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RESURGENCE OF THE EULER-MACLAURIN SUMMATION FORMULA

by Ovidiu COSTIN & Stavros GAROUFALIDIS (*)

ABSTRACT. — The Euler-MacLaurin summation formula compares the sum of a function over the lattice points of an interval with its corresponding integral, plus a remainder term. The remainder term has an asymptotic expansion, and for a typical analytic function, it is a divergent (Gevrey-1) series. Under some decay assumptions of the function in a half-plane (resp. in the vertical strip containing the summation interval), Hardy (resp. Abel-Plana) prove that the asymptotic expansion is a Borel summable series, and give an exact Euler-MacLaurin summation formula.

Using a mild resurgence hypothesis for the function to be summed, we give a Borel summable transseries expression for the remainder term, as well as a Laplace integral formula, with an explicit integrand which is a resurgent function itself. In particular, our summation formula allows for resurgent functions with singularities in the vertical strip containing the summation interval.

Finally, we give two applications of our results. One concerns the construction of solutions of linear difference equations with a small parameter. Another concerns resurgence of 1-dimensional sums of quantum factorials, that are associated to knotted 3-dimensional objects.

RÉSUMÉ. — La formule sommatoire d'Euler-MacLaurin exprime la somme d'une fonction sur un réseau de points d'un intervalle comme l'addition de l'intégrale correspondante et d'un reste. Dans les cas typiques, ce reste est donné par une série asymptotique divergente du type Gevrey-1. Sous des hypothèses adéquates de décroissance de la fonction dans le demi-plan supérieur ou sur une bande verticale contenant l'intervalle de sommation, Hardy et Abel-Plana ont prouvé que cette série asymptotique est Borel sommable. Supposant que la fonction à resommer est résurgente, notre théorème principal fournit une expression, pour le reste, à la fois sous forme d'une trans-série Borel sommable et, à la fois, sous forme d'une transformée de Laplace dont l'intégrand est explicite et lui-même donné par une fonction résurgente. Notre résultat s'applique au problème d'existence de solutions d'équations différentielles linéaires avec petit paramètre, ainsi qu'à celui de la résurgence des sommes unidimensionelles de factorielles quantiques associées à des objets noués en dimension 3.

Keywords: Euler-MacLaurin summation formula, Abel-Plana formula, resurgence, resurgent functions, Bernoulli numbers, Borel transform, Borel summation, Laplace transform, transseries, parametric resurgence, co-equational resurgence, WKB, difference equations with a parameter, Stirling's formula, Quantum Topology.
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1. Introduction

1.1. The Euler-MacLaurin summation formula

The *Euler-MacLaurin summation formula* relates summation to integration in the following way (see [21, Sec.8]):

$$\sum_{k=1}^{N} f\left(\frac{k}{N}\right) = N\int_0^1 f(s)ds + \frac{1}{2}(f(1)-f(0)) + R(f,N) \tag{1.1}$$

where the remainder $R(f,N)$ has an asymptotic expansion

$$R(f,N) \sim \hat{R}(f,N) \tag{1.2}$$

in the sense of Poincaré, where

$$\hat{R}(f,N) = \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}\left(f^{(2n-1)}(1) - f^{(2n-1)}(0)\right)\frac{1}{N^{2n-1}} \in \mathbb{C}[[N^{-1}]]. \tag{1.3}$$

and B_m are the *Bernoulli numbers* defined by the generating series

$$\frac{p}{e^p-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}p^n. \tag{1.4}$$

Typically, the formal power series $\hat{R}(f,N)$ is divergent and Gevrey-1, due to the fact that the n-th derivative in (1.3) is not divided by an $n!$. In the present paper, we discuss an exact form of the Euler-MacLaurin summation formula, under a resurgence hypothesis of the function $f(x)$; see Proposition 1.4 below.

1.2. Two applications of our exact Euler-MacLaurin summation formula

Our exact form of the Euler-MacLaurin summation formula has two applications: in *Quantum Topology* (where one sometimes needs to apply the Euler-MacLaurin summation formula to a resurgent function that has singularities in the vertical strip which is perpendicular to the range of summation), and in Borel summability (with respect to ϵ) of difference equations with a small ϵ-parameter. Let us discuss these applications.

Consider a triple $\mathfrak{t} = (a,b,\epsilon)$ where $a,b \in \mathbb{Z}$, $b>0$, $\epsilon = \pm 1$ and the expression

$$I_{\mathfrak{t}}(q) = \sum_{n=0}^{\infty} q^{a\frac{n(n+1)}{2}}(q)_n^b \epsilon^n \tag{1.5}$$

where $(q)_n$ is the *quantum factorial* defined by:

$$(q)_n = \prod_{k=1}^{n} (1 - q^k), \qquad (q)_0 = 1. \tag{1.6}$$

Although $I_{\mathfrak{t}}(q)$ does not makes sense when q is inside or outside the unit disk, it does makes sense when

(a) q is a complex root of unity; in that case $I_{\mathfrak{t}}(q) \in \mathbb{C}$.
(b) $q = e^{1/x}$; in that case $I_{\mathfrak{t}}(q) \in \mathbb{Q}[[1/x]]$.

Given $\mathfrak{t}$ as above, consider the power series:

$$L^{\mathrm{NP}}_{\mathfrak{t}}(p) = 1 + \sum_{n=0}^{\infty} I_{\mathfrak{t}}(e^{\frac{2\pi i}{n}}) p^n \tag{1.7}$$

$$L^{\mathrm{P}}_{\mathfrak{t}}(p) = \mathcal{B}(I_{\mathfrak{t}}(e^{1/x})) \tag{1.8}$$

where $\mathcal{B}$ is the Borel transform defined below in Definition 1.2. Our present results, together with [11] and additional arguments, imply the following theorem, which will be presented in detail in a forthcoming publication [10].

Theorem 1.1. — [10] *For all $\mathfrak{t}$ as above, the power series $L^{\mathrm{NP}}_{\mathfrak{t}}(p)$ and $L^{\mathrm{P}}_{\mathfrak{t}}(p)$ are resurgent functions.*

In particular, it follows that the generating series $L^{\mathrm{NP}}_{\mathfrak{t}}(p)$ of the Kashaev invariants of the two simplest knots, the 3_1 and 4_1 (corresponding to $\mathfrak{t} = (0, 1, 1)$ and $\mathfrak{t} = (-1, 2, -1)$) are resurgent functions.

