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Perturbative expansions in quantum mechanics

PERTURBATIVE EXPANSIONS IN QUANTUM MECHANICS

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ABSTRACT. — We prove a $D = 1$ analytic versal deformation theorem in the Heisenberg algebra. We define the spectrum of an element in the Heisenberg algebra. The quantised version of the Morse lemma already shows that the perturbation series arising in a perturbed harmonic oscillator become analytic after a formal Borel transform.

RÉSUMÉ. — Nous démontrons un théorème de déformation verselle analytique pour l'algèbre de Heisenberg dans le cas $D = 1$. Nous définissons le spectre d'un élément dans cette algèbre. La quantification du lemme de Morse montre que les séries perturbatives du spectre de l'oscillateur harmonique deviennent analytique après une transformation de Borel formelle.

Introduction

In 1925, Heisenberg showed that the perturbative expansion for the spectrum of the anharmonic oscillator $H(q,p) = 1/2(p^2 + q^2) + 1/4tq^4$ are given by formal power series expansions in t.

He computed the first three terms of these series and found:

$$E(t,h) = (n + 1/2)\frac{h}{2\pi} + \frac{3(n^2 + n + 1/2)}{8}t\left(\frac{h}{2\pi}\right)^2$$
$$- \frac{17n^3 + 51/2n^2 + 59/2 + 21/2}{64}t^2\left(\frac{h}{2\pi}\right)^3 + \dots$$

where h denotes the Planck constant [20].

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Soon after, Born and Jordan proposed the matrix formulation of quantum mechanics in order to interpret and generalise Heisenberg's computations [5]. This approach led its authors and Heisenberg to the "Dreimännerarbeit" in which, among other things, they extended Heisenberg result to arbitrary perturbations [4]. Later, Von Neumann made an attempt to unify matrix quantum mechanics with Schrödinger's approach in terms of operators in Hilbert spaces. This process was pursued by several authors and led to the Kato-Rellich theory. For instance, the perturbative series $E(t, h)$ are asymptotic expansions of spectral values of the operator H for $t > 0$ [35] (see also [33], Chapter XII.3 and historical notes).

Nevertheless, the Born-Jordan approach differs from the one using the Hilbert space formulation in at least two points : there is a parameter $\hbar$ in all computations, there is no condition on boundedness of the matrix coefficients. In particular, rather than vector spaces over $\mathbb{C}$ one has to consider modules over $\mathbb{C}[[\hbar]]$. In the Born-Jordan approach the eigenvalues are obtained by transforming the semi-infinite matrix corresponding to the Hamiltonian to its diagonal form.

This process, originated by Born and carried out by Born Heisenberg and Jordan, is a quantised version of the Birkhoff normal form [4]. It appeared a year before the book of Birkhoff on dynamical systems in which the Birkhoff normal form is presented, but the method was, of course, much older [1] [3].

The difference between the Born-Jordan and the Hilbert space approaches is similar to that encountered in geometry, where one can consider real C^∞ and formal categories. To these two approaches, we add the analytic one. We will prove that the spectrum computed in the analytic and formal categories coincide, and this will prove that the perturbative expansions $E(t, \hbar) = \sum_k \alpha_k(t)\hbar^k$ considered in the Dreimännerarbeit have analytic *formal Borel transforms* $\hat{E}(t, \hbar) = \sum_k \alpha_k(t)\hbar^k/k!$. So, our proof is indirect and reflects the existence of an analytic theory similar to the formal one. For some special polynomial perturbations, it is conjectured that these asymptotic series are resurgent [34, 38, 40]. Part of this work might be used to give a new approach to quantum resurgence by adding to the analytic theory an hypothetic resurgent one.

In this paper, we confine ourselves to one-dimensional quantum mechanics but using deformation theory of singular Lagrangian varieties, one can extend these results to quantum integrable systems [17]. The integrability

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[1] It seems that Born gave the name "quantum mechanics" to the new physical theory precisely because of its relation with perturbation theory in Hamiltonian mechanics.

of the system is needed to establish the Borel convergence. The existence of the formal power series can be established for any perturbation of a quadratic hamiltonian as Born, Heisenberg and Jordan proved [4].

1. Formal quantum mechanics

1.1. The algebra $\widehat{\mathcal{Q}}$

Let $\widehat{\mathcal{Q}}$ be the non-commutative algebra over $\mathbb{C}$ consisting of formal power series in the variables $p, q, \hbar$ which satisfy the commutation relations

$$[p, q] = \hbar, \ [\hbar, p] = 0, \ [\hbar, q] = 0, \ \hbar = \frac{h}{2\pi\sqrt{-1}}.$$

We introduce the variable $\hbar = \frac{h}{2\pi\sqrt{-1}}$ rather than $\hbar = \frac{h}{2\pi}$ in order to simplify the notations and the formulae.

The elements of the algebra $\widehat{\mathcal{Q}}$ can be represented as differential operators acting on $\mathbb{C}[[\hbar, z]]$ by putting[2]

$$p = \hbar\partial_z, \ q = z.$$

Using the basis $1, z, z^2, \ldots$ of the $\mathbb{C}[[\hbar]]$-module $\mathbb{C}[[\hbar, z]]$, we may associate to each element of $\widehat{\mathcal{Q}}$ a semi-infinite matrix. The semi-infinite matrices associated to q, p are given by

$$q = \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}, \ p = \begin{pmatrix} 0 & \hbar & 0 & 0 & \ldots \\ 0 & 0 & 2\hbar & 0 & \ldots \\ 0 & 0 & 0 & 3\hbar & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

One can work abstractly in the algebra $\widehat{\mathcal{Q}}$ or use the semi-infinite matrices [5, 12].

1.2. The Born-Jordan spectrum

Using the basis $1, z, z^2, \ldots$ of the $\mathbb{C}[[\hbar]]$-module $\mathbb{C}[[\hbar, z]]$, we get a representation of the algebra $\widehat{\mathcal{Q}}$
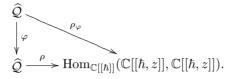
$$\rho : \widehat{\mathcal{Q}} \longrightarrow \mathrm{Hom}_{\mathbb{C}[[\hbar]]}(\mathbb{C}[[\hbar, z]], \mathbb{C}[[\hbar, z]])$$

defined by

$$\rho(p) = \hbar\partial_z, \ \rho(q) = z.$$

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[2] In physics, the usual notation is $a, a^\dagger$ rather than p, q.

An automorphism $\varphi \in \mathrm{Aut}(\widehat{\mathcal{Q}})$ induces a new representation such that the following diagram commutes

$$
\begin{array}{ccc}
\widehat{\mathcal{Q}} & & \\
\downarrow{\scriptstyle \varphi} & \searrow^{\rho_\varphi} & \\
\widehat{\mathcal{Q}} & \xrightarrow{\rho} & \mathrm{Hom}_{\mathbb{C}[[\hbar]]}(\mathbb{C}[[\hbar, z]], \mathbb{C}[[\hbar, z]]).
\end{array}
$$

For instance, the element $H = \frac{1}{2}(p^2 - q^2) \in \widehat{\mathcal{Q}}$ becomes diagonal with diagonal elements $(n + 1/2)\hbar$, $n \geqslant 0$ for the representation

$$
\rho_\varphi(p) = \frac{\hbar\partial_z + z}{\sqrt{2}}, \ \rho_\varphi(q) = \frac{-\hbar\partial_z + z}{\sqrt{2}}.
$$

This quadratic element H is called the *harmonic oscillator*, the unusual appearance of a minus sign is due to our peculiar choice for the variable $\hbar$. An element $H \in \widehat{\mathcal{Q}}$ is called *diagonalisable* if there exists a representation in which it is diagonal in the base $1, z, z^2, \dots$.

DEFINITION 1.1 ([5]). — *The formal spectrum of $H \in \widehat{\mathcal{Q}}$, denoted $\widehat{\mathrm{Sp}}(H)$, consists of the diagonal entries of its diagonal form if it is diagonalisable.*

Remark that any quadratic form in the q, p's is diagonalisable and has therefore a discrete spectrum.

1.3. Comparison between formal and matrix approaches

PROPOSITION 1.2. — *Consider an element of the type $H = H_0 + \hbar H_1 + \dots$, $H_i \in \widehat{\mathcal{Q}}$. If H_0 is diagonalisable then so is H.*

This proposition can be reformulated and easily proved in the abstract approach. We start by some elementary observations.

PROPOSITION 1.3. — *The semi-infinite matrix associated to $H \in \widehat{\mathcal{Q}}$ is diagonal in a representation ρ_φ if and only if it is of the type*

$$
\varphi(H) = \sum_{i \geqslant 0} \alpha_{ii}(qp)^i, \ \alpha_{ii} \in \mathbb{C}[[\hbar]].
$$

Proof. — Write $\rho_\varphi(H)$ has a linear differential operator $\sum_{j,k} a_{j,k} z^j \hbar^k \partial_z^k$. This operator is diagonal if and only if for all values of $n \geqslant 0$, we may find E_n such that

$$
\Big(\sum_{j,k} a_{j,k} z^j \hbar^k \partial_z^k\Big) z^n = E_n z^n.
$$

This condition is equivalent to stating that $a_{j,k}$ is non-zero only for $j = k$. This proves the proposition. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Formal power series can be composed: given any $H \in \mathbb{C}[[\hbar, x, y]]$, we have a $\mathbb{C}[[\hbar]]$-linear mapping

$$\mathbb{C}[[\hbar, z]] \longrightarrow \mathbb{C}[[\hbar, x, y]], \; z \mapsto H$$

which maps $u = \sum a_n z^n \in \mathbb{C}[[\hbar, z]]$ to

$$u \circ H := \sum_{n \geqslant 0} u_n H^n \in \mathbb{C}[[\hbar, x, y]].$$

In the algebra $\widehat{\mathcal{Q}}$ a similar property holds: given any $H \in \widehat{\mathcal{Q}}$, we have a $\mathbb{C}[[\hbar]]$-linear mapping

$$\mathbb{C}[[\hbar, z]] \longrightarrow \widehat{\mathcal{Q}}, \; z \mapsto H$$

which maps $u = \sum a_n z^n$ to

$$u \circ H := \sum_{n \geqslant 0} u_n H^n \in \widehat{\mathcal{Q}}.$$

Thus, the choice of an element $H \in \widehat{\mathcal{Q}}$ induces in the algebra $\widehat{\mathcal{Q}}$ a $\mathbb{C}[[\hbar, z]]$-module structure:

$$z^n F := H^n F, \; u(z)F := (u \circ H)F, \; \forall F \in \widehat{\mathcal{Q}}.$$

If $\rho_\varphi(H)$ is diagonal in the basis $1, \ldots, z^n, \ldots$ with entries $E_0, \ldots, E_n, \ldots$ then $\rho_\varphi(u \circ H)$ becomes diagonal in the same basis with entries $u(E_0), \ldots, u(E_n), \ldots$, i.e.:

$$\widehat{\mathrm{Sp}}(u \circ H) = u(\widehat{\mathrm{Sp}(H)}).$$

Therefore, Proposition 1.2 is a consequence of the following statement.

PROPOSITION 1.4. — *For any element* $H = qp + \hbar H_1 + \hbar^2 H_2 + \ldots$ *there exist an inner automorphism* $\varphi \in \mathrm{Aut}(\widehat{\mathcal{Q}})$ *and a function germ* $u \in \mathbb{C}[[\hbar, z]]$ *such that* $u \circ \varphi(H) = qp$.

We will now give a proof of this proposition.

1.4. The quantum versal deformation space

Following Dirac [12], we define the *quantum Poisson bracket*

$$\{F, G\} := \frac{1}{\hbar}[F, G], \; F, G \in \widehat{\mathcal{Q}}.$$

DEFINITION 1.5. — *The (formal) quantum versal deformation space associated to $H \in \mathcal{Q}$ is the $\mathbb{C}[[\hbar]]$-module $\widehat{M}(H) = \widehat{\mathcal{Q}}/\{H, \widehat{\mathcal{Q}}\}$.*

The map $F \mapsto \{H, F\}$ is $\mathbb{C}[[\hbar, z]]$-linear, indeed

$$\{H, zF\} = \{H, HF\} = H\{H, F\} = z\{H, F\}.$$

Thus, the quantum versal deformation space $\widehat{M}(H)$ has a $\mathbb{C}[[\hbar, z]]$-module structure:

$$\sum_{n \geqslant 0} a_n z^n [m] := \Big[\sum_{n \geqslant 0} a_n H^n m \Big], \ a_n \in \mathbb{C}[[\hbar]], \ m \in \widehat{\mathcal{Q}},$$

where the brackets $[\cdot]$ mean that we project the element in $\widehat{M}(H)$.

PROPOSITION 1.6. — *For $H = qp$, the $\mathbb{C}[[\hbar, z]]$-module $\widehat{M}(H) = \widehat{\mathcal{Q}}/ \{H, \widehat{\mathcal{Q}}\}$ is free of rank 1 generated by the class of $1 \in \widehat{\mathcal{Q}}$.*

Proof. — It is sufficient to prove that the class of any monomial $q^n p^m$ in $\widehat{M}(H)$ lies in the module generated by the class of 1.

For $n \neq m$, we get

$$\Big\{ H, \frac{1}{n-m} q^n p^m \Big\} = q^n p^m.$$

Therefore the class of $q^n p^m$ in $\widehat{M}(H)$ vanishes for $n \neq m$.

We prove by induction on n that the class of any monomial $q^n p^n$ in $\widehat{M}(H)$ lies in the module generated by the class of 1.

For $n = 1$, we have $[qp] = [H] = z[1]$.

By putting the q's before the p's, we get a finite expansion of the type

$$H^n = q^n p^n + r$$

where r is a term of order lower than $2n$ in the q, p variables, thus,

$$[q^n p^n] = z^n [1] - [r].$$

By assumption, the class of r lies on the module generated by [1] and consequently so does $q^n p^n$. This proves the induction step and concludes the proof the proposition. $\qquad \square$

This proposition implies Proposition 1.4 by induction on the degree of $\hbar$. Indeed, put $H = H_0 + \hbar^k H_k (\mathrm{mod}\, \hbar^{k+1})$ with $H_0 = qp$. As the class of H_k in $\widehat{M}(H_0)$ vanishes, there exists a, G such that

$$H_k = a \circ H - \{G, H_0\}, \ a \in \mathbb{C}[[\hbar, z]].$$

Consider the inner automorphism

$$\varphi_k : \widehat{\mathcal{Q}} \longrightarrow \widehat{\mathcal{Q}}, \ F \mapsto \exp(\hbar^{k-1} G) F \exp(-\hbar^{k-1} G)$$

and the map $a_k := 1 - \hbar^k a \in \mathbb{C}[[\hbar, z]]$. We get that

$$a_k \circ \varphi_k(H) = H_0 + \hbar^k H_k - \hbar^k a \circ H_0 + \hbar^k \{G, H_0\} (\mathrm{mod}\, \hbar^{k+1}) = H_0 (\mathrm{mod}\, \hbar^{k+1}).$$

This proves the assertion and concludes the proof of the proposition.

