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ON THE COHOMOLOGY OF VECTOR FIELDS ON PARALLELIZABLE MANIFOLDS

by Yuly BILLIG & Karl-Hermann NEEB

Abstract. — In the present paper we determine for each parallelizable smooth compact manifold M the second cohomology spaces of the Lie algebra $\mathcal{V}_M$ of smooth vector fields on M with values in the module $\overline{\Omega}^p_M = \Omega^p_M/d\Omega^{p-1}_M$. The case of $p = 1$ is of particular interest since the gauge algebra of functions on M with values in a finite-dimensional simple Lie algebra has the universal central extension with center $\overline{\Omega}^1_M$, generalizing affine Kac-Moody algebras. The second cohomology $H^2(\mathcal{V}_M, \overline{\Omega}^1_M)$ classifies twists of the semidirect product of $\mathcal{V}_M$ with the universal central extension of a gauge Lie algebra.

Résumé. — Dans le présent article, nous déterminons, pour chaque variété parallélisable compacte lisse M, les espaces de seconde cohomologie de l'algèbre de Lie $\mathcal{V}_M$ des champs vectoriels lisses sur M à valeurs dans le module $\overline{\Omega}^p_M = \Omega^p_M/d\Omega^{p-1}_M$. Le cas $p = 1$ est d'un intérêt particulier puisque l'algèbre de jauge des fonctions sur M à valeurs dans une algèbre de Lie simple de dimension finie possède l'extension centrale universelle avec le centre $\overline{\Omega}^1_M$, généralisant les algèbres de Kac-Moody affines. L'espace $H^2(\mathcal{V}_M, \overline{\Omega}^1_M)$ classifie des torsions du produit semi-direct de $\mathcal{V}_M$ avec l'extension centrale universelle d'une algèbre de Lie de jauge.

One of the most important insights in the theory of affine Kac–Moody Lie algebras is that they can all be realized as twisted or untwisted loop algebras. In the untwisted case, this realization starts with the Lie algebra of maps from a circle $\mathbb{S}^1$ to a finite-dimensional complex simple Lie algebra $\mathfrak{k}$,

$$\mathcal{L}(\mathfrak{k}) := \mathbb{C}[t, t^{-1}] \otimes \mathfrak{k},$$

then proceeds with the universal central extension and completes the picture by adding a derivation d acting by $t\frac{d}{dt}$ on $\mathcal{L}(\mathfrak{k})$ and, accordingly, on the central extension. In the representation theory of affine algebras, an

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important role is played by the Virasoro algebra, which emerges via the Sugawara construction. At the Lie algebra level, the Witt algebra of vector fields on a circle, $\mathfrak{d} := \mathrm{Der}(\mathbb{C}[t, t^{-1}])$ acts on $\hat{\mathcal{L}}(\mathfrak{k})$, so that we may form the semidirect product $\mathfrak{g} := \hat{\mathcal{L}}(\mathfrak{k}) \rtimes \mathfrak{d}$. To get the Virasoro algebra, we need to twist the above semidirect product by a 2-cocycle $\tau \in Z^2(\mathfrak{d}, \mathbb{C})$, which leads to the affine-Virasoro Lie algebra $\mathfrak{g}_\tau$ with the bracket

$$[(x,d),(x',d')] = ([x,x'] + d.x' - d'.x + \tau(d,d'), [d,d']),$$

which contains both affine Kac-Moody algebra and the Virasoro algebra as subalgebras.

The theory of affine Kac–Moody algebras has an analytic side, where one replaces the algebra $\mathbb{C}[t, t^{-1}]$ of Laurent polynomials by the Fréchet algebra $C^\infty(\mathbb{S}^1, \mathbb{C})$ of complex valued functions on the circle. In this context one also obtains a one-dimensional central extension and the role of the Witt algebra is played by the Fréchet–Lie algebra $\mathcal{V}_{\mathbb{S}^1}$ of smooth vector fields on the circle, which also has a one-dimensional central extension. For an exposition of these ideas we refer to the monograph of Pressley and Segal [29].

Let us discuss the generalization of this construction from $\mathbb{S}^1$ to the case of an arbitrary C^∞-manifold M. The gauge Lie algebra of $\mathfrak{k}$-valued functions on M,

$$\mathcal{L}_M(\mathfrak{k}) = C^\infty(M, \mathfrak{k}),$$

endowed with the pointwise defined Lie bracket and the natural Fréchet topology, has the universal central extension with the central space

$$\overline{\Omega}^1(M, \mathbb{C}) := \Omega^1(M, \mathbb{C})/d\Omega^0(M, \mathbb{C})$$

(*cf.* [21], [24]). The Lie bracket on the Lie algebra

$$\hat{\mathcal{L}}_M(\mathfrak{k}) = C^\infty(M, \mathfrak{k}) \oplus \overline{\Omega}^1(M, \mathbb{C})$$

is given by the formula

$$[f_1 \otimes g_1, f_2 \otimes g_2] = f_1 f_2 \otimes [g_1, g_2] + (g_1|g_2)[f_2 df_1],$$

where $f_1, f_2 \in C^\infty(M, \mathbb{C})$, $g_1, g_2 \in \mathfrak{k}$, $(\cdot|\cdot)$ is the Cartan–Killing form on $\mathfrak{k}$ and $[\alpha]$ denotes the class of a 1-form α in $\overline{\Omega}^1(M, \mathbb{C})$.

The Lie algebra $\mathcal{V}_M$ of smooth vector fields on M acts on both the gauge algebra $C^\infty(M, \mathfrak{k})$ and on $\overline{\Omega}^1(M, \mathbb{C})$ by the Lie derivative. This action is compatible with the above central cocycle, allowing us to consider the semidirect product of $\mathcal{V}_M$ with $\hat{\mathcal{L}}_M(\mathfrak{k})$:

$$\mathfrak{g} = \big(C^\infty(M, \mathfrak{k}) \oplus \overline{\Omega}^1(M, \mathbb{C})\big) \rtimes \mathcal{V}_M.$$

However the interplay between affine Lie algebras and the Virasoro algebra suggests that it will be natural here to twist the Lie bracket in $\mathfrak{g}$ by means of a 2-cocycle $\tau \in Z^2_c(\mathcal{V}_M, \overline{\Omega}^1(M,\mathbb{C}))$, resulting in the family of Lie algebras

$$\mathfrak{g}_\tau = \big(C^\infty(M,\mathfrak{k}) \oplus \overline{\Omega}^1(M,\mathbb{C})\big) \oplus_\tau \mathcal{V}_M,$$

so that the equivalence classes of twistings are classified by the cohomology space

$$H^2_c(\mathcal{V}_M, \overline{\Omega}^1(M,\mathbb{C})) \cong H^2_c(\mathcal{V}_M, \overline{\Omega}^1(M,\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}.$$

In the present paper we calculate this space for parallelizable manifolds M, relying heavily on results on the cohomology of Lie algebras of vector fields with values in differential forms by Gelfand–Fuks, Haefliger and Tsujishita. We calculate this space using the short exact sequences

$$\mathbf{0} \to H^1_{\mathrm{dR}}(M,\mathbb{R}) \hookrightarrow \overline{\Omega}^1(M,\mathbb{R})) \to B^2_{\mathrm{dR}}(M,\mathbb{R}) \to \mathbf{0}$$

and

$$\mathbf{0} \to B^1_{\mathrm{dR}}(M,\mathbb{R}) \hookrightarrow \Omega^1(M,\mathbb{R}) \to \overline{\Omega}^1(M,\mathbb{R}) \to \mathbf{0},$$

and then applying the corresponding long exact sequence in cohomology. Hence we need detailed knowledge on the cohomology of $\mathcal{V}_M$ with values in differential forms.

In the following M denotes an N-dimensional parallelizable compact manifold. The main goal of this paper is to describe for each $p \in \mathbb{N}$ the second cohomology spaces

$$H^2_c(\mathcal{V}_M, \overline{\Omega}^p(M,\mathbb{R})), \text{ where } \overline{\Omega}^p(M,\mathbb{R}) := \Omega^p(M,\mathbb{R})/d\Omega^{p-1}(M,\mathbb{R}),$$

with respect to the natural action of the Lie algebra $\mathcal{V}_M$ of smooth vector fields. For reasons described above, our main interest lies in the case $p = 1$. For $N = 1$ and connected M we have $M \cong \mathbb{S}^1$ and $\overline{\Omega}^1(M,\mathbb{R}) \cong \mathbb{R}$ is a trivial module. Then $H^2_c(\mathcal{V}_{\mathbb{S}^1},\mathbb{R}) \cong \mathbb{R}$ describes the central extensions of $\mathcal{V}_{\mathbb{S}^1}$, and a generator corresponds to the Virasoro algebra. For $N \geqslant 2$, we find that

$$H^2_c(\mathcal{V}_M, \overline{\Omega}^1(M,\mathbb{R})) \cong H^3_{\mathrm{dR}}(M,\mathbb{R}) \oplus \mathbb{R}^2.$$

Here we use that each closed 3-form ω yields an $\overline{\Omega}^1(M,\mathbb{R})$-valued 2-cocycle by $\omega^{[2]}(X,Y) := [i_Y i_X \omega]$ (*cf.* [26]). In terms of a trivializing 1-form $\kappa \in \Omega^1(M,\mathbb{R}^N)$, the other two cocycles can be described as follows. Define $\theta : \mathcal{V}_M \to C^\infty(M,\mathfrak{gl}_N(\mathbb{R}))$ by $\mathcal{L}_X\kappa = -\theta(X)\cdot\kappa$,

$$\overline{\Psi}_1(X) := \mathrm{Tr}(\theta(X)) \in \Omega^0(M,\mathbb{R}) = C^\infty(M,\mathbb{R}),$$
$$\Psi_1(X) := \mathrm{Tr}(d\theta(X)) \in \Omega^1(M,\mathbb{R}),$$

and

$$\overline{\Psi}_2(X,Y) := [\mathrm{Tr}(d\theta(X)\wedge\theta(Y) - d\theta(Y)\wedge\theta(X))] \in \overline{\Omega}^1(M,\mathbb{R}).$$

Then $\overline{\Psi}_1 \wedge \Psi_1$ and $\overline{\Psi}_2$ provide the two additional generators of $H^2_c(\mathcal{V}_M, \overline{\Omega}^1(M,\mathbb{R}))$.

In the case when M is a torus $\mathbb{T}^N$, we give explicit formulas for these cocycles in coordinates. For the toroidal Lie algebras, the cocycles $\overline{\Psi}_2$ and $\overline{\Psi}_1 \wedge \Psi_1$ were discovered in the representation theory of these algebras ([12], [23], see also [6], [8]). It turns out here that it is easier to construct representations of $\mathfrak{g}_\tau$ for a non-trivial cocycle τ, rather than for the semidirect product $\mathfrak{g}$.

For the toroidal Lie algebras it would be natural to work in the algebraic setting, considering the algebra A of Laurent polynomials in N variables as the algebra of functions on $\mathbb{T}^N$ (Fourier polynomials), instead of $C^\infty(\mathbb{T}^N,\mathbb{C})$. Although the cocycles that we get here are well-defined in this algebraic setting, our results do not solve the problem of describing the algebraic cohomology $H^2(\mathrm{Der}(A), \Omega^1(A)/dA)$. The reason for this is that our proof is based on the results of Tsujishita and Haefliger, while the algebraic counterparts of these have not been established.

Toroidal Lie algebras are also closely related to the class of extended affine Lie algebras ([28], [5]). According to a result of [2], most of the extended affine Lie algebras are twisted versions of toroidal Lie algebras. However, instead of the Lie algebra of vector fields $\mathcal{V}_{\mathbb{T}^N}$, in the theory of extended affine Lie algebras one must use the Lie algebra of divergence zero vector fields $\mathcal{V}^{\mathrm{div}}_{\mathbb{T}^N}$ (or a subalgebra). Thus it would be also interesting to describe the space $H^2(\mathcal{V}^{\mathrm{div}}_{\mathbb{T}^N}, \overline{\Omega}^1(\mathbb{T}^N,\mathbb{R}))$. This question remains open. We point out here that the restriction of the cocycle $\overline{\Psi}_1 \wedge \Psi_1$ to $\mathcal{V}^{\mathrm{div}}_{\mathbb{T}^N}$ vanishes, while $\overline{\Psi}_2$ does not. Also we note that the cocycles corresponding to $H^3_{\mathrm{dR}}(M,\mathbb{R})$ are not relevant for the theory of extended affine Lie algebras either, because of the additional restriction that the cocycle should vanish on the subalgebra of degree zero derivations, so that this subalgebra remains abelian.

The cohomology of the subcomplex $C^p_{\mathrm{loc}}(\mathcal{V}_M, \Omega^p)$ of *local cochains*, *i.e.*, cochains $f \in C^p_c(\mathcal{V}_M, \Omega^p_M)$ that are differential operators in each argument is easier accessible than the full complex. In view of [32], the difference fully comes from the case $p = 0$. In this case, the description of $H^\bullet_{\mathrm{loc}}(\mathcal{V}_M, \mathcal{F}_M)$ is facilitated significantly by the results of de Wilde and Lecomte ([11]) who show that the cohomology of the differential graded algebra $C^\bullet_{\mathrm{loc}}(\mathcal{V}_M, \mathcal{F}_M)$ coincides with the cohomology of the subalgebra generated by the image of $\Omega^\bullet_M$, the image of the Chern–Weil homomorphism

$$\chi_1 : \mathrm{Sym}^\bullet(\mathfrak{gl}_N(\mathbb{R}), \mathbb{R})^{\mathfrak{gl}_N(\mathbb{R})} \to C^{2\bullet}_{\mathrm{loc}}(\mathcal{V}(M), \mathcal{F}_M)$$

corresponding to a connection on the frame bundle $J^1(M)$ of M (which yields cocycles depending on first order derivatives), and the image of an algebra homomorphism

$$\chi_2 : C^\bullet(\mathfrak{gl}_N(\mathbb{R}), \mathbb{R}) \to C^\bullet_{\mathrm{loc}}(\mathcal{V}(M), \mathcal{F}_M)$$

whose image consists of cocycles if the connection is flat. If all Prontrjagin classes of M vanish, then this results in an isomorphism

$$\begin{aligned} H^\bullet_{\mathrm{loc}}(\mathcal{V}_M, \mathcal{F}_M) &\cong H^\bullet_{\mathrm{dR}}(M, \mathbb{R}) \otimes H^\bullet(\mathfrak{gl}_N(\mathbb{R}), \mathbb{R}) \\ &\cong H^\bullet_{\mathrm{dR}}(M, \mathbb{R}) \otimes C^\bullet(\mathfrak{gl}_N(\mathbb{R}), \mathbb{R})^{\mathfrak{gl}_N(\mathbb{R})}. \end{aligned}$$

As our results show, for $N > 1$, all classes in $H^2_c(\mathcal{V}_M, \overline{\Omega}^p(M, \mathbb{R}))$ can be represented by local cochains, but the generator of $H^2_c(\mathcal{V}_{\mathbb{S}^1}, \mathbb{R}) \subseteq H^2_c(\mathcal{V}_{\mathbb{S}^1}, C^\infty(M, \mathbb{R}))$ is non-local because it involves an integration.

Some of Tsujishita's results have been generalized by Rosenfeld in [30] to other classes of irreducible primitive Lie subalgebras of $\mathcal{V}_M$. It would be of some interest to see if these results could be used to obtain the cohomology of these Lie algebras with values in $\overline{\Omega}^1_M$.

Notation and conventions

If V is a module of the Lie algebra $\mathfrak{g}$, we write $C^p(\mathfrak{g}, V)$ for the corresponding space p-cochains, $d_{\mathfrak{g}} : C^p(\mathfrak{g}, V) \to C^{p+1}(\mathfrak{g}, V)$ for the Chevalley–Eilenberg differential, $B^p(\mathfrak{g}, V) = d_{\mathfrak{g}}(C^{p-1}(\mathfrak{g}, V))$ for the space of p-coboundaries, $Z^p(\mathfrak{g}, V) = \ker(d_{\mathfrak{g}} : C^p(\mathfrak{g}, V) \to C^{p+1}(\mathfrak{g}, V))$ for the space of p-cocycles and $H^p(\mathfrak{g}, V)$ for the Lie algebra cohomology. If $\mathfrak{g}$ is a topological Lie algebra and V a continuous $\mathfrak{g}$-module, then $C^p_c(\mathfrak{g}, V)$ etc. stands for the space of continuous cochains.

For $x \in \mathfrak{g}$ we have the insertion operator $i_x : C^{p+1}(\mathfrak{g}, V) \to C^p(\mathfrak{g}, V)$ and there is a natural action of $\mathfrak{g}$ on $C^p(\mathfrak{g}, V)$, which is denoted by the operators $\mathcal{L}_x$, satisfying the Cartan relation

$$\mathcal{L}_x = d_{\mathfrak{g}} \circ i_x + i_x \circ d_{\mathfrak{g}}.$$

For a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ we write

$$C^p(\mathfrak{g}, \mathfrak{h}, V) := \{\alpha \in C^p(\mathfrak{g}, V) : (\forall x \in \mathfrak{h})\ i_x\alpha = 0, i_x(d_{\mathfrak{g}}\alpha) = 0\}$$

for the relative cochains modulo $\mathfrak{h}$, and accordingly $B^p(\mathfrak{g}, \mathfrak{h}, V)$, $Z^p(\mathfrak{g}, \mathfrak{h}, V)$ and $H^p(\mathfrak{g}, \mathfrak{h}, V)$.

If G is a group, we denote the identity element by $\mathbf{1}$, and for $g \in G$, we write

$\lambda_g : G \to G, x \mapsto gx$ for the *left multiplication* by g,
$\rho_g : G \to G, x \mapsto xg$ for the *right multiplication* by g,
$m_G : G \times G \to G, (x, y) \mapsto xy$ for the *multiplication map*, and
$\eta_G : G \to G, x \mapsto x^{-1}$ for the *inversion*.

If M is a smooth manifold, we write $\mathcal{V}_M$ for the Lie algebra of smooth vector fields on M, $\mathcal{F}_M := C^\infty(M, \mathbb{R})$ for the algebra of smooth real-valued functions on M, $\Omega^k_M := \Omega^k(M, \mathbb{R})$, the space of smooth real-valued p-forms on M, and

$$Z^k_M := \ker(d\,|_{\Omega^k_M}),\ B^k_M := d\Omega^{k-1}_M,\ H^k_M := H^k_{\mathrm{dR}}(M, \mathbb{R}),$$
$$\text{and}\ \ \overline{\Omega}^k_M := \Omega^k_M / B^k_M.$$

We write $T^{(p,q)}(M) := T(M)^{\otimes p} \otimes T^*(M)^{\otimes q}$ for the tensor bundles over M and $\Gamma(T^{(p,q)}(M))$ for their spaces of smooth sections, *i.e.*, the (p, q)-tensor fields.