Another application of our exact Euler-MacLaurin formula is to prove parametric resurgence (i.e., resurgence with respect to ϵ) of a formal (WKB) solution to a linear difference equation with a small parameter:

$$y(x + \epsilon, \epsilon) = a(x, \epsilon) y(x, \epsilon) \tag{1.9}$$

The formal solution of (1.9) is of the form:

$$y(x, \epsilon) = \exp\left(\frac{1}{\epsilon} \sum_{k=0}^{\infty} F_k(x) \epsilon^k \right). \tag{1.10}$$

Under suitable hypothesis on $a(x, \epsilon)$, Theorem 4.1 below proves resurgence of the above series for x fixed, and constructs an actual solution to (1.9) which is asymptotic to the formal solution (1.10).

1.3. Known forms of the Euler-MacLaurin summation formula

Before we state our results, let us recall what is already known. Suppose that f satisfies the following assumption:

f is analytic and satisfies the following bound:

$$f(x) = O(|x|^{-s}) \tag{1.11}$$

for some $0 < \delta < 1$ and $s > 0$, uniformly in the right-half plane $\Re(x) \geqslant \delta$.

For such functions f, Hardy proved in [18, Sec.13.15] that $\hat{R}(f, N)$ is *Borel summable*, and that the Borel sum agrees with the original sum. In other words, the Borel tranform of $\hat{R}(f, N)$ can be extended to the ray $[0, \infty)$, it is integrable of at most exponential growth, and replacing $\hat{R}(f, N)$ with the corresponding Borel sum replaces the asymptotic relation (1.2) with an exact identity.

In a different direction, suppose that

f is continuous in the vertical strip $0 \leqslant \Re(x) \leqslant 1$, holomorphic in its interior, and $f(x) = o(e^{2\pi|\Im(x)|})$ as $|\Im(x)| \to \infty$ in the strip, uniformly with respect to $\Re(x)$.

Then, the *Abel-Plana* formula states that (see [21, Sec.8.3]):

$$\begin{aligned}\sum_{k=1}^{N} f\left(\frac{k}{N}\right) &= N\int_0^1 f(u)du + \frac{1}{2}(f(1) - f(0)) \\ &\quad + i\int_0^\infty \frac{f\left(\frac{iy}{N}\right) - f\left(1+\frac{iy}{N}\right) - f\left(-\frac{iy}{N}\right) + f\left(1-\frac{iy}{N}\right)}{e^{2\pi y} - 1} dy.\end{aligned} \tag{1.12}$$

1.4. What is a resurgent function?

The notion of a resurgent function was introduced and studied by Écalle; see [14]. For our purposes, a resurgent function is one that admits *endless analytic continuation* (expect at a countable set of non-accumulating singular points) in the complex plane, and is *exponentially bounded*, that is, satisfies an estimate:

$$|f(z)| < Ce^{a|z|} \tag{1.13}$$

for large z. Examples of resurgent functions are meromorphic functions, algebraic functions, or Borel transforms of solutions of generic differential equations with analytic coefficients. The nth coefficient of the Taylor series of a resurgent function around a regular point has a manifest asymptotic expansion with respect to $1/n$ that include small exponential corrections; see for example [9, Sec.7]. This property of resurgent functions is key in applications to quantum topology, where a main problem is to show the existence of asymptotic expansions. For example, an asymptotic expansion of the coefficients of the power series (1.3) is almost trivial (for a fixed function f). On the other hand, the existence of asymptotic expansion for the coefficients of $I_{3_1}(e^{1/x})$ and $I_{4_1}(e^{1/x})$ (or more generally, $I_{\mathfrak{t}}(e^{1/x})$) is a highly non-trivial fact that follows from the resurgence of the Borel transform of $I_{\mathfrak{t}}(e^{1/x})$; see [10].

For an introduction to resurgent functions anèjd their properties, we refer the reader to the survey articles [4, 5, 12, 13, 20, 22] and for a thorough study, the reader may consult Écalle's original work [14, 15]. Let us point out, however, that our main results (Theorems 1.3 and 3.2 below) do not require any substantial knowledge of resurgence.

1.5. Statement of the results

Let us recall a useful definition.

DEFINITION 1.2. — *The* (formal) Borel transform *of a formal power series in* $1/N$ *is a formal power series in* p *defined by:*

$$\mathcal{B} : \mathbb{C}[[N^{-1}]] \longrightarrow \mathbb{C}[[p]], \qquad \mathcal{B}\left(\sum_{n=0}^{\infty} a_n \frac{1}{N^{n+1}}\right) = \sum_{n=0}^{\infty} \frac{a_n}{n!} p^n. \tag{1.14}$$

Let $G_f(p)$ denote the Borel transform of the power series $\hat{R}(f, N)$.

THEOREM 1.3. — *If* $f(x)$ *is resurgent and* $f'(x)$ *is continuous at* $x = 0, 1$ *then* $G_f(p)$ *is given by:*

$$G_f(p) = \frac{1}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(f'\left(1 + \frac{p}{2\pi i n}\right) + f'\left(1 - \frac{p}{2\pi i n}\right) - f'\left(\frac{p}{2\pi i n}\right) - f'\left(-\frac{p}{2\pi i n}\right) \right) \tag{1.15}$$

In particular, $G_f(p)$ *is resurgent with singularities given by*

$$\mathcal{N} = \{2\pi i n \omega,\ 2\pi i n(\omega - 1) \mid n \in \mathbb{Z}^*, \omega = \text{singularity of } f'\}. \tag{1.16}$$

Let us consider a function $f(x)$ that satisfies the following:

(A1) f is resurgent with no singularities in the vertical strip $0 \leqslant \Re(x) \leqslant 1$, and $f(u) = o(e^{2\pi|\Im(u)|})$ as $|\Im(u)| \to \infty$ in the strip, uniformly with respect to $\Re(u)$.

Then, we have the following exact form of the Euler-MacLaurin summation formula.

Proposition 1.4. — *Under the hypothesis (A1), for every $N \in \mathbb{N}$ we have:*

$$\sum_{k=1}^{N} f\left(\frac{k}{N}\right) = N \int_0^1 f(s)ds + \frac{1}{2}(f(1) - f(0)) + \int_0^\infty e^{-Np} G_f(p)dp. \tag{1.17}$$

In particular, the left-hand side of the above equation is the evaluation at N of an analytic function in the right hand plane.

Our proof of Proposition 1.4 allows to generalize to the case that f is resurgent with singularities in the vertical strip $0 \leqslant \Re(x) \leqslant 1$; see Theorem 3.2 in Section 3.2. In that case, every singularity λ of f in the vertical strip gives rise to exponentially small corrections, and the right hand side of Equation (1.17) is replaced by a *transseries.*

Finally, let us give an integral formula for $G_f(p)$ which is useful in studying the behavior of $G_f(p)$ for large p.