1.5. The Born-Jordan-Heisenberg theorem

Proposition 1.4 should be considered as a preliminary exercise rather than a result on its own : in practise, one is interested usually in perturbation with respect to an auxiliary parameter which is certainly not $\hbar$. We now come to such a result.

Let $\widehat{\mathcal{Q}}[[t]]$ be the non-commutative algebra over the ring of formal power series $\mathbb{C}[[\hbar, t]]$ consisting of formal power series in the variables $p, q, \hbar, t$ where the only non-trivial commutation relation is still $[p, q] = \hbar$.

THEOREM 1.7 ([4], §4 of Chapter 1). — *Consider an element* $H = H_0 + tH_1$ *with* $H_1 \in \widehat{\mathcal{Q}}[[t]]$. *If* H_0 *is diagonalisable then so is* H.

Of course Born, Heisenberg and Jordan provided a method for performing this quantised Birkhoff normal form. If we reformulate this theorem we get:

THEOREM 1.8. — *For any element* $H = qp + tH_1$ *with* $H_1 \in \widehat{\mathcal{Q}}[[t]]$ *there exist an automorphism* $\varphi \in \mathrm{Aut}(\widehat{\mathcal{Q}}[[t]])$ *and an element* $u \in \mathbb{C}[[\hbar, t, z]]$ *such that* $\varphi(H) = u \circ (qp)$.

Let us now clarify the notion of perturbative expansion for the spectrum. By substituting $\hbar \partial_z$ and z to p, q, we get a representation

$$\widehat{\mathcal{Q}}[[t]] \longrightarrow \mathrm{Hom}_{\mathbb{C}[[\hbar, t]]}(\mathbb{C}[[\hbar, t, z]], \mathbb{C}[[\hbar, t, z]])$$

and therefore a notion of spectrum. Put $H = H_0 + tH_1 \in \widehat{\mathcal{Q}}[[t]]$ with $H_0 \in \widehat{\mathcal{Q}}$. What is classically called a *perturbative expansion of an eigenvalue* $E_0 \in \widehat{\mathrm{Sp}}(H_0)$ is just a spectral value $E \in \mathbb{C}_\hbar[[t]]$ of H for which $E(t = 0, \cdot) = E_0$. Theorem 1.8 has the following corollary.

COROLLARY 1.9. — *Under the assumptions of the previous theorem, the series* u *maps the spectrum of* $H_0 = H(t = 0, \cdot) \in \widehat{\mathcal{Q}}$ *to that of* H, *i.e., the mapping*

$$\widehat{\mathrm{Sp}}(H) \longrightarrow \widehat{\mathrm{Sp}}(H_0), \ E(t, h) \mapsto E(t = 0, h)$$

is a bijection whose inverse is given by u.

Our aim is to establish similar results in the analytic category.

2. Analytic quantisation deformation

2.1. Majorant series

We will make systematic use of the standard technique of majorant series. Consider the map

$$\mathrm{abs} : \mathbb{C}[[z]] \longrightarrow \mathbb{C}[[z]], \ f(z) = \sum a_i z^i \mapsto \sum |a_i| z^i, \ z = (z_1, \ldots, z_n).$$

The following conditions are equivalent:

(1) the expansion $f \in \mathbb{C}[[z]]$ defines the germ at the origin of a holomorphic function
(2) the expansion $\mathrm{abs}(f)$ defines the germ at the origin of a holomorphic function.

We use the (Poincaré) notation $g \gg f$ if each coefficient appearing in the expansion of g is a real number at least equal to the modulus of the corresponding coefficient in f; the expansion g is then called a *majorant* of the expansion f. Remark that $\mathrm{abs}(f) \gg f$ and that $f \gg 0$ if and only if the coefficients in the expansion of f are real and non-negative.

Given two maps $K, L : \mathbb{C}[[z]] \longrightarrow \mathbb{C}[[z]]$, we say that K *majorates* L and write $K \gg L$ if $K(g) \gg L(f)$ for any $g \gg f$. For instance, abs majorates the identity mapping. We use indifferently the notations $\mathbb{C}\{z\}, \mathbb{C}\{z_1, \ldots, z_n\}$ for the ring $\mathcal{O}_{\mathbb{C}^n, 0}$ in which we specify the labelling of the canonical coordinates.

PROPOSITION 2.1. — *Consider two maps $K, L : \mathbb{C}[[z]] \longrightarrow \mathbb{C}[[z]]$ such that $K \gg L$ then if K maps $\mathbb{C}\{z\}$ to itself then so does L.*

Proof. — For any $f \in \mathbb{C}\{z\}$, we also have $\mathrm{abs}(f) \in \mathbb{C}\{z\}$. As $\mathrm{abs}(f) \gg f$ and $K \gg L$, one has $K(\mathrm{abs}(f)) \gg L(f)$. As K maps $\mathbb{C}\{z\}$ to itself, we have $K(\mathrm{abs}(f)) \in \mathbb{C}\{z\}$ and therefore $L(f)$ is also a convergent power series. $\square$

2.2. Borel analytic functions

The map

$$B : \mathbb{C}[[\hbar]] \longrightarrow \mathbb{C}[[\hbar]], \ \sum_{n \geqslant 0} \alpha_n \hbar^n \mapsto \sum_{n \geqslant 0} \alpha_n \frac{\hbar^n}{n!}.$$

is called the (formal) *Borel transform*. We say that a formal power series is *Borel analytic* provided that its Borel transform is analytic, *i.e.,* convergent in some neighbourhood of the origin. These were originally called *functions*

of class two then became of *expansions of Gevrey class two*, and in more recent texts, they are sometimes called *expansions of Gevrey class one*.

Any Borel analytic formal power series is the asymptotic expansion associated to a holomorphic function (see [26]). The Borel transform maps the commutative algebra $(\mathbb{C}[[\hbar]], \cdot)$ with the standard product to the commutative algebra $(\mathbb{C}[[\hbar]], *)$ with the convolution product $*$ defined by

$$\left(\sum_j a_j \hbar^j\right) * \left(\sum_k b_k \hbar^k\right) := \sum_{j,k} a_j b_k \frac{j!k!}{(j+k)!}\hbar^{j+k},$$

so that $B(fg) = B(f) * B(g)$. The set of formal power series in $\hbar$ which are Borel analytic is denoted by $\mathbb{C}_\hbar$.

PROPOSITION 2.2.

(1) *The set $\mathbb{C}_\hbar$ of Borel analytic functions is a ring for the usual product.*
(2) *If the expansion $\sum_k a_k \hbar^k$, $a_k = \sum_j a_{jk}\hbar^j$, belongs to $\mathbb{C}_\hbar$ then the series $\sum_k B(a_k)\frac{\hbar^k}{k!}$ is analytic.*

Proof. — The first part of the proposition follows from the inequality $j!k! \leqslant (j+k)!$ and the second part from the inequality

$$(j+k)! \leqslant 2^{j+k} j!k!$$

This last inequality is a consequence of the Taylor series expansion (or of the Stirling formula)

$$\frac{1}{1-(z+t)} = \sum_{j,k} \frac{(j+k)!}{j!k!}z^j t^k.$$

$\square$

The notion of Borel convergence extends to any algebra of formal power series $A[[\hbar]]$ over a commutative ring A. We denote the Borel transform in $A[[\hbar]]$ by B without specifying the ring A.

In case $A = \mathbb{C}\{z\}$, $z = (z_1, \ldots, z_n)$ we denote denote by $\mathbb{C}_\hbar\{z\} \subset \mathbb{C}[[\hbar, z]]$ the commutative ring consisting of formal power series $u = \sum_{n,k} a_{nk}\hbar^k z^n$ which are Borel analytic, *i.e.*, such that $Bu := \sum_{n,k} a_{nk}\frac{\hbar^k}{k!}z^n$ is analytic. Remark that there is a vector space isomorphism $\mathbb{C}_\hbar\{z\} \approx \mathbb{C}_\hbar \hat{\otimes}_\mathbb{C} \mathbb{C}\{z\}$ induced by the multiplication, where $\hat{\otimes}$ denotes the topological tensor product [19].

2.3. The normal product

Define the *normal product* $\star$ of two power series $f, g \in \mathbb{C}[[\hbar, x, y]]$ by the formula

$$f \star g := \sum_{k \geqslant 0} \frac{\hbar^k}{k!} \partial_y^k f \partial_x^k g$$

For instance $y \star x = xy + \hbar$, $x \star y = xy$.

The multiplication by $\hbar$ induces an exact sequence of algebras

$$0 \longrightarrow (\mathbb{C}[[\hbar, x, y]], \star) \xrightarrow{\hbar} (\mathbb{C}[[\hbar, x, y]], \star) \longrightarrow (\mathbb{C}[[x, y]], \cdot) \longrightarrow 0$$

where $\cdot$ is the ordinary product.

Thus $(\mathbb{C}[[\hbar, x, y]], \star)$ is a flat deformation of the algebra $(\mathbb{C}[[x, y]], \cdot)$.

Does the normal product also define a flat deformation $(\mathbb{C}_\hbar\{x, y\}, \star)$ of the algebra $(\mathbb{C}\{x, y\}, \cdot)$?

The problem reduces to knowing whether the normal product of two elements in $(\mathbb{C}_\hbar\{x, y\}, \star)$ is again in $(\mathbb{C}_\hbar\{x, y\}, \star)$.

To provide an answer, extend the supremum norm

$$|f|_r = \sup_{|x| < r, \ |y| < r} |f(x, y)|$$

to $\mathbb{C}[[x, y]]$ by putting $|f|_r = +\infty$ if f is not holomorphic inside the disk of radius r centred at the origin and bounded on its boundary. Next, we extend it to $\mathbb{C}[[\hbar, x, y]]$ as a $\mathbb{C}[[\hbar]]$-linear map :

$$\mathbb{C}[[\hbar, x, y]] \longrightarrow \mathbb{R}[[\hbar]] \cup \{+\infty\}, \ \sum_{k \geqslant 0} f_k \hbar^k \mapsto \sum_{k \geqslant 0} |f_k|_r \hbar^k$$

(with the usual rule $+\infty + \alpha = +\infty, \forall \alpha \in \mathbb{R}[[\hbar]] \cup \{+\infty\}$).

An element $f \in \mathbb{C}[[\hbar, x, y]]$ is Borel analytic if and only if $|f|_r \in \mathbb{C}_\hbar$ for r sufficiently small.

PROPOSITION 2.3. — *The subvector space $\mathbb{C}_\hbar\{x, y\} \subset \mathbb{C}[[\hbar, x, y]]$ is stable under multiplication, i.e., it is a subalgebra for the normal product. More precisely, for any $r > 0$ and $\varepsilon \in]0, r[$, we have the estimate*

$$|f \star g|_{r-\varepsilon} \ll \eta |f|_r |g|_r, \ \eta = \sum_{k \geqslant 0} k! \hbar^k \varepsilon^{-2k}.$$

Proof. — As the normal product is $\mathbb{C}[[\hbar]]$-bilinear, it is sufficient to prove the estimate of the proposition in case f, g are independent of $\hbar$.

Denote by $D_r \subset \mathbb{C}$ the closed disk of radius r. The following lemma is known classically as the Cauchy inequalities.

LEMMA 2.4. — *The derivatives of any holomorphic function* $a : D_R \longrightarrow \mathbb{C}$ *satisfy the estimates*

$$\sup_{z \in D_{r-\varepsilon}} |a^{(k)}(z)| \leqslant k! \varepsilon^{-k} \sup_{z \in D_r} |a(z)|$$

for any $r < R$ *and any* $\varepsilon \in]0, r[$.

Proof. — Take $z \in D_{r-\varepsilon}$ and denote by γ be the oriented boundary of the disk centred at z of radius ε. The Cauchy integral formula gives

$$a^{(k)}(z) = \frac{k!}{2i\pi} \int_\gamma \frac{a(\xi)}{(\xi - z)^{k+1}} d\xi.$$

Using the parametrisation $\theta \mapsto z + \varepsilon e^{i\theta}$ for the path (γ), we get the estimate of the lemma. $\qquad \square$

The Cauchy inequalities give the estimates

$$|\partial_y^k f \partial_x^k g|_{r-\varepsilon} \leqslant (k!)^2 \varepsilon^{-k} |f|_r |g|_r$$

and therefore

$$|f \star g|_{r-\varepsilon} \ll \sum_{k \geqslant 0} \frac{\hbar^k}{k!} (k!)^2 \varepsilon^{-2k} |f|_r |g|_r.$$

This proves the proposition. $\qquad \square$

3. The quantum Morse lemma

3.1. The algebra $\mathcal{Q}$

An element $F \in \widehat{\mathcal{Q}}$ can always be *ordered*, i.e., written as a formal sum $F = \sum \alpha_{mnk} q^m p^n \hbar^k$ with the q's before the p's. This ordering is called the *normal ordering*.

The ring $\widehat{\mathcal{Q}}$ is not commutative but the normal ordering allows us to define a Borel transform $\widehat{\mathcal{Q}} \longrightarrow \mathbb{C}[[\hbar, x, y]]$ by setting

$$\sum_{m,n,k \geqslant 0} \alpha_{mnk} q^m p^n \hbar^k \mapsto \sum_{m,n,k \geqslant 0} \frac{\alpha_{mnk}}{k!} x^m y^n \hbar^k.$$

and an "abs" mapping

$$\text{abs} \left(\sum_{m,n,k \geqslant 0} \alpha_{mnk} q^m p^n \hbar^k \right) = \sum_{m,n,k \geqslant 0} |\alpha_{mnk}| q^m p^n \hbar^k.$$

The *total symbol*

$$s : \widehat{\mathcal{Q}} \longrightarrow \mathbb{C}[[\hbar, x, y]]$$

is defined by taking the normal ordering and then replacing the variables q, p with commuting variables x, y:

$$s(F) = \sum_{m,n,k \geqslant 0} \alpha_{mnk} x^m y^n \hbar^k.$$

The *principal symbol*

$$\sigma : \widehat{\mathcal{Q}} \longrightarrow \mathbb{C}[[x,y]] \approx \widehat{\mathcal{Q}}/\hbar\widehat{\mathcal{Q}}$$

is obtained by restricting the total symbol to $\hbar = 0$.

