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The Batalin-Vilkovisky Algebra on Hochschild Cohomology Induced by Infinity Inner Products

THE BATALIN-VILKOVISKY ALGEBRA ON HOCHSCHILD COHOMOLOGY INDUCED BY INFINITY INNER PRODUCTS

by Thomas TRADLER

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ABSTRACT. — We define a BV-structure on the Hochschild cohomology of a unital, associative algebra A with a symmetric, invariant and non-degenerate inner product. The induced Gerstenhaber algebra is the one described in Gerstenhaber's original paper on Hochschild-cohomology. We also prove the corresponding theorem in the homotopy case, namely we define the BV-structure on the Hochschild-cohomology of a unital A_∞-algebra with a symmetric and non-degenerate ∞-inner product.

RÉSUMÉ. — On définit une structure de BV sur la cohomologie de Hochschild d'une algèbre associative unitaire munie d'une forme bilinéaire symétrique non dégénérée. La structure d'algèbre de Gerstenhaber induite est celle introduite dans l'article originel de Gerstenhaber sur la cohomologie de Hochschild. On étend ce résultat au cas d'une algèbre A-infinie unitaire munie d'une forme bilinéaire symétrique A-infinie non dégénérée.

1. Introduction

M. Gerstenhaber [8] showed that for any associative algebra A, the Hochschild-cohomology $H^\bullet(A, A)$ has a Gerstenhaber-structure. More precisely, $H^\bullet(A, A)$ has a Gerstenhaber-bracket $[.,.]$ which is a Lie-bracket of degree -1, and a commutative, associative cup-product $\smile$, such that $[.,.]$ is a graded derivation of the cup-product in each variable. It is an important case, when the Gerstenhaber-structure comes from a Batalin-Vilkovisky (BV) algebra. This means that there is a degree -1 operator Δ on $H^\bullet(A, A)$, such that Δ squares to zero, and so that the deviation of Δ from being a derivation is the bracket.

Our goal in this paper is to define such a Δ-operator on $H^\bullet(A, A)$, under the additional condition that A has a unit 1 and a symmetric, invariant and non-degenerate inner product. We will in fact produce two versions of this theorem; one for associative algebras, and one for homotopy associative algebras. To be precise about our first version, we now state the theorem, that is proved in section 2.

THEOREM 1. — *Let A be a finite dimensional, ungraded, unital, associative algebra with symmetric, invariant and non-degenerate inner product $\langle .,. \rangle : A \otimes A \to R$. For $f \in C^n(A, A)$, let $\Delta f \in C^{n-1}(A, A)$ be given by the equation*

$$\langle \Delta f(a_1, \ldots, a_{n-1}), a_n \rangle = \sum_{i=1}^{n} (-1)^{i(n-1)} \langle f(a_i, \ldots, a_{n-1}, a_n, a_1, \ldots, a_{i-1}), 1 \rangle.$$

Then, Δ is a chain map, such that the induced operation on Hochschild-cohomology $H^\bullet(A, A)$ squares to zero. Furthermore, for $\alpha \in H^n(A, A)$, $\beta \in H^m(A, A)$, the following identity holds,

$$[\alpha, \beta] = (-1)^{(n-1)m+1} \cdot \big(\Delta(\alpha \smile \beta) - \Delta(\alpha) \smile \beta - (-1)^n \alpha \smile \Delta(\beta)\big).$$

Here, $[.,.]$ and $\smile$ denote the Gerstenhaber-bracket and the cup-product defined in [8].

Is it important to remark, that the Δ-operator in the above theorem is just Connes' B-operator on the Hochschild chains $C_\bullet(A, A)$, see [4] or [13, (2.1.9.1)], dualized and transferred to the Hochschild cochains $C^\bullet(A, A)$ via the inner product:

$$C^\bullet(A, A) \qquad \text{Gerstenhaber-structure } [.,.], \smile$$

$\Big\uparrow$ isomorphism induced by $\langle .,. \rangle$

$$C^\bullet(A, A^*) \qquad \text{dual of Connes' } B\text{-operator.}$$

Although this version of the theorem is already useful in certain situations, it is dissatisfactory in more general cases. In the second version of our theorem, we therefore remove the assumptions of finite dimensionality, we include grading, and use infinity versions of the concepts in theorem 1. More precisely, we start with an A_∞-algebra A, and we use the concept of ∞-inner products, which was developed by the author in [20]. Following Stasheff's approach to A_∞-algebras and its Hochschild-cochain complex, see [18] and [19], using the tensor coalgebra and its coderivation complex, we define A_∞-bimodules and the Hochschild-cochain complex with values

in an A_∞-bimodule, via the tensor bi-comodule, and coderivations associated to it. For an A_∞-algebra A, ∞-inner products are A_∞-bimodule maps between the A_∞-bimodules A and its dual A^*, see section 3.1.

It is known, that Hochschild-cohomology of an A_∞-algebra still has a Gerstenhaber-structure, see [9]. In section 3, we generalize theorem 1 to the infinity case as follows.

THEOREM 2. — *Let (A, D) be an A_∞-algebra with strict unit 1 and let F be a symmetric, non-degenerate ∞-inner product. Then, Δ and the generalized cup-product M induce a BV-algebra on the Hochschild-cohomology $H^\bullet(A, A)$. In fact, for $\alpha, \beta \in H^\bullet(A, A)$, we have that*

$$[\alpha, \beta] = \Delta\big(M(\alpha, \beta)\big) - M(\Delta\alpha, \beta) - (-1)^{\|\alpha\|} M(\alpha, \Delta\beta),$$

where [.,.] denotes, up to sign, the Gerstenhaber-bracket from [9].

We remark, that the difference of signs in theorems 1 and 2 stem from the difference of signs of the bracket in [8] and the bracket used in theorem 2.

There are several variations and extensions of the above theorems. For instance, theorem 1 can be lifted to the chain level, which was described by Costello [5], Kaufmann [11], Menichi [16], and Tradler and Zeinalian [21]. In contrast to these chain level approaches, we here give an explicit proof of the BV-algebra in the style of Gerstenhaber's paper [8]. In fact, the idea for theorem 1 is taken from the strong analogy between [1, lemma 4.6] and [8, theorem 8.5], which is applied by transferring Chas and Sullivan's proof [1, lemma 5.2] of the BV-relation in string topology to the Hochschild-cochain level. For cyclic A_∞-algebras, one may find chain level statements in Costello [5], whereas Tradler and Zeinalian [22] describes an action for every A_∞-algebra with ∞-inner product.

It is an interesting question to determine the exact relationship between the BV-algebra given here, and the one from string topology [1]. Using the idea of local constructions of infinity structures from Lawrence and Sullivan in [12], M. Zeinalian and the author constructed in [23] a symmetric, non-degenerate ∞-inner product on the cochains $A = C^\bullet(X)$ of a Poincaré duality space X. If X is simply connected, then it is known that the Hochschild-cohomology $H^\bullet(A, A)$ is isomorphic to the homology of the free loop space $H_\bullet(LX)$ of X, see [10]. Thus, we can ask, if the BV-algebra on $H^\bullet(A, A)$ from theorem 2, and the one on $H_\bullet(LX)$ from string topology coincide under this isomorphism. In fact, it has been shown by Felix-Thomas [6] and Merkulov [17], that in characteristic zero, the BV-algebras on $H^\bullet(A, A)$ and $H_\bullet(LX)$ do coincide. In characteristic non-zero, this does not hold in general, when using the homological Poincaré duality

pairing, cf. Menichi [15] and Yang [24]. It is our hope, that the notion of ∞-inner products provides a suitable notion of chain level Poincaré duality to induce a model of the BV-algebra on the free loop space in the general setting. This question will be addressed in a future paper.

2. The associative case

In this section, we prove theorem 1. For this, we show that the BV-relations from theorem 1 are satisfied on the Hochschild-cohomology of a unital, associative algebra with symmetric, invariant and non-degenerate inner product.

Let R be a commutative ring with unit. As in [8], we assume that all rings A and modules M on which they act, are simultaneously R modules, the operations of A are R module homomorphisms, and all tensor products are assumed to be taken over R.

Let A be an (ungraded) finite dimensional associative algebra over R with unit $1 \in A$, and denote by $\langle .,. \rangle : A \otimes A \to R$ a non-degenerate inner product such that $\langle a_1, a_2 \rangle = \langle a_2, a_1 \rangle$ and $\langle a_1 \cdot a_2, a_3 \rangle = \langle a_1, a_2 \cdot a_3 \rangle$ for all $a_1, a_2, a_3 \in A$. The assumptions imply that the inner product induces a bimodule isomorphism between A and its linear dual $A^* = \mathrm{Hom}(A, R)$, given by $F : A \to A^*, a \mapsto F(a) := \langle a, . \rangle$.

