . According to Merle ([19]) the Newton polygon  $\mathcal{N} = \left\{ \frac{L_1}{M_1} \right\}$  is the jacobian Newton polygon of a branch  $f(x, y) = 0$  if and only if  $1 + M_1 < \frac{L_1}{M_1}$  and  $\text{g. c. d.}(\frac{L_1}{M_1}, 1 + M_1) = 1$ , which follows from the third condition of Theorem 9.5 taking  $r = 1$  and  $i = 0$ .

We suppose now that Theorem 9.5 is true for any special convenient integral Newton polygon with  $r - 1$  compact edges and let  $\mathcal{N} = \sum_{k=1}^r \left\{ \frac{L_k}{M_k} \right\}$  be a special convenient integral Newton polygon satisfying the three conditions of Theorem 9.5. Then

CLAIM 9.6. — *The abrasion operation  $\mathcal{A}(\mathcal{N})$  of  $\mathcal{N}$  also satisfies the hypothesis of Theorem 9.5.*

The second and the third conditions are clearly satisfied. Moreover by hypothesis  $1 + \text{ht}(\mathcal{N}) < \frac{L_1}{M_1}$  which is equivalent to  $1 + \text{ht}(\mathcal{A}(\mathcal{N})) < \frac{\tilde{L}_1}{M_1}$ .

So there is an irreducible plane curve  $\hat{f}$  such that  $\mathcal{N}_J(\hat{f}) = \mathcal{A}(\mathcal{N})$ .

Put  $\gamma_0\mathbf{N} + \gamma_1\mathbf{N} + \dots + \gamma_{r-1}\mathbf{N}$  the semigroup of  $\hat{f} = 0$ , with  $\gamma_i < \gamma_{i+1}$  for  $0 \leq i \leq r - 2$ .

We put  $N = \frac{1+M_1+\dots+M_r}{1+M_1+\dots+M_{r-1}}$  and  $\gamma_r = (1 + M_1 + M_2 + \dots + M_{r-1}) \frac{L_r}{M_r}$ .

CLAIM 9.7. — *The numbers  $N$  and  $\gamma_r$  are integers.*

The second condition of the Theorem gives, for  $i = 1$ , that  $\mathcal{A}^1(\mathcal{N})$  is an integral Newton polygon, so in particular  $N$  divides  $L_i$  for  $i \in \{1, \dots, r - 1\}$  and the third condition of the Theorem gives, for  $i = 1$ ,  $\text{g. c. d.}(1 + M_1 + \dots + M_{r-1}, \frac{L_1}{N}, \dots, \frac{L_{r-1}}{N}) = 1$  so  $N = \text{g. c. d.}(1 + M_1 + \dots + M_r, L_1, \dots, L_{r-1})$ . Moreover  $\gamma_r = (1 + \tilde{M}_1^{(i)} + \tilde{M}_2^{(i)} + \dots + \tilde{M}_{r-i}^{(i)}) \frac{\tilde{L}_{r-i}^{(i)}}{\tilde{M}_{r-i}^{(i)}}$  for  $i = 0$  so  $\gamma_r \in \mathbf{N}$  by hypothesis.

CLAIM 9.8. — *The numbers  $N\gamma_0, N\gamma_1, \dots, N\gamma_{r-1}, \gamma_r$  form the minimal set of generators of the semigroup of an irreducible plane curve.*

According to Bresinsky in [3] about the characterization of the semigroups of plane branches (see the end of Section 2), it is enough to prove the following three conditions:

- (1)  $\text{g. c. d.}(N\gamma_0, N\gamma_1, \dots, N\gamma_{r-1}, \gamma_r) = 1$ .