PROPOSITION 3.1 ([29]). — *The total symbol induces an isomorphism between the algebras $\widehat{\mathcal{Q}}$ and $(\mathbb{C}[[\hbar, x, y]], \star)$, that is,*

$$s(FG) = f \star g, \ f = s(F), \ g = s(G).$$

DEFINITION 3.2 ([31, 36]). — *The algebra $\mathcal{Q}$ is the subalgebra of $\widehat{\mathcal{Q}}$ consisting of power series having a convergent Borel transform:*

$$\mathcal{Q} = \left\{ F \in \widehat{\mathcal{Q}}, \ BF \in \mathbb{C}\{\hbar, x, y\} \right\}.$$

Therefore the total symbol induces an isomorphism between the algebras $\mathcal{Q}$ and $(\mathbb{C}_\hbar\{x, y\}, \star)$. The centre of the algebra $\mathcal{Q}$ is the ring $\mathbb{C}_\hbar$ of Borel analytic functions in $\hbar$.

3.2. The composition property

If $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ and $u : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ are germs of holomorphic mappings then so is $u \circ f$. In algebraic terms, the image of the subalgebra $\mathbb{C}\{z\}$ under the homomorphism

$$\mathbb{C}[[z]] \longrightarrow \mathbb{C}[[x, y]], \ z \mapsto f$$

is contained in the subalgebra $\mathbb{C}\{x, y\}$ provided that f lies in $\mathbb{C}\{x, y\}$.

It is readily seen that this property extends to $\mathbb{C}_\hbar\{x, y\}$: if $f \in \mathbb{C}_\hbar\{x, y\}$ and $u = \sum_{n \geqslant 0} u_n z^n \in \mathbb{C}_\hbar\{z\}$ then the formal power series $u \circ f := \sum_{n \geqslant 0} u_n f^n$ lies in $\mathbb{C}_\hbar\{x, y\}$. In the non-commutative algebra $\mathcal{Q}$ a similar property holds.

PROPOSITION 3.3. — *For any $u \in \mathbb{C}_\hbar\{z\}$, $u = \sum_{n \geqslant 0} u_n z^n$, and any $F \in \mathcal{Q}$, the element $u \circ F := \sum_{n \geqslant 0} u_n F^n$ belongs to the algebra $\mathcal{Q}$.*

Proof. — As $u \circ F \ll \mathrm{abs}(u) \circ \mathrm{abs}(F)$, we may assume, without loss of generality, that $F \gg 0$, $u \gg 0$.

Denote by $f \in \mathbb{C}_\hbar\{x, y\}$ the total symbol of F. The formal power series $u \circ g = \sum_{n \geqslant 0} u_n g^n$ is Borel analytic for any $g \in \mathbb{C}_\hbar\{x, y\}$. The estimate in Proposition 2.3 gives:

$$|s(u \circ F)|_{r-\varepsilon} \ll |u \circ (\eta f)|_r, \quad \eta = \sum_{k \geqslant 0} k! \hbar^k \varepsilon^{-2k}.$$

As $\eta f \in \mathbb{C}_\hbar\{x, y\}$ is Borel analytic, the element $u \circ F$ is also Borel analytic. This proves the proposition. $\qquad\qquad\square$

This proposition shows that the choice of an element $f \in \mathcal{Q}$ induces in the algebra $\mathcal{Q}$ a $\mathbb{C}_\hbar\{z\}$-module structure obtained by substituting f to the variable z.

3.3. The algebra $\mathcal{Q}\{\lambda\}$

There is a variant of the algebra $\mathcal{Q}$ with parameters.

Let $\widehat{\mathcal{Q}}[[\lambda]] := \widehat{\mathcal{Q}}[[\lambda_1, \ldots, \lambda_k]]$ be the non-commutative algebra over the ring of formal power series $\mathbb{C}[[\hbar, \lambda]] := \mathbb{C}[[\hbar, \lambda_1, \ldots, \lambda_k]]$ consisting of formal power series in the variables $p, q, \hbar, \lambda$ where the only non-trivial commutation relation is $[p, q] = \hbar$. Consider the Borel transform

$$B : \widehat{\mathcal{Q}}[[\lambda]] \longrightarrow \mathbb{C}[[\hbar, \lambda, x, y]], \ f \mapsto \sum_{m,n,k \geqslant 0} \frac{\alpha_{mnk}}{k!} x^m y^n \hbar^k, \quad \alpha_{mnk} \in \mathbb{C}[[\lambda]].$$

We denote by $\mathcal{Q}\{\lambda\}$ the algebra of elements having a convergent Borel transform in the $\hbar$ variable:

$$\mathcal{Q}\{\lambda\} := \left\{ f \in \widehat{\mathcal{Q}}, \, Bf \in \mathbb{C}\{\hbar, \lambda, x, y\} \right\} \approx \mathcal{Q} \widehat{\otimes}_\mathbb{C} \mathbb{C}\{\lambda\}.$$

The centre of $\mathcal{Q}\{\lambda\}$ is the ring $\mathbb{C}_\hbar\{\lambda\}$ of formal power series $\sum_{j,k} \alpha_{j,k} \lambda^j \hbar^k$ for which the series $\sum_{j,k} \frac{\alpha_{j,k}}{k!} \lambda^j \hbar^k$ is analytic. If $k = 1$, we often denote the parameter by t or z instead of λ.

Like in the absolute case $\mathcal{Q}\{\lambda\}$ is a ring and the choice of $F \in \mathcal{Q}\{\lambda\}$ induces a $\mathbb{C}_\hbar\{\lambda, z\}$-module structure on $\mathcal{Q}\{\lambda\}$ obtained by substituting F to z.

3.4. Statement of the quantum Morse lemma

The proof of the analyticity of perturbative expansions for the spectrum of a perturbed harmonic oscillator is based on the following theorem.

THEOREM 3.4. — *Consider an element $H = H_0 + tH_1 \in \mathcal{Q}\{t\}$ with $H_0 \in \mathcal{Q}$. If the principal symbol of $H_0 \in \mathcal{Q}$ is a Morse function germ* [3] *then there exist an automorphism $\varphi \in \mathrm{Aut}(\mathcal{Q}\{t\})$ and a function germ $u \in \mathbb{C}_\hbar\{t, z\}$ such that $u \circ \varphi(H) = H_0$.*

The proof of this theorem will be given in Section 6

Remark 3.5. — In the limit $\hbar \longrightarrow 0$, the theorem gives the Vey isochore Morse lemma [37] and if we consider only formal power series in $\hbar$, then the formal variant of the theorem is equivalent to this isochore Morse lemma [9].

By taking a linear interpolation between $H \in \mathcal{Q}$ and its quadratic part, we deduce the following corollary which generalises previous results of Helffer and Sjöstrand ([21], Théorème b1 and Théorème b6).

COROLLARY 3.6. — *For any element $H \in \mathcal{Q}\{t\}$ whose principal symbol is a Morse function germ there exist an automorphism $\varphi \in \mathrm{Aut}(\mathcal{Q})$ and a function germ $u \in \mathbb{C}_\hbar\{z\}$ such that the equality $u \circ \varphi(H) = qp$ holds.*

4. Differential calculus in the algebra $\mathcal{Q}$

We use the old-fashioned notions and notations of quantum mechanics [39]. For notational reasons, we consider the algebra $\mathcal{Q}$ but the results of this section admit straightforward generalisations to the algebra $\mathcal{Q}\{\lambda\}$.

4.1. The evolution operator

The rings $\mathcal{Q}$, $\mathcal{Q}\{t\}$, $\mathcal{Q}[[t]]$ and $\widehat{\mathcal{Q}}[[t]]$ are non-commutative Poisson algebras for the Poisson bracket defined by

$$\{F, H\} := \frac{1}{\hbar}[F, H].$$

For any $H \in \widehat{\mathcal{Q}}[[t]]$, the operator $U \in \widehat{\mathcal{Q}}[[t]]$ satisfying the equation $\dot{U} = HU$ with initial condition $U(t = 0, \cdot) = 1$ is called the *evolution operator* of H.

If H is t-independent then the evolution is given by the exponential series:

$$U = \exp(tH) = \sum_{n \geqslant 0} \frac{t^n H^n}{n!}.$$

By Proposition 3.3, for $H \in \mathcal{Q}$ we also have $U \in \mathcal{Q}\{t\}$.

---

[3] A holomorphic function germ $f : (\mathbb{C}^n, 0) \longrightarrow \mathbb{C}$ is called of *Morse type* if $df(0) = 0$ and if its Hessian $d^2 f(0)$ is a non-degenerate quadratic form.

PROPOSITION 4.1. — *If H lies in $\mathcal{Q}\{t\}$ then so does its evolution operator U. More precisely, put $G = (\mathrm{abs}(H))(t = r, \cdot) \in \mathcal{Q}$ then $U(r, \cdot) \ll \exp(rG)$ provided that r is small enough.*

Proof. — First, we generalise the formula given by the exponential series to the non-autonomous case.

LEMMA 4.2. — *The evolution operator associated to an element $H = \sum_{k \geqslant 0} h_k t^k$, $h_k \in \widehat{\mathcal{Q}}$, is given by the formula*

$$U = 1 + \sum_{k \geqslant 0} \sum_{i \in \mathbb{Z}_{>0}^{k+1}} c_i h_{i_k} \ldots h_{i_1} t^{k+|i|}, \ |i| = i_1 + i_2 + \cdots + i_k$$

with $c_i = (i_1 + 1)^{-1}(i_1 + i_2 + 2)^{-1} \ldots (i_1 + i_2 + \cdots + i_k + k)^{-1}$.

Proof. — Define U as in the lemma. We have

$$\partial_t U = \sum_{k \geqslant 0} \sum_{i \in \mathbb{Z}_{>0}^{k}} c_i(k + |i|) h_{i_k} \ldots h_{i_1} t^{k+|i|-1}$$

and

$$(k + |i|)c_{i_1,\ldots,i_k} = c_{i_1,\ldots,i_{k-1}}$$

therefore relabelling the indices, we get the equality

$$\partial_t U = \sum_{k \geqslant 0} \sum_{j \geqslant 0} \sum_{i \in \mathbb{Z}_{>0}^{k-1}} c_i h_j h_{i_k} \ldots h_{i_1} t^j t^{k+|i|} = HU.$$

This proves the lemma. $\qquad\qquad\square$

We now prove the estimate stated in the proposition.

Take r small enough such that $G := \mathrm{abs}(H)(t = r, \cdot)$ lies in $\mathcal{Q}$. We proceed in two steps: first we show that the evolution operator $V \in \mathcal{Q}[[t]]$ of $\mathrm{abs}(H)$ is a majorant series for the evolution operator U of H and then show that $\exp(rG)$ is a majorant for V.

Put $U = \sum_{n \geqslant 0} u_n t^n$, $V = \sum_{n \geqslant 0} v_n t^n$ and $H = \sum_{n \geqslant 0} h_n t^n$. The function u_n, v_n are defined by the recursions

$$u_n = \frac{\sum_j u_{n-j} h_j}{n}, \ v_n = \frac{\sum_j v_{n-j} \, \mathrm{abs}(h_j)}{n}.$$

The formula for the normal product (Proposition 3.1) implies by induction on n that $u_n \ll v_n$. Thus $U \ll V$, this proves the first step.

Using the notations of the previous lemma, we have the equality :

$$V(r, \cdot) = 1 + \sum_{k \geqslant 0} \sum_{i \in \mathbb{Z}_{>0}^{k+1}} c_i \, \mathrm{abs}(h_{i_k}) \ldots \mathrm{abs}(h_{i_1}) r^{k+|i|}$$

while the exponential series gives :

$$\exp(rG) = 1 + \sum_{k \geqslant 0} \frac{1}{k!} \Big(\sum_{i \geqslant 0} (\mathrm{abs}(h_i)r^i) \Big)^k r^k.$$

Expanding this last series, we get that

$$\exp(rG) = 1 + \sum_{k \geqslant 0} \sum_{i \in \mathbb{Z}_{>0}^{k+1}} \frac{1}{k!} \mathrm{abs}(h_{i_k}) \dots \mathrm{abs}(h_{i_1}) r^{k+|i|}.$$

As $c_i \leqslant \frac{1}{k!}$, this equality implies that $V \ll \exp(rG)$ provided that r is small enough. This proves the second step.

The element e^{tG} belongs to $\mathcal{Q}\{t\}$ therefore so does V. This proves the lemma and concludes the proof of the proposition. $\qquad\square$

4.2. Integration of the Heisenberg equations

By *Heisenberg equations*, we mean a non-autonomous evolution equation of the type

$$\dot{F} = \{H, F\}, \ H, F \in \mathcal{Q}\{t\}, \ F(t=0, \cdot) = f$$

where the dot denotes the derivative with respect to t.

PROPOSITION 4.3. — *If U is the evolution operator of $H \in \mathcal{Q}\{t\}$ then the morphism*

$$\varphi : \mathcal{Q} \longrightarrow \mathcal{Q}\{t\}, f \mapsto U\Big(\frac{t}{\hbar}\Big) f U^{-1}\Big(\frac{t}{\hbar}\Big), \ U \in \mathcal{Q}\{t\},$$

integrates the Heisenberg equations of $H \in \mathcal{Q}\{t\}$, that is:

$$\frac{d}{dt} \varphi(f) = \varphi(\{H, f\}) = \{H, \varphi(f)\}, \ \forall f \in \mathcal{Q}.$$

Proof. — By Proposition 4.1, it is sufficient to consider the case $U = \exp(tH)$, $H \in \mathcal{Q}$ with $H \gg 0$.

Take $f \in \mathcal{Q}$ with $f \gg 0$ and consider the function $F = \exp(tH) f \exp(-tH)$. As $\partial_t^k F_{t=0} = [\cdots \underbrace{[H, \cdots, [H}_{k-times}, f] \dots]$, the element F is given by the expansion

$$F = \sum_{k \geqslant 0} \frac{\hbar^k t^k}{k!} \Big\{ \cdots \Big\{ \underbrace{H, \cdots, \{H}_{k-times}, f\} \cdots \Big\}$$

and therefore the element $\varphi(f)$ is given by the formal expansion

$$\varphi(f) = \sum_{k \geqslant 0} \frac{t^k}{k!} \Big\{ \cdots \Big\{ \underbrace{H, \cdots, \{H}_{k-times}, f\} \cdots \Big\}.$$

A priori the process of dividing t by $\hbar$ guaranties only that $\varphi(f)$ is Borel analytic in t and not necessarily analytic. Let us prove that it is indeed analytic.