For any bimodule M over A, denote by $C^n(A, M) := Hom(A^{\otimes n}, M)$, and $C^\bullet := \bigoplus_{n \geqslant 0} C^n(A, A)$ the Hochschild-cochains of A with values in M. For $f \in C^n(A, A)$, the differential $\delta f \in C^{n+1}(A, M)$ is given by

$$\delta f(a_1, \ldots, a_{n+1}) := a_1 \cdot f(a_2, \ldots, a_{n+1})$$
$$+ \sum_{i=1}^{n} (-1)^i f(a_1, \ldots, a_i \cdot a_{i+1}, \ldots, a_{n+1})$$
$$+ (-1)^{n+1} f(a_1, \ldots, a_n) \cdot a_{n+1}.$$

It is $\delta^2 = 0$, and its homology $H^\bullet(A, M)$ is called Hochschild-cohomology of A with values in M. We are mainly interested in the bimodules $M = A$ and $M = A^*$. Note that the map F induces a chain map

$$F_\sharp : C^\bullet(A, A) \longrightarrow C^\bullet(A, A^*), \quad f \longmapsto F_\sharp(f) := F \circ f.$$

The cup-product $f \smile g \in C^{n+m}(A, A)$ for $f \in C^n(A, A)$ and $g \in C^m(A, A)$ is given by

$$f \smile g(a_1, \ldots, a_{n+m}) := f(a_1, \ldots, a_n) \cdot g(a_{n+1}, \ldots, a_{n+m}).$$

DEFINITION 3 (Δ-operator). — Let $n \geqslant 1$ and $f \in C^n(A, A)$. For i in $\{1, \ldots, n\}$ define $\Delta_i f \in C^{n-1}(A, A)$ by

$$\langle \Delta_i f(a_1, \ldots, a_{n-1}), a_n \rangle = \langle f(a_i, \ldots, a_{n-1}, a_n, a_1, \ldots, a_{i-1}), 1 \rangle.$$

With this, define

$$\Delta := \sum_{i=1}^n (-1)^{i(n-1)} \Delta_i : C^n(A, A) \longrightarrow C^{n-1}(A, A).$$

We also need the maps Δ^1 and Δ^2, which for f in $C^n(A, A)$ and g in $C^m(A, A)$ are elements $\Delta^1(f \otimes g), \Delta^2(f \otimes g) \in C^{n+m-1}(A, A)$,

$$\Delta^1(f \otimes g) := \sum_{i=1}^m (-1)^{i(n+m-1)} \Delta_i(f \smile g),$$

$$\Delta^2(f \otimes g) := \sum_{i=m+1}^{n+m} (-1)^{i(n+m-1)} \Delta_i(f \smile g).$$

We have $\Delta^1(f \otimes g) + \Delta^2(f \otimes g) = \Delta(f \smile g)$. The notation for Δ^1 and Δ^2 is taken from [1, chapter 5].

The following lemma follows by a straightforward calculation.

LEMMA 4. — *In the above notation, we have that*

1) Δ *is a chain map,*
2) *for all* $f \in C^n(A, A), g \in C^m(A, A)$,

$$\Delta^1(f \otimes g) = (-1)^{nm} \Delta^2(g \otimes f).$$

Recall from [8], that for $j \in \{1, \ldots, n\}$ and $f \in C^n(A, A), g \in C^m(A, A)$, $f \circ_j g \in C^{n+m-1}(A, A)$ is defined by

$$f \circ_j g(a_1, \ldots, a_{n+m-1}) := f(a_1, \ldots, a_{j-1}, g(a_j, \ldots, a_{j+m-1}),$$
$$a_{j+m}, \ldots, a_{n+m-1}).$$

From this, one defines for $f \in C^n(A, A), g \in C^m(A, A)$ the pre-Lie product $f \circ g \in C^{n+m-1}(A, A)$ and the Gerstenhaber-bracket $[f, g] \in C^{n+m-1}(A, A)$,

$$f \circ g := \sum_{j=1}^n (-1)^{(j-1)(m-1)} f \circ_j g,$$

$$[f, g] := f \circ g - (-1)^{(n-1)(m-1)} g \circ f.$$

LEMMA 5. — *For closed elements* $f \in C^n(A, A)$ *and* $g \in C^m(A, A)$, $\delta f = \delta g = 0$, *we define* $H \in C^{n+m-2}(A, A)$ *by*

$$H := \sum_{\substack{i \geqslant 1,\, j \geqslant 1 \\ i+j \leqslant n}} (-1)^{(j-1)(m-1)+i(n+m)+1} \Delta_i(f \circ_j g).$$

Then H *satisfies*

$$\delta H = f \circ g - (-1)^{(n-1)m} \Delta(f) \smile g + (-1)^{(n-1)m} \Delta^2(f \otimes g).$$

Before proving lemma 5, we first show how this implies theorem 1.

Proof of theorem 1. — We need to show that the Gerstenhaber-bracket $[.,.]$ is the deviation of Δ from being a derivation of $\smile$,

$$
\begin{aligned}
[f, g] &= f \circ g - (-1)^{(n-1)(m-1)} g \circ f \\
&\cong (-1)^{(n-1)m} \Delta(f) \smile g - (-1)^{(n-1)m} \Delta^2(f \otimes g) \\
&\qquad - (-1)^{m-1} \Delta(g) \smile f + (-1)^{m-1} \Delta^2(g \otimes f) \\
&\cong -(-1)^{(n-1)m} \Delta^2(f \otimes g) - (-1)^{(n-1)m} \Delta^1(f \otimes g) \\
&\qquad + (-1)^{(n-1)m} \Delta(f) \smile g - (-1)^{(n-1)(m-1)} f \smile \Delta(g) \\
&= -(-1)^{(n-1)m} \cdot \big(\Delta(f \smile g) - \Delta(f) \smile g - (-1)^n f \smile \Delta(g) \big),
\end{aligned}
$$

where we have used lemma 4, 2) and the fact that the cup-product is graded commutative on Hochschild-cohomology. Furthermore it is well-known, that on homology the square of Δ vanishes. This can be seen, just as for Connes' B-operator, by considering the normalized Hochschild-cochain complex, which is a quasi-isomorphic subcomplex of $C^\bullet(A, A)$, where $(\Delta)^2 = 0$. $\qquad \square$

Proof of lemma 5. — Let $a_1, \ldots, a_{n+m} \in A$ be fixed. For $i \leqslant j$, we will use the notation from [8], and denote by $a_{i,j} = (a_i, \ldots, a_j) \in A^{\otimes j-i+1}$. For $i \geqslant 1$, $j \geqslant 1$ and $i + j \leqslant n$, we calculate $\delta(\Delta_i(f \circ_j g))$ as follows:

$$
\begin{aligned}
&\big\langle \delta(\Delta_i(f \circ_j g))(a_{1,n+m-1}), a_{n+m} \big\rangle \\
&\quad = \big\langle a_1 \cdot \Delta_i(f \circ_j g)(a_{2,n+m-1}), a_{n+m} \big\rangle \\
&\qquad + \sum_{\lambda=1}^{n+m-2} (-1)^\lambda \big\langle \Delta_i(f \circ_j g)(a_{1,\lambda-1}, (a_\lambda \cdot a_{\lambda+1}), a_{\lambda+2,n+m-1}), a_{n+m} \big\rangle \\
&\qquad + (-1)^{n+m-1} \big\langle \Delta_i(f \circ_j g)(a_{1,n+m-2}) \cdot a_{n+m-1}, a_{n+m} \big\rangle
\end{aligned}
$$

$$= \big\langle f(a_{i+1,i+j-1}, g(a_{i+j,i+j+m-1}), a_{i+j+m,n+m-1}, (a_{n+m} \cdot a_1), a_{2,i}), 1 \big\rangle$$

$$+ \sum_{\lambda=1}^{i-1} (-1)^\lambda \big\langle f(a_{i+1,i+j-1}, g(a_{i+j,i+j+m-1}),$$
$$a_{i+j+m,n+m}, a_{1,\lambda-1}, (a_\lambda \cdot a_{\lambda+1}), a_{\lambda+2,i}), 1 \big\rangle$$

$$+ \sum_{\lambda=i}^{i+j-2} (-1)^\lambda \big\langle f(a_{i,\lambda-1}, (a_\lambda \cdot a_{\lambda+1}), a_{\lambda+2,i+j-1}, g(a_{i+j,i+j+m-1}),$$
$$a_{i+j+m,n+m}, a_{1,i-1}), 1 \big\rangle$$

$$+ \sum_{\lambda=i+j-1}^{i+j+m-2} (-1)^\lambda \big\langle f(a_{i,i+j-2}, g(a_{i+j-1,\lambda-1}, (a_\lambda \cdot a_{\lambda+1}), a_{\lambda+2,i+j+m-1}),$$
$$a_{i+j+m,n+m}, a_{1,i-1}), 1 \big\rangle$$

$$+ \sum_{\lambda=i+j+m-1}^{n+m-1} (-1)^\lambda \big\langle f(a_{i,i+j-2}, g(a_{i+j-1,i+j+m-2}),$$
$$a_{i+j+m-1,\lambda-1}, (a_\lambda \cdot a_{\lambda+1}), a_{\lambda+2,n+m}, a_{1,i-1}), 1 \big\rangle,$$