We have  $\text{g. c. d.}(N\gamma_0, N\gamma_1, \dots, N\gamma_{r-1}, \gamma_r) = \text{g. c. d.}(N, \gamma_r)$ . Moreover  $1 + \text{ht}(\mathcal{N})$  is divisible by  $N$  and  $N = \text{g. c. d.}(1 + M_1 + \dots + M_r, L_1, \dots, L_{r-1})$  so  $\text{g. c. d.}(N, L_r) = \text{g. c. d.}(1 + \text{ht}(\mathcal{N}), L_1, \dots, L_r) = 1$  and consequently  $\text{g. c. d.}(N, \gamma_r) = 1$  since  $\gamma_r = \frac{L_r}{N-1}$ .

- (2) For  $1 \leq i \leq r - 1$ ,  $\text{g. c. d.}(N\gamma_0, N\gamma_1, \dots, N\gamma_{i-1}) > \text{g. c. d.}(N\gamma_0, N\gamma_1, \dots, N\gamma_i)$  by inductive hypothesis and  $\text{g. c. d.}(N\gamma_0, N\gamma_1, \dots, N\gamma_{r-1})$

$= N > 1 = \text{g. c. d.}(N\gamma_0, N\gamma_1, \dots, N\gamma_{r-1}, \gamma_r)$ . This proves the second condition of Bresinski.

- (3) For  $1 \leq i \leq r - 2$ ,  $\frac{\text{g.c.d.}(N\gamma_0, N\gamma_1, \dots, N\gamma_{i-1})}{\text{g.c.d.}(N\gamma_0, N\gamma_1, \dots, N\gamma_i)} N\gamma_i < N\gamma_{i+1}$  by inductive hypothesis. To finish the proof of this third condition observe that  $\mathcal{N}_J(\hat{f}) = \mathcal{A}(\mathcal{N})$ , so  $\text{g. c. d.}(\gamma_0, \gamma_1, \dots, \gamma_{r-2}) = \frac{1+M_1+\dots+M_{r-1}}{1+M_1+\dots+M_{r-2}}$  and

$$\begin{aligned} \frac{\text{g. c. d.}(N\gamma_0, N\gamma_1, \dots, N\gamma_{r-2})}{\text{g. c. d.}(N\gamma_0, N\gamma_1, \dots, N\gamma_{r-1})} N\gamma_{r-1} &= \text{g. c. d.}(\gamma_0, \gamma_1, \dots, \gamma_{r-2}) N\gamma_{r-1} \\ &= (1 + M_1 + \dots + M_{r-1}) \frac{L_{r-1}}{M_{r-1}} \\ &< \gamma_r. \end{aligned}$$

Finally note that the semigroup generated by  $N\gamma_0, \dots, N\gamma_{r-1}, \gamma_r$  is exactly

$$(1 + \text{ht}(\mathcal{N}))\mathbf{N} + \frac{L_1}{M_1}\mathbf{N} + (1 + M_1) \frac{L_2}{M_2}\mathbf{N} + \dots + (1 + M_1 + M_2 + \dots + M_{r-1}) \frac{L_r}{M_r}\mathbf{N}. \quad \square$$

**COROLLARY 9.9.** — *A special convenient integral Newton polygon  $\mathcal{N} = \sum_{k=1}^r \left\{ \frac{L_k}{M_k} \right\}$  is the jacobian Newton polygon of a branch if and only if verifies the next three conditions:*

- (1)  $1 + \text{ht}(\mathcal{N}) < \frac{L_1}{M_1}$ ,
- (2)  $\mathcal{A}^i(\mathcal{N})$  is a special convenient integral Newton polygon,  $\frac{\tilde{L}_1^{(i)}}{\tilde{M}_1^{(i)}} \in \mathbf{N}$  for all  $i \in \{0, \dots, r - 1\}$ , and  $(1 + \tilde{M}_1^{(i)} + \tilde{M}_2^{(i)} + \dots + \tilde{M}_{r-i-1}^{(i)}) \frac{\tilde{L}_{r-i}^{(i)}}{\tilde{M}_{r-i}^{(i)}} \in \mathbf{N}$  for all  $i \in \{0, \dots, r - 2\}$ ,
- (3)  $\text{g. c. d.} \left( 1 + \text{ht}(\mathcal{A}^i(\mathcal{N})), \tilde{L}_1^{(i)}, \dots, \tilde{L}_{r-i}^{(i)} \right) = 1$  for  $0 \leq i \leq r - 1$ .