Theorem 1.5. — *With the assumptions of Theorem 1.3 we have:*

$$G_f(p) = \frac{1}{(2\pi i)^3} \int_0^\infty \int_{\gamma_0} \frac{u}{e^u - 1} \left(\frac{f(s)}{s^2} (e^{\frac{pu}{2\pi i s}} + e^{-\frac{pu}{2\pi i s}}) - \frac{f(1+s)}{(1+s)^2} (e^{\frac{pu}{2\pi i(1+s)}} + e^{-\frac{pu}{2\pi i(1+s)}}) \right) ds du \tag{1.18}$$

where γ_0 is a small circle around 0 oriented counterclockwise.

Let us end the introduction with some remarks.

Remark 1.6. — Theorem 1.3, and especially Theorem 3.2 below provide a new construction of resurgent functions. Best known resurgent functions are those that satisfy a difference or differential equation, linear or not; see for example [2, 3, 7] and [15].

On the other hand, due to the position and shape of their singularities, the resurgent functions $G_f(p)$ of Theorem 1.3 do not seem to satisfy any differential equations with polynomial coefficients.

For example, consider the function $f(x) = (x - \omega)^{-m}$ where $\omega \notin [0, 1]$ which satisfies the linear differential equation with polynomial coefficients:

$$(x - \omega)f'(x) - mf(x) = 0.$$

f is resurgent, with only one singularity at $x = \omega$. The corresponding resurgent function $G_f(p)$ of Theorem 1.3 $G_f(p)$ has infinitely many singularities on the rays $2\pi i\omega\mathbb{R}^+$, $2\pi i\omega\mathbb{R}^-$, $2\pi i(\omega-1)\mathbb{R}^+$, $2\pi i(\omega-1)\mathbb{R}^-$. It seems unlikely that $G_f(p)$ satisfies a linear (or a nonlinear) differential equation with polynomial coefficients.

Remark 1.7. — Let us point out that Theorem 1.3 implies that the shape of the singularities of $G_f(p)$ is the same as that of $f'(x)$. For example, if $f'(x)$ is simply ramified, then so is $G_f(p)$. We recall that a resurgent function $h(x)$ is *simply ramified* if, locally, at each singularity ω of $h(x)$ we have:

$$h(x) = P\left(\frac{1}{x-\omega}\right) + \frac{1}{2\pi i}\log(x-\omega)r(x-\omega) + s(x-\omega) \tag{1.19}$$

where s, r are convergent germs, and P is a polynomial.

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2. Proof of Theorem 1.3

2.1. Computation of the Borel transform $G_f(p)$

Let $\circledast$ denote the *Hadamard product* of power series:

$$\left(\sum_{n=0}^{\infty} b_n p^n\right) \circledast \left(\sum_{n=0}^{\infty} c_n p^n\right) = \sum_{n=0}^{\infty} b_n c_n p^n. \tag{2.1}$$

It is classical, and easy to check, that the Hadamard product $A \circledast B$ of two functions $A(p)$ and $B(p)$ analytic at $p = 0$ is also given by an integral formula:

$$(A \circledast B)(p) = \frac{1}{2\pi i}\int_\gamma A(s)B\left(\frac{p}{s}\right) ds, \tag{2.2}$$

where γ is a suitable contour around the origin. For a detailed explanation of the above formula, see [19, p.302] and also [1, p.245].

Let $G_f(p)$ denote the formal Borel transform of the power series in (1.2). Since $B_m = 0$ for odd $m > 1$, we have:

$$\begin{aligned} G_f(p) &= \mathcal{B}\left(\sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}\left(f^{(2n-1)}(1) - f^{(2n-1)}(0)\right)\frac{1}{N^{2n-1}}\right) \\ &= \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}\left(f^{(2n-1)}(1) - f^{(2n-1)}(0)\right)\frac{p^{2n-2}}{(2n-2)!} \\ &= \sum_{m=2}^{\infty} \frac{B_m}{m!}\left(f^{(m-1)}(1) - f^{(m-1)}(0)\right)\frac{p^{m-2}}{(m-2)!} \\ &= \left(\sum_{m=2}^{\infty} \frac{B_m}{m!} p^{m-2}\right) \circledast \left(\sum_{m=2}^{\infty}\left(f^{(m-1)}(1) - f^{(m-1)}(0)\right)\frac{p^{m-2}}{(m-2)!}\right) \\ &= g_1(p) \circledast g_2(p) \end{aligned}$$

where

$$g_1(p) = \sum_{m=2}^{\infty} \frac{B_m}{m!} p^{m-2} = \frac{1}{p}\left(\frac{1}{e^p - 1} - \frac{1}{p} + \frac{1}{2}\right) \tag{2.3}$$

and

$$\begin{aligned} g_2(p) &= \sum_{m=2}^{\infty}\left(f^{(m-1)}(1) - f^{(m-1)}(0)\right)\frac{p^{m-2}}{(m-2)!} \\ &= \frac{d}{dp}\left(\sum_{m=2}^{\infty}\left(f^{(m-1)}(1) - f^{(m-1)}(0)\right)\frac{p^{m-1}}{(m-1)!}\right) \\ &= \frac{d}{dp}\left(f(1+p) - f(p) - f(1) + f(0)\right) = f'(1+p) - f'(p). \end{aligned} \tag{2.4}$$

Consider positive numbers r_0 and δ such that $g_1(p)$ is analytic for $|p| < r_0$ (eg, $r_0 < 2\pi$) and $g_2(p)$ is analytic for $|p| < \delta$–the latter is possible by Equation (2.4) and our assumptions on f.

Now, Equation (2.2) implies that

$$G_f(p) = \frac{1}{2\pi i}\int_\gamma g_1(s) g_2\left(\frac{p}{s}\right)\frac{ds}{s} \tag{2.5}$$

for all p with $|p| < \delta r$, where γ is a circle around 0 with radius $r < r_0$.

With our assumptions, when $|p| < \delta r$, the function $s \mapsto g_2(p/s)$ has no singularities outside of γ. Thus, outside of γ, the singularities of

$g_1(s)g_2(p/s)/s$ are simple poles at the points $2\pi in$ for $n \in \mathbb{Z}^*$, with residues

$$\begin{aligned} \operatorname{Res}\left(g_1(s)g_2\left(\frac{p}{s}\right)\frac{1}{s}, s = 2\pi in\right) &= \operatorname{Res}(g_1(s), s = 2\pi in)\; g_2\left(\frac{p}{2\pi in}\right)\frac{1}{2\pi in} \\ &= -\frac{1}{4\pi^2 n^2} g_2\left(\frac{p}{2\pi in}\right). \end{aligned}$$

Moreover, $g_1(s) = O(1/s)$ when the distance of s from $2\pi i\mathbb{Z}^*$ is greater than 0.1 and $g_2(p/s) = O(1)$ for s large, thus the integrand vanishes at infinity.