1. Crossed homomorphisms of Lie algebras and pull-backs

In this short first section, we collect some basic facts on crossed homomorphisms of Lie algebras which provide the tools used throughout the forthcoming sections. The main point is Theorem 1.7 on maps in Lie algebra cohomology defined by crossed homomorphisms.

Let $\mathfrak{h}$ and $\mathfrak{n}$ be two Lie algebras, with $\mathfrak{h}$ acting on $\mathfrak{n}$ by derivations, *i.e.*, we are given a Lie algebra homomorphism $\tau : \mathfrak{h} \to \mathrm{Der}(\mathfrak{n})$.

Example 1.1. — (a) Our primary example will be $\mathfrak{h} = \mathcal{V}_M$ and $\mathfrak{n} = C^\infty(M, \mathfrak{g})$, where $\mathfrak{g}$ is a finite-dimensional Lie algebra and the bracket on $\mathfrak{n}$ is defined pointwise. Then

$$(\tau(v).f)(m) := (v.f)(m) := df(m)v(m)$$

defines a Lie algebra homomorphism from $\mathcal{V}_M$ to $\mathrm{Der}(C^\infty(M, \mathfrak{g}))$.

(b) If A is a commutative associative algebra and $\mathfrak{g}$ a finite-dimensional Lie algebra, then $\mathfrak{n} := A \otimes \mathfrak{g}$ carries a Lie algebra structure defined by

$$[a \otimes x, a' \otimes x'] := aa' \otimes [x, x'].$$

Then we obtain an action of $\mathfrak{h} := \mathrm{Der}(A)$ on $\mathfrak{g}$ by $\tau(D).(a\otimes x) := (D.a)\otimes x$.

Definition 1.2. — *A linear map $\theta : \mathfrak{h} \to \mathfrak{n}$ is called a crossed homomorphism if*

$$\theta([x,y]) = [\theta(x),\theta(y)] + \tau(x)\theta(y) - \tau(y)\theta(x) \qquad \text{for all } x,y \in \mathfrak{h}.$$

Remark 1.3. — (a) Note that θ is a crossed homomorphism if and only if the map $(\theta, \mathrm{id}_{\mathfrak{h}}) : \mathfrak{h} \to \mathfrak{n} \rtimes \mathfrak{h}$ is a homomorphism of Lie algebras.

In terms of the Lie algebra cohomology of $\mathfrak{h}$ with values in the $\mathfrak{h}$-module $\mathfrak{n}$, the condition that θ is a crossed homomorphism can be written as $(d_{\mathfrak{h}}\theta)(x,y) + [\theta(x),\theta(y)] = 0$, *i.e.*, θ satisfies the Maurer–Cartan equation

$$d_{\mathfrak{h}}\theta + \tfrac{1}{2}[\theta,\theta] = 0.$$

(b) If $\mathfrak{h} \cong \mathfrak{n} \rtimes_\eta \mathfrak{s}$ is a semidirect product and the action of $\mathfrak{h}$ on $\mathfrak{n}$ is defined by $\tau(n,s) := \eta(s)$, then

$$[(n,s),(n',s')] = ([n,n'] + \eta(s).n' - \eta(s').n, [s,s'])$$

implies that $\theta : \mathfrak{h} \to \mathfrak{n}, (n,s) \mapsto n$ is a crossed homomorphism.

(c) The kernel of any crossed homomorphism is a subalgebra.

Remark 1.4. — Let $\rho : \mathfrak{h} \to \mathfrak{n}$ be a homomorphism of Lie algebras. Then we obtain on $\mathfrak{n}$ an action of $\mathfrak{h}$ by $\tau(x).y := [\rho(x), y]$. Now a linear map $\theta : \mathfrak{h} \to \mathfrak{n}$ is a crossed homomorphism if and only if $\rho' := \rho + \theta$ is a homomorphism of Lie algebras.

This follows directly by comparing

$$\rho'([x,y]) = \rho([x,y]) + \theta([x,y]),$$

with

$$[\rho'(x),\rho'(y)] = [\rho(x),\rho(y)] + [\theta(x),\rho(y)] + [\rho(x),\theta(y)] + [\theta(x),\theta(y)].$$

It follows in particular that $\theta := -\rho$ is a crossed homomorphism.

Let us show how a crossed homomorphism may be used to pull back cocycles on $\mathfrak{n}$ to cocycles on $\mathfrak{h}$. First, we need to define the notion of an equivariant cochain.

Definition 1.5. — *Let V be a module for the Lie algebra $\mathfrak{h}$ and a trivial module for $\mathfrak{n}$. A cochain $\varphi \in C^k(\mathfrak{n}, V)$ is called equivariant if*

$$x\varphi(y_1,\ldots,y_k) = \sum_{j=1}^{k} \varphi(y_1,\ldots,\tau(x)y_j,\ldots,y_k) \text{ for all } \ x \in \mathfrak{h}, y_1,\ldots,y_k \in \mathfrak{n},$$

which is equivalent to $\mathcal{L}_x\varphi = 0$ for each $x \in \mathfrak{h}$.

Since $[\mathcal{L}_x, d_{\mathfrak{n}}] = 0$ holds for each $x \in \mathfrak{h}$ on $C^\bullet(\mathfrak{n}, V)$, we have:

LEMMA 1.6. — *Equivariant cochains form a subcomplex* $C^\bullet_{\rm eq}(\mathfrak{n}, V)$ *in* $C^\bullet(\mathfrak{n}, V)$.

By definition, the subcomplex of equivariant cochains yields the equivariant cohomology $H^\bullet_{\rm eq}(\mathfrak{n}, V)$. Identifying $\mathfrak{n}$ with a subalgebra of the semidirect product $\mathfrak{n} \rtimes \mathfrak{h}$, we may identify $C^p(\mathfrak{n}, V)$ with the subspace

$$\{\alpha \in C^p(\mathfrak{n} \rtimes \mathfrak{h}, V) : (\forall x \in \mathfrak{h})\ i_x\alpha = 0\}.$$

Hence the relative cochain space

$$C^p(\mathfrak{n} \rtimes \mathfrak{h}, \mathfrak{h}, V) := \{\alpha \in C^p(\mathfrak{n} \rtimes \mathfrak{h}, V) : (\forall x \in \mathfrak{h})\ i_x\alpha = 0, i_x(d\alpha) = 0\}$$

can be identified with $C^p_{\rm eq}(\mathfrak{n}, V)$ because if $i_x\alpha = 0$ holds for each $x \in \mathfrak{h}$, then $\mathcal{L}_x\alpha = i_x d\alpha$, so that invariance is equivalent to $\alpha \in C^p(\mathfrak{n} \rtimes \mathfrak{h}, \mathfrak{h}, V)$ (*cf.* [15], p. 16).

THEOREM 1.7. — *Let V be an $\mathfrak{h}$-module and consider it as a trivial $\mathfrak{n}$-module. Further let θ be a crossed homomorphism of Lie algebras $\theta : \mathfrak{h} \to \mathfrak{n}$. Then the map*

$$\theta^* : C^\bullet_{\rm eq}(\mathfrak{n}, V) \to C^\bullet(\mathfrak{h}, V), \quad \varphi \mapsto \varphi \circ (\theta \times \ldots \times \theta)$$

is a morphism of cochain complexes, hence induces a linear map of cohomology spaces $\theta^* : H^\bullet_{\rm eq}(\mathfrak{n}, V) \to H^\bullet(\mathfrak{h}, V)$.

Proof. — We have seen above that we may identify the complex $(C^\bullet_{\rm eq}(\mathfrak{n},V), d_{\mathfrak{n}})$ with the relative Lie algebra subcomplex

$$(C^\bullet(\mathfrak{n} \rtimes \mathfrak{h}, \mathfrak{h}, V), d_{\mathfrak{n}\rtimes\mathfrak{h}}) \subseteq (C^\bullet(\mathfrak{n} \rtimes \mathfrak{h}), d_{\mathfrak{n}\rtimes\mathfrak{h}}).$$

For $\varphi \in C^p_{\rm eq}(\mathfrak{n}, V)$, we write $\tilde\varphi$ for the corresponding element of $C^p(\mathfrak{n}\rtimes\mathfrak{h}, V)$. Then we have

$$\theta^*\varphi = (\theta, \mathrm{id}_{\mathfrak{h}})^*\tilde\varphi,$$

and the assertion follows from the fact that $(\theta, \mathrm{id}_{\mathfrak{h}}) : \mathfrak{h} \to \mathfrak{n} \rtimes \mathfrak{h}$ is a morphism of Lie algebras, hence induces a morphism of cochain complexes $C^\bullet(\mathfrak{n} \rtimes \mathfrak{h}, V) \to C^\bullet(\mathfrak{h}, V)$. □

Some applications to the Lie algebra of vector fields

In this subsection M denotes a finite-dimensional smooth manifold.

PROPOSITION 1.8. — *A $\mathfrak{g}$-valued differential form $\theta \in \Omega^1(M, \mathfrak{g})$ defines an $\mathcal{F}_M$-linear crossed homomorphism*

$$\mathcal{V}_M \to C^\infty(M, \mathfrak{g}), \quad X \mapsto i_X\theta$$

if and only if θ satisfies the Maurer–Cartan equation $d\theta + \frac{1}{2}[\theta, \theta] = 0$.

Proof. — Since the exterior differential on $\Omega^\bullet(M,\mathfrak{g})$ coincides with the $\mathcal{V}_M$-Lie algebra differential, this is an immediate consequence of Remark 1.3 (a). ☐

Recall that the left Maurer–Cartan form $\kappa_G \in \Omega^1(G,\mathfrak{g})$ of a Lie group G with Lie algebra $\mathfrak{g}$ is defined by $\kappa_G(v) := T(\lambda_g^{-1})v$ for $v \in T_g(G)$.

COROLLARY 1.9 (*cf.* [7], Th. 3.1). — *If M is a smooth manifold, $\mathfrak{g}$ a Lie algebra and $\kappa \in \Omega^1(M,\mathfrak{g})$ satisfies the Maurer–Cartan equation, then we have a map*

$$\zeta : C^\bullet(\mathfrak{g},\mathbb{R}) \to C^\bullet_c(\mathcal{V}_M,\mathcal{F}_M), \quad \zeta(\omega)(X_1,\ldots,X_p) := \omega(\kappa(X_1),\ldots,\kappa(X_p)),$$

whose range consists of $\mathcal{F}_M$-multilinear cocycles represented by differential forms. In particular, we obtain an algebra homomorphism

$$H^\bullet(\mathfrak{g},\mathbb{R}) \to H^\bullet_c(\mathcal{V}_M,\mathcal{F}_M).$$

Proof. — In view of Theorem 1.7, it only remains to verify that ζ is compatible with the multiplication of cochains, but this is an immediate consequence of the definition. ☐

COROLLARY 1.10. — *If G is a Lie group with Lie algebra $\mathfrak{g}$ and $\kappa_G \in \Omega^1(G,\mathfrak{g})$ the left Maurer–Cartan form, then*

$$\theta : \mathcal{V}_G \to C^\infty(G,\mathfrak{g}), \quad X \mapsto i_X\kappa_G$$

is a $\mathcal{F}_G$-linear crossed homomorphism which is bijective.

Proof. — In view of Proposition 1.8, we only have to recall that κ_G satisfies the Maurer–Cartan equation. ☐

2. Cocycles with values in differential forms

Let M be a smooth paracompact N-dimensional manifold. In this section we first explain how affine connections can be used to define for each $k \in \mathbb{N}$ cocycles $\Psi_k \in Z^k_c(\mathcal{V}_M,\Omega^k_M)$. If, in addition, the tangent bundle of M is trivial, we obtain two more families of cocycles $\overline{\Psi}_k \in Z^k_c(\mathcal{V}_M,\overline{\Omega}^{k-1}_M)$ and $\varphi_k \in Z^{2k-1}_c(\mathcal{V}_M,\mathcal{F}_M)$. These cocycles satisfy the relation $d \circ \overline{\Psi}_k = \Psi_k$ that will play a crucial role in our determination of the spaces $H^2_c(\mathcal{V}_M,\overline{\Omega}^p_M)$ in Section 4.

The set of affine connections ∇ on M is an affine space whose tangent space is the space $\Gamma(T^{(1,2)}(M))$ of tensors of type $(2,1)$. The elements of this space are $\mathcal{F}_M$-bilinear maps $\mathcal{V}_M \times \mathcal{V}_M \to \mathcal{V}_M$ and any such map can also be considered as an $\mathcal{F}_M$-linear map $\mathcal{V}_M \to \Gamma(\mathrm{End}(TM))$, *i.e.*, as a

1-form with values in the endomorphism bundle $\mathrm{End}(TM) \cong T^{(1,1)}(M)$ of the tangent bundle $T(M)$. In this sense we identify $\Gamma(T^{(1,2)}(M))$ with the space $\Omega^1(M, \mathrm{End}(TM))$ of 1-forms with values in the endomorphism bundle $\mathrm{End}(TM)$. Note that this space carries a natural module structure for the Lie algebra $\mathcal{V}_M$, given by the Lie derivative.

LEMMA 2.1 ([22]). — *Any affine connection ∇ on M defines a 1-cocycle*

$$\zeta : \mathcal{V}_M \to \Omega^1(M, \mathrm{End}(TM)), \quad X \mapsto \mathcal{L}_X \nabla,$$

where

$$(\mathcal{L}_X \nabla)(Y)(Z) := [X, \nabla_Y Z] - \nabla_{[X,Y]} Z - \nabla_Y [X, Z].$$

For any other affine connection ∇' the corresponding cocycle ζ' has the same cohomology class.

To understand the cocycle ζ associated to an affine connection, we first describe it in a local chart.

Remark 2.2. — If U is an open subset of the vector space V, then any affine connection on U is given by

$$\nabla_X Y = dY \cdot X + \Gamma(X, Y)$$

with a $(2,1)$-tensor Γ. We then have

$$\begin{aligned}
\mathcal{L}_X(\nabla - \Gamma)(Y, Z) &= [X, dZ(Y)] - dZ([X, Y]) - d[X, Z](Y) \\
&= d^2 Z(X, Y) + dZ(dY(X)) - dX(dZ(Y)) \\
&\quad - dZ(dY(X)) + dZ(dX(Y)) - d(dZ(X) - dX(Z))(Y) \\
&= d^2 Z(X, Y) - dX(dZ(Y)) + dZ(dX(Y)) \\
&\quad - d^2 Z(Y, X) - dZ(dX(Y)) + d^2 X(Y, Z) + dX(dZ(Y)) \\
&= d^2 X(Y, Z).
\end{aligned}$$

This means that

$$\mathcal{L}_X \nabla = d^2 X + \mathcal{L}_X \Gamma.$$

Since the Lie derivative of any symmetric tensor is symmetric, we see that $\mathcal{L}_X \nabla$ is symmetric if and only if $\mathcal{L}_X \Gamma$ is symmetric, which is the case if Γ is symmetric, and this in turn means that ∇ is torsion free.

As a consequence of the preceding discussion, we can associate to any smooth manifold a canonical cohomology class $[\zeta] \in H^1_c(\mathcal{V}_M, \Omega^1(M, \mathrm{End}(TM)))$ (*cf.* [22]). That this class is always non-zero can be seen in local coordinates by showing that there is no $(1,2)$-tensor Γ with $d^2 X = \mathcal{L}_X \Gamma$ for all smooth vector fields on a 0-neighborhood in $\mathbb{R}^N$. Indeed, for constant vector fields X, the preceding relation means that Γ is constant and for linear vector fields X we see that $\Gamma : \mathbb{R}^N \otimes \mathbb{R}^N \to \mathbb{R}^N$ should be $\mathrm{GL}_N(\mathbb{R})$-equivariant, so that $\lambda^2\Gamma(v,w) = \Gamma(\lambda v, \lambda w) = \lambda\Gamma(v,w)$ for each $\lambda \in \mathbb{R}$ yields $\Gamma = 0$.

Example 2.3. — Let $V := \mathbb{R}^N$ and assume that M is parallelizable of dimension N. Then there exists some $\kappa \in \Omega^1(M,V)$ such that each κ_m is invertible, *i.e.*, κ defines a trivialization of the tangent bundle of M via $M \times V \to TM, (m,v) \mapsto \kappa_m^{-1}v$. Then, for each $X \in \mathcal{V}_M$, $\mathcal{L}_X\kappa$ (the Lie derivative on 1-forms) also is an element of $\Omega^1(M,V)$, and since all the linear maps κ_m are invertible, it can be written as $\mathcal{L}_X\kappa = -\theta(X)\cdot\kappa$ for some smooth function $\theta(X) \in C^\infty(M,\mathfrak{gl}(V))$. Clearly, $C^\infty(M,\mathfrak{gl}(V)) \cong C^\infty(M,\mathbb{R}) \otimes \mathfrak{gl}(V)$ is a Lie algebra with respect to the pointwise bracket and $\mathcal{V}_M$ acts naturally by derivations.

We claim that θ is a crossed homomorphism:

$$\begin{aligned}
\theta([X,Y])\cdot\kappa &= -\mathcal{L}_{[X,Y]}\kappa = -\mathcal{L}_X(\mathcal{L}_Y\kappa) + \mathcal{L}_Y(\mathcal{L}_X\kappa)\\
&= \mathcal{L}_X(\theta(Y)\cdot\kappa) - \mathcal{L}_Y(\theta(X)\cdot\kappa)\\
&= (\mathcal{L}_X\theta(Y) - \mathcal{L}_Y\theta(X))\kappa + \theta(Y)\mathcal{L}_X\kappa - \theta(X)\mathcal{L}_Y\kappa\\
&= (\mathcal{L}_X\theta(Y) - \mathcal{L}_Y\theta(X))\kappa + (\theta(Y)\theta(X) - \theta(X)\theta(Y))\kappa\\
&= (\mathcal{L}_X\theta(Y) - \mathcal{L}_Y\theta(X) + [\theta(X),\theta(Y)])\kappa.
\end{aligned}$$

Now let

$$\nabla_X Y := \kappa^{-1}(X.\kappa(Y))$$

denote the corresponding affine connection. Then

$$\kappa(\nabla_X Y) = X.\kappa(Y) = -\theta(X)\kappa(Y) + \kappa([X,Y]) \tag{2.1}$$

is an immediate consequence of the definition of θ. Moreover, the map

$$\tilde{\kappa} : \Gamma(\mathrm{End}(TM)) \to C^\infty(M,\mathfrak{gl}(V)), \quad \tilde{\kappa}(\varphi)(m) = \kappa_m \circ \varphi_m \circ \kappa_m^{-1}$$

satisfies

$$\tilde{\kappa} \circ \mathcal{L}_X\nabla = -d\theta(X) \in \Omega^1(M,\mathfrak{gl}(V)). \tag{2.2}$$

Indeed, we have

$$\begin{aligned}\kappa((\mathcal{L}_X\nabla)(Y)(Z)) &= \kappa([X,\nabla_Y Z] - \nabla_{[X,Y]}Z - \nabla_Y[X,Z]) \\ &= \kappa(\nabla_X\nabla_Y Z) + \theta(X)\kappa(\nabla_Y Z) \\ &\qquad - [X,Y].\kappa(Z) - Y.\kappa([X,Z]) \\ &= XY.\kappa(Z) + \theta(X)(Y.\kappa(Z)) - [X,Y].\kappa(Z) \\ &\qquad - Y.(X.\kappa(Z) + \theta(X)\cdot\kappa(Z)) \\ &= \theta(X)(Y.\kappa(Z)) - Y.(\theta(X).\kappa(Z)) \\ &= -Y.(\theta(X))\cdot\kappa(Z) = -d\theta(X)(Y)\cdot\kappa(Z),\end{aligned}$$

and this calculation shows that $\tilde{\kappa}((\mathcal{L}_X\nabla)(Y)) = -d\theta(X)(Y)$ for each vector field Y, which is (2.2).