Using the fact that $\delta(g) = 0$, we see that we can write this as

$$(2.1) \qquad \big\langle \big(\delta(\Delta_i(f \circ_j g)) \big)(a_{1,n+m-1}), a_{n+m} \big\rangle = h_{i,j} + h'_{i,j} + h''_{i,j},$$

where we define

$$h_{i,j} := (-1)^{i+1} \big\langle a_i \cdot f(a_{i+1,i+j-1}, g(a_{i+j,i+j+m-1}),$$
$$a_{i+j+m,n+m}, a_{1,i-1}), 1 \big\rangle$$
$$+ \sum_{\lambda=i}^{i+j-2} (-1)^\lambda \big\langle f(a_{i,\lambda-1}, (a_\lambda \cdot a_{\lambda+1}), a_{\lambda+2,i+j-1}, g(a_{i+j,i+j+m-1}),$$
$$a_{i+j+m,n+m}, a_{1,i-1}), 1 \big\rangle$$
$$+ (-1)^{i+j-1} \big\langle f(a_{i,i+j-2}, (a_{i+j-1} \cdot g(a_{i+j,i+j+m-1})),$$
$$a_{i+j+m,n+m}, a_{1,i-1}), 1 \big\rangle,$$

$$h'_{i,j} := (-1)^{i+j+m-2} \big\langle f(a_{i,i+j-2}, (g(a_{i+j-1,i+j+m-2}) \cdot a_{i+j+m-1}),$$
$$a_{i+j+m,n+m}, a_{1,i-1}), 1 \big\rangle$$

$$+ \sum_{\lambda=i+j+m-1}^{n+m-1} (-1)^\lambda \big\langle f(a_{i,i+j-2}, g(a_{i+j-1,i+j+m-2}),$$
$$a_{i+j+m-1,\lambda-1}, (a_\lambda \cdot a_{\lambda+1}), a_{\lambda+2,n+m}, a_{1,i-1}), 1 \big\rangle,$$

$$h''_{i,j} := \big\langle f(a_{i+1,i+j-1}, g(a_{i+j,i+j+m-1}), a_{i+j+m,n+m-1}, (a_{n+m} \cdot a_1), a_{2,i}), 1 \big\rangle$$

$$+ \sum_{\lambda=1}^{i-1} (-1)^\lambda \big\langle f(a_{i+1,i+j-1}, g(a_{i+j,i+j+m-1}), a_{i+j+m,n+m},$$
$$a_{1,\lambda-1}, (a_\lambda \cdot a_{\lambda+1}), a_{\lambda+2,i}), 1 \big\rangle$$

$$+ (-1)^i \big\langle f(a_{i+1,i+j-1}, g(a_{i+j,i+j+m-1}), a_{i+j+m,n+m}, a_{1,i-1}) \cdot a_i, 1 \big\rangle.$$

On the other hand, if $i \geqslant 2$, $j \geqslant 1$ and $i + j \leqslant n - 1$, we may use the fact that $\delta(f) = 0$, to obtain

$$(2.2) \qquad (-1)^{i+1} h_{i,j} + (-1)^{i+m} h'_{i,j+1} + (-1)^{i+n} h''_{i-1,j+1}$$
$$= \big\langle (\Delta_i((\delta f) \circ_j g))(a_{1,n+m-1}), a_{n+m} \big\rangle = 0.$$

If we set furthermore for $i, j \in \{1, \ldots, n\}$,

$$h_{i,0} := (-1)^{i+1} \big\langle g(a_{i,i+m-1}) \cdot f(a_{i+m,n+m}, a_{1,j-1}), 1 \big\rangle,$$

$$h'_{i,n-i+1} := (-1)^{n+m+1} \big\langle f(a_{i,n-1}, (g(a_{n,n+m-1}) \cdot a_{n+m}), a_{1,i-1}), 1 \big\rangle,$$

$$h''_{0,j} := \big\langle f(a_{1,j-1}, g(a_{j,j+m-1}), a_{j+m,n+m-1}) \cdot a_{n+m}, 1 \big\rangle,$$

then all three quantities $h_{i,j}$, $h'_{i,j+1}$, $h''_{i-1,j+1}$ are defined for all i, j with $i \geqslant 1$, $j \geqslant 0$ and $i + j \leqslant n$. Furthermore equation (2.2) holds for each of these i, j, so that we obtain

$$0 = \sum_{\substack{i \geqslant 1, j \geqslant 0 \\ i+j \leqslant n}} (-1)^{(j-1)(m-1)+i(n+m-1)}$$
$$\times \big((-1)^{i+1} h_{i,j} + (-1)^{i+m} h'_{i,j+1} + (-1)^{i+n} h''_{i-1,j+1} \big).$$

Using equation (2.1) and rearranging the terms in this sum gives

$$0 = \sum_{\substack{i \geqslant 1, j \geqslant 1 \\ i+j \leqslant n}} (-1)^{(j-1)(m-1)+i(n+m)+1} \big(h_{i,j} + h'_{i,j} + h''_{i,j} \big)$$
$$- \sum_{i=1}^{n} (-1)^{m-1+i(n+m)} h_{i,0}$$
$$- \sum_{i=1}^{n} (-1)^{n(m+1)+i(n+1)} h'_{i,n-i+1}$$
$$- \sum_{j=1}^{n} (-1)^{(j-1)(m-1)} h''_{0,j}$$
$$= \big\langle \delta(H)(a_{1,n+m-1}), a_{n+m} \big\rangle$$
$$- (-1)^{m(n+1)} \big\langle (\Delta^2(f \otimes g))(a_{1,n+m-1}), a_{n+m} \big\rangle$$
$$- (-1)^{m(n+1)+1} \big\langle (\Delta(f) \smile g)(a_{1,n+m-1}), a_{n+m} \big\rangle$$
$$- \big\langle (f \circ g)(a_{1,n+m-1}), a_{n+m} \big\rangle.$$

This implies the claim of the lemma.   □

We demonstrate theorem 1 in the easiest non-trivial example.

Example 6. — Let $A = R^2$ with generators u and v, where the algebra structure is so that u is the unit in A, and $v^2 = 0$. The inner product is defined by letting $\langle v, u \rangle = \langle u, v \rangle = 1$, and zero in the other cases. Recall, that the normalized Hochschild cochain complex consists, in degree n, of

maps of the form $\bar{A}^{\otimes n} \to A$, where $\bar{A} = A/R$ is isomorphic to R with generator v. For $n \geqslant 0$, denote by

$$f_n : \bar{A}^{\otimes n} \longrightarrow A, \quad f_n(v, \ldots, v) = u,$$

$$g_n : \bar{A}^{\otimes n} \longrightarrow A, \quad g_n(v, \ldots, v) = v.$$

One checks that $\delta f_{2n} = 0$, $\delta f_{2n+1} = 2g_{2n+2}$, and $\delta g_n = 0$ for all n. Assuming that our ring R contains an element $\frac{1}{2}$, we can now calculate Hochschild cohomology as,

$$H^\bullet(A, A) = \frac{\mathrm{Ker}(\delta)}{\mathrm{Im}(\delta)} \cong \bigoplus_{n \geqslant 0} R f_{2n} \oplus R g_0 \oplus \bigoplus_{n \geqslant 0} R g_{2n+1}.$$

The algebra structure is given by

$$f_n \smile f_m = f_{n+m}, \quad f_n \smile g_m = g_m \smile f_n = g_{n+m}, \quad g_n \smile g_m = 0,$$

so that, as an algebra, we may write

$$H^\bullet(A, A) \cong R[f_2, g_0, g_1]/(g_0^2, g_0 f_2, g_0 g_1, g_1^2).$$

From the definition, one also calculates the Δ-operator on the chain level as $\Delta(f_n) = \Delta(g_{2n}) = 0$, and $\Delta(g_{2n+1}) = (2n + 1)f_{2n}$.

In the case that A is a graded associative algebra, we may obtain theorem 1 in the same way as above, except for an appropriate change in signs. For this case, example 6, with the modification of letting v be in even degree $2d$, gives the calculation for the BV-algebra of Hochschild cohomology of the homology of the $2d$-sphere S^{2d}, cf. [3], [7], [17]. The next section includes graded associative algebras as a special case.