We now enlarge the circle γ and collect the corresponding residues by Cauchy's theorem. Using the above calculation of the residue and Equation (2.4), it follows that

$$\begin{aligned} G_f(p) &= \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(g_2\left(\frac{p}{2\pi in}\right) + g_2\left(-\frac{p}{2\pi in}\right)\right) \\ &= \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\Bigg(f'\left(1+\frac{p}{2\pi in}\right) + f'\left(1-\frac{p}{2\pi in}\right) - f'\left(\frac{p}{2\pi in}\right) \\ &\qquad\qquad -f'\left(-\frac{p}{2\pi in}\right)\Bigg) \end{aligned}$$

Since $f'(x)$ is regular at $x = 0, 1$, it follows that the above series is convergent for $p \in \mathbb{C} - \mathcal{N}$, where $\mathcal{N}$ is defined in (1.16). In addition, we conclude that $G_f(p)$ has endless analytic continuation with singularities in $\mathcal{N}$.

It remains to prove that $G_f(p)$ is exponentially bounded, assuming that f is. If f is exponentially bounded, Cauchy's formula implies that f' is exponentially bounded. Then, we have:

$$\left|f'\left(1+\frac{p}{2\pi in}\right)\right| \leqslant C\exp\left(a\left|1+\frac{p}{2\pi in}\right|\right) \leqslant Ce^a \exp\left(a\frac{|p|}{2\pi}\right).$$

Thus,

$$|G_f(p)| \leqslant \frac{C(e^a+1)}{2\pi^2}\exp\left(a\frac{|p|}{2\pi}\right)\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{C(e^a+1)}{12}\exp\left(a\frac{|p|}{2\pi}\right).$$

This completes the proof of Theorem 1.3. □

3. An exact form of Euler-Maclaurin summation formula

3.1. Proof of Proposition 1.4

Proposition 1.4 follows easily from the Abel-Plana formula; see Appendix A. However, we give a proof of Proposition 1.4 that allows us to generalize to Theorem 3.2 below.

Consider a resurgent function f that satisfies the assumptions (A1), and let us introduce the function

$$h(u) = \frac{N}{2} f(u) \frac{e^{\pi i N u} + e^{-\pi i N u}}{e^{\pi i N u} - e^{-\pi i N u}}$$

and the contour $\Gamma_{R,\delta}$ which is a rectangle oriented counterclockwise with vertices $-iR, 1-iR, 1+iR, iR$ that excludes the points 0, 1 together with small semicircles of radius δ at the points 0 and 1.

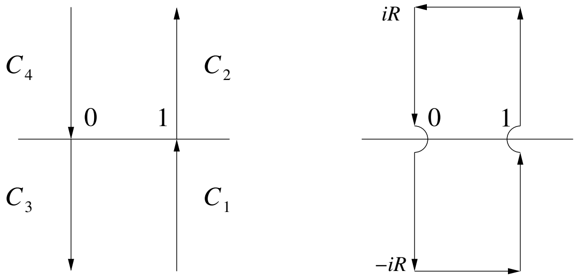

Figure 3.1. The contours C_1, C_2, C_3, C_4 of the critical strip, and a truncated contour $\Gamma_{R,\delta}$

Due to our assumptions on f, the singularities of $h(u)$ inside $\Gamma_{R,\delta}$ are simple poles at k/N with residue $f(k/N)/(2\pi i)$ for $k = 1, \dots, N-1$. The residue theorem implies that

$$\sum_{k=1}^{n} f\left(\frac{k}{N}\right) = \frac{N}{2} \int_{\Gamma_{R,\delta}} f(u) \frac{e^{\pi i N u} + e^{-\pi i N u}}{e^{\pi i N u} - e^{-\pi i N u}} du. \tag{3.1}$$

Let $\Gamma^+_{R,\delta}$ (resp. $\Gamma^-_{R,\delta}$) denote the upper (resp. lower) part of the contour Γ. Since $f(x)$ has no singularities in $\Re(u) \in [0,1]$, the residue theorem implies that

$$-N \int_0^1 f(u) du = \frac{N}{2} \int_{\Gamma^+_{R,\delta}} f(u) du - \frac{N}{2} \int_{\Gamma^-_{R,\delta}} f(u) du. \tag{3.2}$$

Adding up Equations (3.1), (3.2) and using

$$\begin{aligned} \frac{1}{2}\frac{z+z^{-1}}{z-z^{-1}}+\frac{1}{2} &= \frac{1}{1-z^{-2}} \\ \frac{1}{2}\frac{z+z^{-1}}{z-z^{-1}}-\frac{1}{2} &= \frac{1}{z^2-1} \end{aligned}$$

we obtain that

$$\sum_{k=1}^{N-1} f\left(\frac{k}{N}\right) - N\int_0^1 f(u)du = N\int_{\Gamma^+_{R,\delta}} \frac{f(u)}{1-e^{-2\pi i N u}}du + N\int_{\Gamma^-_{R,\delta}} \frac{f(u)}{e^{2\pi i N u}-1}du \tag{3.3}$$

Now let $R \to \infty$. Due to assumption (A1), the integrals over the horizontal parts of $\Gamma^{\pm}_{R,\delta}$ approach zero. Next, let $\delta \to 0$. Since f is continuous, the integral around the quarter circle that links δ to $i\delta$ tends to $-f(0)/4$. The other quarter circles are treated similarly.

Thus, we have:

(3.4)
$$\sum_{k=1}^{N} f\left(\frac{k}{N}\right) - N\int_0^1 f(u)du = \frac{1}{2}(f(1)-f(0)) + \int_{C_2} \frac{f(u)-f(1)}{1-e^{-2\pi i N u}}du$$
$$+\int_{C_4} \frac{f(u)-f(0)}{1-e^{-2\pi i N u}}du + \int_{C_1} \frac{f(u)-f(1)}{e^{2\pi i N u}-1}du + \int_{C_3} \frac{f(u)-f(0)}{e^{2\pi i N u}-1}du.$$

Consider now the corresponding function $G_f(p)$ from Theorem 1.3. We have:

$$G_f(p) = G_1(p) + G_2(p) + G_3(p) + G_4(p)$$

where

$$\begin{aligned} G_1(p) &= \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} f'\left(1+\frac{p}{2\pi i n}\right) \\ G_2(p) &= \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} f'\left(1-\frac{p}{2\pi i n}\right) \\ G_3(p) &= -\frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} f'\left(\frac{p}{2\pi i n}\right) \\ G_4(p) &= -\frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} f'\left(-\frac{p}{2\pi i n}\right). \end{aligned}$$

Consider the contours C_1, C_2, C_3, C_4 on the boundary of our strip, as shown in Figure 3.1.