Remark 2.4. — For any affine connection ∇, the operator $\eta(X) := \nabla_X - \operatorname{ad} X$ on $\mathcal{V}_M$ is $\mathcal{F}_M$-linear, hence defines a section of $\operatorname{End}(TM)$. We thus obtain a map

$$\eta : \mathcal{V}_M \to \Gamma(\operatorname{End}(TM)), \quad X \mapsto \nabla_X - \operatorname{ad} X.$$

The space $\Gamma(\operatorname{End}(TM))$ carries a natural associative algebra structure, hence in particular the structure of a Lie algebra (the gauge Lie algebra of the vector bundle TM) and $\mathcal{V}_M$ acts via the Lie derivative by derivations. The map η is a crossed homomorphism for this structure if and only if $X \mapsto \nabla_X$ defines a representation of $\mathcal{V}_M$ on itself, *i.e.*, if ∇ is a flat connection (Remark 1.4). This is in particular the case if ∇ is obtained from a trivialization $\kappa \in \Omega^1(M,V)$ of the tangent bundle. In the latter case, (2.1) implies that

$$\tilde{\kappa}(\eta(X)) = -\theta(X).$$

In view of $\tilde{\kappa}(\nabla_X) = \mathcal{L}_X$, we now see that

$$\tilde{\kappa}(\operatorname{ad} X) = \tilde{\kappa}(\mathcal{L}_X) = \mathcal{L}_X + \theta(X),$$

which is a representation of $\mathcal{V}_M$ on $C^\infty(M,V)$ if θ is a crossed homomorphism (Remark 1.4).

We likewise see for the action of $\mathcal{V}_M$ on $\Gamma(\operatorname{End}(TM))$ and $C^\infty(M,\mathfrak{gl}(V))$ that $\tilde{\kappa}(\varphi) := \kappa\circ\varphi\circ\kappa^{-1}$ leads to

$$\tilde{\kappa}\circ\mathcal{L}_X = (\mathcal{L}_X + \operatorname{ad}(\theta(X)))\circ\tilde{\kappa},$$

i.e., the representation of $\mathcal{V}_M$ on $\Gamma(\operatorname{End}(TM))$ by the Lie derivative is transformed by $\tilde{\kappa}$ into the representation given by $\mathcal{L}_X + \operatorname{ad}(\theta(X))$ on $C^\infty(M,\mathfrak{gl}(V))$. Similarly, the map

$$\tilde{\kappa}_1 : \Omega^1(M,\operatorname{End}(TM)) \to \Omega^1(M,\mathfrak{gl}(V)), \quad \omega \mapsto \tilde{\kappa}\circ\omega$$

intertwines the Lie derivative on the left hand side with the representation on the right hand side given by $\mathcal{L}_X + \mathrm{ad}(\theta(X))$.

DEFINITION 2.5. — *Next we use the cocycle ζ, associated to the affine connection ∇, to define k-cocycles $\Psi_k \in Z^k_c(\mathcal{V}_M, \Omega^k_M)$ depending on second order partial derivatives of vector fields. Let $V := \mathbb{R}^N$. For each $p \in \mathbb{N}$, we have a polynomial of degree p on $\mathfrak{gl}(V)$, invariant under conjugation, given by $A \mapsto \mathrm{Tr}(A^p)$. The corresponding invariant symmetric p-linear map is given by*

$$\beta(A_1, \ldots, A_p) = \sum_{\sigma \in S_p} \mathrm{Tr}(A_{\sigma(1)} \cdots A_{\sigma(p)}),$$

and we consider it as a linear $\mathrm{GL}(V)$-equivariant map $\mathrm{End}(V)^{\otimes p} \to \mathbb{R}$, where $\mathrm{GL}(V)$ acts trivially on $\mathbb{R}$. This $\mathrm{GL}(V)$-equivariant map leads to a vector bundle map

$$\beta_M : \mathrm{End}(TM)^{\otimes p} \to M \times \mathbb{R},$$

where $M \times \mathbb{R}$ stands for the trivial vector bundle with fiber $\mathbb{R}$. On the level of bundle-valued differential forms, this in turn yields an alternating p-linear map

$$\beta^1_M : \Omega^1(M, \mathrm{End}(TM))^p \to \Omega^p(M, \mathbb{R}) = \Omega^p_M.$$

To see that this map is $\mathcal{V}_M$-equivariant, we note that in local coordinates we have on an open subset $U \subseteq V \cong \mathbb{R}^N$ the corresponding linear map

$$\Omega^1(U, \mathrm{End}(TU))^{\otimes p} \cong \Omega^1(U, \mathbb{R})^{\otimes p} \otimes \mathrm{End}(V)^{\otimes p} \to \Omega^p(U, \mathbb{R}),$$

acting on $\Omega^1(U, \mathbb{R})^{\otimes p}$ as the p-fold exterior product and on $\mathrm{End}(V)^{\otimes p}$ as β. If $\varphi : U_1 \to U_2$ is a local diffeomorphism, and $\alpha \in \Omega^1(U_2, \mathrm{End}(T(U_2)))$, then $\varphi^*\alpha \in \Omega^1(U_1, \mathrm{End}(T(U_1)))$ is given by

$$(\varphi^*\alpha)_m(v) := T_m(\varphi)^{-1} \circ \alpha_{\varphi(m)}(T_m(\varphi)v) \circ T_m(\varphi).$$

From this it follows that β^1_M is equivariant under diffeomorphisms, and hence with respect to the infinitesimal action of vector fields.

We conclude that we can use β^1_M to multiply Lie algebra cocycles (cf. [15], *App. F in* [25]*). In particular, this leads with the cocycle ζ from Lemma 2.1 for each $k \in \mathbb{N}$ to a Lie algebra cocycle*

$$\Psi_k \in Z^k_c(\mathcal{V}_M, \Omega^k_M),$$

defined by

$$\Psi_k(X_1, \ldots, X_k) := (-1)^k \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \beta^1_M(\zeta(X_{\sigma(1)}), \ldots, \zeta(X_{\sigma(k)}))$$

(cf. [15]*).*

For any other affine connection ∇', the corresponding cocycle ζ', and the associated cocycles Ψ'_k, we first note that the difference $\zeta' - \zeta$ is a coboundary, and since products of cocycles and coboundaries are coboundaries, $\Psi'_k - \Psi_k$ is a coboundary. Hence its cohomology class in $H^k_c(\mathcal{V}_M, \Omega^k_M)$ does not depend on the choice of the affine connection.

Remark 2.6. — If the connection ∇ is defined by a trivialization of TM as in Example 2.3, then $\zeta(X)$ is transformed into $-d\theta(X) \in \Omega^1(M, \mathfrak{gl}(V))$, which leads to

$$\Psi_k(X_1, \ldots, X_k) = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \operatorname{Tr}\left(d\theta(X_{\sigma(1)}) \wedge \ldots \wedge d\theta(X_{\sigma(k)})\right).$$

Next we construct several equivariant cocycles on a gauge Lie algebra.

Theorem 2.7. — *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, $\mathcal{F}_M \otimes \mathfrak{g}$ be the corresponding gauge Lie algebra, consider Ω^p_M as a trivial module of $\mathcal{F}_M \otimes \mathfrak{g}$, and let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional representation.*

(a) *By $\mathcal{F}_M$-linear extension, we get a morphism of cochain complexes, $C^\bullet(\mathfrak{g}, \mathbb{R}) \to C^\bullet_{\mathrm{eq}}(\mathcal{F}_M \otimes \mathfrak{g}, \mathcal{F}_M)$, where equivariance refers to the action of $\mathcal{V}_M$, which gives a homomorphism of cohomology algebras*

$$H^\bullet(\mathfrak{g}, \mathbb{R}) \to H^\bullet_{\mathrm{eq}}(\mathcal{F}_M \otimes \mathfrak{g}, \mathcal{F}_M).$$

(b) *The following expression is a $\mathcal{V}_M$-equivariant cocycle with values in k-forms, $\psi_k \in Z^k_{\mathrm{eq}}(\mathcal{F}_M \otimes \mathfrak{g}, \Omega^k_M)$:*

$$\psi_k(f_1 \otimes x_1, \ldots, f_k \otimes x_k) = \Big(\sum_{\sigma \in S_k} \operatorname{Tr}\left(\rho(x_{\sigma(1)}) \cdots \rho(x_{\sigma(k)})\right) \Big) df_1 \wedge \cdots \wedge df_k,$$

where $f_1, \ldots, f_k \in \mathcal{F}_M$, $x_1, \ldots, x_k \in \mathfrak{g}$.

(c) *The following expression is an equivariant cocycle with values in $\overline{\Omega}^{k-1}_M$, $k \geqslant 1$:*

$$\overline{\psi}_k(f_1 \otimes x_1, \ldots, f_k \otimes x_k) = \Big(\sum_{\sigma \in S_k} \operatorname{Tr}\left(\rho(x_{\sigma(1)}) \cdots \rho(x_{\sigma(k)})\right) \Big) [f_1 df_2 \wedge \ldots \wedge df_k],$$

where $f_1, \ldots, f_k \in \mathcal{F}_M$, $x_1, \ldots, x_k \in \mathfrak{g}$.

Proof. — Verification of part (a) is straightforward. Assertions (b) and (c) easily follow from Proposition A.3, which is first used with $\Omega^p = \Omega^p_M$ to obtain the cocycle property on $\mathcal{F}_M \otimes \mathfrak{gl}(V)$, and then we pull it back by the homomorphism $\mathrm{id}_{\mathcal{F}_M} \otimes \rho : \mathcal{F}_M \otimes \mathfrak{g} \to \mathcal{F}_M \otimes \mathfrak{gl}(V)$. □

Definition 2.8. — *Combining Theorems 1.7 and 2.7 with Remark 2.4, we get for each trivialization of the tangent bundle $T(M)$ of an N-dimensional manifold M by pulling back with the corresponding crossed homomorphism $\theta : \mathcal{V}_M \to \mathcal{F}_M \otimes \mathfrak{gl}(V)$, $V = \mathbb{R}^N$, the following cocycles*

$$\varphi_k \in Z_c^{2k-1}(\mathcal{V}_M, \mathcal{F}_M),$$

$$\varphi_k(X_1, \dots, X_{2k-1}) = \sum_{\sigma \in S_{2k-1}} \operatorname{sgn}(\sigma) \operatorname{Tr}\big(\theta(X_{\sigma(1)}) \cdots \theta(X_{\sigma(2k-1)})\big)$$

and $\overline{\Psi}_k \in Z_c^k(\mathcal{V}_M, \overline{\Omega}_M^{k-1})$, *defined by*

$$\overline{\Psi}_k(X_1, \dots, X_k) = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) [\operatorname{Tr}\big(\theta(X_{\sigma(1)}) d\theta(X_{\sigma(2)}) \wedge \dots \wedge d\theta(X_{\sigma(k)})\big)].$$

Note that we have for each $k \geqslant 1$ the relation

$$d \circ \overline{\Psi}_k = \Psi_k. \tag{2.3}$$

3. Cohomology of smooth vector fields with values in differential forms

In this section, we first recall Tsujishita's Theorem ([32], Thm. 5.1.6, see also 3.3.4), describing the continuous cohomology of $\mathcal{V}_M$ with the values in differential forms.

To explain its statement, we recall that each compact manifold M satisfies Tsujishita's condition (F): $H_M^\bullet$ is finite-dimensional and, for each $p \in \mathbb{N}$, the subspace Z_M^p of closed p-forms in Ω_M^p has a closed complement. For compact oriented manifolds, the latter assertion is a direct consequence of the Hodge Decomposition Theorem (*cf.* [1], Thm. 7.5.3) and the general case can be reduced to this one via the 2-sheeted orientation covering. Moreover, Beggs shows in [4] (Theorems 4.4 and 6.6) that for all finite-dimensional paracompact smooth manifolds the space Z_M^p of closed p-forms in Ω_M^p has a closed complement, so that condition (F) is equivalent to all cohomology spaces H_M^p, resp., the cohomology algebra $H_M^\bullet$, being finite-dimensional.

The following theorem is a key result of this paper because it provides a bridge between Tsujishita's results and the cocycles Ψ_k. It will be proved in Appendix C.

Theorem 3.1. — *Let M be an N-dimensional manifold for which $H_M^\bullet$ is finite-dimensional. Then $H_c^q(\mathcal{V}_M, \Omega_M^p)$ vanishes if $p > q$ and for $p, n \in \mathbb{N}_0$, we have*

$$H_c^{p+n}(\mathcal{V}_M, \Omega_M^p) \cong H_c^n(\mathcal{V}_M, \mathcal{F}_M) \otimes E_p,$$

where

$$E_p := \operatorname{span}\Big\{[\Psi_1]^{m_1}[\Psi_2]^{m_2}\cdots[\Psi_p]^{m_p} : \sum_{j=1}^{p} jm_j = p\Big\} \subseteq H^p_c(\mathcal{V}_M, \Omega^p_M).$$

In particular, $H^\bullet_c(\mathcal{V}_M, \Omega^\bullet_M)$ is a free $H^\bullet_c(\mathcal{V}_M, \mathcal{F}_M)$-module, generated by the non-zero products of the classes $[\Psi_k]$, $k = 1, \ldots, N$.

From the preceding theorem, we immediately derive with $H^0_c(\mathcal{V}_M, \mathcal{F}_M) = H^0_M \cong \mathbb{R}$:

Corollary 3.2.

(a) *If $p < k$, then $H^p_c(\mathcal{V}_M, \Omega^k_M) = 0$.*
(b) *If M is connected, then $H^p_c(\mathcal{V}_M, \Omega^p_M) \cong E_p$.*

Lemma 3.3. — *Let M be a compact connected orientable manifold and μ a volume form on M. Then*

$$\operatorname{div} : \mathcal{V}_M \to \mathcal{F}_M, \quad \mathcal{L}_X\mu = \operatorname{div}(X)\mu$$

defines a continuous Lie algebra cocycle and furthermore, each closed 1-form $\alpha \in Z^1_M$ defines a Lie algebra cocycle $\mathcal{V}_M \to \mathcal{F}_M, X \mapsto \alpha(X)$. These two types of cocycles span the cohomology space

$$H^1_c(\mathcal{V}_M, \mathcal{F}_M) \cong H^1_M \oplus \mathbb{R}[\operatorname{div}].$$

If, in addition, the volume form is defined by a trivialization of the tangent bundle $T(M)$, then $-\operatorname{div} = \varphi_1 = \overline{\Psi}_1$.

Proof. — In view of [15], Th. II.4.11, we only have to verify the last part (*cf.* also [14] for the local cohomology of $\mathcal{V}_M$ with values in Ω^p_M). So let us assume that the volume form can be written as

$$\mu = \kappa_1 \wedge \ldots \wedge \kappa_N,$$

where the κ_i are the components of a trivializing 1-form $\kappa \in \Omega^1(M, \mathbb{R}^N)$. From $\mathcal{L}_X\kappa = -\theta(X)\kappa$ we derive $\mathcal{L}_X\kappa_j = -\sum_{i=1}^N \theta(X)_{ji}\kappa_i$ and further

$$\begin{aligned}\mathcal{L}_X\mu &= \sum_{i=1}^{N} \kappa_1 \wedge \ldots \wedge \mathcal{L}_X\kappa_i \wedge \ldots \wedge \kappa_N = -\Big(\sum_{i=1}^{N} \theta(X)_{ii}\Big)\kappa_1 \wedge \ldots \wedge \kappa_N \\ &= -\operatorname{Tr}(\theta(X)) \cdot \mu.\end{aligned}$$

This proves that $\operatorname{div} X = -\varphi_1(X) = -\overline{\Psi}_1(X)$. □

Remark 3.4. — We take a closer look at the second cohomology of the $\mathcal{V}_M$-modules Ω^p_M.

From Corollary 3.2 (a) we know that $H^2_c(\mathcal{V}_M, \Omega^p_M)$ vanishes for $p > 2$, and for $p = 2$ the space

$$H^2_c(\mathcal{V}_M, \Omega^2_M) = \mathrm{span}\{[\Psi_1^2], [\Psi_2]\}$$

is 2-dimensional. For $p = 1$ we also know that

$$H^2_c(\mathcal{V}_M, \Omega^1_M) = H^1_c(\mathcal{V}_M, \mathcal{F}_M).[\Psi_1] = H^1_M \cdot [\Psi_1] \oplus \mathbb{R}[\overline{\Psi}_1 \wedge \Psi_1]$$

is of dimension $b_1(M) + 1$ (Lemma 3.3). The most intricate case is $p = 0$, *i.e.*, $H^2_c(\mathcal{V}_M, \mathcal{F}_M)$ (*cf.* Theorem 4.8 below).

Example 3.5. — Let $M = G$ be an abelian Lie group of dimension N and $\kappa_1, \ldots, \kappa_N$ a basis of the space of invariant $\mathbb{R}$-valued 1-forms on G. Then $\kappa := (\kappa_1, \ldots, \kappa_N)$ is a closed trivializing 1-form (the Maurer–Cartan form of G). Let $X_1, \ldots, X_N \in \mathcal{V}(G)$ be the dual basis of left invariant vector fields. Writing $X \in \mathcal{V}(G)$ as $X = \sum_{i=1}^N f_i X_i$, we obtain

$$\mathcal{L}_X \kappa_i = d(i_X \kappa_i) = df_i.$$

Therefore $\theta(X) \in C^\infty(G, \mathfrak{gl}_N(\mathbb{R}))$ is given by $\theta(X)_{ij} = -df_i(X_j)$, *i.e.*, $-\theta(X)$ is the Jacobian matrix of X with respect to the basis $X_1, \ldots, X_N$.