3. The homotopy associative case

We now consider the more general situation from theorem 2. We recall the notion of ∞-inner products from [20] in subsection 3.1, define the operations on the Hochschild-cochain complex in subsection 3.2, and prove theorem 2 in subsection 3.3.

3.1. Infinity-inner products

This section briefly recalls the definitions of A_∞-algebra, ∞-inner-products and the Hochschild-cochain complex as defined in [20].

Informally, an A_∞-algebra is an algebra over a certain operad, the Stasheff operad, cf. [14, chap. 1.8]. This operad is a particular cofibrant replacement of the operad of associative algebras, and its cellular structure allows A_∞-algebras to be stated in terms of coderivations, as we will do now.

DEFINITION 7 (A_∞-algebra). — Let $A = \bigoplus_{j \in \mathbb{Z}} A_j$ be a graded module over a given ground ring R. Define its *suspension* sA to be the graded module

$$sA = \bigoplus_{j \in \mathbb{Z}} (sA)_j \quad \text{with} \quad (sA)_j := A_{j-1}.$$

The suspension map $s : A \longrightarrow sA$, $a \mapsto sa := a$ is an isomorphism of degree $+1$. We define

$$BA := TsA = \bigoplus_{i \geqslant 0} sA^{\otimes i}$$

to be the tensor coalgebra of sA, with the comultiplication

$$\Delta : BA \longrightarrow BA \otimes BA, \quad \Delta(a_1, \ldots, a_n) := \sum_{i=0}^{n} (a_1, \ldots, a_i) \otimes (a_{i+1}, \ldots, a_n).$$

A *coderivation* on BA is a map $D : BA \to BA$ such that

$$
\begin{array}{ccc}
BA & \xrightarrow{\Delta} & BA \otimes BA \\
{\scriptstyle D}\downarrow & & \downarrow{\scriptstyle D \otimes \mathrm{Id} + \mathrm{Id} \otimes D} \\
BA & \xrightarrow{\Delta} & BA \otimes BA
\end{array}
$$

The space of coderivation on BA is denoted by $\mathrm{Coder}(BA)$.

An A_∞ *algebra structure on* A is a coderivation $D \in \mathrm{Coder}(BA)$ of degree -1 with $D^2 = 0$.

Every A_∞-algebra (A, D) is uniquely given by its components

$$D = \sum_{i \geqslant 0} D_i,$$

where $D_i : sA^{\otimes i} \to sA$ are lifted to coderivations, which, by abuse of notation, are still denoted by D_i, cf. [20, section 2]. We follow the usual convention and assume that the lowest level D_0 vanishes. Graphically, we represent D by a filled circle, which takes many inputs and gives one output, here from left to right:
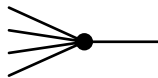
DEFINITION 8 (A_∞-bimodule). — Now, let (A, D) be an A_∞-algebra, and let $M = \bigoplus_{j \in \mathbb{Z}} M_j$ be a graded module. Define

$$B^M A := R \oplus \bigoplus_{k \geqslant 0, \ell \geqslant 0} sA^{\otimes k} \otimes sM \otimes sA^{\otimes \ell}$$

to be the tensor bi-comodule with the bi-comodule map

$$\Delta^M : B^M A \longrightarrow (BA \otimes B^M A) \oplus (B^M A \otimes BA),$$

$$\Delta^M(a_1, \ldots, a_k, m, a_{k+1}, \ldots, a_{k+\ell})$$
$$:= \sum_{i=0}^{k} (a_1, \ldots, a_i) \otimes (a_{i+1}, \ldots, m, \ldots, a_n)$$
$$+ \sum_{i=k}^{k+\ell} (a_1, \ldots, m, \ldots, a_i) \otimes (a_{i+1}, \ldots, a_{k+\ell}).$$

Now, define a coderivation on $B^M A$ over D to be a map $D^M : B^M A \to B^M A$ such that

$$
\begin{array}{ccc}
B^M A & \xrightarrow{\Delta^M} & (BA \otimes B^M A) \oplus (B^M A \otimes BA) \\
\downarrow{\scriptstyle D^M} & & \downarrow{\scriptstyle (D \otimes \mathrm{Id} + \mathrm{Id} \otimes D^M) \oplus (D^M \otimes \mathrm{Id} + \mathrm{Id} \otimes D)} \\
B^M A & \xrightarrow{\Delta^M} & (BA \otimes B^M A) \oplus (B^M A \otimes BA)
\end{array}
$$

The space of all coderivations on $B^M A$ over D is denoted by $\mathrm{Coder}_D(B^M A)$. An A_∞-*bimodule structure on M over A* is defined to be a coderivation $D^M \in \mathrm{Coder}_D(B^M A)$ of degree -1 with $(D^M)^2 = 0$.

Finally, a *coderivation from BA to $B^M A$* is a map $f : BA \to B^M A$ such that

$$
\begin{array}{ccc}
BA & \xrightarrow{\quad \Delta \quad} & BA \otimes BA \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{Id} \otimes f + f \otimes \mathrm{Id}} \\
B^M A & \xrightarrow{\Delta^M} & (BA \otimes B^M A) \oplus (B^M A \otimes BA)
\end{array}
$$

The space of all coderivations from BA to $B^M A$ is denoted by $C^\bullet(A, M)$, and is called the *Hochschild-cochain complex of A with values in M*. The space $C^\bullet(A, M)$ has a differential

$$\delta^M(f) := D^M \circ f - (-1)^{|f|} f \circ D$$

satisfying $(\delta^M)^2 = 0$. The homology $H^\bullet(A, M)$ is called the *Hochschild-cohomology of A with values in M*.

Similar to before, we can uniquely decompose an element $f \in C^\bullet(A, M)$ in its components $f = \sum_{i \geqslant 0} f_i$, where $f_i : sA^{\otimes i} \to sM$ is lifted to a coderivation $f_i : BA \to B^M A$, see [20, section 3]. Here is the graphical representation for f, where the inputs on the left are from sA and the output on the right lies in sM:
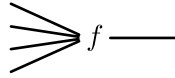
DEFINITION 9 (∞-inner-product). — Let (A, D) be an A_∞ algebra and let (M, D^M) and (N, D^N) be two A_∞ bimodules over A. An A_∞-bimodule map between M and N is a map $F : B^M A \to B^N A$ such that

$$
\begin{array}{ccc}
B^M A & \xrightarrow{\ \Delta^M\ } & (BA \otimes B^M A) \oplus (B^M A \otimes BA) \\
\Big\downarrow{\scriptstyle F} & & \Big\downarrow{\scriptstyle (\mathrm{Id}\, \otimes F) \oplus (F \otimes \mathrm{Id})} \\
B^N A & \xrightarrow{\ \Delta^N\ } & (BA \otimes B^N A) \oplus (B^N A \otimes BA)
\end{array}
$$

and $D^N \circ F = F \circ D^M$. Every A_∞-bimodule map $F : B^M A \to B^N A$ induces a chain map on the Hochschild-cochain complexes $F_\sharp : C^\bullet(A, M) \to C^\bullet(A, N)$ by $f \mapsto F \circ f$.

Let (A, D) be an A_∞ algebra. Then both A and its linear dual $A^* = \mathrm{Hom}(A, R)$ have induced A_∞-bimodule structures $D^A \in \mathrm{Coder}_D(B^A A)$ and $D^{A^*} \in \mathrm{Coder}_D(B^{A^*} A)$, see [20, lemma 3.9]. An ∞-inner-product is by definition an A_∞-bimodule map $F : B^A A \to B^{A^*} A$ between A and A^* of degree 0. The ∞-inner-product is called *non-degenerate* if there exists an A_∞-bimodule map $G : B^{A^*} A \to B^A A$, such that

$$
F_\sharp : C^\bullet(A, A) \longrightarrow C^\bullet(A, A^*) \quad \text{and} \quad G_\sharp : C^\bullet(A, A^*) \longrightarrow C^\bullet(A, A)
$$

are quasi-inverse to each other, so that their induced maps on Hochschild-cohomology are inverses.

Again, one can show, that the ∞-inner product F is given by components $F : \sum_{i,j \geqslant 0} F_{i,j}$, where the $F_{i,j} : sA^{\otimes i} \otimes sA \otimes sA^{\otimes j} \to sA^*$ are lifted according to the property in definition 9, see [20, section 4]. If we apply arguments to this map, $F_{i,j}(a_1, \ldots, a_i, a_{i+1}, a_{i+2}, \ldots, a_{i+j+1})(a_{i+j+2}) \in R$, then we can represent the ∞-inner-product graphically as in [20, section 5]

by an open circle:
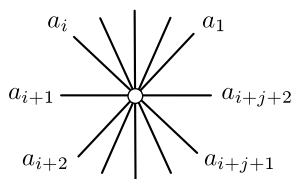
Here we included the arguments $a_1, \ldots, a_{i+j+2}$ to show their chosen position; the first i arguments are placed on top, the special element a_{i+1} on the left, the next j arguments on the bottom and the output argument on the right.