We claim that the Laplace transform of the $G_j(p)$ for $j = 1, \dots, 4$ is given by:

$$\int_0^\infty e^{-Np} G_j(p) = \begin{cases} N \displaystyle\int_{C_j} \frac{f(u) - f(1)}{e^{2\pi i N u} - 1} du & \text{for } j = 1 \\ N \displaystyle\int_{C_j} \frac{f(u) - f(0)}{e^{2\pi i N u} - 1} du & \text{for } j = 3 \\ N \displaystyle\int_{C_j} \frac{f(u) - f(1)}{1 - e^{-2\pi i N u}} du & \text{for } j = 2 \\ N \displaystyle\int_{C_j} \frac{f(u) - f(0)}{1 - e^{-2\pi i N u}} du & \text{for } j = 4. \end{cases} \tag{3.5}$$

Let us show this for $j = 3$; the other integrals are treated in the same way. We compute as follows:

$$\begin{aligned} \int_0^\infty e^{-Np} G_3(p) &= -\frac{1}{4\pi^2} \int_0^\infty \sum_{n=1}^\infty e^{-Np} \frac{1}{n^2} f'\left(\frac{p}{2\pi i n}\right) dp && \text{by interchanging sum and integral} \\ &= \frac{1}{2\pi i} \int_{C_3} \sum_{n=1}^\infty \frac{e^{-2\pi i N n u}}{n} f'(u) du && \text{by } p = 2\pi i n u \\ &= -\frac{1}{2\pi i} \int_{C_3} \log(1 - e^{-2\pi i N u}) f'(u) du && \text{by (3.6)} \\ &= N \int_{C_3} \frac{f(u) - f(0)}{e^{2\pi i N u} - 1} du && \text{by integration by parts} \end{aligned}$$

where

$$\sum_{n=1}^\infty \frac{e^{-2\pi i N n u}}{n} = -\log(1 - e^{-2\pi i N u}) \tag{3.6}$$

This concludes the proof of Proposition 1.4 in case f satisfies (A1). □

Let us end this section with a remark.

Remark 3.1. — If $f(x) = e^{cx}$, one may verify Equation (1.15) directly by using the Mittag-Leffler decomposition of the function $x/(e^x - 1)$.

3.2. Euler-MacLaurin summation for functions with singularities in the vertical strip

In this section we consider a function f that satisfies the following assumptions:

(A2) f is resurgent, and let Λ denote its set of singularities on the critical strip $0 \leqslant \Re(x) \leqslant 1$. We assume that $\lambda \notin [0,1]$ for all $\lambda \in \Lambda$, and $f(u) = o(e^{2\pi|\Im(u)|})$ as $|\Im(u)| \to \infty$ in the strip, uniformly with respect to $\Re(u)$. We also assume that on a vertical ray $\lambda + i\mathbb{R}^+$, we have $f(u)e^{-2\pi\Im(u)} \in L^1(\lambda + i\mathbb{R}^+)$.

(A3) For every $\lambda \in \Lambda$ there exist a holomorphic germ $h_\lambda(u)$ and real numbers $\alpha_\lambda, \beta_\lambda$ so that for u near 0 we have:

$$f(u+\lambda) = u^{\alpha_\lambda}(\log u)^{\beta_\lambda} h_\lambda(u). \tag{3.7}$$

(A4) For simplicity, let us also assume that $\Re(\lambda) \neq \Re(\lambda')$ for $\lambda \neq \lambda'$, and that Λ is a finite set.

Let

$$(\mathcal{L}G)(x) = \int_0^x e^{-xp} G(p)dp \tag{3.8}$$

denote the *Laplace transform* of $G(p)$. We denote by $f_\lambda(u)$ the *variation* (or jump) of the multivalued function $f(u+\lambda)$ at u; where u lies the vertical ray starting at 0 (see for example, [20]). We also define:

$$G_{f,\lambda,m}(p) = i\frac{1}{2\pi m} f_\lambda\left(\frac{ip}{2\pi m}\right). \tag{3.9}$$

In case $f(u+\lambda)$ is single-valued then $G_{f,\lambda,m}(p)$ is a distribution supported at $p = 0$.

Then, we have the following exact form of the Euler-MacLaurin summation formula.

Theorem 3.2. — (a) If f satisfies (A1-A4) and $\alpha_\lambda > -1$ for all $\lambda \in \Lambda$, then for every $N \in \mathbb{N}$ we have:

$$\begin{aligned}\sum_{k=1}^N f\left(\frac{k}{N}\right) &= N\int_0^1 f(s)ds + \frac{1}{2}(f(1)-f(0)) + (\mathcal{L}G_f)(N) \\ &+ Ne^{2\pi i\lambda N}\sum_{\lambda:\Im(\lambda)>0}\sum_{m=0}^\infty e^{2\pi i\lambda m N}(\mathcal{L}G_{f,\lambda,m})(N) \\ &+ Ne^{-2\pi i\lambda N}\sum_{\lambda:\Im(\lambda)<0}\sum_{m=0}^\infty e^{-2\pi i\lambda m N}(\mathcal{L}G_{f,\lambda,m})(N)\end{aligned} \tag{3.10}$$

(b) If some $\alpha_\lambda \leqslant -1$, Equation (3.10) is true after integration by parts M-times where $M \geqslant \max_{\lambda:\alpha_\lambda \leqslant -1}[-\alpha_\lambda]$.

Proof. — Without loss of generality, let us assume that f has a single singularity λ in the vertical strip $0 \leqslant \Re(x) \leqslant 1$ with $\Im(\lambda) > 0$.

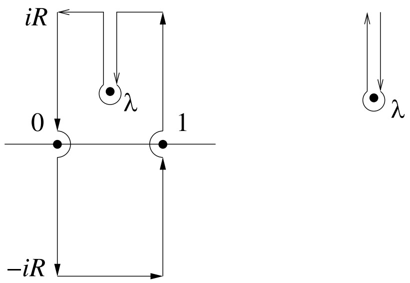

Figure 3.2. The modified truncated contour $\Gamma_{R,\delta,\lambda}$ on the left and a Hankel contour H on the right.