4. Cohomology with values in differential forms modulo exact forms

Let M be a parallelizable connected compact manifold of dimension N. Since the subspace H^k_M of $\overline{\Omega}^k_M$ is a trivial $\mathcal{V}_M$-module, we derive from Corollary D.5 that

$$H^k_c(\mathcal{V}_M, H^m_M) = \mathbf{0} \quad \text{for} \quad 0 < k \leqslant N, m \in \mathbb{N}_0. \tag{4.1}$$

Therefore the short exact sequence of $\mathcal{V}_M$-modules

$$\mathbf{0} \to H^m_M \to \overline{\Omega}^m_M \xrightarrow{d} B^{m+1}_M \to \mathbf{0}$$

induces a long exact sequence in cohomology, hence leads to isomorphisms

(4.2a)
$$d_* : H^k_c(\mathcal{V}_M, \overline{\Omega}^m_M) \to H^k_c(\mathcal{V}_M, B^{m+1}_M) \quad \text{for} \quad m \in \mathbb{N}_0, 0 < k \leqslant N-1$$

and an exact sequence

$$\begin{aligned} \mathbf{0} \to H^N_c(\mathcal{V}_M, \overline{\Omega}^m_M) \xrightarrow{d_*} H^N_c(\mathcal{V}_M, B^{m+1}_M) \to H^{N+1}_c(\mathcal{V}_M, H^m_M) \\ \cong H^{N+1}_c(\mathcal{V}_M, \mathbb{R}) \otimes H^m_M. \end{aligned} \tag{4.2b}$$

In particular, we get

$$\dim H_c^N(\mathcal{V}_M, \overline{\Omega}_M^m) \leqslant \dim H_c^N(\mathcal{V}_M, B_M^{m+1}).$$

LEMMA 4.1 ([26] Lemma 23, [20]). — *For any smoothly paracompact manifold M we have*

$$H_c^0(\mathcal{V}_M, \overline{\Omega}\,{}_M^p) = (\overline{\Omega}\,{}_M^p)^{\mathcal{V}_M} = H_M^p.$$

LEMMA 4.2. — *If M is parallelizable, then for each $k \in \mathbb{N}$ the exterior differential d induces a surjective map*

$$d_* : H_c^k(\mathcal{V}_M, \overline{\Omega}\,{}_M^{k-1}) \to H_c^k(\mathcal{V}_M, \Omega_M^k), \quad [\alpha] \mapsto [d \circ \alpha],$$

and the natural map

$$H_c^k(\mathcal{V}_M, B_M^k) \to H_c^k(\mathcal{V}_M, \Omega_M^k)$$

is surjective.

Proof. — First we recall from Definition 2.8 that $d \circ \overline{\Psi}_k = \Psi_k$.

Next we observe that the exterior product of differential forms induces maps

$$Z_M^p \times B_M^q \to B_M^{p+q}, \quad (\alpha, \beta) \mapsto \alpha \wedge \beta,$$

and hence maps

$$Z_M^p \times \overline{\Omega}\,{}_M^q \to \overline{\Omega}\,{}_M^{p+q}, \quad (\alpha, [\beta]) \mapsto [\alpha \wedge \beta].$$

Since Ψ_j has values in $B_M^1 \wedge \ldots \wedge B_M^1 \subseteq B_M^j$, each product $\Psi_{j_1} \wedge \ldots \wedge \Psi_{j_r}$ has values in $B_M^{j_1+\ldots+j_r}$, so that

$$\overline{\Psi}_{j_1} \wedge \Psi_{j_2} \wedge \ldots \wedge \Psi_{j_r}$$

is a well-defined cocycle with values in $\overline{\Omega}_M^{j_1+\ldots+j_r}$, satisfying

$$d \circ \left(\overline{\Psi}_{j_1} \wedge \Psi_{j_2} \wedge \ldots \wedge \Psi_{j_r}\right) = (d \circ \overline{\Psi}_{j_1}) \wedge \Psi_{j_2} \wedge \ldots \wedge \Psi_{j_r} = \Psi_{j_1} \wedge \Psi_{j_2} \wedge \ldots \wedge \Psi_{j_r}.$$

For $j_1 + \ldots + j_r = k$, this implies that the image of d contains all products $[\Psi_{j_1} \wedge \Psi_{j_2} \wedge \ldots \wedge \Psi_{j_r}]$ with $\sum j_i = k$, and, in view of Theorem 3.1, these products span $H_c^k(\mathcal{V}_M, \Omega_M^k)$.

This proves the first part, and the second part of the assertion is an immediate consequence of the first one. □

In view of

$$H_c^q(\mathcal{V}_M, \Omega_M^p) = \mathbf{0} \quad \text{for} \quad q < p. \tag{4.3}$$

(Corollary 3.2 (a)), the long exact cohomology sequence associated to the short exact sequence

$$\mathbf{0} \to B_M^p \to \Omega_M^p \to \overline{\Omega}\,{}_M^p \to \mathbf{0}$$

(which splits topologically by condition (F)), leads to isomorphisms

$$H^q_c(\mathcal{V}_M, \overline{\Omega}^p_M) \cong H^{q+1}_c(\mathcal{V}_M, B^p_M) \quad \text{for} \quad q < p-1. \tag{4.4a}$$

For $q = p-1$, Lemma 4.2 leads to an exact sequence

$$\begin{aligned} \mathbf{0} = H^{p-1}_c(\mathcal{V}_M, \Omega^p_M) \to H^{p-1}_c(\mathcal{V}_M, \overline{\Omega}^p_M) \to H^p_c(\mathcal{V}_M, B^p_M) \\ \twoheadrightarrow H^p_c(\mathcal{V}_M, \Omega^p_M) \to \mathbf{0}. \end{aligned} \tag{4.4b}$$

From [26], Prop. 6, we recall:

Lemma 4.3. — *For each closed $(p+q)$-form $\omega \in \Omega^{p+q}_M$, the prescription*

$$\omega^{[p]}(X_1, \dots, X_p) := [i_{X_p} \dots i_{X_1}\omega] \in \overline{\Omega}^q_M$$

defines a continuous p-cocycle in $Z^p_c(\mathcal{V}_M, \overline{\Omega}^q_M)$.

Calculating $H^2_c(\mathcal{V}_M, \overline{\Omega}^m_M)$

In this subsection we address the problem to determine the second cohomology space $H^2_c(\mathcal{V}_M, \overline{\Omega}^m_M)$ for a parallelizable manifold M of dimension N.

We start by observing that we have the following isomorphisms

$$\begin{aligned} H^2_c(\mathcal{V}_M, \overline{\Omega}^m) &\cong H^2_c(\mathcal{V}_M, B^{m+1}_M) \quad \text{for } N \geqslant 3 \quad \text{by (4.2}a\text{)} \\ &\cong H^1_c(\mathcal{V}_M, \overline{\Omega}^{m+1}_M) \quad \text{for } m > 1 \quad \text{by (4.4}a\text{)} \\ &\cong H^1_c(\mathcal{V}_M, B^{m+2}_M) \quad \text{for } N \geqslant 2 \quad \text{by (4.2}a\text{)} \\ &\cong H^0_c(\mathcal{V}_M, \overline{\Omega}^{m+2}_M) \quad \text{for each } m \quad \text{by (4.4a)} \\ &= H^{m+2}_M \quad \text{by Lemma 4.1} \end{aligned} \tag{4.5}$$

We thus obtain

Proposition 4.4. — (a) *For $N \geqslant 3$ and $m \geqslant 2$ the map*

$$H^{m+2}_M \to H^2_c(\mathcal{V}_M, \overline{\Omega}^m_M), \quad [\omega] \mapsto [\omega^{[2]}]$$

is an isomorphism.

(b) *For $N \geqslant 2$ and $m \geqslant 1$ the map*

$$H^{m+1}_M \to H^1_c(\mathcal{V}_M, \overline{\Omega}^m_M), \quad [\omega] \mapsto [\omega^{[1]}]$$

is an isomorphism.

Proof. — In view of (4.5), we only have to see that the isomorphisms are implemented by the maps $\omega \mapsto \omega^{[2]}$, resp., $\omega \mapsto \omega^{[1]}$.

(a) We start with a closed $(m+2)$-form ω, representing an element of H_M^{m+2}, which we consider as a $\mathcal{V}_M$-fixed element in $\overline{\Omega}_M^{m+2}$. A corresponding 1-cocycle with values in B_M^{m+2} is given by $\alpha(X) := \mathcal{L}_X\omega = d(i_X\omega) = (d \circ \omega^{[1]})(X)$. This already shows that the corresponding $\overline{\Omega}_M^{m+1}$-valued 1-cocycle is $\omega^{[1]}$ (*cf.* Lemma 4.3).

To distinguish it from the exterior differential d, we write $d_{\mathcal{V}(M)}$ for the Chevalley–Eilenberg differential on $C^\bullet(\mathcal{V}_M, \Omega_M^\bullet)$. To see the corresponding 2-cocycle with values in B_M^{m+1}, we recall from [26], Lemma 5 that for $\tilde{\omega}^{[1]}(X) := i_X\omega$ we have

$$(d_{\mathcal{V}_M}\tilde{\omega}^{[1]})(X,Y) = d(i_X i_Y \omega) = -(d \circ \omega^{[2]})(Y,X). \tag{4.6}$$

Hence $-\omega^{[2]} \in Z_c^2(\mathcal{V}_M, \overline{\Omega}_M^m)$ is the 2-cocycle corresponding to $[\omega]$ under the isomorphisms $H_M^{m+2} \cong H_c^2(\mathcal{V}_M, \overline{\Omega}^{\,m})$ in (4.5).

(b) is proved as in (a). □

Now we turn to the cases not covered by Proposition 4.4. The following proposition was the original motivation for the present paper:

Theorem 4.5. — *For $N \geqslant 2$ we have*

$$H_c^2(\mathcal{V}_M, \overline{\Omega}{}^1_M) \cong H_M^3 \oplus \mathbb{R}[\overline{\Psi}_1 \wedge \Psi_1] \oplus \mathbb{R}[\overline{\Psi}_2],$$

where H_M^3 embeds via the map $[\omega] \mapsto [\omega^{[2]}]$.

Proof. — First we assume that $N \geqslant 3$. For $m = 1$, we first get from (4.2a) that $H_c^2(\mathcal{V}_M, \overline{\Omega}{}^1_M) \cong H_c^2(\mathcal{V}_M, B_M^2)$ and further from (4.4b) the exact sequence

$$\mathbf{0} \to H_c^1(\mathcal{V}_M, \overline{\Omega}_M^2) \hookrightarrow H_c^2(\mathcal{V}_M, B_M^2) \to H_c^2(\mathcal{V}_M, \Omega_M^2) = \mathbb{R}[\Psi_1^2] \oplus \mathbb{R}[\Psi_2],$$

where the inclusion on the left maps $[\omega^{[1]}]$ to $[\omega^{[2]}]$ (*cf.* the proof of Proposition 4.4). Since we know from Proposition 4.4 that the map

$$H_M^3 \to H_c^1(\mathcal{V}_M, \overline{\Omega}_M^2), \quad [\omega] \mapsto [\omega^{[1]}]$$

is an isomorphism, the assertion follows because the classes $[\Psi_1^2]$ and $[\Psi_2]$ in $H_c^2(\mathcal{V}_M, \Omega_M^2)$ are contained in the image of $H_c^2(\mathcal{V}_M, B_M^2)$ (4.2) and satisfy

$$[d \circ (\overline{\Psi}_1 \wedge \Psi_1)] = [\Psi_1^2] \quad \text{and} \quad [d \circ \overline{\Psi}_2] = [\Psi_2].$$

Now we consider the case $N = 2$. For $m = 1$ we get from (4.2a) and Proposition 4.4

$$H_c^1(\mathcal{V}_M, \overline{\Omega}_M^1) \cong H_c^1(\mathcal{V}_M, B_M^2) \cong H_M^2 \text{ and } H_c^1(\mathcal{V}_M, \overline{\Omega}_M^2) \cong H_c^1(\mathcal{V}_M, B_M^3) = \{0\}.$$

Therefore the exact sequence $B_M^2 \hookrightarrow \Omega_M^2 \to \overline{\Omega}_M^2$ leads to an exact sequence

$$\mathbf{0} = H_c^1(\mathcal{V}_M, \overline{\Omega}_M^2) \to H_c^2(\mathcal{V}_M, B_M^2) \to H_c^2(\mathcal{V}_M, \Omega_M^2) \to H_c^2(\mathcal{V}_M, \overline{\Omega}_M^2) = \mathbf{0},$$

showing that the map

$$H_c^2(\mathcal{V}_M, B_M^2) \to H_c^2(\mathcal{V}_M, \Omega_M^2) \cong \mathbb{R}[\Psi_1^2] \oplus \mathbb{R}[\Psi_2]$$

is an isomorphism.

On the other hand, (4.2b) yields an embedding

$$d\circ : H_c^2(\mathcal{V}_M, \overline{\Omega}_M^1) \hookrightarrow H_c^2(\mathcal{V}_M, B_M^2) \cong H_c^2(\mathcal{V}_M, \Omega_M^2) \cong \mathbb{R}[\Psi_1^2] \oplus \mathbb{R}[\Psi_2],$$

and Lemma 4.2 asserts that this embedding is surjective. This completes the proof. □

Proposition 4.6. — *For $N = 2$ we have $H_c^2(\mathcal{V}_M, \overline{\Omega}_M^m) = \mathbf{0}$ for $m \geqslant 2$ and*

$$\dim H_c^2(\mathcal{V}_M, \overline{\Omega}\,_M^1) = 2.$$

Proof. — For $N = 2$ we have $\Omega_M^m = 0$ for $m \geqslant 3$, so that $\overline{\Omega}_M^m$ vanishes in these cases. For $m = 2$ we get from (4.2b) an inclusion $H_c^2(\mathcal{V}_M, \overline{\Omega}_M^2) \hookrightarrow H_c^2(\mathcal{V}_M, B_M^3) = \mathbf{0}$. Therefore $H_c^2(\mathcal{V}_M, \overline{\Omega}_M^m)$ vanishes for $m \geqslant 2$. The case $m = 1$ follows from Theorem 4.5. □

So far we have covered all cases $m \geqslant 1$ for $N \geqslant 2$, and we are left with the case $m = 0$ or $N = 1$. First we recall the results for $N = 1$:

Example 4.7. — (a) (*cf.* [13], Th. 30) For $N = 1$ and $M = \mathbb{S}^1$ the algebra $H^\bullet(\mathcal{V}_{\mathbb{S}^1}, \mathcal{F}_{\mathbb{S}^1})$ is a free graded commutative algebra generated by two elements α_1, α_2 of degree 1 and one element β of degree 2. In particular

$$H_c^1(\mathcal{V}_{\mathbb{S}^1}, \mathcal{F}_{\mathbb{S}^1}) = \mathbb{R}\alpha_1 \oplus \mathbb{R}\alpha_2 \quad \text{and} \quad H_c^2(\mathcal{V}_{\mathbb{S}^1}, \mathcal{F}_{\mathbb{S}^1}) = \mathbb{R}\alpha_1\alpha_2 \oplus \mathbb{R}\beta.$$

Furthermore $\overline{\Omega}\,_{\mathbb{S}^1}^1 = H_{\mathbb{S}^1}^1$ is one-dimensional and a trivial $\mathcal{V}_{\mathbb{S}^1}$-module, so that

$$H^2(\mathcal{V}_{\mathbb{S}^1}, \overline{\Omega}\,_{\mathbb{S}^1}^1) = H^2(\mathcal{V}_{\mathbb{S}^1}, \mathbb{R}) \cong \mathbb{R}$$

(Theorem 29 in [13], p. 195).

(b) For $M = \mathbb{R}^N$ the cohomology algebra $H^\bullet(\mathcal{V}_{\mathbb{R}^N}, \mathcal{F}_{\mathbb{R}^N}) \cong H^\bullet(\mathfrak{gl}_N(\mathbb{R}), \mathbb{R})$ is the free exterior algebra generated by the classes $[\varphi_k]$, $k = 1, \ldots, N$ (*cf.* Definition II.7; [31], § 4).

Now we turn to the case $m = 0$.

Theorem 4.8. — *For $N \geqslant 2$ the map*

$$H_M^2 \oplus H_M^1 \to H_c^2(\mathcal{V}_M, \mathcal{F}_M), \quad ([\alpha], [\beta]) \mapsto [\alpha + \beta \wedge \overline{\Psi}_1]$$

is a linear isomorphism.