The ∞-inner product is called *symmetric*, if it is invariant under a $180°$ rotation, i.e. for all i, j,

$$F_{i,j}(a_1, \ldots, a_i, a_{i+1}, a_{i+2}, \ldots, a_{i+j+1})(a_{i+j+2})$$
$$= (-1)^\varepsilon F_{j,i}(a_{i+2}, \ldots, a_{i+j+1}, a_{i+j+2}, a_1, \ldots, a_i)(a_{i+1}),$$

where $\varepsilon = \left(\sum_{k=1}^{i+1}(|a_k| + 1) \right)\left(\sum_{k=i+2}^{i+j+2}(|a_k| + 1) \right)$.

In our generalization of theorem 1, we use a symmetric, non-degenerate ∞-inner product in the above sense. We call such a structure F an ∞-*Poincaré-duality structure*.

In general, one may not avoid higher homotopies for the inner product, even if the underlying space is a differential graded associative algebra. In fact, in [23, Appendix B], the following ∞-Poincaré-duality structure was determined for the simplicial cochains on the circle.

Example 10. — We represent the circle S^1 using the 0-simplicies a, b, and the 1-simplicies σ, τ as follows,
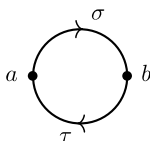
Let $sA = sC^\bullet S^1$ be the shifted cochain model of the circle with generators, denoted by the same symbols, a, b, σ and τ. The differential d on sA is defined by,

$$d(a) = \sigma - \tau, \quad d(b) = \tau - \sigma, \quad d(\sigma) = d(\tau) = 0,$$

and the multiplication is non-zero only on the following generators,

$$a \cdot a = a, \quad b \cdot b = b, \quad b \cdot \sigma = -\sigma \cdot a = \sigma, \quad a \cdot \tau = -\tau \cdot b = \tau.$$

Note, that with these definitions, $(A, d, .)$ becomes a (non-commutative) differential graded associative algebra, and one may check that there is no non-degenerate, invariant inner product associated to A. But, assuming that the underlying ring R contains $\frac{1}{2}$, it was shown in [23, Appendix B and Proposition 3.1.3], that the following maps define an ∞-Poincaré-duality structure, representing Poincaré duality on the chain level of S^1,

$$F_{k,0}(\sigma, \cdots, \sigma, \sigma)(a) = +\tfrac{1}{2}, \qquad F_{0,k}(a, \sigma, \ldots, \sigma)(\sigma) = +\tfrac{1}{2},$$
$$F_{k,0}(\tau, \cdots, \tau, a)(\tau) = -\tfrac{1}{2}, \qquad F_{0,k}(\tau, \tau, \ldots, \tau)(a) = -\tfrac{1}{2},$$
$$F_{k,0}(\sigma, \cdots, \sigma, b)(\sigma) = -\tfrac{1}{2}, \qquad F_{0,k}(\sigma, \sigma, \ldots, \sigma)(b) = -\tfrac{1}{2},$$
$$F_{k,0}(\tau, \cdots, \tau, \tau)(b) = +\tfrac{1}{2}, \qquad F_{0,k}(b, \tau, \ldots, \tau)(\tau) = +\tfrac{1}{2},$$

for $k \geqslant 0$, and $F_{i,j} = 0$, for $i, j > 0$.

3.2. Definition of the operations

We now describe the operations that are needed to obtain the BV-algebra on the Hochschild-cohomology of an A_∞-algebra. To this end, we first recall the Gerstenhaber-bracket and the cup-product from [9], and the Δ-operator from the previous section. These operations will only be up to sign the needed operations, and we denote them by $[.,.]'$, M' and Δ'. With the help of the ∞-inner product, we can rewrite these operations as maps of the form $C^\bullet(A, A)^{\otimes p} \to C^\bullet(A, A^*)$. Using a graphical representation of a large class of operations $C^\bullet(A, A)^{\otimes p} \to C^\bullet(A, A^*)$, which we call symbols, we may find symbols which represent the above operations up to sign. The operations of these symbols, toegther with their naturally given signs will be denoted by $[.,.]$, M and Δ, and will be used in subsection 3.3 to prove theorem 2.

DEFINITION 11 (Gerstenhaber-bracket, cup-product). — Fix an A_∞-algebra (A, D), which in components is given by $D = \sum_{i \geqslant 1} D_i$, where $D_i : sA^{\otimes i} \to sA$. Let $f, g \in C^\bullet(A, A)$ be represented in components by $f = \sum_{n \geqslant 0} f_n$, and $g = \sum_{m \geqslant 0} g_m$ as in definition 8, where $f_n : sA^{\otimes n} \to sA$ and $g_m : sA^{\otimes m} \to sA$. For these components, we define the components of the Gerstenhaber-bracket

$$[f_n, g_m]' : sA^{\otimes n+m-1} \longrightarrow sA$$

and the components of the generalized cup-product

$$M(f_n, g_m) : \bigoplus_{k \geqslant 0} sA^{\otimes n+m+k} \longrightarrow sA$$

by

$$[f_n, g_m]' = f_n \circ g_m - (-1)^{\|f_n\| \cdot \|g_m\|} g_m \circ f_n,$$

where

$$f_n \circ g_m(a_1, \ldots, a_{n+m-1})$$
$$= \sum_{i=1}^{n} (-1)^{\|g_m\| \cdot \sum_{\ell=1}^{i-1}(|a_\ell|+1)} f_n(a_1, \ldots, g_m(a_i, \ldots, a_{i+m-1}), \ldots, a_{n+m-1}),$$

$$M'(f_n, g_m)(a_1, \ldots, a_{n+m+k})$$
$$= \sum_{\substack{1 \leqslant i \\ i+n \leqslant j \leqslant n+k}} (-1)^{\|f_n\| \cdot \sum_{\ell=1}^{i-1}(|a_\ell|+1) + \|g_m\| \cdot (\|f_n\| + \sum_{\ell=1}^{j-1}(|a_\ell|+1))}$$
$$\cdot D_{k-2}(a_1, \ldots, f_n(a_i, \ldots, a_{i+n-1}),$$
$$\ldots, g_m(a_j, \ldots, a_{j+m-1}), \ldots, a_{n+m+k})$$

where $\|f_n\|$ and $\|g_m\|$ are the total degrees of the maps $f_n : sA^{\otimes n} \to sA$ and $g_m : sA^{\otimes m} \to sA$. Both the bracket $[.,.]'$ and the product M' are extended linearly to operations $C^\bullet(A, A) \otimes C^\bullet(A, A) \to C^\bullet(A, A)$.

The following is observed by E. Getzler and J. Jones in [9].

THEOREM 12. — *Let (A, D) be an A_∞-algebra. Then $[.,.]'$ and M' induce a Gerstenhaber-structure on Hochschild-cohomology $H^\bullet(A, A)$.*

Next, we consider the unit and the Δ'-operator.

DEFINITION 13 (Unit, Δ'-operator). — An element

$$1 \in A_0 \cong (sA)_{+1} \subset BA$$

is called a *strict unit of A*, or simply a *unit of A*, if D_n applied to any element of the form $(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n) \in (sA)^{\otimes n}$ vanishes, except for the case $n = 2$,

$$D_n(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n) = 0 \quad \text{for } n \neq 2,$$
$$-D_2(1, a) = (-1)^{|a|+1} D_2(a, 1) = a.$$

The sign is chosen so that the unshifted multiplication $m_2 = s^{-1} \circ D_2 \circ (s \otimes s)$ satisfies $m_2(1, a) = m_2(a, 1) = a$, see [20, prop. 2.4].

We also have the Δ'-operator from section 2, which we write as an operation $\Delta' : C^\bullet(A, A^*) \to C^\bullet(A, A^*)$, which is the dual of Connes' B-operator. For $f \in C^\bullet(A, A^*)$ with components $f = \sum_{n \geqslant 0} f_n$, where

$f_n : sA^{\otimes n} \to sA^*$, we define the component $\Delta' f_n : sA^{n-1} \to sA^*$ by

$$\Delta' f_n(a_1, \ldots, a_{n-1})(a_n)$$
$$= \sum_{i=1}^{n} (-1)^{\sum_{\ell=1}^{i-1}(|a_\ell|+1) \cdot \sum_{\ell=i}^{n}(|a_\ell|+1)} f_n(a_i, \ldots, a_{n-1}, a_n, a_1, \ldots, a_{i-1})(1).$$

It is straightforward to check that in the A_∞ setting with strict unit 1, Δ' still forms a chain map. Furthermore, the operation on Hochschild-cohomology induced by the square of Δ' vanishes, which can be seen by considering the normalized Hochschild-cochain subcomplex, where $(\Delta')^2 = 0$ holds identically.