*Proof.* — It is enough to show that if  $\mathcal{N} = \mathcal{N}_J(f)$  for an irreducible distinguished Weierstrass polynomial  $f(x, y) \in \mathbf{C}\{x\}[y]$  then  $\mathcal{N}$  satisfies conditions 1–3.

Let  $\langle \bar{\beta}_0, \bar{\beta}_1, \dots, \bar{\beta}_r \rangle$  be the semigroup of  $\{f = 0\}$ . Then by (4.2)

$$\mathcal{N} = \sum_{k=1}^r \left\{ \frac{L_k}{M_k} \right\} = \sum_{k=1}^r \left\{ \frac{(n_k - 1)\bar{\beta}_k}{(n_k - 1)n_1 \dots n_{k-1}} \right\}.$$

By above equality  $L_1/M_1 = \bar{\beta}_1$ ,  $(1 + M_1 + M_2 + \dots + M_{r-1})L_r/M_r = \bar{\beta}_r$  and  $\text{g. c. d.}(1 + \text{ht}(\mathcal{N}), L_1, \dots, L_r) = \text{g. c. d.}(\bar{\beta}_0, (n_1 - 1)\bar{\beta}_1, \dots, (n_r - 1)\bar{\beta}_r) = 1$  (see Property 2.1). Hence conditions 2 and 3 are satisfied for  $i = 0$ .

In order to check that conditions 2 and 3 are satisfied for  $i > 1$  it is enough to observe that  $\mathcal{A}(\mathcal{N}) = \mathcal{N}_J(f^{(r-1)})$  and apply an induction because an

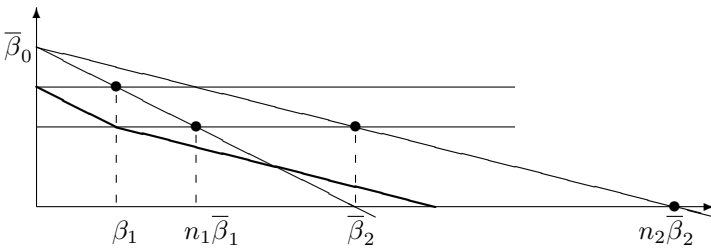


approximate root  $f^{(r-1)}$  is again an irreducible distinguished Weierstrass polynomial. □

**COROLLARY 9.10.** — *Let  $\mathcal{N} = \sum_{k=1}^r \left\{ \frac{L_k}{M_k} \right\}$  be the jacobian Newton polygon of a plane curve  $f(x, y) = 0$ . Put  $\gamma_0 := 1 + \text{ht}(\mathcal{N})$ , and  $\gamma_i := (1 + M_1 + M_2 + \dots + M_{i-1}) \frac{L_i}{M_i}$  for  $1 \leq i \leq r$ . Then  $f$  is irreducible if and only if the numbers  $\gamma_0, \gamma_1, \dots, \gamma_r$  verify the three conditions of Bresinsky. In such case  $\gamma_0, \dots, \gamma_r$  generate the semigroup of  $f$ .*

*Example 9.11 (Kuo’s example).* — Let  $f(x, y) = (y^2 - x^3)^2 - x^7$  be Kuo’s example in [12]. Then  $\frac{\partial f}{\partial y} = 4y(y^2 - x^3)$ , so  $Q(f) = \langle 6 : 1, 7 : 2 \rangle$ , since  $(f, y)_0 = 6$  and  $(f, y^2 - x^3)_0 = 14$ . That is  $\mathcal{N}_J(f) = \left\{ \frac{6}{1} \right\} + \left\{ \frac{14}{2} \right\}$ , so  $1 + \text{ht}(\mathcal{N}_J(f)) = 4$ ,  $\frac{L_1}{M_1} = 6$  and  $(1 + M_1) \frac{L_2}{M_2} = 14$ . Using Corollary 9.10 we show that  $f$  is not irreducible since  $\text{g. c. d.}(4, 6, 14) \neq 1$ . We can also show that  $f$  is not irreducible using the reduction operation since  $\mathcal{R}\mathcal{N}_J(f) = \left\{ \frac{8}{1} \right\}$  and the third condition, in Corollary 8.6, is not true for  $i = 1$ .