Proof. — In view of $\overline{\Omega}^0_M = \mathcal{F}_M$, (4.2a, 4.2b) provide an embedding

$$d_* : H^2_c(\mathcal{V}_M, \mathcal{F}_M) \hookrightarrow H^2_c(\mathcal{V}_M, B^1_M), \quad [\omega] \mapsto [d \circ \omega]$$

which is an isomorphism for $N > 2$. We also get from Proposition 4.4 for $N \geqslant 2$ an isomorphism:

$$H^2_M \to H^1_c(\mathcal{V}_M, \overline{\Omega}^1_M), \quad [\omega] \mapsto [\omega^{[1]}].$$

In view of Lemma 3.3, we have

$$H^2_c(\mathcal{V}_M, \Omega^1_M) = H^1_c(\mathcal{V}_M, \mathcal{F}_M) \cdot [\Psi_1] \cong H^1_M \oplus \mathbb{R}[\overline{\Psi}_1 \wedge \Psi_1].$$

We now consider the exact sequence

$$\begin{aligned} H^1_c(\mathcal{V}_M, B^1_M) \twoheadrightarrow H^1_c(\mathcal{V}_M, \Omega^1_M) \xrightarrow{0} H^1_c(\mathcal{V}_M, \overline{\Omega}^1_M) \cong H^2_M \\ \xrightarrow{-d_*} H^2_c(\mathcal{V}_M, B^1_M) \xrightarrow{f} H^2_c(\mathcal{V}_M, \Omega^1_M) \xrightarrow{g} H^2_c(\mathcal{V}_M, \overline{\Omega}^1_M), \end{aligned}$$

in which we know all terms except the middle one, the one we are interested in. The one-dimensional space $H^1_c(\mathcal{V}_M, \Omega^1_M)$ is generated by $[\Psi_1]$, which has values in B^1_M, so that its image in $\overline{\Omega}^1_M$ vanishes. This yields the 0-arrow in the upper row of the diagram (Lemma 4.2). To calculate the connecting map $H^1_c(\mathcal{V}_M, \overline{\Omega}^1_M) \cong H^2_M \to H^2_c(\mathcal{V}_M, B^1_M)$, we recall from (4.6) that we have for each closed 2-form ω the relation

$$\delta([\omega^{[1]}]) = [d_{\mathcal{V}_M} \tilde{\omega}^{[1]}] = -[d \circ \omega^{[2]}] = -d_*[\omega].$$

Let β be a closed 1-form on M, considered as a 1-cocycle $\mathcal{V}_M \to \mathcal{F}_M$. Then

$$\beta \wedge \overline{\Psi}_1 \in Z^2_c(\mathcal{V}_M, \mathcal{F}_M)$$

satisfies

$$d \circ (\beta \wedge \overline{\Psi}_1) = (d \circ \beta) \wedge \overline{\Psi}_1 + \beta \wedge d \circ \overline{\Psi}_1 = (d \circ \beta) \wedge \overline{\Psi}_1 + \beta \wedge \Psi_1.$$

The 1-cochain

$$\gamma : \mathcal{V}_M \to \Omega^1_M, \quad \gamma(X) := \overline{\Psi}_1(X) \cdot \beta = (\overline{\Psi}_1 \cdot \beta)(X)$$

satisfies

$$d_{\mathcal{V}(M)}\gamma = (d_{\mathcal{V}(M)}\overline{\Psi}_1) \wedge \beta - \overline{\Psi}_1 \wedge (d_{\mathcal{V}(M)}\beta) = -\overline{\Psi}_1 \wedge d_{\mathcal{V}(M)}\beta.$$

Since the Cartan formula implies that

$$(d_{\mathcal{V}(M)}\beta)(X) = \mathcal{L}_X\beta = i_X(d\beta) + d(i_X\beta) = d(\beta(X)),$$

we obtain $d_{\mathcal{V}(M)}\gamma = -\overline{\Psi}_1 \wedge (d \circ \beta)$, showing that $(d \circ \beta) \wedge \overline{\Psi}_1 \in B^2_c(\mathcal{V}_M, \Omega^1_M)$. This implies that

$$d_*[\beta \cdot \overline{\Psi}_1] = [d \circ (\beta \wedge \overline{\Psi}_1)] = [\beta \wedge \Psi_1].$$

We also recall that

$$H^2_c(\mathcal{V}_M, \Omega^1_M) = (H^1_M \wedge [\Psi_1]) \oplus \mathbb{R}(\overline{\Psi}_1 \wedge \Psi_1) \cong H^1_M \oplus \mathbb{R}.$$

From the preceding considerations, we see that the subspace $H^1_M \wedge [\Psi_1]$ lies in the range of f. Since $g([\overline{\Psi}_1 \wedge \Psi_1]) = [\Psi_1^2]$ is non-zero, we derive

$$\mathrm{im}(f) = \ker(g) = H^1_M \wedge [\Psi_1].$$

From the exactness in $H^1_c(\mathcal{V}_M, B^1_M)$ we now see that the natural map

$$H^2_M \oplus H^1_M \to H^2_c(\mathcal{V}_M, B^1_M), \quad ([\alpha],[\beta]) \mapsto [d\circ\alpha + \beta\wedge\Psi_1] = [d\circ(\alpha + \beta\cdot\overline{\Psi})]$$

is a linear isomorphism. As the elements $\alpha + \beta \cdot \overline{\Psi}$ form $\mathcal{F}_M$-valued 2-cocycles, we finally conclude that the inclusion d_* is also surjective for $N = 2$. This completes the proof. □

Problem 1. — *Determine the spaces $H^2(\mathcal{V}_M, \overline{\Omega}{}^1_M)$ for all connected smooth manifolds M, without assuming that M is parallelizable.*

Appendix A. Lie algebra cocycles with values in associative algebras

Let A be a unital associative algebra and $A_L := (A, [\cdot,\cdot])$ be the underlying Lie algebra. We consider A as an A_L-module with respect to the adjoint representation $x.y := [x,y]$.

We write $m_A : A \times A \to A$ for the product map and $b_A : A \times A \to A$ for the commutator bracket and observe that both are A_L-invariant.

We take a closer look at the Lie algebra complex $(C^\bullet(A_L, A), d)$, where we use the multiplication on A, which is A_L-equivariant, to define a multiplication on $C^\bullet(A_L, A)$ by

$$\begin{aligned}&(\alpha\wedge\beta)(x_1,\ldots,x_{p+q})\\ &:= \frac{1}{p!q!}\sum_{\sigma\in S_{p+q}} \mathrm{sgn}(\sigma) m_A\big(\alpha(x_{\sigma(1)},\ldots,x_{\sigma(p)}), \beta(x_{\sigma(p+1)},\ldots,x_{\sigma(p+q)})\big)\end{aligned}$$

for $\alpha \in C^p(A_L, A)$ and $\beta \in C^q(A_L, A)$ (*cf.* [15], Sect. I.3.2). With

$$\mathrm{Alt}(\gamma)(x_1,\ldots,x_r) := \sum_{\sigma\in S_r} \mathrm{sgn}(\sigma)\gamma(x_{\sigma(1)},\ldots,x_{\sigma(r)}),$$

this means that

$$\alpha\wedge\beta = \frac{1}{p!q!}\mathrm{Alt}(\alpha\cdot\beta) = \frac{1}{p!q!}\mathrm{Alt}(m_A\circ(\alpha,\beta)).$$

An easy induction implies that n-fold products in $C^\bullet(A_L, A)$ are given by

$$\alpha_1 \wedge \cdots \wedge \alpha_n = \frac{1}{\prod_{j=1}^n p_j!} \operatorname{Alt}(\alpha_1 \cdots \alpha_n) \quad \text{for} \quad \alpha_i \in C^{p_i}(A_L, A).$$

We thus obtain an associative differential graded algebra $(C^\bullet(A_L, A), d)$ (*cf.* [25], App. F; see also [32], p. 30).

Note that $\mathrm{id}_A \in C^1(A_L, A)$ is a 1-cochain with

$$d_{A_L}(\mathrm{id}_A)(x,y) = x.y - y.x - [x,y] = [x,y] - [y,x] - [x,y] = [x,y],$$

so that $d_{A_L}(\mathrm{id}_A) = b_A$. It follows in particular that b_A is a 2-coboundary. We therefore obtain a sequences of coboundaries

$$b_A^n = d_{A_L}(\mathrm{id}_A \wedge b_A^{n-1}) \in B^{2n}(A_L, A) \subseteq Z^{2n}(A_L, A).$$

Lemma A.1. — *If $T : A \to \mathfrak{z}$ is a linear map vanishing on all commutators, then $T \circ b_A^k$ vanishes for each $k \in \mathbb{N}$, and if $\mathfrak{z}$ is considered as a trivial A_L-module, then*

$$\varphi_k := T \circ (\mathrm{id}_A \wedge b_A^{k-1}) = T \circ (\mathrm{id}_A)^{2k-1} \in Z^{2k-1}(A_L, \mathfrak{z}),$$

is a cocycle given by

$$\varphi_k(x_1, \ldots, x_{2k-1}) = \sum_{\sigma \in S_{2k-1}} \operatorname{sgn}(\sigma) T(x_{\sigma(1)} \cdots x_{\sigma(2k-1)}).$$

Proof. — Arguing as in Remark I.2 of [27], we see that for each $n \in \mathbb{N}_0$, we have

$$b_A^n = \frac{1}{2^n} \operatorname{Alt}(b_A \cdots b_A) = \operatorname{Alt}(m_A \cdots m_A) = \operatorname{Alt}(\mathrm{id}_A^{2n}) = (\mathrm{id}_A)^{2n},$$

i.e.,

$$b_A^n(x_1, \ldots, x_{2n}) = \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(2n)}.$$

Let τ be the cyclic permutation $\tau = (1\ 2\ \ldots\ 2n)$. Then $\operatorname{sgn}(\tau) = -1$, but, for each $\sigma \in S_{2n}$, we have

$$T(x_{\sigma\tau(1)} \cdots x_{\sigma\tau(2n)}) = T(x_{\sigma(2)} \cdots x_{\sigma(2n)} x_{\sigma(1)}) = T(x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(2n)}).$$

This leads to $T \circ b_A^n = -T_A \circ b_A^n$, which implies that $T \circ b_A^n$ vanishes.

Since b_A is a cocycle, we now obtain

$$d\varphi_k = d(T \circ (\mathrm{id}_A \wedge b_A^{k-1})) = T \circ (d(\mathrm{id}_A) \wedge b_A^{k-1}) = T \circ (b_A^k) = 0.$$

□

The following theorem describes an important application of the preceding construction, namely that for $A = \mathfrak{gl}_n(\mathbb{R})$ we thus obtain a set of generators of the cohomology algebra ([15], Th. 2.1.6):

Theorem A.2. — *The cohomology algebra $H^\bullet(\mathfrak{gl}_N(\mathbb{R}),\mathbb{R})$ is generated by the cohomology classes of the cocycles $\varphi_k \in Z^{2k-1}(\mathfrak{gl}_N(\mathbb{R}),\mathbb{R}), k = 1,\dots,N$, given by*

$$\varphi_k(x_1,\dots,x_{2k-1}) = \sum_{\sigma\in S_{2k-1}} \mathrm{sgn}(\sigma)\,\mathrm{Tr}\left(x_{\sigma(1)}\dots x_{\sigma(2k-1)}\right) \qquad \textit{for} \quad x_1,\dots,x_{2k-1}\in\mathfrak{gl}_N(\mathbb{R}).$$

Lie algebra cocycles with values in differential forms

Let $(\Omega^\bullet, d)$ be a differential graded algebra whose multiplication is denoted $\alpha\wedge\beta$, V a finite-dimensional vector space, and put $A := \Omega^0\otimes\mathrm{End}(V)$ and $B := \Omega^\bullet\otimes\mathrm{End}(V)$. Then B also carries the structure of a differential graded algebra with differential $d_B := d\otimes\mathrm{id}_{\mathrm{End}(V)}$ which is not graded commutative. Note that $A_L \cong \Omega^0\otimes\mathfrak{gl}(V)$ as Lie algebras.

Proposition A.3. — *For each $k\in\mathbb{N}$, we obtain cocycles $\psi_k\in Z^k(A_L, \Omega^k)$ and $\overline{\psi}_k\in Z^k(A_L,\Omega^{k-1}/d\Omega^{k-2})$ satisfying $d\circ\overline{\psi}_k = \psi_k$ by*

$$\psi_k(f_1\otimes x_1,\dots,f_k\otimes x_k) := \sum_{\sigma\in S_k}\mathrm{Tr}(x_{\sigma(1)}\cdots x_{\sigma(k)})df_1\wedge\ldots\wedge df_k$$

and

$$\overline{\psi}_k(f_1\otimes x_1,\dots,f_k\otimes x_k) := \sum_{\sigma\in S_k}\mathrm{Tr}(x_{\sigma(1)}\cdots x_{\sigma(k)})[f_1\cdot df_2\wedge\ldots\wedge df_k] \in \Omega^{k-1}/d\Omega^{k-2}.$$

Proof. — We first consider the linear map

$$\varphi = d_B : A_L \to \Omega^1\otimes\mathrm{End}(V), \quad f\otimes x\mapsto df\otimes x$$

and note that

$$(d_{A_L}\varphi)(a,b) = a.db - b.da - d[a,b] = [a,db]-[b,da]-[da,b]-[a,db] = 0$$

implies that φ is a 1-cocycle. This implies that its $\wedge$-powers $\varphi^k\in Z^k(A_L,\Omega^k \otimes\mathrm{End}(V))$ are k-cocycles and hence that

$$\psi_k = \mathrm{Tr}\circ\varphi^k \in Z^k(A_L,\Omega^k),$$

because $\mathrm{Tr} : \Omega^k\otimes\mathrm{End}(V)\to\Omega^k$ is a morphism of the A_L-module $\Omega^k\otimes \mathrm{End}(V)$ onto the trivial module Ω^k.

That $\overline{\psi}_k$ is alternating follows from

$$\begin{aligned}[f_1 \wedge df_2 \wedge \ldots \wedge df_k] + [f_2 \wedge df_1 \wedge df_3 \wedge \ldots \wedge df_k] &= [d(f_1 f_2) \wedge \ldots \wedge df_k] \\ &= [d(f_1 f_2 \wedge df_3 \wedge \ldots df_k)] = 0.\end{aligned}$$

It remains to show that $\overline{\psi}_k$ is a k-cocycle. To this end, we consider

$$\mathrm{id}_A \wedge \varphi^{k-1} \in C^k(A_L, \Omega^{k-1} \otimes \mathrm{End}(V))$$

and observe that

$$d_{A_L}(\mathrm{id}_A \wedge \varphi^{k-1}) = d_{A_L}(\mathrm{id}_A) \wedge \varphi^{k-1} = b_A \wedge \varphi^{k-1} = \mathrm{id}_A \wedge \mathrm{id}_A \wedge \varphi^{k-1}.$$

If $q : \Omega^k \to \Omega^k / d\Omega^{k-1}, \beta \mapsto [\beta]$ is the quotient map, we have

$$\begin{aligned}&\big(q \circ \mathrm{Tr} \circ (\mathrm{id}_A \wedge \varphi^{k-1})\big)(f_1 \otimes x_1, \ldots, f_k \otimes x_k) \\ &= \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \,\mathrm{Tr}(x_{\sigma(1)} \cdots x_{\sigma(k)})[f_{\sigma(1)} df_{\sigma(2)} \wedge \ldots \wedge df_{\sigma(k)}] \\ &= \sum_{\sigma \in S_k} \mathrm{Tr}(x_{\sigma(1)} \cdots x_{\sigma(k)})[f_1 df_2 \wedge \ldots \wedge df_k] = \overline{\psi}_k(f_1 \otimes x_1, \ldots, f_k \otimes x_k).\end{aligned}$$

To see that $\overline{\psi}_k$ is a cocycle, it therefore suffices to show that

$$d\overline{\psi}_k = q \circ \mathrm{Tr} \circ (\mathrm{id}_A \wedge \mathrm{id}_A \wedge \varphi^{k-1}) = 0.$$

Explicitly, we have

$$\begin{aligned}&\big(q \circ \mathrm{Tr} \circ (\mathrm{id}_A^2 \wedge \varphi^{k-1})\big)(f_1 \otimes x_1, \ldots, f_{k+1} \otimes x_{k+1}) \\ &\qquad = \sum_{\sigma \in S_{k+1}} \mathrm{sgn}(\sigma) \,\mathrm{Tr}(x_{\sigma(1)} \cdots x_{\sigma(k+1)})[f_{\sigma(1)} f_{\sigma(2)}\ df_{\sigma(3)} \wedge \ldots \wedge df_{\sigma(k+1)}].\end{aligned}$$

If $\tau = (1\ 2\ \ldots\ k+1)$ is the cyclic permutation, we obtain

$$\begin{aligned}&\mathrm{sgn}(\sigma\tau) \,\mathrm{Tr}(x_{\sigma\tau(1)} \cdots x_{\sigma\tau(k+1)})[f_{\sigma\tau(1)} f_{\sigma\tau(2)}\ df_{\sigma\tau(3)} \wedge \ldots \wedge df_{\sigma\tau(k+1)}] \\ &= (-1)^k \mathrm{sgn}(\sigma) \,\mathrm{Tr}(x_{\sigma(1)} \cdots x_{\sigma(k+1)})[f_{\sigma(2)} f_{\sigma(3)}\ df_{\sigma(4)} \wedge \ldots \wedge df_{\sigma(k+1)} \wedge df_1] \\ &= \mathrm{sgn}(\sigma) \,\mathrm{Tr}(x_{\sigma(1)} \cdots x_{\sigma(k+1)})[f_{\sigma(2)} f_{\sigma(3)}\ df_{\sigma(1)} \wedge df_{\sigma(4)} \wedge \ldots \wedge df_{\sigma(k+1)}].\end{aligned}$$

Now the relation

$$\begin{aligned}[f_1 f_2 df_3 \wedge d(\cdots)] + [f_2 f_3 df_1 \wedge d(\cdots)] + [f_3 f_1 df_2 \wedge d(\cdots)]& \\ = [d(f_1 f_2 f_3) \wedge d(\cdots)] = [d(f_1 f_2 f_3 \cdots)] &= 0\end{aligned}$$

implies that $3\overline{\psi}_k$ is a cocycle, so that $\overline{\psi}_k$ is a cocycle. □

Appendix B. Cohomology of formal vector fields

In this section, we will review results of Gelfand–Fuks on the cohomology of formal vector fields because this is needed to reduce Theorem 3.1 to Tsujishita's result.

Fix $N \in \mathbb{N}$ and write $\mathcal{F}_N = \mathbb{R}[[x_1, \ldots, x_N]]$ for the commutative algebra of formal power series in the variables $x_1, \ldots x_N$, endowed with the projective limit topology, and let W_N be the Lie algebra of continuous derivations of $\mathcal{F}_N$:

$$W_N = \bigoplus_{j=1}^{N} \mathcal{F}_N \frac{\partial}{\partial x_j}.$$

Consider also the subalgebra L_0 in W_N consisting of vector fields that vanish at the origin:

$$L_0 = \Big\{ \sum_{j=1}^{N} f_j(x) \frac{\partial}{\partial x_j} : f_j(0) = 0,\, j = 1, \ldots, N \Big\}, \text{ satisfying } W_N \cong L_0 \rtimes \mathbb{R}^N,$$

where the second factor corresponds to the abelian Lie subalgebra of constant (formal) vector fields.

For $X \in W_N$, we write

$$J(X) \in \mathcal{F}_N \otimes \mathfrak{gl}_N(\mathbb{R})$$

for the *Jacobian of* X, defined for $X = \sum_i f_i \frac{\partial}{\partial x_i}$ by $J(X)_{ij} = \frac{\partial f_i}{\partial x_j}$. Then

$$-J : W_N \to \mathcal{F}_N \otimes \mathfrak{gl}_N(\mathbb{R})$$

is a crossed homomorphism whose kernel is the subalgebra of constant vector fields. From the relation

$$-J([X, Y]) = -X.J(Y) + Y.J(X) + [-J(X), -J(Y)],$$

it follows that

$$-J_0 : L_0 \to \mathfrak{gl}_N(\mathbb{R}), \quad X \mapsto -J(X)(0)$$

is a surjective homomorphism of Lie algebras, restricting to an isomorphism on the subalgebra of linear vector fields.

Let Ω_N^p be the space of formal p-forms in N variables, considered as a free $\mathcal{F}_N$-module and also as a W_N-module, with respect to the action defined by the Lie derivative.