Use the modified contour $\Gamma_{R,\delta,\lambda}$ in Figure 3.2.

Let H_R denote the portion of $\Gamma_{R,\delta,\lambda}$ that consists of the truncated Hankel contour around λ, and $S_{R,\delta} = \Gamma_{R,\delta,\lambda} - H_R$. Equations (3.1) and (3.2) become:

$$-\frac{N}{2}\int_{H_R} f(u)\frac{e^{\pi iNu}+e^{-\pi iNu}}{e^{\pi iNu}-e^{-\pi iNu}}du+\sum_{k=1}^{n} f\left(\frac{k}{N}\right) = \frac{N}{2}\int_{S_{R,\delta}} f(u)\frac{e^{\pi iNu}+e^{-\pi iNu}}{e^{\pi iNu}-e^{-\pi iNu}}du \tag{3.11}$$

and

$$-\frac{N}{2}\int_{H_R} f(u)du - N\int_0^1 f(u)du = \frac{N}{2}\int_{S^+_{R,\delta}} f(u)du - \frac{N}{2}\int_{S^-_{R,\delta}} f(u)du. \tag{3.12}$$

Adding up, the extra contribution from H_R becomes:

$$-N\int_{H_R}\frac{f(u)}{1-e^{-2\pi iNu}}du \tag{3.13}$$

Now let $R\to\infty$. Notice that $f(u+\lambda)$ is uniformly L^1 for u near 0 iff $\alpha_\lambda > -1$ for all λ. Using this and our integrability assumption (A2), it follows that in the limit the above integral equals to

$$\begin{aligned} I_\lambda &= -N\int_H \frac{f(u)}{1-e^{-2\pi iNu}}du \\ &= -Ni\int_0^\infty \frac{f_\lambda(is)}{1-e^{-2\pi iN(\lambda+is)}}ds. \end{aligned}$$

Now, $\lambda = \lambda_1 + i\lambda_2$ with $\lambda_2 > 0$, and we may write

$$\begin{aligned} \frac{1}{1-e^{-2\pi i N(\lambda+is)}} &= \frac{1}{1-\omega^{-N}e^{2\pi N s}} \\ &= -\omega^N e^{-2\pi N s} \sum_{m=0}^{\infty} \omega^{Nm} e^{-2Nms} \end{aligned}$$

where $\omega = e^{2\pi i\lambda}$ satisfies $|\omega| < 1$. Thus,

$$I_\lambda = -N\omega^N \sum_{m=0}^{\infty} \omega^{Nm} (\mathcal{L}G_{\lambda,m})(N).$$

Part (a) of Theorem 3.2 follows. Part (b) follows from the fact that if $f(u+\lambda)$ has a local expansion of the form (3.7), and $F^{(s)}(u) = f(u)$, then $F(u+\lambda)$ as a local expansion of the form:

$$F(u+\lambda) = u^{\alpha_\lambda + s} (\log u)^{\beta_\lambda} H_\lambda(u) \tag{3.14}$$

for a holomorphic germ $H_\lambda(u)$. Cf. also [7, Thm.1]. □

3.3. Euler-MacLaurin with logarithmic singularities at $x = 0$

In this section we consider functions $f(x)$ that have a logarithmic singularity at $x = 0$. Motivated by our applications to quantum topology, we consider functions f of the form:

$$f(x) = c \log x + g(x) \tag{3.15}$$

where g that satisfies (A1), and $c \in \mathbb{C}$. Let us define

$$H(p) = \frac{1}{p^2}\left(\frac{p}{e^p - 1} - 1 + \frac{p}{2}\right) \tag{3.16}$$

It is easy to see that $H(p)$ is analytic at $p = 0$. In fact, the Taylor series of H at $p = 0$ is given by:

$$\sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} p^{2n-2} \tag{3.17}$$

THEOREM 3.3. — *Under the above hypothesis, for every $N \in \mathbb{N}$ we have:*

$$\begin{aligned} \sum_{k=1}^{N} f\left(\frac{k}{N}\right) &= N \int_0^1 f(s)ds + \frac{c}{2}\log N + \frac{c}{2}\log(2\pi) \\ &\quad + \frac{1}{2}(g(1) - g(0)) + \mathcal{L}(G_g + cH)(N). \end{aligned} \tag{3.18}$$

Proof. — Since f is given by (3.15), we have:

$$\sum_{k=1}^{N} f\left(\frac{k}{N}\right) = c\log\left(\frac{N!}{N^N}\right) + \sum_{k=1}^{N} g\left(\frac{k}{N}\right)$$

Recall now from [18, Sec.13.15] the following exact form of *Stirling's formula*:

$$\log\left(\frac{N!}{N^N}\right) = \frac{1}{2}\log N - N + \frac{1}{2}\log(2\pi) + (\mathcal{L}H)(N). \tag{3.19}$$

Applying Proposition 1.4 to g gives:

$$\sum_{k=1}^{N} g\left(\frac{k}{N}\right) = N\int_0^1 g(s)ds + \frac{1}{2}(g(1) - g(0)) + (\mathcal{L}G_g)(N).$$

Adding up, and using

$$N\int_0^1 f(s)ds = N\int_0^1 g(s)ds + Nc\int_0^1 \log s ds = N\int_0^1 g(s)ds - Nc$$

we obtain (3.18). The result follows. □

4. Parametric resurgence of difference equations with a parameter

Consider the first order linear difference equation with a small parameter ϵ:

$$y(x+\epsilon, \epsilon) = a(x, \epsilon)y(x, \epsilon) \tag{4.1}$$

where $a(x,\epsilon)$ is smooth. (4.1) has a unique *formal solution* (often called a *WKB solution*) of the form:

$$y(x, \epsilon) = e^{\frac{1}{\epsilon}\sum_{k=0}^{\infty} F_k(x)\epsilon^k} \tag{4.2}$$

where $F_j(0) = 0$. See for example, [6] and [16]. For simplicity, suppose that $a(x,\epsilon) = a(x)$ is independent of ϵ. Under the stated assumptions, the next theorem gives an exact solution to (4.1) which is asymptotic to the formal solution (4.2).