Consider the endomorphism of $\widehat{\mathcal{Q}}$ defined by

$$\phi_1 : f \mapsto \sum_{k \geqslant 0} t^k \left[\cdots \left[\underbrace{H, \cdots, [H, f]}_{k-times} \right] \cdots \right].$$

I assert that ϕ_1 maps the ring $\mathcal{Q}$ to $\mathcal{Q}\{t\}$. To see it put

$$\xi_k(f) = \sum_{j \geqslant 0} \binom{k}{j} H^{k-j} f H^j, \ \tilde{\xi}_k(f) = \sum_{j \geqslant 0} (-1)^j \binom{k}{j} H^{k-j} f H^j$$

and define

$$\phi_2 : f \mapsto \sum_{k \geqslant 0} t^k \xi_k(f).$$

As $\phi_1 = \sum_{k \geqslant 0} t^k \tilde{\xi}_k$ and $H \gg 0$, we have $\phi_2 \gg \phi_1$ and using the estimate of Proposition 2.3, we get that

$$|s(\xi_k(f))|_{r-\varepsilon} \ll |2^k \eta^k s(H)^k s(f)|_r, \ \eta = \sum_{k \geqslant 0} k! \hbar^k \varepsilon^{-2k}$$

for any r, ε small enough. Here as usual $s(\cdot)$ stands for the total symbol. Using this estimate, we get that

$$|s(\phi_2(f))|_{r-\varepsilon} \ll |s(f) \sum_{k \geqslant 0} 2^k t^k \eta^k s(H)^k|_r = \left| \frac{s(f)}{1 - 2t\eta s(H)} \right|_r.$$

This proves that ϕ_2 and consequently ϕ_1 map the ring $\mathcal{Q}$ to $\mathcal{Q}\{t\}$.

Write

$$\phi_1(f) = \sum_{k \geqslant 0} \hbar^k t^k \left\{ \cdots \left\{ \underbrace{H, \cdots, \{H, f\}}_{k-times} \right\} \cdots \right\}.$$

By Proposition 2.2, the series

$$\sum_{k \geqslant 0} \frac{t^k \hbar^k}{k!} B\left(\left\{ \cdots \left\{ \underbrace{H, \cdots, \{H, f\}}_{k-times} \right\} \cdots \right\} \right).$$

is analytic, thus the series $B\varphi(f)$ is also analytic. This proves the proposition. $\qquad\square$

Remark 4.4. — This proposition implies that a change of polarisation induces an automorphism of the $\mathcal{Q}$-algebra. For instance, if the series $\sum \alpha_{mn} q^m p^n$ lies in $\mathcal{Q}$ then so does the series $\sum \alpha_{mn} p^m q^n$.

Remark 4.5. — This proposition yields a formula for the integration of ordinary differential equations. Consider, for instance, the differential equation $\dot{x} = v(x)$ where $v : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ is a holomorphic function germ. Define the function $H = v(q)p$. The flow of the vector field $v(x)\partial_x$ is given by the formula

$$\varphi(t) = \sigma\left(\exp\left(t\frac{H}{\hbar}\right) q \exp\left(-t\frac{H}{\hbar}\right) \right).$$

This formula is just a different way of writing the Taylor type formula $\varphi(t) = e^{tL_v}x$ where L_v denotes the Lie derivative along v.

4.3. Derivations in $\mathcal{Q}\{t\}$

Following Born, Jordan and Heisenberg [4], we define partial derivatives

$$\partial_q f := -\{f, p\}, \ \partial_p f := \{f, q\}, \ f \in \mathcal{Q}$$

with $\{.,.\} = \frac{1}{\hbar}[.,.]$. We denote by $\int f dq$ (resp. $\int f dp$) the only function germ $F \in \mathcal{Q}$ such that

(1) $\partial_q F = f$ (resp. $\partial_p F = f$),
(2) F is divisible by q, i.e., there exists $G \in \mathcal{Q}$ such that $F = qG$ (resp. $F = pG$).

For instance, if we write $f = \sum_{m,n \geqslant 0} \alpha_{mn} q^m p^n$ we get

$$\int f dq = \sum_{m,n \geqslant 0} \frac{\alpha_{mn}}{m+1} q^{m+1} p^n, \quad \partial_q f = \sum_{m,n \geqslant 0} m\alpha_{mn} q^{m-1} p^n.$$

A *derivation* $D : \mathcal{Q}\{t\} \longrightarrow \mathcal{Q}\{t\}$ of the algebra $\mathcal{Q}\{t\}$ over the ring $\mathbb{C}_\hbar\{t\}$ is a $\mathbb{C}_\hbar\{t\}$-linear mapping satisfying the Leibniz rule. Due to the non-commutativity of the algebra $\mathcal{Q}\{t\}$, the space of $\mathcal{Q}\{t\}$-derivations is not a $\mathcal{Q}\{t\}$-module but only a $\mathbb{C}_\hbar\{t\}$-module.

PROPOSITION 4.6. — *For any derivation D of the algebra $\mathcal{Q}\{t\}$, there exists an element $G \in \mathcal{Q}\{t\}$ such that $D = \{G, \cdot\}$. The function germ G is related to the derivation D by the formula*

$$G = \int (Dq)dp - \int (Dp)dq + \int \int \{Dq, p\}dpdq.$$

Proof. — Define $F = \int (Dq)dp$, then we have the equality $Dq = \partial_p F = \{F, q\}$. I assert that the function germ $Dp - \{F, p\}$ does not depend on p, that is, $\{Dp, q\} = \{\{F, p\}, q\}$. As D is a derivation, we have the equalities

$$D\{p, q\} = \{Dp, q\} + \{p, Dq\} = 0;$$

from which we deduce that $\{Dp, q\} = \{Dq, p\} = \{\{F, q\}, p\}$. Finally, using the Jacobi identity, we get that $\{\{F, q\}, p\} = \{\{F, p\}, q\}$. This proves the assertion.

The assertion implies that $F' = \int (Dp - \{F, p\})dq$ is a function of q independent on p. We put $G = F - F'$, then $Dq = \{G, q\}$ and $Dp = \{G, p\}$. This concludes the proof of the proposition. $\square$

Remark 4.7. — Define the non-commutative de Rham complex by putting

$$\Omega^0 := \mathcal{Q}\{t\}, \quad \Omega^1 := \text{ free } \mathcal{Q}\{t\}\text{module generated by } dq, dp$$

$\Omega^2 = \bigwedge^2 \Omega^1$ and the differential is given by the usual formula. One can prove a Poincaré lemma for this complex. The proof of the existence of G is a consequence of this fact. Nevertheless such a complex is hard to handle because, due to the non-commutativity of the ring $\mathcal{Q}$, it behaves badly under automorphism of the ring $\mathcal{Q}\{t\}$.

Remark 4.8. — The proposition yields a proof of the Poincaré lemma. Take, for instance, a holomorphic differential one form $\alpha(x)dx \in \Omega^1_{\mathbb{C},0}$. To this form, we associate the derivation $D = \alpha(q)\partial_p$ in $\mathcal{Q}$. We have $D = \{G, \cdot\}$ and therefore $\alpha = d\sigma(G)$ where σ denotes the principal symbol. This proof can be easily extended to arbitrary dimensions.

4.4. Non-commutative derivatives in $\mathbb{C}_\hbar\{z\}$

Let $f, v : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be holomorphic function germs. For any holomorphic function germ $u : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$, the chain rule gives:

$$u \circ (f + \varepsilon v) = u \circ f + \varepsilon(\partial_z u \circ f)v + o(\varepsilon).$$

In the non-commutative ring $\mathcal{Q}$ such a formula does not hold[4]. Take for instance $u = z^2$, $f = q$, $v = p$ then

$$u \circ (q + \varepsilon p) = q^2 + \varepsilon(qp + pq) + \varepsilon^2 = q^2 + 2\varepsilon qp + \varepsilon\hbar + o(\varepsilon)$$

while

$$u \circ f + \varepsilon(\partial_z u \circ f)v = q^2 + 2\varepsilon qp.$$

DEFINITION 4.9. — The $\mathcal{Q}$-derivative of a map $u \in \mathbb{C}_\hbar$ at $f \in \mathcal{Q}$ is the linear map $Du(f) : \mathcal{Q} \longrightarrow \mathcal{Q}$ defined by

$$u \circ (f + \varepsilon v) = u \circ f + \varepsilon Du(f) \cdot v + o(\varepsilon).$$

_____

[4] This fact was pointed out to me by B. Malgrange.

PROPOSITION 4.10. — *Write $u(z) = \sum u_n z^n$, $u_n \in \mathbb{C}_\hbar$, then the $\mathcal{Q}$-derivative of $u \in \mathbb{C}_\hbar\{z\}$ at $f \in \mathcal{Q}$ is given by the formula*

$$Du(f) \cdot v = \sum_n u_n (f^{n-1}v + f^{n-2}vf + \cdots + vf^{n-1}), \ v \in \mathcal{Q}.$$

Proof. — This is a direct consequence of the expansion

$$(f + \varepsilon v)^n = f^n + \varepsilon(f^{n-1}v + f^{n-2}vf + \cdots + vf^{n-1})\mathrm{mod}\,\varepsilon^2.$$

$\square$

If u is invertible for the composition law then the inverse of its $\mathcal{Q}$-derivative satisfies the equality $Du(f)Du^{-1}(u \circ f) = \mathrm{Id}$.

The case with parameters $\mathcal{Q}\{t\}$ is similar. For instance, given two elements $f, v \in \mathcal{Q}\{t\}$, we put

$$u \circ (f + \varepsilon v) = u \circ f + \varepsilon Du(f) \cdot v + o(\varepsilon).$$

We have the chain rule formula

$$\frac{\partial}{\partial t}(u \circ f) = \left(\frac{\partial}{\partial t}u\right) \circ f + Du(f) \cdot \frac{\partial}{\partial t}f.$$

5. The analytic quantum versal deformation module

5.1. Basic facts

DEFINITION 5.1. — *The (analytic) quantum versal deformation module associated to H is the $\mathbb{C}_\hbar\{z\}$-module $M(H) := \mathcal{Q}/\{H, \mathcal{Q}\}$.*

As the map $F \mapsto \{H, F\}$ is $\mathbb{C}_\hbar\{z\}$-linear:

$$\{H, zF\} = \{H, HF\} = H\{H, F\} = z\{H, F\}.$$

Thus, the space $M(H)$ inherits a $\mathbb{C}_\hbar\{z\}$-module structure.

Given a ring R, let us denote by (f) the ideal generated by $f \in R$.

As shown in the following proposition, the quantum versal deformation module parametrise deformations over $\mathrm{Spec}(\mathbb{C}_\hbar[\varepsilon]/(\varepsilon^2))$ modulo the ones given by automorphisms. To simplify our notations, we denote the class of ε in $\mathcal{Q}[\varepsilon]/(\varepsilon^2)$ simply by ε.

PROPOSITION 5.2. — *An element $H_v = H + \varepsilon v \in \mathcal{Q}[\varepsilon]/(\varepsilon^2)$ is of the type $H_v = u \circ \varphi(H)$ with $u \in \mathbb{C}\{\varepsilon, z\}/(\varepsilon^2)$ and $\varphi \in \mathrm{Aut}(\mathcal{Q}[\varepsilon]/(\varepsilon^2))$ provided that v lies in the $\mathbb{C}\{z\}$-module generated by 1.*

Proof. — Assume that $[v] = r[1]$, this means that there exists $F \in \mathcal{Q}$ such that
$$v = r \circ H + \{F, H\}.$$
Take $\varphi = \mathrm{Id} + \varepsilon\{F, \cdot\}$ and $u(z) = z + \varepsilon r(z)$, then
$$u \circ \varphi(H) = u \circ (H + \varepsilon\{F, H\}) = H + \varepsilon r \circ H + \varepsilon\{F, H\}.$$
This proves the proposition. □

Consider the complex
$$C_H^{\cdot} : 0 \longrightarrow \mathcal{Q} \longrightarrow \mathcal{Q} \longrightarrow 0$$
where the only non zero boundary map is given by $G \mapsto \{G, H\}$. The module $H^0(C_H^{\cdot}) \approx \mathbb{C}_\hbar\{z\}$ is freely generated by the class of 1 provided that the principal symbol of H has an isolated critical point at the origin (or no critical point at all) and $H^1(C_H^{\cdot}) = M(H)$.

The differential of the complex $C_H^{\cdot}$ being only $\mathbb{C}_\hbar\{z\}$-linear and not $\mathcal{Q}$-linear the cohomology space have only $\mathbb{C}_\hbar\{z\}$-module structure.

(The module $M(H)/\hbar M(H)$ is the *Lagrange complex* of $f = \sigma(H)$ which is in this case isomorphic to its Brieskorn lattice of f [17]).

5.2. Finiteness theorem

THEOREM 5.3. — *Assume that the principal symbol $f = \sigma(H)$ of H has an isolated critical critical point at the origin in $\mathbb{C}^2$, then the module $M(H)$ is a free finite type module of rank $\dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(\partial_x f, \partial_y f)$.*

Example 5.4. — Take $H = p^2 - q^2$, then according to the theorem, the module $M(H)$ is free of rank $\dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(x, y) = 1$. As the class of 1 is non-zero, it generates this module. More generally if the principal symbol of H is a Morse function germ then, according to the theorem, the module $M(H)$ is generated by the class of 1.

The theorem might be seen as a quantisation of results obtained by Brieskorn and Deligne ([7], Satz 1 for the coherence and [7], Proposition 1.8, Bemerkungen 2 for the freeness).

There is a variant of the quantum versal deformation space with parameters: given $H \in \mathcal{Q}\{\lambda\}$ the space $M(H) = \mathcal{Q}\{\lambda\}/\{H, \mathcal{Q}\{\lambda\}\}$ has a $\mathbb{C}_\hbar\{z, \lambda\}$-module structure :
$$\sum_{n \geqslant 0} a_n z^n \circ [m] := \left[\sum_{n \geqslant 0} a_n H^n m\right], \ a_n \in \mathbb{C}_\hbar\{\lambda\}, \ m \in \mathcal{Q}\{\lambda\}.$$
The following theorem is the main technical result of the paper.