DEFINITION B.1. — *Using the crossed homomorphism $\theta := -J$, we get with Lemma A.1, Proposition A.3 and Theorem 1.7 the following cocycles*

$$\varphi_k^W \in Z_c^{2k-1}(W_N, \mathcal{F}_N),\ \varphi_k^W(X_1,\ldots,X_{2k-1}) = \sum_{\sigma\in S_{2k-1}} \mathrm{sgn}(\sigma)\,\mathrm{Tr}\left(\theta(X_{\sigma(1)})\ldots\theta(X_{\sigma(2k-1)})\right),$$

$$\Psi_k^W \in Z_c^k(W_N, \Omega_N^k),\ \Psi_k^W(X_1,\ldots,X_k) = \sum_{\sigma\in S_k} \mathrm{sgn}(\sigma)\,\mathrm{Tr}\left(d\theta(X_{\sigma(1)})\wedge\ldots\wedge d\theta(X_{\sigma(k)})\right),$$

and $\overline{\Psi}_k^W \in Z_c^k(W_N, \Omega_N^{k-1}/d\Omega_N^{k-2})$, *defined by*

$$\overline{\Psi}_k^W(X_1,\ldots,X_k) = \sum_{\sigma\in S_k} \mathrm{sgn}(\sigma)[\mathrm{Tr}\left(\theta(X_{\sigma(1)})d\theta(X_{\sigma(2)})\wedge\ldots\wedge d\theta(X_{\sigma(k)})\right)].$$

The following theorem is a re-formulation of the results of Gelfand-Fuks ([17]), describing the cohomology of W_N with values in the modules Ω_N^p. Let $V = \mathbb{R}^N$, considered as the canonical module of $\mathfrak{gl}_N(\mathbb{R})$ and write

$$\mathrm{ev}_0 : \Omega_N^p \to \Lambda^p(V'), \quad \omega \mapsto \omega(0)$$

for the evaluation map. We consider V as a module of L_0 by pulling back the module structure from $\mathfrak{gl}_N(\mathbb{R})$, so that ev_0 is a morphism of L_0-modules.

THEOREM B.2.

(a) *For each $p, q \in \mathbb{N}_0$, the map $C_c^p(W_N, \Omega_N^q) \to C_c^p(L_0, \Lambda^q(V')), \omega \mapsto \mathrm{ev}_0 \circ\omega\,|_{L_0}$ induces an isomorphism*

$$H_c^p(W_N, \Omega_N^q) \to H_c^p(L_0, \Lambda^q(V')).$$

(b) *$H_c^\bullet(L_0, \Lambda^\bullet(V')) \cong H_c^\bullet(L_0, \mathbb{R}) \otimes H_c^\bullet(L_0, \mathfrak{gl}_N(\mathbb{R}), \Lambda^\bullet(V'))$ as bigraded algebras.*

(c) *The inclusions $\mathfrak{gl}_N(\mathbb{R}) \hookrightarrow L_0 \hookrightarrow W_N$ induce isomorphisms of graded algebras*

$$H_c^\bullet(W_N, \mathbb{R}) \to H_c^\bullet(L_0, \mathbb{R}) \to H_c^\bullet(\mathfrak{gl}_N(\mathbb{R}), \mathbb{R}).$$

In particular, $H_c^\bullet(W_N, \mathbb{R})$ is an exterior algebra with generators of degree $2k-1$, $k = 1, \ldots, N$.

(d) *Let $\Psi_k^L := \mathrm{ev}_0 \circ \Psi_k^W\,|_{L_0} \in Z_c^k(L_0, \mathfrak{gl}_N(\mathbb{R}), \Lambda^k(V'))$. Then $H_c^\bullet(L_0, \mathfrak{gl}_N(\mathbb{R}), \Lambda^\bullet(V'))$ is a quotient of the commutative algebra generated by the cohomology classes $[\Psi_k^L]$, $k = 1, \ldots, N$, by the ideal spanned by the elements of degree exceeding N.*

Proof. — We only explain how this can be derived from [15].

(a) Write $U(\mathfrak{g})$ for the enveloping algebra of a Lie algebra $\mathfrak{g}$. First we observe that the map

$$\zeta : \Omega^p_N \to \mathrm{Hom}_{L_0}(U(W_N), \Lambda^p(V')) \cong \mathrm{Hom}(S(\mathbb{R}^N), \Lambda^p(V')) \cong \mathcal{F}_N \otimes \Lambda^p(V'),$$

$$\zeta(\alpha)(D) := (D.\alpha)(0)$$

is an isomorphism of W_N-modules. Hence Ω^p_N is coinduced, as a W_N-module, from the L_0-module $\Lambda^p(V')$. Note that, since L_0 is of finite codimensional in W_N, no problems arise from continuity requirements. Therefore (a) follows from a general result on coinduced modules ([15], Th. 1.5.4).

(b) The proof of part (b) is based on the Hochschild–Serre Spectral Sequence associated with the filtration on $C^\bullet(L_0, \Lambda^\bullet(V'))$ relative to the subalgebra $\mathfrak{gl}_N(\mathbb{R}) \subset L_0$. By Theorem 1.5.1(ii) in [15], $H^p(L_0, \Lambda^\bullet(V'))$ is the $E_2^{p,0}$-term in the spectral sequence, while in the proof of Theorem 2.2.7' in [15], it is shown that $E_2^{p,0} = E_1^{p,0}$, and the term $E_1^{p,0}$ is calculated explicitly.

(c), (d) follow from (a), combined with [15], Thms. 2.2.7 and 2.2.7'.

From the formulas in [15] it is not completely obvious that our formula for Ψ^L_k describes the same cocycle (up to a scalar factor). Fuks describes Ψ^L_k as an element of

$$\Lambda^k(V') \otimes \Lambda^k(S^2(V) \otimes V').$$

In our context, the cocycle $\Psi^L_k \in Z^k_c(L_0, \mathfrak{gl}_N(\mathbb{R}), \Lambda^k(V'))$ is given by the formula

$$\Psi^L_k(X_1, \ldots, X_k) = (-1)^k \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, \mathrm{Tr}\,\big(dJ(X_{\sigma(1)})(0) \\ \wedge \ldots \wedge dJ(X_{\sigma(k)})(0)\big) \in \Lambda^k(V').$$

For each $X \in L_0$, the constant term of the 1-form $dJ(X) \in \Omega^1_N \otimes \mathfrak{gl}_N(\mathbb{R})$ corresponds to the quadratic term in X, which can be identified with an element of $S^2(V') \otimes V \cong \mathrm{Sym}^2(V) \otimes V$.

For the basis element $X_{i_1,i_2,\ell} := x_{i_1} x_{i_2} \partial_\ell$, we have, in terms of the matrix basis E_{ij} of $\mathfrak{gl}_N(\mathbb{R})$:

$$J(X) = x_{i_1} E_{\ell i_2} + x_{i_2} E_{\ell i_1} \text{ and } d(J(X)) = dx_{i_1} E_{\ell i_2} + dx_{i_2} E_{\ell i_1} \\ = \sum_{\tau \in S_2} dx_{i_{\tau(1)}} E_{\ell i_{\tau(2)}}.$$

This leads to

$$\begin{aligned}
&(-1)^k \Psi_k^L(X_{i_{11},i_{12},\ell_1},\ldots,X_{i_{k1},i_{k2},\ell_k})\\
&= \sum_{\sigma\in S_k} \sum_{\substack{\tau_i\in S_2\\ i=1,\ldots,k}} \mathrm{Tr}\Big(dx_{i_{\sigma(1),\tau_1(1)}} E_{\ell_{\sigma(1)} i_{\sigma(1)\tau_1(2)}} \wedge\cdots\wedge dx_{i_{\sigma(k),\tau_k(1)}} E_{\ell_{\sigma(k)} i_{\sigma(k)\tau_k(2)}}\Big)\\
&= \sum_{\sigma\in S_k} \sum_{\substack{\tau_i\in S_2\\ i=1,\ldots,k}} \delta_{i_{\sigma(1)\tau_1(2)},\ell_{\sigma(2)}} \delta_{i_{\sigma(2)\tau_2(2)},\ell_{\sigma(3)}} \cdots \delta_{i_{\sigma(k)\tau_k(2)},\ell_{\sigma(1)}}\\
&\qquad\qquad \cdot dx_{i_{\sigma(1),\tau_1(1)}} \wedge\cdots\wedge dx_{i_{\sigma(k),\tau_k(1)}}.
\end{aligned}$$

From this formula it is not hard to verify that our Ψ_k^L are multiples of those in [15]. □

Appendix C. Higher frame bundles and differential forms

In this appendix we shall prove our key Theorem 3.1. The major part of the work consists in explaining why Theorem 3.1 can be derived from Tsujishita's [32]. For that we need the passage from the cohomology of formal vector fields to the cohomology of $\mathcal{V}_M$ and finally to link it to Theorem 3.1.

Let M be an N-dimensional smooth manifold. For $k \in \mathbb{N}_0 \cup \{\infty\}$ we write $J^k(M)$ for its k-frame bundle whose elements are k-jets $[\alpha]$ of local diffeomorphism $\alpha : (\mathbb{R}^N, 0) \to M$. Then $J^0(M) = M$ and $J^1(M)$ is the frame bundle of M whose elements are the isomorphisms $\mathbb{R}^N \to T_m(M)$, $m \in M$. Evaluating $[\alpha]$ in 0 leads to a natural map $J^k(M) \to M$ which exhibits $J^k(M)$ as a fiber bundle over M. We write $J^k_m(M)$ for the fiber over $m \in M$. For $\ell \leqslant k$ we have projection maps

$$\pi_{\ell,k} : J^k(M) \to J^\ell(M), \quad [\alpha] \mapsto j^\ell_0(\alpha).$$

In particular, we have $\pi_{0,k}([\alpha]) = \alpha(m)$ and $\pi_{1,k}([\alpha]) = T_0(\alpha)$.

Let G^k denote the group of k-jets of local diffeomorphisms of $\mathbb{R}^N$ fixing 0. As a set, G^k is the set of all polynomial maps $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ of degree $\leqslant k$ without constant term for which $d\varphi(0)$ is invertible. The multiplication is given by

$$\varphi_1\varphi_2 := j^k_0(\varphi_1 \circ \varphi_2).$$

For $\ell \leqslant k$ we have natural surjective homomorphisms

$$q_{\ell,k} : G^k \to G^\ell, \quad \alpha \mapsto j^\ell_0(\alpha),$$

cutting off all terms of order $> \ell$. We put

$$G_k := \ker(q_{k,\infty}) \trianglelefteq G^\infty$$

and note that

$$G^\infty \cong G_1 \rtimes \mathrm{GL}_N(\mathbb{R}) \tag{C.1}$$

in a natural way.

For $k < \infty$, the group G^k is a finite-dimensional Lie group, $G^1 \cong \mathrm{GL}_N(\mathbb{R})$, and

$$G^k \cong G_1^k \rtimes \mathrm{GL}_N(\mathbb{R}),$$

where $G_1^k := \ker(q_{1,k})$ is a simply connected nilpotent Lie group. Identifying G^∞ with the projective limit of all groups $G^k \cong G^\infty/G_k$, we obtain a topology for which it actually is a Lie group with Lie algebra $\mathrm{Lie}(G^\infty) \cong L_0$ and $L_k := \mathrm{Lie}(G_k)$ is a finite-codimensional ideal of L_0 (*cf.* B). The normal subgroup G_1 is pro-nilpotent, its exponential function $\exp_{G_1} : L_1 \to G_1$ is a diffeomorphism, and (C.1) is a semidirect product of Lie groups.

The group G^k acts on $J^k(M)$ from the right by $[\alpha].\varphi := [\alpha \circ \varphi]$. This action is transitive on the fibers and defines on $J^k(M)$ the structure of a smooth G^k-principal bundle.

The group $\mathrm{Diff}(M)$ acts on each frame bundle $J^k(M)$ by $\varphi.[\alpha] := [\varphi \circ \alpha]$, and the corresponding homomorphism of Lie algebras is given by

$$\gamma^k : \mathcal{V}_M \to \mathcal{V}_{J^k(M)}, \quad \gamma^k(X)_{[\alpha]} = \frac{d}{dt}\Big|_{t=0} [\mathrm{Fl}_t^X \circ \alpha]. \tag{C.2}$$

Since the action of $\mathrm{Diff}(M)$ on $J^k(M)$ commutes with the action of the structure group G^k,

$$\gamma^k(\mathcal{V}_M) \subseteq \mathcal{V}_{J^k(M)}^{G^k}. \tag{C.3}$$

Lemma C.1 ([32], Lemma 4.2.2). — *Let $k \in \mathbb{N} \cup \{\infty\}$, ∇ be an affine connection on M, and $D \subseteq TM$ the open domain of the corresponding exponential function $\mathrm{Exp}^\nabla : D \to M$, which is an open neighborhood of the zero section. Then*

$$s : J^1(M) \to J^k(M), \quad s(\alpha) := j_0^k(\mathrm{Exp}_m^\nabla \circ \alpha)$$

is a smooth section.

Note that for each $m \in M$ the intersection $D_m := D \cap T_m(M)$ is an open zero neighborhood and that $\mathrm{Exp}_m^\nabla : D_m \to M$ is a smooth map with $T_0(\mathrm{Exp}_m^\nabla) = \mathrm{id}_{T_m(M)}$, hence a local diffeomorphism. Now the map $J^1(M) \times G_1^k \to J^k(M), (\alpha, g) \mapsto s(\alpha)g$ is a diffeomorphism and the map

$$F : J^k(M) \to G_1^k, \quad s(\alpha)g \mapsto g, \quad \alpha \in J^1(M),$$

is smooth and G_1^k-equivariant. For $g \in \mathrm{GL}_N(\mathbb{R})$ we have $s(\alpha \circ g) = s(\alpha).g$, which implies that

$$F(\alpha.g) = g^{-1}F(\alpha)g \quad \text{for all} \quad \alpha \in J^k(M). \tag{C.4}$$

Hence F is equivariant with respect to the action of $G^k \cong G_1^k \rtimes \mathrm{GL}_N(\mathbb{R})$ on G_1^k from the right by $g.(g_1, g_0) = g_0^{-1}gg_1g_0$.

Since Tsujishita uses a realization of Lie algebra cohomology in terms of right invariant, resp., equivariant differential forms on the corresponding group, we briefly discuss the relevant identifications in the following remark.

Remark C.2. — (a) Let G be a Lie group. A *smooth G-module* is a topological vector space V on which G acts smoothly by linear maps. We write

$$\rho_V : G \times V \to V, \quad (g, v) \mapsto \rho_V(g)(v) =: g.v$$

for the action map.

Further, let M be a smooth manifold on which G acts from the right by $M \times G \to M, (m, g) \mapsto m.g =: \rho_g^M(m)$. We call a p-form $\alpha \in \Omega^p(M, V)$ *equivariant* if we have for each $g \in G$ the relation

$$\left(\rho_g^M\right)^* \alpha = \rho_V(g)^{-1} \circ \alpha.$$

We write $\Omega^p(M, V)^G$ for the space of G-equivariant p-forms on M. This is the space of G-fixed elements with respect to the action of G on $\Omega^p(M, V)$, given by $g.\alpha := \rho_V(g) \circ \left(\rho_g^M\right)^* \alpha$.

(b) For the right action of G on $M = G$ by left multiplication $x.g := g^{-1}x$ we obtain the space $\Omega_\ell^p(G, V)$ of left equivariant forms, *i.e.*, forms satisfying

$$\lambda_g^* \omega = \rho_V(g) \circ \omega, \quad g \in G.$$

In [9] it is shown that the evaluation map

$$\mathrm{ev}_{\mathbf{1}} : (\Omega_\ell^\bullet(G, V), d) \to (C_c^\bullet(\mathfrak{g}, V), d_{\mathfrak{g}}), \quad \omega \mapsto \omega_{\mathbf{1}}$$

yields an isomorphism of cochain complexes (*cf.* [25] for the unproblematic extension to infinite-dimensional Lie groups).

(c) There is also a realization of the complex $(C_c^\bullet(\mathfrak{g}, V), d_{\mathfrak{g}})$ by right equivariant differential forms on G: If $\eta_G : G \to G, g \mapsto g^{-1}$ is the inversion map, then for each left equivariant p-form $\alpha \in \Omega_\ell^p(G, V)$ the p-form $\tilde{\alpha} := \eta_G^* \alpha$ is *right equivariant*, *i.e.*,

$$\rho_g^* \tilde{\alpha} = \rho_V(g)^{-1} \circ \tilde{\alpha} \quad \text{for each} \quad g \in G.$$

We thus obtain an isomorphism of cochain complexes

$$\eta_G^* : (\Omega_\ell^\bullet(G, V), d) \to (\Omega_r^\bullet(G, V), d).$$

Since $T_{\mathbf{1}}(\eta_G) = -\operatorname{id}_G$, we also obtain an isomorphism

$$\tilde{\operatorname{ev}}_{\mathbf{1}} : (\Omega_r^\bullet(G,V), d) \to (C_c^\bullet(\mathfrak{g}, V), d_{\mathfrak{g}}), \quad \omega \mapsto (-\operatorname{id}_{\mathfrak{g}})^*\omega_{\mathbf{1}}.$$

For each $g \in G$ and $\omega \in \Omega_r^p(G,V)$ the form $\lambda_g^*\omega$ is also right equivariant and satisfies

$$\begin{aligned}(\lambda_{g^{-1}}^*\omega)_{\mathbf{1}}(x_1,\dots,x_p) &= \omega_{g^{-1}}(g^{-1}.x_1,\dots,g^{-1}.x_p)\\ &= \rho_V(g).\omega(\operatorname{Ad}(g)^{-1}.x_1,\dots,\operatorname{Ad}(g)^{-1}.x_p),\end{aligned}$$

showing that $\tilde{\operatorname{ev}}_{\mathbf{1}}$ intertwines the action of G by left translation on $\Omega_r^p(G,V)$ with the natural action of G on the cochain space $C_c^p(\mathfrak{g},V)$.

(d) There is an alternative realization of the complex $(C^\bullet(\mathfrak{g},V), d_{\mathfrak{g}})$ as the space of left, resp., right invariant V-valued differential forms on G.

First we write the Chevalley–Eilenberg differential as

$$d_{\mathfrak{g}} = d_{\mathfrak{g}}^0 + \rho_V \wedge,$$

where $d_{\mathfrak{g}}^0$ is the differential corresponding to the trivial module structure on V and $\rho_V : \mathfrak{g} \to \mathfrak{gl}(V)$ is the homomorphism of Lie algebras, defining the $\mathfrak{g}$-module structure on V.

If $\kappa_G \in \Omega^1(G,\mathfrak{g})$ denotes the left Maurer–Cartan form of G, we obtain $\kappa_V := \rho_V \circ \kappa_G \in \Omega^1(G,\mathfrak{gl}(V))$ and a corresponding covariant differential on $\Omega^\bullet(M,V)$:

$$d_\kappa\omega = d\omega + \kappa_V \wedge \omega.$$

Since κ is left invariant, this differential preserves the subspace of left invariant forms, and it is easy to see that evaluation in $\mathbf{1}$ intertwines it with $d_{\mathfrak{g}}$ on $C_c^\bullet(\mathfrak{g},V)$.

In a similar fashion, one obtains a realization by right invariant differential forms with the appropriate differential.