Theorem 4.1. — (a) For all x such that $s \to \log a(sx)$ satisfies (A1) we have:

$$\frac{1}{\epsilon}\sum_{k=0}^{\infty} F_k(x)\epsilon^k \sim \frac{1}{\epsilon}\int_0^x \log a(q)dq - \frac{1}{2}\log a(x) + \frac{1}{2}\log a(0) + \int_0^{\infty} e^{-q/\epsilon}G(q,x)dq \tag{4.3}$$

where

(4.4)

$$G(q,x) = \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{a'\left(x+\frac{q}{2\pi in}\right)}{a\left(x+\frac{q}{2\pi in}\right)} + \frac{a'\left(x-\frac{q}{2\pi in}\right)}{a\left(x-\frac{q}{2\pi in}\right)} - \frac{a'\left(\frac{q}{2\pi in}\right)}{a\left(\frac{q}{2\pi in}\right)} - \frac{a'\left(-\frac{q}{2\pi in}\right)}{a\left(-\frac{q}{2\pi in}\right)}\right)$$

$$= \frac{1}{(2\pi i)^3}\int_0^\infty\int_{\gamma_0}\frac{u}{e^u-1}\left(\frac{\log a(s)}{s^2}\left(e^{\frac{pu}{2\pi is}}+e^{-\frac{pu}{2\pi is}}\right) - \frac{\log a(x+s)}{(x+s)^2}\left(e^{\frac{pu}{2\pi i(x+s)}}+e^{-\frac{pu}{2\pi i(x+s)}}\right)\right)dsdu.$$

where γ_0 is a small circle around 0 oriented counterclockwise.
(b) Moreover, (4.1) has a solution $y(x,\epsilon)$ of the form:

$$y(x,\epsilon) = \sqrt{\frac{a(0)}{a(x)}}\exp\left(\frac{1}{\epsilon}\int_0^x \log a(q)dq + \int_0^\infty e^{-q/\epsilon}G(q,x)dq\right). \tag{4.5}$$

Remark 4.2. — It follows that the singularities of $G(q,x)$ are of the form $2\pi in\lambda$ or $2\pi in(\lambda-x)$ where $n\in\mathbb{Z}^*$ and λ is a singularity of $\log a$. These type of singularities appear in parametric (i.e., co-equational) resurgence of Écalle; see [15].

The proof of Theorem 4.1 indicates the close relation between the Euler-MacLaurin summation formula and the formal solutions of a linear difference equation with a parameter.

From that point of view, resurgence of $G_f(p)$ translates to parametric resurgence of formal solutions of linear difference equations. In the case of formal solutions of linear differential equations with a parameter, Écalle shows that their singularities are of the form $n(\alpha_i - x)$ for $n=-1,1,2,3,\dots$; see [15, Eqn.(6.9)].

Proof. — (a) Let $z(x,\epsilon)=\log y(x,\epsilon)$. Taking the logarithm of (4.1), it follows that

$$z(k\epsilon+\epsilon,\epsilon) = \log a(k\epsilon) + z(k\epsilon,\epsilon).$$

Summing up for $k=0,\dots,N-1$ and using the variable

$$x = N\epsilon \tag{4.6}$$

we obtain that:

$$\begin{aligned} z(x,\epsilon)-z(0,\epsilon) &= \sum_{k=0}^{N-1}\log a(k\epsilon) \\ &= -\log a(x)+\log a(0)+\sum_{k=1}^{N}\log a(xk/N). \end{aligned}$$

Let us fix x and apply Proposition 1.4 to the function $s\to\log a(sx)$. We obtain that

$$\begin{aligned} &z(x,\epsilon)-z(0,\epsilon)= \\ &= N\int_0^1\log a(xs)ds-\frac{1}{2}\log a(x)+\frac{1}{2}\log a(0)+\int_0^\infty e^{-Np}H(p,x)dp \\ &= \frac{1}{\epsilon}\int_0^x\log a(s)ds-\frac{1}{2}\log a(x)+\frac{1}{2}\log a(0)+\int_0^\infty e^{-xp/\epsilon}H(p,x)dp \\ &= \frac{1}{\epsilon}\int_0^x\log a(s)ds-\frac{1}{2}\log a(x)+\frac{1}{2}\log a(0)+\int_0^\infty e^{-q/\epsilon}H\left(\frac{q}{x},x\right)\frac{dq}{x} \end{aligned}$$

where by Theorems 1.3 and 1.5 we have:

$$\begin{aligned} H(p,x) &= \frac{x}{4\pi^2}\sum_{n=1}^\infty\frac{1}{n^2}\left(\frac{a'\left(x\left(1+\frac{p}{2\pi in}\right)\right)}{a\left(x\left(1+\frac{p}{2\pi in}\right)\right)}+\frac{a'\left(x\left(1-\frac{p}{2\pi in}\right)\right)}{a\left(x\left(1-\frac{p}{2\pi in}\right)\right)}-\frac{a'\left(x\frac{p}{2\pi in}\right)}{a\left(x\frac{q}{2\pi in}\right)}\right. \\ &\qquad\left.-\frac{a'\left(-x\frac{p}{2\pi in}\right)}{a\left(-x\frac{p}{2\pi in}\right)}\right) \\ &= \frac{1}{(2\pi i)^3}\int_0^\infty\int_{\gamma_0}\frac{u}{e^u-1}\left(\frac{\log a(sx)}{s^2}\left(e^{\frac{pu}{2\pi is}}+e^{-\frac{pu}{2\pi is}}\right)\right. \\ &\qquad\left.-\frac{\log a((1+s)x)}{(1+s)^2}\left(e^{\frac{pu}{2\pi i(1+s)}}+e^{-\frac{pu}{2\pi i(1+s)}}\right)\right)dsdu. \end{aligned}$$

Since $H(q/x,x)/x=G(q,x)$ (where $G(q,x)$ is given by (4.4)) and $z(0,\epsilon)=0$, it follows that for all $\epsilon>0$ and $k\in\mathbb{Z}$ we have:

$$z(k\epsilon+\epsilon,\epsilon)=\log a(k\epsilon)+z(k\epsilon,\epsilon). \tag{4.7}$$

To prove (b), let us consider the difference

$$E(x,\epsilon)=\epsilon(\log y(x+\epsilon,\epsilon)-\log a(x)-\log y(x,\epsilon)).$$

It follows by definition that $E(x,\epsilon)$ is analytic in (x,ϵ). Thus,

$$E(x,\epsilon)=\sum_{i,j=0}^\infty c_{ij}x^i\epsilon^j.$$

Moreover, (4.7) implies that for all $\epsilon > 0$ and all $k \in \mathbb{Z}$ we have:

$$0 = F(k\epsilon, \epsilon) = \sum_{i,j=0}^{\infty} c_{ij} k^i \epsilon^{i+j}.$$

Thus $c_{i,j} = 0$ for all i, j and $E(x, \epsilon) = 0$. This completes the proof of (b).