THEOREM 5.5. — *For any germ $H \in \mathcal{Q}\{\lambda\}$, $\lambda = (\lambda_1, \ldots, \lambda_k)$, such that the principal symbol f of $H_{|\lambda=0}$ has an isolated critical point at the origin, the space $M(H) = \mathcal{Q}\{\lambda\}/\{H, \mathcal{Q}\{\lambda\}\}$ is a finite type free $\mathbb{C}_\hbar\{z, \lambda\}$-module of rank $\dim_\mathbb{C} \mathbb{C}\{x, y\}/(\partial_x f, \partial_y f)$.*

Example 5.6. — Take $k = 1$ and write t instead of λ for the parameter. According to the theorem $M(H)$ is a free module of rank one for any H of the type $p^2 - q^2 + tH_1$ with $H_1 \in \mathcal{Q}\{t\}$. As the class of 1 is non-zero, it generates the module $M(H)$. More generally, if the principal symbol of H restricted to $t = 0$ has a non-degenerate quadratic part then the class of 1 generates the module $M(H)$.

To understand the proof of this theorem, let us recall the formulation of the finiteness theorem we gave in [14] for the commutative case.

5.3. Finiteness and constructibility for the sheaf $\mathcal{O}_X$

Consider a map $F : X \longrightarrow S$, $S \subset \mathbb{C}^l$ between Whitney stratified manifolds and assume that it satisfies Thom's a_F condition.

DEFINITION 5.7. — *A sheaf $\mathcal{F}$ on X is called F-constant if $\mathcal{F} \approx F^{-1}(F)_*\mathcal{F}$.*

DEFINITION 5.8. — *A sheaf $\mathcal{F}$ is called F-constructible if at each point $x \in X$ there exists a neighbourhood U inside the stratum of x such that the restriction of $\mathcal{F}$ to U is $F_{|U}$-constant.*

A complex of coherent sheaves is called *F-constructible* if its cohomology sheaves are F-constructible and if its differential is $F^{-1}\mathcal{O}_S$-linear. Similar notions hold for germs of mappings.

THEOREM 5.9 ([14]). — *Let $F : (\mathbb{C}^k \times \mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^l, 0)$ be a holomorphic map germ satisfying the a_F-condition. The cohomology spaces $H^k(K^\cdot)$ associated to a complex of F-constructible $\mathcal{O}_{\mathbb{C}^{k+n},0}$-coherent modules are $\mathcal{O}_{\mathbb{C}^l,0}$-modules of finite type.*

Typical examples of applications are the absolute and relative de Rham complex of an isolated hypersurface singularity.

5.4. The sheaf $\mathcal{Q}_{\mathbb{C}^2}$

Consider the map

$$ j : \mathbb{C}^2 \longrightarrow \mathbb{C}^3 = \{(\hbar, x, y)\}, \ (x, y) \mapsto (0, x, y). $$

We denote by $\mathcal{O}_{\mathbb{C}^3|\mathbb{C}^2}$ the sheaf $j^{-1}\mathcal{O}_{\mathbb{C}^3}$ consisting of restriction of the holo-morphic functions in $\mathbb{C}^3$ to the hyperplane $\hbar = 0$. Let $U \subset \mathbb{C}^2$ be an open subset.

PROPOSITION 5.10. — *If F, G are Borel analytic in $U \subset \mathbb{C}^2$, i.e., $B(F)$, $B(G) \in \mathcal{O}_{\mathbb{C}^3|\mathbb{C}^2}(U)$ then their normal product $F \star G \in \mathcal{Q}$ is also Borel analytic in U.*

The proof is similar to that of Proposition 2.3. This proposition shows that the algebras $\mathcal{Q}$ induces a sheaf of algebras $\mathcal{Q}_{\mathbb{C}^2}$ in $\mathbb{C}^2$ defined by the presheaf:
$$f \in \mathcal{Q}_{\mathbb{C}^2}(U) \iff Bf \in \mathcal{O}_{\mathbb{C}^3|\mathbb{C}^2}(U).$$
The total symbol maps isomorphically the sheaf of algebras $\mathcal{Q}_{\mathbb{C}^2}$ to the sheaf $(\mathcal{O}_{\mathbb{C}^3|\mathbb{C}^2}, \star)$ where $\star$ denotes the normal product.

Consider the sheaf $\mathcal{B}_{\mathbb{C}}$ of *analytic functions with Borel analytic coeffi-cients* defined by the presheaf:
$$U \longrightarrow \mathcal{B}_{\mathbb{C}}(U) = \{f \in \mathbb{C}[[\hbar, z]], Bf \in \mathcal{O}_{\mathbb{C}^2|\mathbb{C}}(U)\}.$$
The algebra $\mathbb{C}_\hbar\{z\}$ is the stalk at the origin of the sheaf $\mathcal{B}_{\mathbb{C}}$.

PROPOSITION 5.11. — *Given any section $H \in \mathcal{Q}_{\mathbb{C}^2}(U)$ over an open subset $U \subset \mathbb{C}^2$ and any $u = \sum_n a_n z^n \in \mathcal{B}_{\mathbb{C}}(f(U))$ where f is the principal symbol of H, the element $u \circ H$ belongs to $\mathcal{Q}_{\mathbb{C}^2}(U)$.*

The proof is similar to that of Proposition 3.3.

The sheaf $\mathcal{Q}_{\mathbb{C}^2}$ was considered in [31] (see also [32]). The formal version of this sheaf $\widehat{\mathcal{Q}}_{\mathbb{C}^2} \approx (\mathcal{O}_{\mathbb{C}^2}[[\hbar]], \star)$ is standard in deformation quantisation (see [10] and references therein).

5.5. The sheaf $\mathcal{Q}_{\mathbb{C}^{k+2}/\mathbb{C}^k}$

We now introduce auxiliary parameters.

Denote by $\mathcal{O}_{\mathbb{C}^{k+3}|\mathbb{C}^{k+2}}$ the restriction of the sheaf of holomorphic func-tions in $\mathbb{C} \times \mathbb{C}^{k+2} = \{(\hbar, \lambda, x, y)\}$ to the vector subspace $\{0\} \times \mathbb{C}^{k+2}$.

The sheaf $\mathcal{Q}_{\mathbb{C}^{k+2}/\mathbb{C}^k}$, is defined by the presheaf:
$$U \longrightarrow \mathcal{Q}_{\mathbb{C}^{k+2}/\mathbb{C}^k}(U) = \{f \in \widehat{\mathcal{Q}}[[\lambda]], Bf \in \mathcal{O}_{\mathbb{C}^{k+3}|\mathbb{C}^{k+2}}(U)\}$$
where $U \subset \mathbb{C}^{k+2}$ denotes an open subset. The situation is similar to the one without parameters : one may identify $\mathcal{Q}_{\mathbb{C}^{k+2}/\mathbb{C}^k}$ with the sheaf of algebras $(\mathcal{O}_{\mathbb{C}^{k+3}|\mathbb{C}^{k+2}}, \star)$ where $\star$ denotes the normal product.

Consider the sheaf $\mathcal{B}_{\mathbb{C}^l}$ on $\mathbb{C}^l = \{\lambda\}$ defined by the presheaf:

$$U \longrightarrow \mathcal{B}_{\mathbb{C}^l}(U) = \{f \in \mathbb{C}[[\hbar, \lambda_1, \ldots, \lambda_l]], Bf \in \mathcal{O}_{\mathbb{C}^{l+1}|\mathbb{C}^l}(U)\}.$$

The algebra $\mathbb{C}_\hbar\{\lambda\}$ is the stalk at the origin of the sheaf $\mathcal{B}_{\mathbb{C}^l}$.

5.6. The finiteness theorem for the ring $\mathcal{Q}$

The notion of F-constructibility extends in an obvious manner to complexes of $\mathcal{Q}_{\mathbb{C}^{k+2}/\mathbb{C}^k}$-coherent sheaves.

THEOREM 5.12. — *Let* $F : (\mathbb{C}^k \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^{k+1}, 0)$ *be a holomorphic map germ satisfying the* a_F*-condition. The cohomology spaces* $H^k(K^\cdot)$ *associated to a complex of F-constructible $\mathcal{Q}_{\mathbb{C}^{k+2}/\mathbb{C}^k,0}$-coherent free modules are* $F^{-1}\mathcal{B}_{\mathbb{C}^{k+1},0}$*-coherent modules.*

The algebra structure of $\mathcal{Q}$ plays no essential role as the results are of functional analytic nature. The freeness is also unessential. The proof of this theorem is given in the appendix (see also [14]).

5.7. Proof of Theorem 5.5, Part 1 (finiteness)

Consider the unfolding of the plane curve singularity associated to the principal symbol $f = \sigma(H)$ of H

$$F : (\mathbb{C}^k \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^k \times \mathbb{C}, 0), \ (\lambda, x, y) \mapsto (\lambda, f(\lambda, x, y)).$$

As F defines an isolated complete intersection singularity it admits standard representatives (sometimes called good or Milnor representatives), which trivially satisfies the Thom a_F condition for any Whitney stratification which refines the stratification by the rank (see [2, 24]).

Let $F : X \longrightarrow S, \ (\lambda, x, y) \longrightarrow (\lambda, f(\lambda, x, y))$ be such a representative.

We consider the complex of sheaves on X:

$$\mathcal{C}_H^\cdot : 0 \longrightarrow \mathcal{Q}_{X/S} \longrightarrow \mathcal{Q}_{X/S} \longrightarrow 0$$

where the only non zero boundary map is given by $G \mapsto \{G, H\}$. Here $\mathcal{Q}_{X/S}$ denote the restriction of the sheaf $\mathcal{Q}_{\mathbb{C}^{k+2}/\mathbb{C}^k}$ to X.

According to Theorem 5.12, it suffices to prove the following lemma.

LEMMA 5.13. — *The sheaf complex* $\mathcal{C}_H^\cdot$ *is F-constructible, i.e, its cohomology sheaves are locally constant along the fibres of* $F : X \longrightarrow S$.

Proof. — The fibres of F are either smooth or with isolated singular points, therefore it suffices to prove the lemma at regular points of F (any sheaf restricted to a point is constant).

At the level of zero cohomologies, there is nothing to prove, indeed a cocycle $m \in \mathcal{C}_H^0(X)$ satisfies $\{m, H\} = 0$ and is therefore constant along the fibres of F.

Denote by Φ be the automorphism of $\mathcal{Q}_{\mathbb{C}^{k+2}/\mathbb{C}^k}\{t\}$ obtained by integrating the Heisenberg equations of H. The principal symbol φ of Φ is the flow of the Hamilton vector field associated to $f = \sigma(H)$.

Now, take a cocycle $m \in \mathcal{C}_H^1(U)$ where U is a sufficiently small open neighbourhood of a regular point of F, so that:

(1) it does not contain the origin,
(2) the map $\psi : U \longrightarrow \mathbb{C} \times S, \ z \mapsto (t, F(z))$ with $\varphi(t, w) = z$ is biholomorphic onto its image, i.e, t is a local coordinate on the fibres of the map $F_{|U}$.

Define $m_t = \Phi_t(m) \in \mathcal{C}_H^1(\varphi_t(X))$. We differentiate m_t with respect to t and use the fact that $\Phi_t(H) = H$, we get

$$\frac{d}{dt}(m_t) = \Phi_t(\{m, H\}) = \{\Phi_t(m), H\} = \delta\Phi_t(m).$$

Thus, the cocycles m and m_t define the same class in $H^1(\mathcal{C}_H(X \cap \varphi_t(X)))$. This shows that $\mathcal{H}^1(\mathcal{C}_H)_{|U} = F^{-1}(F_{|U})_* \mathcal{H}^1(\mathcal{C}_H)$ and hence the complex $\mathcal{C}_H$ is F-constructible. This concludes the proof of the lemma. If we write abusively m as a function of the value of F and t, then the same computation can be written

$$\frac{d}{dt}m(F, t) = \delta a_t, \ a_t = \Phi_t(m(F, 0))$$

and consequently $m(F, t) = m(F, 0) + \delta(\int a_t dt)$ which shows that $[m(F, t)] = [m(0, t)]$. $\square$

5.8. Proof of Theorem 5.5, Part 2 (freeness)

We put $\hbar = \lambda_{k+1}$, $z = \lambda_{k+2}$ and define the complex $C_j^\cdot, \ j = 0, \ldots, k+2$ inductively by $C_{j+1}^\cdot := C_j^\cdot/\lambda_{j+1}C_j^\cdot$ and $C_0^\cdot$ is defined by a unique differential

$$\mathcal{Q}\{\lambda\} \longrightarrow \mathcal{Q}\{\lambda\}, \ F \mapsto \{H, F\}.$$

The multiplication by λ_{j+1} induces an exact sequence of complexes

$$0 \longrightarrow C_j^\cdot \longrightarrow C_j^\cdot \longrightarrow C_{j+1}^\cdot \longrightarrow 0$$

which induces in turn a long exact sequence in cohomology. There are canonical isomorphisms

$$H^0(C_j^{\cdot}) \approx \mathbb{C}_\hbar\{\lambda_{j+1}, \ldots, \lambda_k\}, \ H^0(C_{k+1}^{\cdot}) \approx \mathbb{C}\{z\}, \ H^0(C_{k+2}^{\cdot}) \approx \mathbb{C}$$

for $j = 0, \ldots, k$. Therefore the exact sequences split and we get short exact sequences

$$0 \longrightarrow H^1(C_j^{\cdot}) \longrightarrow H^1(C_j^{\cdot}) \longrightarrow H^1(C_{j+1}^{\cdot}) \longrightarrow 0$$

which shows that $(\lambda_1, \ldots, \lambda_{k+2})$ is a regular sequence of maximal length, therefore the finite type module $M = H^1(C^{\cdot})$ has depth $k+2$; consequently the Auslander-Buchsbaum formula implies that M is a free module (see e.g. [13]). The map

$$m \mapsto m dx \wedge dy$$

is an isomorphism between $H^1(C_{k+1}^{\cdot})$ and the Brieskorn lattice $\Omega^2_{\mathbb{C}^2,0}/df \wedge d\mathcal{O}_{\mathbb{C}^2}$ of f which is of rank $\mu := \dim \mathbb{C}\{x,y\}/(\partial_x f, \partial_y f)$ [7] (see also [15, 25]). Therefore the rank of the module $M = H^1(C_0^{\cdot})$ also equals μ.

6. Deformation theory in $\mathcal{Q}$

6.1. Proof of the quantum Morse lemma (Theorem 3.4)

As in the case of singularity theory for mappings, we start by using the path method [1].