Let A be a finite-dimensional smooth G^∞-module, where we assume that the action of the non-connected Lie group $\mathrm{GL}_N(\mathbb{R})$ on A is the one obtained by restricting the action of $\mathrm{GL}_N(\mathbb{C})$ on the complexification $A_{\mathbb{C}}$, hence completely determined by the action of $\mathfrak{gl}_N(\mathbb{R})$ on A. We then have natural identifications

$$C_c^\bullet(L_0, \mathfrak{gl}_N(\mathbb{R}), A) \cong C_c^\bullet(L_1, A)^{\mathfrak{gl}_N(\mathbb{R})} \cong \Omega_r^\bullet(G_1, A)^{\mathrm{GL}_N(\mathbb{R})} \tag{C.5}$$

([32], Lemma 3.3.4, Remark C1b(c)). Although $\mathrm{GL}_N(\mathbb{R})$ is not connected, this follows from a complexification argument since $\mathrm{GL}_N(\mathbb{C})$ is connected.

Further, let $\omega \in C_c^p(L_1, A)^{\mathrm{GL}_N(\mathbb{R})} \cong C_c^p(L_1, A)^{\mathfrak{gl}_N(\mathbb{R})}$ and $\tilde{\omega} \in \Omega_r^p(G_1, A)$ the corresponding right equivariant A-valued p-form with $\tilde{\omega}_{\mathbf{1}} = (-1)^p\omega$ (*cf.*

Remark C.2(c)). For $g_1 \in G_1$ we then have

$$\rho_{g_1}^* F^* \tilde{\omega} = (F \circ \rho_{g_1})^* \tilde{\omega} = (\rho_{g_1} \circ F)^* \tilde{\omega} = \rho_A(g_1) \circ F^* \tilde{\omega}$$

and for $g_0 \in \mathrm{GL}_N(\mathbb{R})$ we obtain with (C.4)

$$\rho_{g_0}^* F^* \tilde{\omega} = (F \circ \rho_{g_0})^* \tilde{\omega} = (c_{g_0}^{-1} \circ F)^* \tilde{\omega} = F^* \big((c_{g_0}^{-1})^* \tilde{\omega}\big) = \rho_A(g_0) \circ F^* \tilde{\omega},$$

hence the $\mathrm{GL}_N(\mathbb{R})$-equivariance of $\tilde{\omega}$. We thus obtain for $k = \infty$ a morphism of chain complexes

$$\tilde{F}^* : C_c^\bullet(L_1, A)^{\mathrm{GL}_N(\mathbb{R})} \to \Omega^\bullet(J^\infty(M), A)^{G^\infty}, \quad \omega \mapsto F^* \tilde{\omega}. \tag{C.6}$$

The map γ_A defined below is the one used by Tsujishita in [32], p. 62.

Proposition C.3. — *Let $\mathcal{A} := J^\infty(M) \times_{G^\infty} A \to M$ denote the bundle associated to $J^\infty(M)$ by the representation of G^∞ on A and $\Gamma(\mathcal{A})$ be its space of smooth sections on which $\mathcal{V}_M$ acts via γ^∞. We identify $\Gamma(\mathcal{A})$ with $C^\infty(J^\infty(M), A)^{G^\infty}$. Then the map*

(C.7)

$$\gamma_A := (\gamma^\infty)^* \circ \tilde{F}^* : C_c^\bullet(L_1, A)^{\mathrm{GL}_N(\mathbb{R})} \cong C_c^\bullet(L_0, \mathfrak{gl}_N(\mathbb{R}), A) \to C_c^\bullet(\mathcal{V}_M, \Gamma(\mathcal{A}))$$

is a morphism of chain complexes. Different choices of affine connections on M yield the same maps

$$H_c^\bullet(L_1, A)^{\mathrm{GL}_N(\mathbb{R})} \cong H_c^\bullet(L_0, \mathfrak{gl}_N(\mathbb{R}), A) \to H_c^\bullet(\mathcal{V}_M, \Gamma(\mathcal{A}))$$

in cohomology.

Proof. — For the continuity of the so obtained cochains $\gamma_A(\omega)$ of $\mathcal{V}_M$, we note that the continuous representation of G^∞ on the finite-dimensional space A factors through some finite-dimensional quotient group G^k. Hence the bundle $\mathcal{A}$ is also associated to $J^k(M)$, $\Gamma(\mathcal{A})$ can be realized as A-valued G^k-equivariant functions on $J^k(M)$, and the action of $\mathcal{V}_M$ on this space via $\gamma^k : \mathcal{V}_M \to \mathcal{V}_{J^k(M)}$ is continuous because the finite dimensionality of $J^k(M)$ implies that γ^k is a continuous morphism of topological Lie algebras.

To see that the choice of connection has no effect on the corresponding map in cohomology, let ∇' be another affine connection on M and observe that $\nabla_t := t\nabla' + (1-t)\nabla$ defines a smooth family of affine connections on M with $\nabla_0 = \nabla$ and $\nabla_1 = \nabla'$. It is easy to see that the corresponding functions $s_1 : J^1(M) \to J^\infty(M)$ depend smoothly on t, and so do the corresponding G_1-equivariant functions $F_t : J^\infty(M) \to G_1$. Hence, for each $\omega \in C_c^p(L_1, A)^{\mathrm{GL}_N(\mathbb{R})}$ the differential forms $F_1^* \tilde{\omega} - F_0^* \tilde{\omega}$ are equivariantly exact, *i.e.*, the differential of a G^∞-equivariant A-valued $(p-1)$-form (*cf.* [22], p.143). This implies the assertion. □

Remark C.4. — We also note that the 1-form $\delta(F) \in \Omega^1(J^\infty(M), L_1)$ satisfies the Maurer–Cartan equation, hence defines a crossed homomorphism

$$\delta(F) : \mathcal{V}(J^\infty(M)) \to C^\infty(J^\infty(M), L_1)$$

(Proposition 1.8). Composing with the homomorphism $\gamma^\infty : \mathcal{V}_M \to \mathcal{V}_{J^\infty(M)}$, we thus obtain a crossed homomorphism

$$\delta(F) \circ \gamma^\infty : \mathcal{V}_M \to C^\infty(J^\infty(M), L_1)^{G^\infty}.$$

That $\delta(F) \circ \gamma^\infty$ maps into G^∞-equivariant functions is due to (C.3) and the G^∞-equivariance of F, resulting from (C.4). We further note that any element $\omega \in C^p_c(L_1, A)^{\mathrm{GL}_N(\mathbb{R})}$ defines by composition a $\mathcal{V}_M$-equivariant p-cochain of the Lie algebra $C^\infty(J^\infty(M), L_1)^{G^\infty}$ with values in $C^\infty(J^\infty(M), A)^{G^\infty} \cong \Gamma(\mathcal{A})$. Now Theorem 1.7 shows that pulling back with the crossed homomorphism $\delta(F) \circ \gamma^\infty$ yields $\Gamma(\mathcal{A})$-valued Lie algebra cochains and that this is compatible with the mutual differentials.

Example C.5. — For $A = \Lambda^k(V')$, $V = \mathbb{R}^N$, and the canonical representation of $G^1 \cong \mathrm{GL}_N(\mathbb{R}) \cong G^\infty/G_1$ on this space, we obtain for the space of smooth sections $\Gamma(\mathcal{A}) \cong \Omega^k_M$, so that we get a morphism of cochain complexes

$$\gamma_A : C^\bullet_c(L_1, \Lambda^k(V'))^{\mathrm{GL}_N(\mathbb{R})} \to C^\bullet_c(\mathcal{V}_M, \Omega^k_M).$$

In this case the bundle $\mathcal{A} = \Lambda^k(T^*(M))$ is associated to the frame bundle $J^1(M)$ which creates a simpler picture than working with the infinite-dimensional manifold $J^\infty(M)$.

Example C.6. — If $A = \mathbb{R}$ is the trivial module, we obtain in particular $\mathcal{F}_M$-valued cocycles on $\mathcal{V}_M$ from any map

$$(\gamma^k)^* : (\Omega^p_{J^k(M)})^{G^k} \to C^p_c(\mathcal{V}(M), \mathcal{F}_M).$$

Here is a concrete example: For $k = 1$ we consider the 1-form $\omega \in \Omega^1(J^1(M), \mathbb{R})^{\mathrm{GL}_N(\mathbb{R})}$ defined as follows. From the homomorphism

$$\chi : \mathrm{GL}_N(\mathbb{R}) \to \mathbb{R}^\times_+, \quad g \mapsto |\det(g)|$$

we obtain an associated bundle $J^1(M) \times_\chi \mathbb{R}^\times_+$, and since $\mathbb{R}^\times_+$ is contractible, this bundle has a global section (we could also take $\log \circ \chi$ and obtain an affine bundle), which means that there is a smooth function

$$F : J^1(M) \to \mathbb{R}^\times_+ \text{ with } F(\alpha.g) = F(\alpha)|\det(g)| \text{ for } g \in \mathrm{GL}_N(\mathbb{R}), \alpha \in J^1(M).$$

Then

$$\delta(F) \in \Omega^1(J^1(M), \mathbb{R})^{\mathrm{GL}_N(\mathbb{R})}$$

is a $\mathrm{GL}_N(\mathbb{R})$-invariant 1-form.

If μ is a volume form on M, then we can construct F directly from μ by

$$F(v_1, \dots, v_N) := |\mu_m(v_1, \dots, v_N)|$$

for any basis $(v_1, \dots, v_N)$ on $T_m(M)$.

For a diffeomorphism $\varphi \in \mathrm{Diff}(M)$, we then have

$$\begin{aligned}(\varphi.F)(v_1, \dots, v_N) &= F(T(\varphi^{-1}).(v_1, \dots, v_N)) = |\mu(T(\varphi^{-1}).(v_1, \dots, v_N))| \\ &= |(\varphi^{-1})^*\mu(v_1, \dots, v_N)|.\end{aligned}$$

Dividing by $F(v_1, \dots, v_N)$, we obtain a smooth function

$$(\varphi.F)F^{-1} \in C^\infty(M, \mathbb{R}_+^\times),$$

and passing to the Lie derivative, we obtain for each vector field X on M a smooth function

$$(\mathcal{L}_X.F)F^{-1} \in C^\infty(M, \mathbb{R}).$$

Now we turn to Tsujishita's construction of the homomorphism

$$H_c^\bullet(L^1, \Lambda^k(V'))^{\mathrm{GL}(V)} \to H_c^\bullet(\mathcal{V}_M, \Omega_M^k)$$

in term of the cocycles $\mathcal{L}_X\nabla$ associated to affine connections (Lemma 2.1).

Let ∇ be an affine connection on M. As in Lemma C.1, we obtain from ∇ a smooth section

$$s : J^1(M) \to J^2(M), \quad s(\alpha) := j_0^2(\mathrm{Exp}_m^\nabla \circ \alpha) = j_0^2(\mathrm{Exp}_m^\nabla) \circ \alpha.$$

From $s(\alpha g) = s(\alpha)g$ for $\beta \in \mathrm{GL}(V)$ it follows that s is $\mathrm{GL}(V)$-equivariant.

Let $F : J^2(M) \to G_1/G_2 \cong \mathrm{Sym}^2(V, V)$ denote the unique smooth G^1-equivariant smooth function vanishing on $s(J^1(M))$. Identifying $\mathrm{Sym}^2(V, V) \cong L_1/L_2$ with the corresponding subspace of $V' \otimes \mathfrak{gl}(V) \cong (V' \otimes V') \otimes V$ corresponds to composition with the map $\mathrm{ev}_0\, dJ : L_1 \to V' \otimes \mathfrak{gl}(V)$.

The map

$$\delta(F) \circ \gamma^2 : \mathcal{V}_M \to C^\infty(J^2(M), L_1/L_2)^{G^2}$$

is a crossed homomorphism (Proposition 1.8), and since L_1/L_2 is abelian, it is a 1-cocycle. Composing with $\mathrm{ev}_0\, dJ : L_1/L_2 \to V' \otimes \mathfrak{gl}(V)$, we thus get a 1-cocycle

$$\begin{aligned}\mathrm{ev}_0\, dJ\delta(F) \circ \gamma^2 : \mathcal{V}_M &\to C^\infty(J^2(M), V' \otimes \mathfrak{gl}(V))^{G^2} \\ &\cong C^\infty(J^1(M), V' \otimes \mathfrak{gl}(V))^{\mathrm{GL}(V)} \cong \Omega^1(M, \mathrm{End}(TM)).\end{aligned}$$

The following theorem is the link between the approaches in [32] and [22].

Theorem C.7. — *If ∇ is torsion free, then*

$$\mathrm{ev}_0\, dJ\delta(F)(\gamma^2(X)) = \mathcal{L}_X\nabla \in \Omega^1(M, \mathrm{End}(TM)).$$

Proof. — Since this is a local assertion, it suffices to assume that M is an open subset of $V \cong \mathbb{R}^N$. Then we write

$$\nabla_X Y = dY(X) + \Gamma(X,Y)$$

for a symmetric $(1,2)$-tensor Γ (recall that Γ is symmetric if and only if ∇ is torsion free). We then identify

$$J^1(M) = M \times \mathrm{GL}(V) \text{ and } J^2(M) = M \times G^2 \cong M \times (\mathrm{Sym}^2(V,V) \rtimes \mathrm{GL}(V)).$$

Here we write elements of G^2 as $\alpha = \alpha_1 + \alpha_2$ with $\alpha_1 \in \mathrm{GL}(V)$ linear and α_2 quadratic. Then the group structure of G^2 is given by

$$\alpha \bullet \beta = \alpha_1\beta_1 + (\alpha_1\beta_2 + \alpha_2\beta_1) = \alpha_1\beta + \alpha_2\beta_1 = \alpha\beta_1 + \alpha_1\beta_2.$$

From this we see that

$$\alpha^{-1} = \alpha_1^{-1} - \alpha_1^{-1}\alpha_2\alpha_1^{-1}. \tag{C.8}$$

To describes the section $s : J^1(M) \to J^2(M)$ explicitly, let $\gamma_v(t) = \mathrm{Exp}^\nabla(tv)$ be the geodesic with $\gamma_v(0) = m$ and $\gamma_v'(0) = v$ (which is defined for t sufficiently close to 0). Then the relation

$$\gamma_v''(t) = -\Gamma(\gamma_v'(t), \gamma_v'(t))$$

leads to $\gamma_v''(0) = -\Gamma(v,v)$, so that

$$j_0^2(\mathrm{Exp}_m^\nabla)(v) = v + \tfrac{1}{2}\gamma''(0) = v - \tfrac{1}{2}\Gamma(v,v).$$

We thus obtain

$$s(m,\alpha_1) = (m, (\mathbf{1} - \tfrac{1}{2}\Gamma)\alpha_1).$$

Accordingly, we find

$$F(m,\alpha) = \alpha_1^{-1}(\alpha + \tfrac{1}{2}\Gamma \circ \alpha_1) = \mathbf{1} + \alpha_1^{-1}\alpha_2 + \tfrac{1}{2}\alpha_1^{-1}\Gamma \circ \alpha_1$$

because

$$F(m, \alpha(\mathbf{1} + \beta_2)) = F(m, \alpha + \alpha_1\beta_2) = F(m,\alpha) + \beta_2 = F(m,\alpha).\beta_2$$

and

$$F(m, \alpha\beta_1) = \beta_1^{-1}F(m,\alpha)\beta_1.$$

From the relation

$$(\beta \bullet \alpha)_1^{-1}(\beta \bullet \alpha)_2 = \alpha_1^{-1}\beta_1^{-1}(\beta_1\alpha_2 + \beta_2\alpha_1) = \alpha_1^{-1}\alpha_2 + \alpha_1^{-1}\beta_1^{-1}\beta_2\alpha_1,$$

we obtain the following formula for the differential of F in $(v, \beta \bullet \alpha) \in T_{(m,\alpha)}(M \times G^2)$ with $\beta \in T_{\mathbf{1}}(G^2) \cong L_0/L_2$:

$$\begin{aligned} dF(m,\alpha)(v, \beta \bullet \alpha) &= \tfrac{1}{2}\alpha_1^{-1} d\Gamma(v) \circ \alpha_1 + \alpha_1^{-1}\beta_2\alpha_1 - \tfrac{1}{2}\alpha_1^{-1}(\beta_1.\Gamma) \circ \alpha_1 \\ &= \alpha_1^{-1}\Big(\tfrac{1}{2}d\Gamma(v) + \beta_2 - \tfrac{1}{2}(\beta_1.\Gamma)\Big) \circ \alpha_1. \end{aligned}$$

The natural lift of $X \in \mathcal{V}_M$ to $J^2(M)$ is given by

$$\gamma^2(X)(m,\alpha) = j_0^2(X \circ \alpha) = (X(m), J(X)_m \circ \alpha + \tfrac{1}{2}(d^2X)_m \circ \alpha_1)$$
$$= \Big(X(m), \big(J(X)_m + \tfrac{1}{2}(d^2X)_m\big) \bullet \alpha\Big)$$

($J(X)_m v = (dX)_m(v)$ denoting the Jacobian of X), so that

$$\gamma^2(X)(s(m,\alpha_1)) = \Big(X(m), J(X)_m \circ \alpha_1 - \tfrac{1}{2}J(X)_m\Gamma \circ \alpha_1 + \tfrac{1}{2}(d^2X)_m \circ \alpha_1\Big).$$

This leads to

$$dF(\gamma^2(X))(s(m,\alpha_1)) = \delta(F)(\gamma^2(X))(s(m,\alpha_1))$$
$$= \tfrac{1}{2}\alpha_1^{-1}\Big(d\Gamma(X(m)) + (d^2X)_m - J(X)_m.\Gamma\Big) \circ \alpha_1.$$

This is a $\mathrm{GL}(V)$-equivariant function $J^1(M) \to \mathrm{Sym}^2(V,V)$, so that the corresponding $(1,2)$-tensor field is given in the canonical coordinates by the smooth function $M \to \mathrm{Sym}^2(V,V)$ given by

$$m \mapsto dF(\gamma^2(X))(s(m,\mathbf{1})) = \tfrac{1}{2}\big(d\Gamma(X(m)) + (d^2X)_m - J(X)_m\Gamma\big).$$

Suppose that $\beta : V \to V$ is a quadratic map and $\tilde{\beta}$ the corresponding symmetric bilinear map determined by $\beta(v) = \frac{1}{2}\tilde{\beta}(v,v)$. Considering β as an element of L_0/L_2, we have

$$J(\beta)(v) = (d\beta)_v = \tilde{\beta}(v,\cdot) \in \mathfrak{gl}(V)$$

and thus

$$(dJ(\beta))_0(v)v = \mathrm{ev}_0(dJ(\beta))(v)(v) = \tilde{\beta}(v,v) = 2\beta(v).$$

Applying $\mathrm{ev}_0 \circ dJ$ to the smooth function above thus leads to

$$\mathrm{ev}_0\, dJ\delta(F)(\gamma^2(X))(s(m,\mathbf{1})) = d\Gamma(X(m)) + (d^2X)_m - J(X)_m\Gamma \in V' \otimes \mathfrak{gl}(V).$$

In view of Remark 2.2, it remains to show that

$$\mathcal{L}_X\Gamma = d\Gamma(X) - J(X).\Gamma.$$

It suffices to verify this relation by evaluating on constant vector fields Y and $Z \in V$:

$$(\mathcal{L}_X\Gamma)(Y,Z) = [X,\Gamma(Y,Z)] - \Gamma([X,Y],Z) - \Gamma(Y,[X,Z])$$
$$= (d\Gamma(X))(Y,Z) - dX(\Gamma(Y,Z)) + \Gamma(dX(Y),Z) + \Gamma(Y,dX(Z))$$
$$= (d\Gamma(X))(Y,Z) - (J(X).\Gamma)(Y,Z).$$

□

Now we can identify the image of the $\Lambda^k(V')$-valued cocycle Ψ_k^L (Theorem B.2) under Tsujishita's map $\gamma_{\Lambda^k(V')}$, described in Proposition C.3, which maps it into an element of $Z_c^k(\mathcal{V}_M, \Omega_M^k)$.