Now, fix a symmetric, non-degenerate ∞-inner product F. The non-degeneracy of F implies that $F_\sharp : C^\bullet(A, A) \to C^\bullet(A, A^*)$ induces an isomorphism of Hochschild-cohomologies $H^\bullet(A, A) \to H^\bullet(A, A^*)$. We use $F_\sharp : C^\bullet(A, A) \to C^\bullet(A, A^*)$ and its quasi-inverse $G_\sharp : C^\bullet(A, A^*) \to C^\bullet(A, A)$ from definition 9, to write all necessary operations as maps $C^\bullet(A, A)^{\otimes p} \to C^\bullet(A, A^*)$. For example, we have the induced operations

$$[.,.]' : C^\bullet(A, A)^{\otimes 2} \xrightarrow{[.,.]'} C^\bullet(A, A) \xrightarrow{F_\sharp} C^\bullet(A, A^*),$$

$$M' : C^\bullet(A, A)^{\otimes 2} \xrightarrow{M'} C^\bullet(A, A) \xrightarrow{F_\sharp} C^\bullet(A, A^*),$$

$$\Delta' : C^\bullet(A, A) \xrightarrow{F_\sharp} C^\bullet(A, A^*) \xrightarrow{\Delta'} C^\bullet(A, A^*),$$

$$\Delta' \circ M' : C^\bullet(A, A)^{\otimes 2} \xrightarrow{M'} C^\bullet(A, A) \xrightarrow{F_\sharp} C^\bullet(A, A^*) \xrightarrow{\Delta'} C^\bullet(A, A^*),$$

$$M' \circ (\mathrm{Id} \otimes \Delta') : C^\bullet(A, A)^{\otimes 2} \xrightarrow{\mathrm{Id} \otimes (G_\sharp \circ \Delta' \circ F_\sharp)} C^\bullet(A, A)^{\otimes 2}$$
$$\xrightarrow{M'} C^\bullet(A, A) \xrightarrow{F_\sharp} C^\bullet(A, A^*).$$

We will now introduce a convenient graphical representation of a large class of operations $C^\bullet(A, A)^{\otimes p} \to C^\bullet(A, A^*)$.

DEFINITION 14 (Symbols). — A symbol is a finite, planar tree, where each vertex is labeled by one of the following labels:

    ▷ an open circle $\bigcirc$,
    ▷ a filled circle $\bullet$,
    ▷ the unit 1,
    ▷ a numbered circle $\textcircled{1}$, $\textcircled{2}$, $\textcircled{3}$, ...

We require the following restrictions for a symbol:

    ▷ There is exactly one vertex which is labeled by an open circle. This vertex has two distinguished edges attached, which in the plane will be drawn as horizontally incoming from the left and right. There

are any amount of additional edges attached to this vertex, placed above or below the horizontal line.

▷ Each vertex labeled by a filled circle has at least three edges attached.

▷ Each vertex labeled by the unit has exactly one edge attached.

▷ There is exactly one distinguished external edge in the symbol. This edge will be denoted by a fat edge ▬▬▬.

Every symbol σ with numbered circles from 1 to p uniquely determines an operation $\rho_\sigma : C^\bullet(A, A)^{\otimes p} \to C^\bullet(A, A^*)$ in the following way. Let $f^1, \ldots, f^p \in C^\bullet(A, A)$. We describe the element $\rho_\sigma(f^1, \ldots, f^p) \in C^\bullet(A, A^*)$ by determining its value on arguments $\rho_\sigma(f^1, \ldots, f^p)(a_1, \ldots, a_n)(a_{n+1}) \in R$.

This value is obtained by cyclically attaching the arguments $a_1, \ldots, a_{n+1}$ to vertices which are labeled by an open circle, a filled circle, or a numbered circle, such that the last argument a_{n+1} lands on the distinguished external fat edge. Note, that each vertex labeled by a filled circle or a numbered circle has a unique path to the vertex labeled by the open circle. Thus there is one preferred edge at every filled and numbered circle, which lies on the path to the open circle. We call this edge the outgoing edge of this vertex, and all other edges attached to the vertex are considered as incoming to this vertex. We use this notion of incoming and outgoing edges at filled and numbered circles, to successively evaluate the arguments and the unit 1 toward the open circle. More precisely, to every filled vertex, we evaluate the arguments and the incoming edges using the A_∞-structure D, where $D : \bigoplus_{i \geqslant 1} sA^{\otimes i} \to sA$, and record the answer of this evaluation at the outgoing edge. Similarly, to every vertex with the circle numbered by r, we evaluate using the Hochschild-cochain $f^r : \bigoplus_{i \geqslant 0} sA^{\otimes i} \to sA$. We finally end at the vertex labeled by the open circle, where we evaluate all remaining arguments and incoming edges using the ∞-inner product $F : \bigoplus_{j,i} sA^{\otimes i} \otimes sA \otimes sA^{\otimes j} \to sA^*$. Here, the horizontal edge from the left is identified with the special $(i + 1)$th argument, and the horizontal edge from the right is identified with the argument plugged into sA^*. We thus obtain an overall number in R for each possible way of attaching the arguments $a_1, \ldots, a_{n+1}$.

With this, we define $\rho_\sigma(f^1, \ldots, f^p)(a_1, \ldots, a_n)(a_{n+1})$ to be the sum of all numbers in R, obtained by the above procedure from attaching the arguments $a_1, \ldots, a_{n+1}$ in all possible ways, where each number is multiplied by the following sign factor. We use the sign $(+1)$ for the unique term which is obtained by applying all arguments $a_1, \ldots, a_{n+1}$ to the position specified by the distinguished external fat edge. All other terms will be compared to

this unique term, by looking at its linear expression using the $f^1, \ldots, f^r$, D, and F, with arguments $a_1, \ldots, a_{n+1}$ and units 1 applied, such as

$$F_{i,j}\big(\ldots, D(\ldots, f^k(\ldots)), \ldots, f^m(\ldots, f^n(\ldots), \ldots), \ldots\big) \in R.$$

We then apply the usual Koszul sign rule of a factor of $(-1)^{|x| \cdot |y|}$ for each variable x that has moved across the variable y when comparing it to the expansion of the above term with sign $(+1)$.

The following examples show some symbols and their associated operations.

Example 15. — All examples except for the last one will be studied without signs.

1) Denote by σ the following symbol:
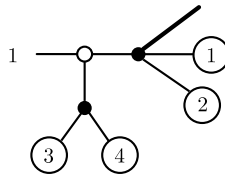
The induced operation $\rho_\sigma : C^\bullet(A, A)^{\otimes 4} \to C^\bullet(A, A^*)$ is given for f^1, f^2, f^3, f^4 in $C^\bullet(A, A)$ and $a_1, \ldots, a_{n+1}$ in A by

$$\rho_\sigma(f^1, f^2, f^3, f^4)(a_1, \ldots, a_n)(a_{n+1})$$
$$= \sum \pm F\big(\ldots, 1, \ldots, D(\ldots, f_3(\ldots), \ldots, f_4(\ldots), \ldots), \ldots\big)$$
$$\big(D(\ldots, f_2(\ldots)), \ldots, f_1(\ldots), \ldots, a_{n+1}, \ldots\big)\big),$$

where the unit 1 is placed at the special spot of F, and the sum is over all possible ways of placing $a_1, \ldots, a_{n+1}$ at the dots, such that their cyclic order is preserved and a_{n+1} is at the indicated position.

2) Our next example will be the Δ-operator. We define Δ to be the sum of operation associated to the following symbols:
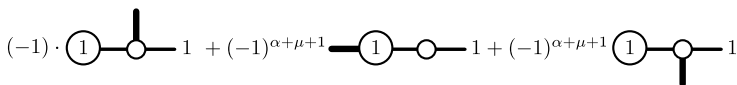
Note that its induced operation $C^\bullet(A, A) \to C^\bullet(A, A^*)$ is given for f in $C^\bullet(A, A)$ and $a_1, \ldots, a_{n+1}$ by

$$\sum \pm F(\ldots, a_{n+1}, \ldots, f(\ldots), \ldots)(1)$$
$$+ \sum \pm F(\ldots, f(\ldots, a_{n+1}, \ldots), \ldots)(1)$$
$$+ \sum \pm F(\ldots, f(\ldots), \ldots, a_{n+1}, \ldots)(1),$$

where f is at the special spot of F. We see that (up to sign) this sum is Δ' applied to $F_\sharp(f) \in C^\bullet(A, A^*)$. In order to obtain the correct signs for Δ, we let $\mu = \|f\|$ denote the degree of f, and $\alpha = \sum_{\ell=1}^{n+1}(|a_\ell| + 1)$ the total degree of the shifted arguments a_ℓ. Also, note that Δ is an operator of degree -1, since we apply the unit $1 \in (sA)_{+1}$.