We search for an automorphism $\varphi \in \text{Aut}(\mathcal{Q}\{t\})$ with $\varphi(t) = t$ and a map $u \in \mathbb{C}_\hbar\{z, t\}$ such that $u \circ \varphi(H) = H(t = 0, \cdot)$. We differentiate this equality with respect to t and get the equation

$$(6.1) \qquad \frac{\partial u}{\partial t} \circ \varphi(H) + Du(\varphi(H)) \cdot \varphi\left(\{G, H\} + \frac{\partial H}{\partial t}\right) = 0$$

where, according to Proposition 4.6, the operator G is defined by the equality $[G, \cdot] + \partial_t = \varphi^{-1}(\frac{d}{dt}\varphi(\cdot))$. Here $Du(\varphi(H))$ stands for the Q-derivative of u at $\varphi(H)$. Applying the map $Du^{-1}(u \circ \varphi(H))$ to Equation (6.1) and then acting by the automorphism φ^{-1}, we get an equation of the type

$$(6.2) \qquad g \circ H + \{G, H\} = \gamma, \ \gamma = -\frac{\partial H}{\partial t} \in \mathcal{Q}\{t\}, \ g \in \mathbb{C}_\hbar\{z, t\}.$$

The automorphism φ is obtained from G by integration of the Heisenberg equations (Proposition 4.3). I assert that the map germ u can also be recovered from the map germ g. Indeed, as $g \circ H$ commutes with H, the relation

$$\frac{\partial u}{\partial t} \circ H = Du(H) \cdot (g \circ H)$$

reduces to

$$\frac{\partial u}{\partial t} \circ H = \left(\frac{\partial u}{\partial z} g\right) \circ H.$$

Now, the assertion follows from :

LEMMA 6.1. — *The initial value problem*

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial u}{\partial z} g \\ u(t = 0, z) = z \end{cases}$$

can be solved in $\mathbb{C}_\hbar\{z, t\}$.

Proof. — The Borel transform Bu of u satisfies the equation

(6.3) $$\frac{\partial Bu}{\partial t} = \frac{\partial Bu}{\partial z} * Bg,$$

with initial condition $Bu(t = 0, z) = z$, where $*$ denotes the convolution product in the $\hbar$ variable.

We chose $r \in \mathbb{R}$, so that, in the series expansions

$$Bg = \sum_{n,m \geqslant 0} g_{n,m} z^n t^m, \quad Bu = \sum_{n,m \geqslant 0} u_{n,m} z^n t^m, \quad g_{n,m}, \ u_{n,m} \in \mathbb{C}\{\hbar\}$$

the coefficients $g_{n,m}, u_{n,m}$ are holomorphic in the disk $D = \{\hbar \in \mathbb{C}, |\hbar| < 2r\}$ and both functions are holomorphic in some polydisk $D \times D' \subset \mathbb{C}^3$.

We define the holomorphic function

$$\tilde{g} = \sum_{n,m \geqslant 0} c_{n,m} z^n t^m, \quad c_{n,m} = \text{abs}(g_{n,m})(\hbar = r) \in \mathbb{C}.$$

The integro-differential equation (6.3) gives the recursion

$$(m + 1)u_{n,m+1} = \sum_{j+j'=n, k+k'=m} (j + 1)u_{j+1,k} g_{j',k'}.$$

Therefore an induction on m shows that the solution v of the partial differential equation

$$\frac{\partial v}{\partial t} = \frac{\partial v}{\partial z} \tilde{g}$$

with initial condition $v(t = 0, z) = z$ is a majorant series for Bu evaluated at $\hbar = r$. By the Cauchy-Kovalevskaïa theorem, the function v is holomorphic in some neighbourhood of the origin and therefore so is Bu. This proves the lemma.   □

This lemma implies that there exist u, φ such that $u \circ \varphi(H) = H_0$ provided that there exist g, G satisfying Equation (6.2).

In the notations introduced at the beginning of this subsection, Equation (6.2) becomes $g \circ [1] = [\gamma]$.

Therefore it can be solved provided that $[1]$ generates the module $M(H)$.

We apply Theorem 5.5. As the module $M(H)$ is free of rank one and the class of 1 is non-zero this shows that [1] generates the module $M(H)$. This concludes the proof of the theorem.

(The freeness of the module was used by convenience for the reader, in fact, because of Nakayama's lemma, the finiteness of the module is sufficient to conclude the proof).

6.2. The quantum versal deformation theorem

We show that the finiteness of the deformation module (Theorem 5.5) implies the versal deformation theorem in the algebra $\mathcal{Q}\{\lambda\}$. The proof is similar to the one we gave in the isochore case [15].

We recall some standard definitions adapted to our setting. An element $F \in \mathcal{Q}\{\lambda\}$ is called a *deformation* of $H = F(0, \cdot) \in \mathcal{Q}$. A deformation $G \in \mathcal{Q}\{\mu\}$ of H is called induced from F is there exist homomorphisms of algebras $\varphi : \mathcal{Q}\{\lambda\} \longrightarrow \mathcal{Q}\{\mu\}$, $u \in \mathbb{C}_\hbar\{\mu\}$ such that $u \circ G = \varphi(H)$.

A deformation of $H \in \mathcal{Q}$ is called *versal* if any other deformation of H can be induced from it.

THEOREM 6.2 (compare [8], Theorems 6,7,8,9 and [30]). — *A deformation F of an element $H \in \mathcal{Q}$ is versal provided that the classes of the $\partial_{\lambda_j} \sigma(F)_{|\lambda=0}$'s and of 1 generate the $\mathbb{C}$-vector space $\mathbb{C}\{x, y\}/(\{\mathbb{C}\{x, y\}, \sigma(H)\} + \mathbb{C}\{x, y\} \sigma(H))$.*

Remark 6.3. — The converse statement of the above theorem holds trivially.

Example 6.4. — The deformation $F = p^2 + q^{k+1} + \sum_{j=1}^{k-1} \lambda_j q^j$ is versal. Indeed, here $\sigma(H) = y^2 + x^{k+1}$ and the $\mathbb{C}$-vector space $\mathbb{C}\{x, y\}/(\{\mathbb{C}\{x, y\}, \sigma(H)\} + \mathbb{C}\{x, y\} \sigma(H))$ can be identified with the algebra $\mathbb{C}\{x, y\}/(y, x^k)$ of $\sigma(H)$ which is generated by the classes of $1, x, \ldots, x^{k-1}$ (see [15], Example 2 for details).

Proof. — We use a standard method introduced by Martinet in the context of singularity theory for differentiable mappings [27].

Let G be an arbitrary deformation of H depending on the parameters $\mu_1, \ldots, \mu_l$.

Define the deformation $\Phi = F + G - H$ and let Φ_j be the restriction of Φ to $\mu_1 = \cdots = \mu_j = 0$ with $\Phi_0 = \Phi$.

Assertion. The deformation Φ_{j-1} is induced by the deformation Φ_j.

We put $t = \mu_j$, $\alpha = (\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_j)$ and differentiate with respect to t the equation $u_t \circ \varphi_t(\Phi_{j-1}) = \Phi_j$. Proceeding like in the proof of the quantum Morse lemma, we get the equation

$$(6.4) \qquad g \circ \Phi_{j-1} + \{\Phi_{j-1}, H\} + \sum_{l=1}^{k+j-1} a_l \partial_{\alpha_l} \Phi_{j-1} = \gamma.$$

with $\gamma \in \mathcal{Q}\{\alpha\}$, $g \in \mathbb{C}_\hbar\{z, \alpha\}$, $a_l \in \mathbb{C}_\hbar\{\alpha\}$. This equation can be solved provided that [1] and the $[\partial_{\alpha_l} \Phi_{l-1}]$'s generate the $\mathbb{C}_\hbar\{\alpha, z\}$-module $M = \mathcal{Q}\{\alpha\}/\{\Phi_{j-1}, \mathcal{Q}\{\alpha\}\}$.

Theorem 5.5 implies that the module M is of finite type, therefore the Nakayama lemma implies the equivalence:

(i) the classes of 1 and of the $\partial_{\alpha_l} \Phi_{j-1}$'s generate the $\mathbb{C}_\hbar\{z, \alpha\}$-module M,

(ii) the classes of 1 and of the $\partial_{\alpha_l} \Phi_{j-1}$'s generate the $\mathbb{C}$-vector space $V = M/\mathcal{M}M$ where $\mathcal{M}$ is the maximal ideal of the local ring $\mathbb{C}_\hbar\{z, \alpha\}$.

The assumption on F implies the last statement. This proves the assertion.

Applying successively the assertion from $j = 0$ to $j = l$, we get that $\Phi_0 = F + G - H$ is induced by $\Phi_l = F$. This concludes the proof of the theorem. $\qquad \square$

6.3. Miniversality implies Universality

Recall that a versal deformation depending on a minimal number of parameter is called *miniversal*. In [8], Colin de Verdière conjectured the following result[5].

THEOREM 6.5. — *Let $F \in \mathcal{Q}\{\lambda\}$ be a miniversal deformation of an element $H \in \mathcal{Q}$. Let G be another deformation of H, so that G is induced from F, that is, $u \circ G = \varphi(F)$, then the function germs $\varphi(\lambda_j) \in \mathbb{C}_\hbar\{\mu\}$ and $u \in \mathbb{C}_\hbar\{\mu, z\}$ are uniquely determined by the choices of F and G.*

Proof. — We use the same notations as in the proof of Theorem 6.2. Equation (6.4) can be written as

$$(6.5) \qquad g \circ [1] + \sum_{l=1}^{k+j-1} a_l [\partial_{\alpha_l} \Phi_{j-1}] = [\gamma]$$

---

[5] In the initial conjecture, there was no condition on the analyticity of the expansions and of the data. The case of formal power series is simpler since the finiteness of the deformation module is in that case obvious.

where the bracket denotes the class in the module $M_j = \mathcal{Q}\{\lambda\}/\{\Phi_{j-1}, \mathcal{Q}\{\lambda\}\}$.

Since F is miniversal and the module M_j is free of finite type, the classes $[\partial_{\alpha_l}\Phi_{j-1}]$ and $[1]$ freely generate the module M for $l \in \{1, \ldots, k\}$. Therefore the solution of Equation (6.4) with $a_l = 0$ for $l > k$ is unique.

This shows that the functions germs $\varphi_t^{-1}(\lambda_k)$ obtained after integrating the coefficients a_l are uniquely determined. By a finite induction on $j \in \{0, \ldots, k\}$, we get that the function germs $\varphi(\lambda_k)$ are uniquely determined by F and G. This proves the theorem. $\qquad\square$

Remark 6.6. — In "standard singularity theory", the symmetry group of the singularity defines an action on the base of the versal deformation which prevents the deformation from being universal. Such a situation does not occur in our context. From the point of view of analytic geometry, the Colin de Verdière conjecture was therefore completely "unexpected". One can show *mutatis mutandis* the universality of isochore miniversal deformations [15].

7. Basics of analytic spectral analysis

7.1. The operator representation.

We adapt the Born-Jordan matrix approach to the analytic case.

Denote by $\mathcal{Q}p$ the left ideal generated by p and put $\mathcal{H} = \mathcal{Q}/\mathcal{Q}p$. The map $\mathcal{H} \longrightarrow \mathbb{C}_\hbar\{z\}$ sending the class of q to z is an isomorphism of $\mathbb{C}_\hbar$-modules.

The left multiplication by $H \in \mathcal{Q}$ induces a homomorphism of $\mathbb{C}_\hbar$-modules

$$\rho^{an} : \mathcal{Q} \longrightarrow \mathrm{Hom}_{\mathbb{C}_\hbar}(\mathcal{H}, \mathcal{H})$$

representing the elements in $\mathcal{Q}$ as $\mathbb{C}_\hbar$-linear operators in $\mathcal{H}$. Via the isomorphism $\mathcal{H} \approx \mathbb{C}_\hbar\{z\}$, the operators associated to q and p are mapped respectively to the multiplication by z and to $\hbar\partial_z$.

Let us now introduce, the Dirac notation in this setting. The projection of $q^i \in \mathcal{Q}$ to $\mathcal{H}$ is denoted by $|i\rangle$. The image of the vector $|i\rangle$ under an operator A is denoted by $A|i\rangle$.

It is useful, although not essential, to introduce a pairing in $\mathcal{H}$. According to the standard commutation rules of quantum mechanics our choice $\hbar$ equals $\hbar = \frac{h}{2\sqrt{-1}\pi}$. where h is the Planck constant. Therefore we define the conjugate of $\alpha := \sum_{n \geqslant 0} \alpha_k \hbar^k$, $\alpha_k \in \mathbb{C}$ by

$$\bar{\alpha} := \sum_{n \geqslant 0} (-1)^k \bar{\alpha}_k \hbar^k \in \mathbb{C}_\hbar$$

Consider the "restriction to zero" mapping

$$\pi : \mathcal{Q} \longrightarrow \mathbb{C}_\hbar, \quad \sum_{m,n \geqslant 0} \alpha_{mn} q^m p^n \mapsto \alpha_{0,0}.$$

We define hermitian conjugation in $\mathcal{Q}$ by

$$\dagger : \mathcal{Q} \longrightarrow \mathcal{Q}, \quad \sum_{m,n \geqslant 0} \alpha_{mn} q^m p^n \mapsto \sum_{m,n \geqslant 0} \bar{\alpha}_{mn} p^m q^n;$$

and a pairing $P : \mathcal{Q} \times \mathcal{Q} \longrightarrow \mathbb{C}_\hbar$, $(f,g) \mapsto \pi(f^\dagger g)$.

The inner product $\langle \cdot | \cdot \rangle$ is defined by the commutative diagram

$$
\begin{array}{ccc}
\mathcal{Q} \times \mathcal{Q} & & \\
\Big\downarrow & \searrow^{P} & \\
\mathcal{H} \times \mathcal{H} & \xrightarrow[\langle \cdot | \cdot \rangle]{} & \mathbb{C}_\hbar
\end{array}
$$

where the vertical arrow denotes the canonical projection.

We have $\langle i | j \rangle = i! \hbar^i \delta_{ij}$ where δ_{ij} stands for the Kronecker symbol.

PROPOSITION 7.1. — *The pairing* $\langle - | - \rangle : \mathcal{Q} \times \mathcal{Q} \longrightarrow \mathbb{C}_\hbar$ *satisfies the following properties*

    (1) $\langle u | u \rangle = 0 \iff u = 0$,
    (2) $\langle u | v \rangle = \overline{\langle v | u \rangle}$.