THEOREM C.8. — *For each torsion free connection* ∇ *on the manifold* M*, we have*

$$\gamma_{\Lambda^k(V')}(\Psi_k^L) = \Psi_k.$$

Proof. — We first observe that

$$\Psi_k^L = (\mathrm{ev}_0\, dJ)^* f_k, \qquad \text{where} \qquad f_k : (V' \otimes \mathrm{End}(V))^k \to \Lambda^k(V')$$

is given by

$$((\alpha_1 \otimes A_1), \dots, (\alpha_k \otimes A_k)) \mapsto \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, \mathrm{Tr}(A_{\sigma(1)} \cdots A_{\sigma(k)}) \alpha_{\sigma(1)} \wedge \cdots \wedge \alpha_{\sigma(k)}$$
$$= \Big(\sum_{\sigma \in S_k} \mathrm{Tr}(A_{\sigma(1)} \cdots A_{\sigma(k)}) \Big) \cdot \alpha_1 \wedge \cdots \wedge \alpha_k.$$

In this sense, we have $f_k = m_k \otimes \beta_k$, where $m_k : (V')^{\otimes k} \to \Lambda^k(V')$ is the alternating multiplication map and $\beta_k : \mathrm{End}(V)^k \to \mathbb{R}$ is the symmetric k-linear map defined by

$$\beta_k(A_1, \dots, A_k) = \sum_{\sigma \in S_k} \mathrm{Tr}(A_{\sigma(1)} \cdots A_{\sigma(k)}).$$

According to Proposition C.3, we have for $X_1, \dots, X_k \in \mathcal{V}_M$:

$$\gamma_{\Lambda^k(V')}(\Psi_k^L)(X_1, \dots, X_k) = (F^* \tilde{\Psi}_k^L)(\gamma^2(X_1), \dots, \gamma^2(X_k)),$$

where we identify

$$\Omega_M^k \cong C^\infty(J^\infty(M), \Lambda^k(V'))^{G^\infty} \cong C^\infty(J^2(M), \Lambda^k(V'))^{G^2}$$
$$\cong C^\infty(J^1(M), \Lambda^k(V'))^{\mathrm{GL}(V)}.$$

This identification is based on the canonical map

$$J^\infty(M) \times \Lambda^k(V') \to \Lambda^k(T^*(M)), \quad ([\alpha], \omega) \mapsto \omega \circ \alpha_1^{-1}$$

which is constant on the diagonal G^∞-action and factors through the corresponding map on $J^1(M)$, resp., $J^2(M)$.

Since the cocycles Ψ_k^L factor through the quotient algebra L_0/L_2, the map $\gamma_{\Lambda^k(V')}$ can be constructed with $J^2(M)$ instead of $J^\infty(M)$ and to pass from G^2-equivariant smooth functions on $J^2(M)$ to $\mathrm{GL}(V)$-equivariant smooth functions on $J^1(M)$, we may simply restrict to $s(J^1(M))$, the **1**-level set of the function F. We thus obtain with Theorem C.7 for any torsion free connection ∇:

$$\begin{aligned}\gamma_{\Lambda^k(V')}(\Psi_k^L)(X_1,\ldots,X_k) &= (F^*\tilde{\Psi}_k^L)(\gamma^2(X_1),\ldots,\gamma^2(X_k))\\ &= (-1)^k\Psi_k^L(\delta(F)(\gamma^2(X_1)),\ldots,\delta(F)(\gamma^2(X_k)))\\ &= (-1)^k f_k\Big(\operatorname{ev}_0 dJ\delta(F)(\gamma^2(X_1)),\ldots,\operatorname{ev}_0 dJ\delta(F)(\gamma^2(X_k))\Big)\\ &= (-1)^k f_k(\mathcal{L}_{X_1}\nabla,\ldots,\mathcal{L}_{X_k}\nabla)\\ &= \Psi_k(X_1,\ldots,X_k),\end{aligned}$$

where we identify $\mathcal{L}_X\nabla$ with an element of $\Omega^1(M,\mathrm{End}(TM)) \cong C^\infty(J^1(M), V'\otimes\mathrm{End}(V))^{\mathrm{GL}(V)}$ and use $\Omega^k_M \cong C^\infty(J^1(M),\Lambda^k(V'))^{\mathrm{GL}(V)}$. □

Proof of Theorem 3.1. — We choose an affine torsion free connection ∇. From the formulation of Tsujishita's Theorem in [32], Thm. 5.1.6, we get (with $V=\mathbb{R}^N$) an isomorphism of bigraded algebras:

$$H_c^\bullet(\mathcal{V}_M,\Omega_M^\bullet)\cong H_c^\bullet(\mathcal{V}_M,\mathcal{F}_M)\otimes H_c^\bullet(L_0,\mathfrak{gl}_N(\mathbb{R}),\Lambda^\bullet(V')).$$

We further know from Theorem B.2, that $H_c^\bullet(L_0,\mathfrak{gl}_N(\mathbb{R}),\Lambda^\bullet(V'))$ is generated by the classes of the cocycles Ψ_k^L. Hence it suffices to observe with Theorem C.8 that Tsujishita's map

$$\gamma_{\Lambda^k(V')}: C_c^\bullet(L_1,\Lambda^k(V'))^{\mathrm{GL}_N(\mathbb{R})}\cong C_c^\bullet(L_0,\mathfrak{gl}_N(\mathbb{R}),\Lambda^k(V'))\to C_c^\bullet(\mathcal{V}_M,\Omega_M^k)$$

(Proposition C.3) maps Ψ_k^L to Ψ_k. □

Appendix D. Cohomology of vector fields with values in the trivial module

Here we shall review the results of Haefliger [19].

We begin by recalling the definition of the Weil algebra $\widehat{W}_N$. Let $E(u_1,\ldots,u_N)$ be the exterior algebra in the generators $u_1,\ldots,u_N$, and let $\mathbb{R}[c_1,\ldots,c_N]$, be the polynomial algebra in the generators $c_1,\ldots,c_N$. Introduce gradings on these algebras by assigning degrees to the generators:

$$\deg(u_k)=2k-1 \quad\text{and}\quad \deg(c_k)=2k.$$

Consider the quotient $\widehat{\mathbb{R}}[c_1,\ldots,c_N]$ of the polynomial algebra $\mathbb{R}[c_1,\ldots,c_N]$ by the ideal spanned by the elements of degrees exceeding $2N$. The *Weil algebra* is the differential graded algebra

$$\widehat{W}_N = E(u_1,\ldots,u_N)\otimes\widehat{\mathbb{R}}[c_1,\ldots,c_N]$$

with the differential defined by

$$d(u_k)=c_k \quad\text{and}\quad d(c_k)=0.$$

Let $H^\bullet(\widehat{W}_N)$ be the cohomology of this differential graded algebra.

Gelfand–Fuks used the Weil algebra to describe the cohomology of the Lie algebra W_N of formal vector fields in dimension N:

Theorem D.1 ([16]). — *$H_c^\bullet(W_N, \mathbb{R}) \cong H^\bullet(\widehat{W}_N)$.*

The explicit description of the cohomology of the Weil algebra was given by Vey:

Theorem D.2 ([18]). — *$H^\bullet(\widehat{W}_N)$ is spanned by 1 and the cocycles*

$$(u_{i_1} \wedge \ldots \wedge u_{i_r}) \otimes (c_{j_1} \ldots c_{j_s}) \tag{D.1}$$

satisfying the following conditions: $1 \leqslant i_1 < \ldots < i_r \leqslant N$; $1 \leqslant j_1 \leqslant \ldots \leqslant j_s \leqslant N$; $i_1 \leqslant j_1$; $r > 0$; $j_1 + \ldots + j_s \leqslant N$; $i_1 + j_1 + \ldots + j_s > N$.

Combining the preceding two theorems, we derive:

Corollary D.3.

(a) *$H^q(W_N, \mathbb{R}) = 0$ for $1 \leqslant q \leqslant 2N$ and for $q \geqslant (N+1)^2$.*
(b) *$\dim H^{2N+1}(W_N, \mathbb{R}) = p(N+1) - 1$, where p is the partition function.*
(c) *$\dim H^{2N+2}(W_N, \mathbb{R}) = 0$.*

Proof. — (a) The degree of the generator (D.1) is at least $2i_1 - 1 + 2(j_1 + \ldots + j_s) > 2N - 1$, and since it is odd, it is at least $2N + 1$. On the other hand, it is at most

$$1 + 3 + \ldots + (2N-1) + 2(j_1 + \ldots + j_s) \leqslant N^2 + 2N < (N+1)^2.$$

(b) If the degree is exactly $q = 2N + 1$, then the argument in (a) implies

$$2i_1 - 1 + 2(j_1 + \ldots + j_s) = 2N + 1,$$

which leads to $r = 1$ and $i_1 + j_1 + \ldots + j_s = N + 1$. Only the partition

$$N + 1 = (N+1) + 0 + \ldots + 0$$

does not contribute, and this proves (b).

(c) If the degree q of the generator in (D.1) is even and greater than N, then $j_1 + \ldots + j_s \leqslant N$ implies $r \geqslant 2$ and hence

$$q = (2i_1 - 1) + (2i_2 - 1) + 2(j_1 + \ldots + j_s) > 2N - 1 + 3 = 2N + 2.$$

□

Let $\widetilde{W}(N)$ be the graded vector space

$$\widetilde{W}(N) = \bigoplus_{q=2N}^{(N+1)^2-2} \widetilde{W}(N)_q,$$

where $\dim \widetilde{W}(N)_q = \dim H^{q+1}(\widehat{W}_N)$. We view $\widetilde{W}(N)$ as a super vector space, where the parity of $\widetilde{W}(N)_q$ is equal to the parity of q.

Let $\mathcal{L}_N$ be the free Lie superalgebra generated by $\widetilde{W}(N)$ (see *e.g.*, [3] for the description of the basis of a free Lie superalgebra). The $\mathbb{Z}$-grading on $\widetilde{W}(N)$ extends to a $\mathbb{Z}$-grading of $\mathcal{L}_N$.

This Lie superalgebra was introduced by Haefliger [19] in order to give the description of the cohomology of the Lie algebra of smooth vector fields on a manifold with coefficients in the trivial module.

Let $\tilde{V}$ be a graded vector space obtained from $\mathcal{L}_N$ by a shift in the grading: $\tilde{V}_q = (\mathcal{L}_N)_{q-1}$. Since the grading of $\mathcal{L}_N$ starts at component $2N$, the grading of $\tilde{V}$ starts at component $2N+1$.

Let $\Omega^\bullet_M$ be the differential graded algebra of smooth differential forms on M, and consider the tensor product space $\Omega^\bullet_M \otimes \tilde{V}$ with the grading

$$\deg(\omega \otimes v) = \deg v - \deg \omega, \quad \omega \in \Omega^\bullet_M, v \in \tilde{V}.$$

Since Ω^p_M vanishes for $p > N$, the grading on the space $\Omega^\bullet_M \otimes \tilde{V}$ starts at degree $N+1$. Next we consider the graded algebra $S^\bullet(\Omega^\bullet_M \otimes \tilde{V})$ of supersymmetric multilinear forms on the graded superspace $\Omega^\bullet_M \otimes \tilde{V}$. We have

$$S^\bullet(\Omega^\bullet_M \otimes \tilde{V}) = \mathop{\oplus}_{p=0}^{\infty} S^p(\Omega^\bullet_M \otimes \tilde{V}),$$

where $S^p(\Omega^\bullet_M \otimes \tilde{V})_q = 0$ for $q < (N+1)p$. Hence, apart from the component of degree 0, the grading on $S^\bullet(\Omega^\bullet_M \otimes \tilde{V})$ starts in degree $N+1$.

Theorem D.4 ([19], Theorem A). — *Let M be an N-dimensional smooth manifold. On the graded algebra $S^\bullet(\Omega^\bullet_M \otimes \tilde{V})$ there exists a differential of degree 1, which depends on a choice of the representatives of the Pontryagin classes in $\Omega^\bullet_M$, and a homomorphism of differential graded algebras*

$$S^\bullet(\Omega^\bullet_M \otimes \tilde{V}) \to C^\bullet_c(\mathcal{V}_M, \mathbb{R}),$$

which induces an isomorphism in cohomology.

Corollary D.5. — $H^s_c(\mathcal{V}_M, \mathbb{R}) = 0$ *for* $1 \leqslant s \leqslant N$.

If all Pontryagin classes of M vanish and there is a splitting algebra homomorphism $H^\bullet_M \hookrightarrow \Omega^\bullet_M$ (which is, for example, the case when M is a torus or a compact Lie group), Haefliger gives a more explicit realization for the cohomology of $\mathcal{V}_M$, which we described now.

Let $H^\bullet_M$ be the cohomology algebra of the manifold M. We view $H^\bullet_M$ as a commutative superalgebra and form the graded Lie superalgebra

$$\mathcal{L}(M) = H^\bullet(M, \mathbb{R}) \otimes \mathcal{L}_N$$

with the bracket of homogeneous elements given by

$$[\omega \otimes x, \omega' \otimes x'] = (-1)^{\deg x \cdot \deg \omega'} \omega\omega' \otimes [x, x'],$$

and the grading by

$$\mathcal{L}(M)_r = \sum_{p-q=r} H_M^p \otimes (\mathcal{L}_N)_q.$$

It follows in particular that $\mathcal{L}(M)_q = 0$ for $q < N$.

Consider the cohomology $H^\bullet(\mathcal{L}(M))$ of the Lie superalgebra $\mathcal{L}(M)$ (see [15], Sect. 1.6.3 for the definition of the cohomology for the Lie superalgebras, *cf.* Sect. 1.2 in [19]). Since $\mathcal{L}(M)$ is graded, its cohomology inherits a grading $H^p(\mathcal{L}(M), \mathbb{R}) = \oplus_q H^p(\mathcal{L}(M), \mathbb{R})_q$ (see [15], Sect. 1.3.7). Since the grading of $\mathcal{L}(M)$ starts at degree N, we get that

$$H^p(\mathcal{L}(M), \mathbb{R})_q = 0 \quad \text{for } q < p \cdot N. \tag{D.2}$$

Theorem D.6 ([19], Theorem 3.4). — *Let M be a smooth manifold with a finite-dimensional cohomology $H_M^\bullet$ for which all Pontryagin classes vanish. Suppose, moreover, that there exists a homomorphism of differential graded algebras $H_M^\bullet \to \Omega_M^\bullet$, which induces an isomorphism in cohomology (this is the case if M is a compact Lie group). Then*

$$H_c^s(\mathcal{V}_M, \mathbb{R}) = \bigoplus_{p+q=s} H^p(\mathcal{L}(M), \mathbb{R})_q.$$

Corollary D.7. — *Under the assumptions of the preceding theorem, we have*

(a) $H_c^s(\mathcal{V}_M, \mathbb{R}) = 0$ *for* $1 \leqslant s \leqslant N$.

(b) *For* $N+1 \leqslant s \leqslant 2N+1$,

$$\dim H_c^s(\mathcal{V}_M, \mathbb{R}) = \sum_{k=0}^{N} \dim H_M^k \cdot \dim \widetilde{W}(N)_{s+k-1}.$$

(c) $\dim H_c^{N+1}(\mathcal{V}_M, \mathbb{R}) = p(N+1) - 1$ *and only* $H_M^N \cong \mathbb{R}$ *contributes in the decomposition above.*

(d) *If* $b_1(M) := \dim H_M^1 = \dim H_M^{N-1}$, *then* $\dim H_c^{N+2}(\mathcal{V}_M, \mathbb{R}) = b_1(M) \cdot (p(N+1) - 1)$ *and only* $H_M^{N-1} \cong \mathbb{R}^{b_1(M)}$ *contributes in the decomposition.*

Proof. — (a) If $H^p(\mathcal{L}(M), \mathbb{R})_q$ is non-zero, then $q \geqslant pN$, so that (D.2) implies $s = p + q \geqslant (N+1)p \geqslant N+1$ if $s \neq 0$.

(b) We note that (D.2) implies $H_c^s(\mathcal{V}_M, \mathbb{R}) \cong H^1(\mathcal{L}(M), \mathbb{R})_{s-1}$ for $N + 1 \leqslant s \leqslant 2N+1$. However,

$$H^1(\mathcal{L}(M), \mathbb{R}) = (\mathcal{L}(M)/[\mathcal{L}(M), \mathcal{L}(M)])^* = \left(H_M^\bullet \otimes \widetilde{W}(N)\right)^*$$

and the claim (b) follows. □

If M is an N-dimensional torus $\mathbb{T}^N$, its cohomology $H^\bullet(\mathbb{T}^N)$ is an exterior algebra in N generators of degree 1. Combining the two previous corollaries, we get with $b_1(\mathbb{T}^N) = N$:

Corollary D.8. — *For the torus $T = \mathbb{T}^N$, we have:*

(a) $H_c^s(\mathcal{V}_T, \mathbb{R}) = 0$ *for* $1 \leqslant s \leqslant N$.
(b) $\dim H_c^{N+1}(\mathcal{V}_T, \mathbb{R}) = p(N+1) - 1$.
(c) $\dim H_c^{N+2}(\mathcal{V}_T, \mathbb{R}) = N \cdot (p(N+1) - 1)$.

Remark D.9. — For $M = \mathbb{S}^n$, the algebra $H^\bullet(\mathcal{V}_{\mathbb{S}^n}, \mathbb{R})$ is calculated by Cohen and Taylor in [10]; Theorems 1.3 and 3.2.

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