3) Consider the symbol
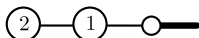
Its action on $f^1, f^2 \in C^\bullet(A, A)$ and $a_1, \ldots, a_{n+1} \in A$ is

$$\sum \pm F(\ldots, f^1(\ldots, f^2(\ldots), \ldots), \ldots)(a_{n+1}),$$

where $a_1, \ldots, a_n$ are placed at the dots in all possible ways. Note, that (up to sign) we obtain $F_\sharp(f^1 \circ f^2)$, where we used the operation "$\circ$" from definition 11. With this, we define $[.,.] : C^\bullet(A, A)^{\otimes 2} \to C^\bullet(A, A^*)$ to be the operation associated to
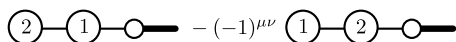
where $\mu = \|f^1\|$ and $\nu = \|f^2\|$ are the degrees of the Hochschild cochains plugged into the bracket.

4) Define $M : C^\bullet(A, A)^{\otimes 2} \to C^\bullet(A, A^*)$ to be the operation associated to
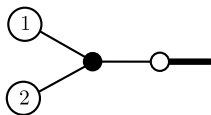
5) Let σ be the following symbol:
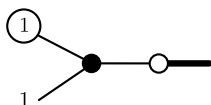
Then its action on $f \in C^\bullet(A, A)$ and $a_1, \ldots, a_{n+1} \in A$ is given by

$$\rho_\sigma(f)(a_1, \ldots, a_n)(a_{n+1})$$
$$= \sum \pm F(\ldots, D(\ldots, f(\ldots), \ldots, 1, \ldots), \ldots)(a_{n+1})$$
$$= \sum \pm F(\ldots, f(\ldots), \ldots)(a_{n+1}),$$

where used the fact that 1 is a strict unit and $D(\ldots, f(\ldots), \ldots, 1, \ldots)$ is therefore only non-zero, for $D(f(\ldots), 1)$, which is equal to $\pm f(\ldots)$. We thus see that the operation ρ_σ is (up to sign) the same as the operation $\rho_{\widetilde{\sigma}}$, where the symbol $\widetilde{\sigma}$ is defined by the following graph:
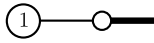
In fact, there are other symbols which also induce the same operations. Here is a third symbol with the same induced operation:
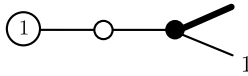
6) As a final example, consider the following symbol:
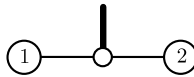
Its action on $f^1, f^2 \in C^\bullet(A, A)$ and $a_1, \ldots, a_{21} \in A$ is

$$\sum \pm F(\ldots, a_{21}, \ldots, f^1(\ldots), \ldots)(f^2(\ldots)).$$

If we assume that the Hochschild cochains f^k, for $k = 1, 2$, are in components $f^k = \sum_{i \geq 0} f_i^k$, where $f_i^k : sA^{\otimes i} \to sA$, and $F = \sum_{i,j \geq 0} F_{i,j}$, with $F_{i,j} : sA^{\otimes i} \otimes sA \otimes sA^{\otimes j} \to sA^*$, then some of the terms of the above sum with their proper signs are given by

$$(+1) \cdot F_{21,0}(a_1, \ldots, a_{21}, f_0^1)(f_0^2)$$
$$+ (-1)^{(|a_1|+1) \cdot (\|f^1\| + \|f^2\| + \sum_{l=2}^{21}(|a_l|+1))} \cdot F_{20,0}(a_2, \ldots, a_{21}, f_0^1)(f_1^2(a_1))$$
$$+ \cdots$$
$$+ (-1)^\epsilon F_{6,7}(a_{18}, \ldots, a_{21}, a_1, a_2, f_5^1(a_3, \ldots, a_7), a_8, \ldots, a_{14})(f_3^2(a_{15}, a_{16}, a_{17}))$$
$$+ \cdots,$$

where

$$\epsilon = \Big(\sum_{\ell=15}^{17} \big(|a_\ell| + 1 \big) \Big) \cdot \Big(\|f^1\| + \|f^2\| + \sum_{\ell=18}^{21} \big(|a_\ell| + 1 \big) \Big)$$

$$+ \Big(\sum_{\ell=3}^{14} \big(|a_\ell| + 1 \big) \Big) \cdot \Big(\|f^1\| + \sum_{\ell=18}^{21} \big(|a_\ell| + 1 \big) \Big)$$

$$+ \Big(\sum_{\ell=1}^{2} \big(|a_\ell| + 1 \big) \Big) \cdot \Big(\sum_{\ell=18}^{21} \big(|a_\ell| + 1 \big) \Big).$$

We now define a differential δ on symbols.

DEFINITION 16 (Differential δ). — Recall from definition 8, that there are differentials δ^A and δ^{A^*} on $C^\bullet(A, A)$ and $C^\bullet(A, A^*)$, respectively. From this, we obtain an induced differential δ on the space of operations

$$C^\bullet(A, A)^{\otimes p} \longrightarrow C^\bullet(A, A^*),$$

given for an operation ρ, by setting $\delta\rho(f^1, \ldots, f^p)$ to be

$$\sum_{i=1}^{p} (-1)^{\|\rho\| + \sum_{\ell=1}^{i-1} \|f^\ell\|} \rho(f^1, \ldots, \delta^A f^i, \ldots, f^p) + \delta^{A^*}\big(\rho(f^1, \ldots, f^p) \big).$$

Since every symbol σ induces an operation $\rho_\sigma : C^\bullet(A, A)^{\otimes p} \to C^\bullet(A, A^*)$, we can define the differential $\delta\sigma$ to be the sum of symbols that represent $\delta\rho_\sigma$. To describe this differential, we need to consider symbols σ' together with a chosen vertex v labeled by a filled circle, and a chosen edge e attached to v. We denote by $\sigma'/(v, e)$ the symbol obtained by contracting the edge e and replacing v by the other endpoint of e. Then, an analysis similar to [20, section 5] shows, that the differential $\delta\sigma$ consists of a sum $\sum (-1)^{\epsilon_{\sigma'}} \sigma'$ over all σ', such that $\sigma = \sigma'/(v, e)$ for some v and e. The sign $\epsilon_{\sigma'}$ is obtained by comparing the linear order of the operation associated to σ with the one from σ'. Examples for the differential can be found in the proofs of lemma 17 and lemma 18 below.

We end this subsection with an alternative description for Δ on cohomology.

LEMMA 17. — Δ is cohomologous to the operation obtained by the following symbol
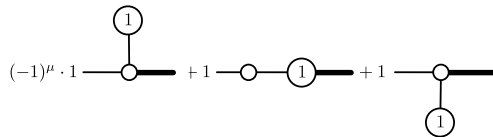
where μ is the degree of the Hochschild cochain plugged into the symbol.

Proof. — The proof consists in calculating the differential of a sum of symbols H, which will turn out to be the difference of the sum of symbols for Δ from example 15, 2) and the sum of symbols stated in the lemma. More precisely, let

$$H = (-1) \cdot 1 \;\text{[diagram]}\; 1 \; - \; 1 \;\text{[diagram]}\; 1 \; - \; (-1)^{\alpha \cdot \mu} \cdot 1 \;\text{[diagram]}$$

where $\alpha = \sum_{i=1}^{n+1}(|a_i| + 1)$ is the total degree of all elements $a_1, \ldots, a_{n+1}$, which are plugged into the Hochschild cochain. One calculates, that

$$\delta\left((-1) \cdot 1 \;\text{[diagram]}\; 1\right) = (-1)^{\alpha+\mu} \cdot \;\text{[diagram]}\; 1 + (-1)^{\alpha+\alpha \cdot \mu} \cdot \;\text{[diagram]}\; + 1 \;\text{[diagram]}\; 1$$

where, in the first two terms, we used the fact described in example 15(15), that the strict unit can only multiply with one term, where it yields the identity. Similarly, we get the other terms as

$$\delta\left(-(-1)^{\alpha \cdot \mu} \cdot 1 \;\text{[diagram]}\; 1\right) = (-1)^{\alpha \cdot \mu + \alpha + \mu} \cdot \;\text{[diagram]}\; 1 + (-1)^{\mu} \cdot 1 \;\text{[diagram]}\; + (-1)^{\alpha \cdot \mu} \cdot 1 \;\text{[diagram]}\; 1$$

$$\delta\left((-1) \cdot 1 \;\text{[diagram]}\; 1\right) = 1 \;\text{[diagram]}\; + (-1)^{\alpha+\mu} \cdot \;\text{[diagram]}\; 1 - 1 \;\text{[diagram]}\; 1 - (-1)^{\alpha \cdot \mu} \cdot \;\text{[diagram]}\; 1$$