Proof. — Write

$$u = \sum_{i \in I} (\alpha_i \hbar^{j_i} | i \rangle + \dots), \quad \alpha_i \in \mathbb{C}$$

where $I \subset \mathbb{Z}_{\geqslant 0}$ denotes the set of multi-indices for which $\alpha_i \neq 0$ and the dots denote higher order terms in $\hbar$. Then, the hermitian product

$$\langle u | u \rangle = \sum_{i \in I} (-1)^{j_i} |\alpha_i|^2 i! \hbar^{i+2j_i} + \dots$$

vanishes if and only if $I = \emptyset$. The second part of the Proposition is obvious. $\qquad\square$

PROPOSITION 7.2. — *The homomorphism of* $\mathbb{C}_\hbar$*-rings* $\rho^{an} : \mathcal{Q} \longrightarrow \mathrm{Hom}_{\mathbb{C}_\hbar}(\mathcal{H}, \mathcal{H})$, *is a ring monomorphism.*

Proof. — The kernel I of the homomorphism ρ^{an} is a left-ideal invariant under right multiplication by a and q.

Define the map $v : I \longrightarrow \mathbb{Z}_{\geqslant 0}$ sending $H \in I$ to the smallest $k \in \mathbb{Z}_{\geqslant 0}$ for which there exists j such that the coefficient of $q^j p^k$ in the expansion $H = \sum_{jk} \alpha_{jk} q^j p^k$ is non zero.

Assume that $I \neq 0$, then there exists a non-zero element H for which v is minimal.

We have necessarily $v(H) = 0$ otherwise $v([H, q])$ would be smaller than $v(H)$. Evaluating $H_k = [p^k, H]$ with $H = \sum_j \alpha_j q^j$ on the class of $|0\rangle \in \mathcal{Q}$, we get that $H_k|0\rangle = \hbar^k \alpha_k |0\rangle = 0$. This contradicts the fact that $H \neq 0$. $\square$

Proposition 7.2 allows us to identify $\mathcal{Q}$ with a subspace of $\mathrm{Hom}_{\mathbb{C}_\hbar}(\mathcal{H}, \mathcal{H})$. This is the analytic variant of the Born-Jordan matrix formulation of quantum mechanics. This representation induces a notion of analytic spectrum, denoted $\mathrm{Sp}(\cdot)$, similar to that introduced in the formal case.

7.2. The harmonic oscillator

PROPOSITION 7.3.
  (1) The analytic spectrum of the function $H = qp$ is equal to $\hbar \mathbb{Z}_{\geqslant 0}$.
  (2) The analytic spectrum of the operator $H = \frac{1}{2}(p^2 - q^2)$ is equal to $\hbar \mathbb{Z}_{\geqslant 0} + \hbar/2$.

Proof. — The first part of the Proposition follows from the equality

$$qp|n\rangle = n\hbar|n\rangle.$$

Consider the automorphism $\varphi \in \mathrm{Aut}(\mathcal{Q})$ defined by

$$\varphi(q) = p + q, \ \varphi(p) = \frac{1}{2}(p - q).$$

Using the fact that $p^2 - q^2 = (p + q)(p - q) + [p, q]$, we get that

$$\varphi\left(qp + \frac{\hbar}{2}\right) = \frac{1}{2}(p^2 - q^2).$$

Therefore

$$\mathrm{Sp}\left(\frac{1}{2}(p^2 - q^2)\right) = \mathrm{Sp}(qp) + \frac{\hbar}{2} = \hbar \mathbb{Z}_{\geqslant 0} + \hbar/2.$$

This proves the proposition. $\square$

Remark 7.4. — It is a rather puzzling fact that the Born-Jordan spectrum is discrete without imposing any boundary conditions. The analytic theory, gives an hint to the understanding of this phenomenon. Via the isomorphism $\mathcal{H} \approx \mathbb{C}_\hbar\{z\}$, the operator $H = qp$ might be identified with the operator $\hbar z \partial_z$. For any λ, the function $z^{\lambda/\hbar}$ lies in the kernel of the operator $\hbar z \partial_z - \lambda$ but only for $\lambda \in \hbar \mathbb{Z}_{\geqslant 0}$ is this solution an unbranched holomorphic function germ.

7.3. Borel analyticity of perturbative expansions

The notion of spectrum extends naturally to the algebras $\mathcal{Q}\{t\}$, $\mathcal{Q}\{\lambda\}$, $\lambda = (\lambda_1, \ldots, \lambda_k)$.

The inclusion $\mathcal{Q}\{t\} \subset \widehat{\mathcal{Q}}[[t]]$ induces a commutative diagram

$$
\begin{array}{ccc}
\mathcal{Q}\{t\} & \xrightarrow{\ \rho^{an}\ } & \mathrm{Hom}_{\mathbb{C}_\hbar\{t\}}(\mathbb{C}_\hbar\{z,t\}, \mathbb{C}_\hbar\{z,t\}) \\
\downarrow & & \downarrow \\
\widehat{\mathcal{Q}}[[t]] & \xrightarrow{\ \rho\ } & \mathrm{Hom}_{\mathbb{C}[[\hbar,t]]}(\mathbb{C}[[\hbar, t, z]], \mathbb{C}[[\hbar, t, z]])
\end{array}
$$

and therefore we get a forgetful map

$$
\widehat{\mathrm{Sp}}(H) \longrightarrow \mathrm{Sp}(H)
$$

for any $H \in \mathcal{Q}\{t\}$.

Take a deformation $H = H_0 + tH_1 \in \mathcal{Q}\{t\}$ of $H_0 \in \mathcal{Q}$. That the perturbative expansion E is analytic means exactly that E lies in the image of the forgetful mapping.

THEOREM 7.5. — *If the principal symbol of an element $H \in \mathcal{Q}\{t\}$ evaluated at $t = 0$ is a Morse function germ, then the forgetful map*

$$
\widehat{\mathrm{Sp}}(H) \longrightarrow \mathrm{Sp}(H)
$$

is an isomorphism, that is, the perturbative expansions of the spectrum are Borel analytic.

Proof. — The quantum Morse lemma (Theorem 3.4) asserts that there exist an automorphism $\varphi \in \mathrm{Aut}(\mathcal{Q}\{t\})$ and an element $u \in \mathbb{C}_\hbar\{t, z\}$ such that the equality $H = u \circ \varphi(H_0)$ holds with $H_0 = H(t = 0, \cdot)$ and $\mathrm{Sp}(H) = u(\mathrm{Sp}(H_0))$ In particular, the Borel transform of the perturbative expansion for the spectrum are holomorphic function germs. This proves the theorem.
$\square$

7.4. The Heisenberg formula for the anharmonic oscillator

We apply the recipe given in the proof of Theorem 3.4 (Subsection 6.1). First we solve the equation

$$
[G, H] + g \circ H = -\frac{\partial H}{\partial t}
$$

up to some order in t. We solve the partial differential equation $\partial_t u = g \partial_z u$ with initial condition $u(t = 0, \cdot) = z$ and invert u. This can be done easily using elementary mathematical programming[6] .

PROPOSITION 7.6. — *Let $G \in \mathcal{Q}\{t\}$ and $u \in \mathbb{C}_\hbar\{z, t\}$ be defined by*

$$G = -\frac{3}{32}qp^3 - \frac{5}{32}q^3p + \frac{13}{128}tqp^5 + \frac{13}{48}tq^3p^3 + \frac{19}{128}tq^5p - \frac{3}{32}\hbar q^2 + \frac{91}{128}q^2p^2\hbar t$$
$$+ \frac{3}{16}q^4\hbar t + \frac{53}{128}qp\hbar^2 t,$$

and

$$u = z + \frac{3}{32}(4z^2 - \hbar^2)t - \frac{1}{256}(68z^2 - 67\hbar^2)zt^2$$

then

$$e^{\frac{t}{\hbar}G}\left(\frac{1}{2}(p^2 - q^2) + \frac{t}{4}q^4\right)e^{-\frac{t}{\hbar}G} = u \circ \left(\frac{1}{2}(p^2 - q^2)\right) + o(t^2).$$

In particular the asymptotic expansion of the spectral values admit the expansions

$$E(t, \hbar) = u\left(\left(n + \frac{1}{2}\right)\sqrt{-1}\hbar\right) + o(t^2)$$

which coincides with Heisenberg's formula.

7.5. A pairing in $\mathcal{Q}$

We use the notation $\langle i|H|j \rangle = \langle i|Hj \rangle$.

PROPOSITION 7.7. — *For any function germ $H \in \mathcal{Q}$, the series $v(H) = \sum_{i \geqslant 0}\langle i|H|i \rangle$ defines an element of $\mathbb{C}_\hbar$.*

Proof. — Write $H = \sum_{jk} \alpha_{jk}p^j q^k$. Then

$$v(H) = \sum_{i,j,k \geqslant 0} \alpha_{jk}\langle i|p^j q^k|i \rangle.$$

Now, $\langle i|p^j q^k|i \rangle = \langle i + j|i + k \rangle$. Thus, we get the formula

$$v(H) = \sum_{i,j \geqslant 0} \alpha_{jj}(i + j)!\hbar^{i+j},$$

therefore

$$Bv(H) \ll \sum_{i,j \geqslant 0}(B\alpha_{jj})\hbar^{i+j} = \frac{1}{1 - \hbar}\sum_{j \geqslant 0}(B\alpha_{jj})\hbar^j.$$

---

[6] Computations were done using Xcas and Maple.

As the Borel transform H is convergent, the series $\sum_j \alpha_{jj} r^{2j}$ obtained by substituting q, p by r is Borel analytic for r small enough, therefore $\sum_{j \geqslant 0} B\alpha_{jj} \hbar^j$ is analytic. This proves the Proposition.   □

Therefore the algebra $\mathcal{Q}$ possesses a pairing

$$\mathcal{Q} \times \mathcal{Q} \longrightarrow \mathbb{C}_\hbar, \ (A, B) \mapsto v(AB^\dagger).$$

Appendix A. Proof of the finiteness theorem

An elementary and detailed exposition of this appendix is given in [14].

A.1. Construction of the contraction

We use the notations of Theorem 5.12 denote by B_r the ball of radius r centred at the origin in $\mathbb{C}^{k+2}$.

Let $F : X \longrightarrow S$ be a standard representative of a germ $F : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^s, 0)$. Recall that such a representative is obtained by restriction of a representative $F : B_R \longrightarrow \mathbb{C}^s$ first to a closed ball B_ε by choosing ε such that the boundary of the ball B_r is transverse to the strata of $F^{-1}(0)$ for $r \leqslant \varepsilon$ and then above a polycylinder $S \subset F(B_\varepsilon)$ centred at the origin such that the fibres of F above S are transverse to the boundary of B_ε.

The aim of this subsection is to prove the following proposition (the proof follows an argument due to Brieskorn [7]).

PROPOSITION A.1. — *For any $\varepsilon' < \varepsilon$, the restriction mapping $r : \mathcal{K}^\cdot(X) \to \mathcal{K}^\cdot(X')$ is a quasi-isomorphism with $X' = X \cap B_{\varepsilon'}$.*

Proof. — It is sufficient to prove the proposition for ε' sufficiently close to ε.

As the map F satisfies Thom's a_F condition, Thom' isotopy theorem implies the existence of an isotopy

$$\varphi_{\varepsilon'} : X \longrightarrow X' := X \cap B_{\varepsilon'}$$

tangent to the fibres of F which preserve the strata, provided ε' is sufficiently close to ε (see for instance [28], Proposition 11.3, Proposition 11.5 and Proposition 11.6).

Chose a covering $U = (U_i)$ of X such that the sheaves $\mathcal{H}^\cdot(\mathcal{K}^\cdot)$ are F-constant on U_i and denote by U' the covering defined by $U_i' = U_i \cap B_{\varepsilon'}$.

The spectral sequences $E_0^{p,q}(X) = \mathcal{C}^p(U, \mathcal{K}^q)$, $E_0^{p,q}(X') = \mathcal{C}^p(U', \mathcal{K}^q)$ compute respectively the hypercohomology of the complex $\mathcal{K}^{\cdot}$ in X and in X'.

The map $\varphi_{\varepsilon'}$ induces a homeomorphism between each strata in U_i and the corresponding stratum in U_i' for each i. As the cohomology sheaves are F-constant in U_i and $F(U_i) = F(U_i')$, we get isomorphisms of vector spaces

$$\mathcal{H}^q(\mathcal{K}^{\cdot})(U_i) \approx F_*\mathcal{H}^q(\mathcal{K}^{\cdot})(F(U_i)) \approx \mathcal{H}^q(\mathcal{K}^{\cdot})(U_i')$$

on each small open subset U_i. Therefore, the restriction mapping induces an isomorphism between the E_1-terms of the spectral sequences:

$$E_1^{p,q}(X) = \mathcal{C}^p(U, \mathcal{H}^q(\mathcal{K}^{\cdot})) \approx \mathcal{C}^p(U', \mathcal{H}^q(\mathcal{K}^{\cdot})) = E_1^{p,q}(X').$$

This shows that the inclusion $X' \subset X$ gives an isomorphism in hypercohomology

$$\mathbb{H}^{\cdot}(X, \mathcal{K}^{\cdot}) \approx \mathbb{H}^{\cdot}(X', \mathcal{K}^{\cdot}).$$

As X, X' are Stein, by Cartan's theorem B, for any $p \geqslant 0$, we have the isomorphisms

$$\mathbb{H}^{\cdot}(X, \mathcal{K}^{\cdot}) \approx H^{\cdot}(\mathcal{K}^{\cdot}(X)), \ \ \mathbb{H}^{\cdot}(X', \mathcal{K}^{\cdot}) \approx H^{\cdot}(\mathcal{K}^{\cdot}(X')),$$

therefore the restriction mapping is a quasi-isomorphism. This proves the proposition. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

A.2. Proof of Theorem 5.12

We apply Houzel's variant of the Schwartz finiteness theorem for a mapping between two modules[7].

THEOREM A.2 ([22]). — *Let A be a multiplicatively convex, complete bornological algebra and let $u : M^{\cdot} \longrightarrow N^{\cdot}$ be an A-linear bounded mapping between complexes of complete bornological A modules. We make the following assumptions:*

  (1) *M^n, N^n satisfy the homomorphism property for all n,*
  (2) *u is a quasi-isomorphism and it is A-nuclear.*

Then the complexes $M^{\cdot}$ are pseudo-coherent.

---

[7] Houzel considers the more general situation of a sequence of modules.

We briefly give some explanations on the terminology used in the theorem.

A *bornology* in E is a collection of subsets called *bounded subsets* which satisfy natural axioms (the union of two bounded subsets is bounded, a subset of a bounded subset is bounded). In the cases we consider, the vector spaces have a topology defined by a set of semi-norms and the bounded subsets are the subset on which any continuous semi-norm defines a bounded function.