*Remark 9.12.* — In the case of a germ of plane irreducible curve (i.e., a branch), Merle shows in [19] that the datum of the jacobian Newton polygon determines and is determined by the equisingularity class of the curve. Note that if  $\mathcal{N}$  is the jacobian Newton polygon of a branch  $f(x, y) = 0$  of semigroup generated by  $\bar{\beta}_0, \dots, \bar{\beta}_g$  then  $\mathcal{N}$  has  $g$  compact faces  $\Gamma_1, \dots, \Gamma_g$  where the inclinations of these faces form an increasing sequence. Note by  $P_i$  the segment which is the projection of  $\Gamma_i$  on the vertical axis. In particular the intersection point of  $\Gamma_1$  and the vertical axis is  $(0, \bar{\beta}_0 - 1)$ . If  $l_i$  is the line containing the point  $(0, \bar{\beta}_0)$  and parallel to  $\Gamma_i$ ,  $\mathcal{B}_i = P_i \times \mathbf{R}$ , and  $r_i$  (resp.  $r_{i+1}$ ) is the top line (resp. the bottom line) of  $\mathcal{B}_i$  then the abscissa of the intersection point of  $l_i$  and  $r_i$  is  $\bar{\beta}_i$  and the abscissa of the intersection point of  $l_i$  and  $r_{i+1}$  is  $n_i \bar{\beta}_i$ , where  $n_i = \frac{\text{g.c.d.}(\bar{\beta}_0, \dots, \bar{\beta}_{i-1})}{\text{g.c.d.}(\bar{\beta}_0, \dots, \bar{\beta}_i)}$ . Finally note that the length of  $P_i$  is exactly  $n_1 \dots n_{i-1} (n_i - 1)$ .



*Conclusion.* — In this paper we characterize the special convenient Newton polygons which are jacobian Newton polygons of a branch. This allows us to give combinatorial criteria of irreducibility of complex series in two variables. Their natures are different from the classical criterion in [1]: we draw the Newton polygon in the coordinates  $(u, v)$  of the discriminant  $D(u, v)$  of the morphism defined by

$$(9.3) \quad \begin{aligned} (x, f) : (\mathbf{C}^2, 0) &\longrightarrow (\mathbf{C}^2, 0) \\ (x, y) &\longrightarrow (u, v) := (x, f(x, y)), \end{aligned}$$

and we check whether it verifies the geometrical conditions of Corollary 8.6 or Corollary 9.9 or arithmetical conditions of Corollary 9.10.

By Corollary 5.3 we may assume that  $f$  is a Weierstrass polynomial. In this case it is not difficult to compute an equation of the discriminant  $D(u, v)$ . We get this equation by eliminating  $x$  and  $y$  from

$$\left\{ \begin{aligned} x &= u \\ f(x, y) &= v \\ \frac{\partial f(x, y)}{\partial y} &= 0 \end{aligned} \right.$$

and using the classical notion of resultant of two polynomials in one variable we have:

$$\begin{aligned} D(u, v) &= \text{Result}_y \left( f(u, y) - v, \frac{\partial f(u, y)}{\partial y} \right) \\ &= \text{Result}_y \left( f(u, y) - v, \frac{\partial (f(u, y) - v)}{\partial y} \right) \\ &= \text{Disc}_y (f(u, y) - v). \end{aligned}$$

We think this approach provides effective methods of checking the irreducibility of complex series in two variables.

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