By definition 16 of the differential, we have that $\delta(\rho_H) = \rho_{\delta(H)}$. Thus the operation $\rho_{\delta(H)}$ associated to the sum of symbols in $\delta(H)$ is a boundary in the operation complex of maps $C^\bullet(A, A) \to C^\bullet(A, A^*)$, and must therefore vanish on Hochschild-cohomology. We see that the following operations are cohomologous, i.e. they induce the same operation on Hochschild-cohomology

$$\left((-1)^{\mu} \cdot 1 \;\text{[diagram]}\; + 1 \;\text{[diagram]}\; + 1 \;\text{[diagram]}\;\right)$$

$$- \left((-1)^{\alpha \cdot \mu + \alpha + \mu + 1} \cdot \;\text{[diagram]}\; 1 + (-1)^{\alpha+\mu+1} \cdot \;\text{[diagram]}\; 1 + (-1)^{\alpha+\mu+1} \cdot \;\text{[diagram]}\; 1\right)$$

$$\cong (-1)^{\alpha+\mu} \cdot \;\text{[diagram]}\; 1 + 1 \;\text{[diagram]}\; + (-1)^{\alpha \cdot \mu + \mu + 1} \cdot 1 \;\text{[diagram]}\; + (-1)^{\alpha+\mu+1} \cdot \;\text{[diagram]}\; 1$$

The operations associated to the symbols on the right vanish since our ∞-inner product was assumed to be symmetric, and thus is invariant under a $180°$ rotation. The first parenthesis on the left is the sum of symbols stated in the lemma, the second parenthesis is the sum of symbols representing Δ from example 15, 2). This completes the proof of the lemma. $\square$

3.3. Proof of the BV-relations

This subsection is devoted to the proof of theorem 2. Here, it is crucial, that we can identify $H^\bullet(A, A)$ with $H^\bullet(A, A^*)$ via the map $F_\sharp$ and its quasi-inverse $G_\sharp$ induced by the ∞-inner product.

First, notice that $[.,.]$ and M induce a Gerstenhaber-structure on $H^\bullet(A, A)$, since all the homotopies needed in the proof in [8], and more generally in [9], still can be applied here. Also, the Δ-operator squares to zero on Hochschild-cohomology, which again can be seen by considering the normalized Hochschild-cochain complex. It is left to show that on Hochschild-cohomology, the deviation of Δ from being a derivation is given by $[.,.]$. The following symbols X, Y and Z will be important ingredients.

LEMMA 18. — *For $f, g \in C^\bullet(A, A)$ with degrees $\|f\|$ and $\|g\|$ respectively, the following Hochschild-cochains in $C^\bullet(A, A^*)$ are cohomologous*

1) $X(f, g) \cong Y(f, g) + Z(f, g)$,
2) $Y(f, g) \cong -M(\Delta f, g)$,
3) $Z(f, g) - (-1)^{\|f\| \cdot \|g\|} Z(g, f) \cong \Delta(M(f, g))$.

Before proving lemma 18, we show how it implies the required BV-relation from theorem 2.

Proof of the BV-relation. — From example 15, 3) we see that $[f, g]$ is given as $X(f, g) - (-1)^{\|f\| \cdot \|g\|} X(g, f)$. Thus, with lemma 18, we get

$$[f, g] = X(f, g) - (-1)^{\|f\| \cdot \|g\|} X(g, f)$$
$$\cong Y(f, g) - (-1)^{\|f\| \cdot \|g\|} Y(g, f) + Z(f, g) - (-1)^{\|f\| \cdot \|g\|} Z(g, f)$$
$$\cong -M(\Delta f, g) + (-1)^{\|f\| \cdot \|g\|} M(\Delta g, f) + \Delta\big(M(f, g)\big)$$
$$\cong \Delta\big(M(f, g)\big) - M(\Delta f, g) - (-1)^{\|f\|} M(f, \Delta g),$$

where we used the graded commutativity of M in the last step. $\square$

It remains to prove lemma 18.

Proof of lemma 18. — We use the following abbreviation for the relevant degrees:

$\triangleright$ $\alpha = \sum_\ell (|a_\ell| + 1)$ denotes the total shifted degree of the elements a_ℓ to be applied as arguments,

$\triangleright$ $\mu = \|f\|$ and $\nu = \|g\|$, where f and g will be placed into the first, respectively second spot of the corresponding operation of a symbol.

1) Let

We will show that the differential of H is exactly $\delta(H) = -X + Y + Z$. Before doing so, we want to remark on the motivation for H. As it was mentioned in the introduction, there is a close connection to the Chas-Sullivan BV-structure from [1]. In this analogy, H should be compared to the homotopy of [1, figure 7].

We now calculate the boundary $\delta(H)$ term by term:

A close investigation shows that all terms in $\delta(H)$ cancel, except for the terms in $-X + Y + Z$. This implies the claim.

2) In order to see that $Y(f,g) \cong -M(\Delta f, g)$, we consider the following differential:

$$\delta\left((-1)\cdot \text{①—○—▬} \atop \quad\quad ②\right) = \text{①} \atop ② \text{—●—○—▬} - (-1)^\mu \cdot \text{①—○—◀} \atop \qquad ②$$

Since Δ is a chain map, we can pre-compose this equation by $\Delta \otimes \mathrm{Id}$, which shows that the following induced operations are cohomologous:

$$\Delta\text{(①)} \atop ② \text{—●—○—▬} \cong (-1)^\mu \cdot \Delta\text{(①)—○—◀} \atop \qquad ②$$

The left term is exactly $F \circ (M(\Delta f, g))$, when applying f and g to it. The term on the right will be seen to be equal to $Y(f,g)$, after applying the description for Δ from lemma 17:

$$(-1)^\mu \cdot \Delta\text{(①)—○—◀} \atop \qquad\qquad ② \cong (-1)\cdot 1 \text{—○—◀} \atop \quad ① \atop\quad ②$$

$$- (-1)^\mu \cdot 1 \text{—○—①—◀} \atop \qquad\qquad ② \quad - (-1)^\mu \cdot 1 \text{—○—◀} \atop \quad ① \atop\quad ②$$

3) We now show that $\Delta(M(f,g)) \cong Z(f,g) - (-1)^{\|f\|\cdot\|g\|} \cdot Z(g,f)$. For this, we calculate the differential of

$$H' = -(-1)^\mu \cdot 1 \text{—✛—▬} \atop \quad ① \atop ② \quad - (-1)^{\mu\cdot\nu} \cdot 1 \text{—○—①—▬} \atop \qquad\qquad ②$$

$$- (-1)^{\mu\cdot\nu} \cdot 1 \text{—○—▬} \atop \quad ② \;① \quad - (-1)^\mu \cdot 1 \text{—✛—②—▬} \atop \quad ① \quad - (-1)^{\mu+\nu+\mu\cdot\nu} \cdot 1 \text{—○—✓} \atop \qquad\qquad ①②$$

We remark, that the first three terms were already used in part 1) as terms in H, and will now be subtracted again. We chose this way of proceeding, in order to have a closer analogy to the proof in [1]. So, for the differential of H', we will only perform the calculation for the last two terms, and refer

to the calculation in 1) for the first three terms.

A thorough investigation shows that all terms in $\delta(H')$ cancel, except for

which is equal to $Z(f,g) - (-1)^{\mu \cdot \nu} \cdot Z(g,f)$ when applied to elements f, g in $C^\bullet(A, A)$, together with the remaining terms

which are equal to

$$-(-1)^{\mu\cdot\nu}\cdot\left((-1)^{\mu+\nu+1}\cdot 1 \; \overset{M(g,f)}{\multimap\!\!-} \; + 1 \; \multimap M(g,f)\blacktriangleright + 1 \; \underset{M(g,f)}{\multimap\!\!-}\right)$$

In the last line we indicated that for $f, g \in C^\bullet(A, A)$, we apply $M(g, f) \in C^\bullet(A, A)$ to the operations associated to the sum of symbols. Using lemma 17, and the graded commutativity of M, we see that this is exactly $-(-1)^{\mu\cdot\nu}\cdot \Delta(M(g, f)) \cong -\Delta(M(f, g))$. $\square$

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