For instance, if $U \subset \mathbb{C}$ is an open subset then by endowing the vector space $\mathcal{O}(U)$ with the compact open topology, we get a locally convex vector space. The open subsets

$$B_{K,\varepsilon} = \left\{ f \in \mathcal{O}(U) : \sup_{z \in K} \mid f(z) \mid \leqslant \varepsilon \right\}$$

where K runs over the compact subsets of U generate the topology. If these subset are disks, *i.e.*, they are convex subsets of $\mathcal{O}(U)$ also invariant under multiplication by a number of modulus 1.

As the vector space $\mathbb{C}\{z\}$, is the direct limit of the $\mathcal{O}(U)$'s where U runs over the neighbourhoods of the origin it inherits a direct limit topology. A basis for the bounded subsets is given by the subsets

$$B_{r,\varepsilon} = \{ f \in \mathbb{C}\{z\} : \mid f \mid_r \leqslant \varepsilon \}$$

where $\mid \cdot \mid_r$ denotes the supremum norm in the disk of radius r. In the general case, the direct limit

$$\mathcal{O}_{\mathbb{C}^n}(K) := \varinjlim \mathcal{O}_{\mathbb{C}^n}(U), \ K \subset U$$

has a topological vector space structure induced from the $\mathcal{O}_{\mathbb{C}^n}(U)$'s and therefore inherits a bornology. Any bounded subset is a bounded subset of some $\mathcal{O}_{\mathbb{C}^n}(U)$ where U contains K in its interior (see [18], Chapter 3, for more details).

The product bornology on $E \times F$ induces a bornology on $E \otimes_{\mathbb{C}} F$; if E, F are complete we denote by $E \hat{\otimes}_{\mathbb{C}} F$ the completion of $E \otimes_{\mathbb{C}} F$ for this bornology. A bounded morphism of $u : E \longrightarrow F$ is called *nuclear* if it lies in the image of the morphism

$$E' \hat{\otimes}_{\mathbb{C}} F \longrightarrow L(E, F), \ \sum \lambda_i \xi_i \otimes y_i \mapsto \left[x \mapsto \sum \lambda_i \xi_i(x) y_i \right], \ \lambda_i \in \mathbb{C},$$

with $\sum_i \mid \lambda_i \mid < +\infty$. The prototype of a nuclear mapping is the restriction mapping $\mathcal{O}(U) \longrightarrow \mathcal{O}(U')$. These notions are standard [6, 11, 19].

Now assume E, F are modules over a bornological algebra A, the topo-logical tensor product over A is the cokernel of the map

$$(E\hat{\otimes}_{\mathbb{C}}A\hat{\otimes}_{\mathbb{C}}F) \longrightarrow (E\hat{\otimes}_{\mathbb{C}}F), m \otimes a \otimes n \longrightarrow ma \otimes n - m \otimes an.$$

A morphism of $u : E \longrightarrow F$ is called A-*nuclear* if it lies in the image of the morphism

$$L_A(E, A)\hat{\otimes}F \longrightarrow L_A(E, F), \ \sum \lambda_i \xi_i \otimes y_i \mapsto \left[x \mapsto \sum \lambda_i \xi_i(x)y_i\right], \ \lambda_i \in \mathbb{C}$$

where $L_A(\cdot, \cdot)$ denotes the space of bounded A-linear mappings.

A bornology is called *convex* if any bounded subset is contained in a bounded disk. A bornological algebra A is called *multiplicatively convex* if any bounded subset is absorbed by a bounded disk stable under multipli-cation.

We leave to the reader to check that Proposition 3.3 implies that Q is multiplicatively convex.

A bornologically convex vector space E has the *homomorphism property* if any surjective bounded linear mapping $u : E \longrightarrow F$ to a convex com-plete bornological space F, any bounded sequence of F lifts to a bounded sequence of E. These notions are due to Houzel [22].

In our situation, we start from a standard representative $F : X \longrightarrow S$ of a germ

$$F : (\mathbb{C}^k \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^{k+1}, 0)$$

and a complex of constructible $\mathcal{Q}_{\mathbb{C}^{k+2}/\mathbb{C}^k}$-modules. We denote by X' a con-traction of X like in Proposition A.1.

Take $A = \mathcal{B}(S)$, where S is the closure of a Stein neighbourhood, $M = \mathcal{Q}_{\mathbb{C}^{k+2}/\mathbb{C}^k}(X)$, $N = \mathcal{Q}_{\mathbb{C}^{k+2}/\mathbb{C}^k}(X')$. These vector spaces are respectively isomorphic to $\mathcal{O}_{\mathbb{C}\times S|S}(S)$, $\mathcal{O}_{\mathbb{C}\times X|X}(X)$, $\mathcal{O}_{\mathbb{C}\times X|X}(X')$ and moreover the restriction mapping $r : \mathcal{K}^{\cdot}(X) \longrightarrow \mathcal{K}^{\cdot}(X')$ is $\mathcal{B}(S)$-nuclear (see again [14] for details). By Proposition A.1, the restriction mapping is a quasi-isomorphism, therefore Theorem A.2 applies to our situation. This shows that the cohomology spaces of the modules $\mathcal{K}^{\cdot}(X)$ are $\mathcal{B}_S(S)$-coherent.

Denote by $\mathcal{L}^{\cdot}$ a complex of free coherent $\mathcal{B}_S$-sheaves so that $\mathcal{L}^{\cdot}(S)$ is quasi-isomorphic to $\mathcal{K}^{\cdot}(X)$.

LEMMA A.3. — *The sheaf complexes $\mathcal{L}^{\cdot}$, $f_*\mathcal{K}^{\cdot}_{|X}$ are quasi-isomorphic.*

Proof. — A mapping $u : M^{\cdot} \longrightarrow L^{\cdot}$ of complexes induces a quasi-isomorphism between two complexes if and only if its mapping cone $C^{\cdot}(u)$ is exact.

We apply this fact to the mapping cone of the quasi-isomorphism

$$u : \mathcal{L}^{\cdot}(S) \longrightarrow \mathcal{K}^{\cdot}(X).$$

As the vector space $\mathcal{B}_S(P)$ is nuclear for any polydisk $P \subset S$, the functor $\hat{\otimes}\mathcal{B}_S(P)$ is exact ([23]). Therefore, the complex $C^{\cdot}(u)\hat{\otimes}\mathcal{B}_S(P)$ is also exact.

The complex $C^{\cdot}(u)\hat{\otimes}\mathcal{B}_S(P)$ is the mapping cone of the mapping $u' : \mathcal{L}^{\cdot}(P) \longrightarrow \mathcal{K}^{\cdot}(X \cap f^{-1}(P))$. Therefore, the complexes of sheaves $\mathcal{L}^{\cdot}$ and $f_*\mathcal{K}_{|X}^{\cdot}$ are quasi-isomorphic. This proves the lemma. $\square$

I assert that the complex $K^{\cdot} = \mathcal{K}_0^{\cdot}$ is quasi-isomorphic to the stalk of the complex $\mathcal{L}^{\cdot}$ at the origin.

Let (B_{ε_n}) be a fundamental sequence of neighbourhoods of the origin in $\mathbb{C}^n$, so that their intersection with the special fibre of F is transverse. As the map F satisfies the a_F-condition, we may find a fundamental sequence (S_n) of neighbourhoods of the origin in $\mathbb{C}^k$ so that the fibres of F intersect B_{ε_n} transversally above S_n. Put $X_n = f^{-1}(S_n)$, we have the isomorphism:

$$\mathcal{L}_{|S_n}^{\cdot} \approx F_*\mathcal{K}_{|X_n}^{\cdot} \approx F_*\mathcal{K}_{|X_n \cap B_{\varepsilon_n}}^{\cdot}.$$

The first isomorphism is a consequence of the previous lemma and the second one follows from the fact that the contraction is a quasi-isomorphism (Proposition A.1). In the limit $n \longrightarrow \infty$, we get that the complex $K^{\cdot} = \mathcal{K}_0^{\cdot}$ is quasi-isomorphic to the complex $\mathcal{L}_0^{\cdot}$. This concludes the proof of Theorem 5.12.

BIBLIOGRAPHY

[1] V. I. ARNOLD, A. N. VARCHENKO & S. GOUSSEIN-ZADE, "Singularity of differentiable mapping, vol. I", Nauka:Moscow, 1982, English transl.: Birkhauser, 382p., Basel (1986).

[2] ———, "Singularity of differentiable mapping, vol. II", Nauka:Moscow, 1982, English transl.: Birkhauser, 382p., Basel(1986).

[3] G. D. BIRKHOFF, *Dynamical systems*, Colloquium Publications, vol. IX, American Mathematical Society, Providence R.I., 1927.

[4] M. BORN, W. HEISENBERG & P. JORDAN, "Zur Quantenmechaniks II", *Z. Phys.* **35** (1926), p. 557-615.

[5] M. BORN & P. JORDAN, "Zur Quantenmechaniks", *Zeit.für Phys.* **34** (1925), p. 858-888.

[6] N. BOURBAKI, "Espaces vectoriels topologiques", Hermann, 1966.

[7] E. BRIESKORN, "Die Monodromie der isolierten Singularitäten von Hyperflächen", *Manuscr. Math.* **2** (1970), p. 103-161.

[8] Y. COLIN DE VERDIÈRE, "Singular lagrangian manifolds and semi-classical analysis", *Duke Math. Journal* **116** (2003), no. 2, p. 263-298.

[9] Y. COLIN DE VERDIÈRE & B. PARISSE, "Equilibres instables en régime semi-classique I: concentration micro-locale", *Comm. PDE* **19** (1994), p. 1535-1564.

[10] P. DELIGNE, "Déformations de l'algèbre des fonctions d'une variété symplectique: comparaison entre Fedosov et De Wilde, Lecomte", *Selecta Math. (N.S.)* **1** (1995), no. 4, p. 667-697.

[11] J. DIEUDONNÉ & L. SCHWARTZ, "La dualité dans les espaces ($\mathcal{F}$) et ($\mathcal{LF}$)", *Annales de l'Institut Fourier* **1** (1949), p. 61-101.

[12] P. A. M. DIRAC, "The fundamental equations of quantum mechanics", *Proc. Roy. Soc. A* **109** (1926), p. 642-653.

[13] D. EISENBUD, *Commutative algebra with a view towards algebraic geometry*, Springer, 1999, 797 pp.

[14] M. D. GARAY, "Finiteness and constructibility in local analytic geometry", math.AG/0610409, To appear in L'Enseignement Mathématique.

[15] ———, "An isochore versal deformation theorem", *Topology* **43** (2004), no. 5, p. 1081-1088.

[16] ———, "Analytic quantum mechanics", 2005, math-ph/0502027.

[17] ———, "Analytic geometry and semi-classical analysis", *Proceedings of the Steklov Insitute of Mathematics* **259** (2007), p. 35-59.

[18] A. GROTHENDIECK, "Topological vector spaces", Gordon and Breach, 1973, 245 p., English Translation: Espaces vectoriels topologiques, São Paulo 1954.

[19] ———, "Résumé des résultats essentiels dans la théorie des produits tensoriels topologiques et des espaces nucléaires", *Annales de l'Institut Fourier* (1952), p. 73-112.

[20] W. HEISENBERG, "Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen", *Zeitschrift für Physik* **33** (1925), p. 879-893.

[21] B. HELFFER & J. SJÖSTRAND, "Semiclassical analysis for Harper's equation. III. Cantor structure of the spectrum", *Mémoire de la Société Mathématique de France* **39** (1989), p. 1-124.

[22] C. HOUZEL, "Espaces analytiques relatifs et théorème de finitude", *Math. Annalen* **205** (1973), p. 13-54.

[23] R. KIEHL & J. L. VERDIER, "Ein Einfacher Beweis des Kohärenzsatzes von Grauert", *Math. Annalen* **195** (1971), p. 24-50.

[24] E. J. N. LOOIJENGA, "Isolated singular points on complete intersections", in *Lect. Notes Series* (C. U. Press, ed.), no. 77, London Math. Society, 1984, p. 200 pp.

[25] B. MALGRANGE, "Intégrales asymptotiques et monodromie", *Ann. Scient. École Norm. Sup.* **7** (1974), no. 4, p. 405-430.

[26] ———, "Sommation des séries divergentes", *Expositiones Mathematicae* **13** (1995), no. 2/3, p. 163-222.

[27] J. MARTINET, "Singularities of smooth functions and maps", in *Lecture Notes Series*, vol. 58, Cambridge University Press, 1982, p. 272 pp.

[28] J. MATHER, "Stratifications and mappings", Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971), Academic Press, 1973, pp. 195–232.

[29] J. E. MOYAL, "Quantum mechanics as a statistical theory", *Proc. Cambridge Philos. Soc.* **45** (1949), p. 99-124.

[30] F. PHAM, "Multiple turning points in exact WKB analysis (variations on a theme of Stokes)", Towards the exact WKB analysis of differential equations, linear or non linear (C. Howls, T. Kawai, and Y. Takei, eds.), Kyoto University Press, 2000, pp. 71–85.

[31] ———, "Resurgence, quantized canonical transformations, and multi-instanton expansions", Algebraic analysis (M. Kashiwara and T. Kawai, eds.), vol. II, Academic Press, Boston, MA, 1988, Papers dedicated to Professor Mikio Sato on the occasion of his sixtieth birthday, pp. 699–726.

[32] P. POLESELLO & P. SCHAPIRA, "Stacks of quantization-deformation modules on complex symplectic manifolds", *Int. Math. Research Notices* **49** (2004), p. 2637-2664.

[33] M. REED & B. SIMON, "Methods of modern mathematical physics, vol. IV", Academic Press, 1978.

[34] B. SIMON, "Borel summability of the ground state energy in spatially cutoff $(\varphi^4)_2$", *Physical Review letters* **25** (1970), no. 22, p. 1583-1586.

[35] ———, "Determination of eigenvalues by divergent perturbation series", *Advances in Mathematics* **7** (1971), p. 240-253.

[36] J. SJÖSTRAND, "Singularités analytiques microlocales", *Astérisque* **95** (1982), p. 1-166.

[37] J. VEY, "Sur le lemme de Morse", *Invent. Math.* **40** (1977), no. 1, p. 1-9.

[38] A. VOROS, "Exact quantization condition for anharmonic oscillators (in one dimension)", *J. Phys. A* **27** (1994), p. 4653-4661.

[39] VAN DER WAERDEN (ED.), *Sources of quantum mechanics*, Dover, 1968.

[40] J. ZINN-JUSTIN, "Multi-instanton contributions in quantum mechanics, 2", *Nucl.Phys. B* **218** (1983), p. 333-348.

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