The definition of $y(x, \epsilon)$ by a Laplace integral and Watson's lemma (see [21, Sec.4.3.1]) implies that

$$y(x, \epsilon) \sim \frac{1}{\epsilon} \sum_{k=0}^{\infty} \phi_k(x) \epsilon^k$$

for analytic functions $\phi_k(x)$ that satisfy $\phi_k(0) = 0$. Since a formal WKB solution given by (4.2) is unique, it follows that $F_k(x) = \phi_k(x)$ for all k. Thus, (a) follows. □

Remark 4.3. — Theorem 4.1 can be generalized when

$$a(x, \epsilon) = \sum_{k=0}^{\infty} a_k(x) \epsilon^k$$

is analytic with respect to (x, ϵ), and the coefficients $a_k(x)$ are resurgent functions. It may also be generalized to the case of higher order linear difference equations with a parameter. This will be explained elsewhere.

Remark 4.4. — The reader may compare Theorem 4.1 with the results of the last section of [22].

5. An integral formula for $G_f(p)$

In this section we give a proof of Theorem 1.5. We follow the ideas of [8] to convert the sum of Equation (1.15) into an integral. Let us show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} f'\left(\frac{p}{2\pi i n}\right) = \frac{1}{2\pi i} \int_0^{\infty} \int_{\gamma_0} \frac{u f(s)}{s^2 (e^u - 1)} e^{\frac{pu}{2\pi i s}} ds du \tag{5.1}$$

and similarly for the sum of the other three terms in (1.15).

To prove Equation (5.1), we first expand f' at $p = 0$, then take a Laplace transform with respect to the summation variable n, interchange the order of summation and sum the geometric series. We obtain that:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^2}f'\left(\frac{p}{2\pi i n}\right) &= \sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\frac{f^{(j+1)}(0)}{j!}\left(\frac{p}{2\pi i}\right)^j\frac{1}{n^{j+2}}\\ &= \sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\frac{f^{(j+1)}(0)p^j}{j!(j+1)!(2\pi i)^j}\int_0^{\infty}e^{-nu}u^{j+1}du\\ &= \sum_{j=0}^{\infty}\int_0^{\infty}\frac{f^{(j+1)}(0)p^j}{j!(j+1)!(2\pi i)^j}u^{j+1}\frac{1}{e^u-1}du\end{aligned}$$

Using Cauchy's formula

$$\frac{f^{(j+1)}(0)}{(j+1)!}=\frac{1}{2\pi i}\int_{\gamma_0}\frac{f(s)}{s^{j+2}}ds$$

and interchanging summation and integration it follows that:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^2}f'\left(\frac{p}{2\pi i n}\right) &= \frac{1}{2\pi i}\sum_{j=0}^{\infty}\int_0^{\infty}\int_{\gamma_0}\frac{f(s)p^j}{j!s^{j+2}}\frac{u^{j+1}}{(2\pi i)^j}\frac{1}{e^u-1}dsdu\\ &= \frac{1}{2\pi i}\int_0^{\infty}\int_{\gamma_0}\frac{uf(s)}{s^2(e^u-1)}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{pu}{2\pi i s}\right)^j dsdu\\ &= \frac{1}{2\pi i}\int_0^{\infty}\int_{\gamma_0}\frac{uf(s)}{s^2(e^u-1)}e^{\frac{pu}{2\pi i s}}dsdu.\end{aligned}$$

The interchanges of summation and integration are justified by dominated convergence. This concludes the proof of (5.1) and Theorem 1.5. □

Appendix A.

For completeness, let us show how the Abel-Plana formula implies Proposition 1.4. With the notation as in Proposition 1.4, we claim that for every $N\in\mathbb{N}$ we have:

$$-i\int_0^{\infty}\frac{f\left(1+\frac{iy}{N}\right)-f(1)}{e^{2\pi y}-1}=\frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^{\infty}e^{-Np}f'\left(1-\frac{p}{2\pi i n}\right)dp \tag{A.1}$$

$$i\int_0^{\infty}\frac{f\left(1-\frac{iy}{N}\right)-f(1)}{e^{2\pi y}-1}=\frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^{\infty}e^{-Np}f'\left(1+\frac{p}{2\pi i n}\right)dp \tag{A.2}$$

$$i\int_0^{\infty}\frac{f\left(\frac{iy}{N}\right)-f(0)}{e^{2\pi y}-1}=-\frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^{\infty}e^{-Np}f'\left(-\frac{p}{2\pi i n}\right)dp \tag{A.3}$$

$$-i\int_0^\infty \frac{f\left(-\frac{iy}{N}\right)-f(0)}{e^{2\pi y}-1} = -\frac{1}{4\pi^2}\sum_{n=1}^\infty \frac{1}{n^2}\int_0^\infty e^{-Np} f'\left(\frac{p}{2\pi i n}\right) dp. \tag{A.4}$$

Adding up, and using the Abel-Plana formula (1.12), gives a proof of Proposition 1.4. Let us give the proof of (A.1) and leave the rest as an exercise. For $y > 0$, we have $e^{-2\pi y} < 1$ and the geometric series gives:

$$\frac{1}{e^{2\pi y}-1} = \sum_{n=1}^\infty e^{-2\pi n y}. \tag{A.5}$$

Interchanging summation and integration, changing variables $2\pi n y = Np$ and integrating by parts (justified by the hypothesis (A1)), we obtain that

$$\begin{aligned}
-i\int_0^\infty \frac{f\left(1+\frac{iy}{N}\right)-f(1)}{e^{2\pi y}-1} &= -i\sum_{n=1}^\infty \int_0^\infty e^{-2\pi n y}\left(f\left(1+\frac{iy}{N}\right)-f(1)\right) dy \\
&= -\frac{iN}{2\pi}\sum_{n=1}^\infty \frac{1}{n}\int_0^\infty e^{-Np}\left(f\left(1-\frac{p}{2\pi i n}\right)-f(1)\right) dp \\
&= \frac{i}{2\pi}\sum_{n=1}^\infty \frac{1}{n}\int_0^\infty (e^{-Np})'\left(f\left(1-\frac{p}{2\pi i n}\right)-f(1)\right) dp \\
&= \frac{1}{4\pi^2}\sum_{n=1}^\infty \frac{1}{n^2}\int_0^\infty e^{-Np} f'\left(1-\frac{p}{2\pi i n}\right) dp.
\end{aligned}$$

BIBLIOGRAPHY

[1] E. Borel, “Sur les singularités des séries de Taylor”, *Bull. Soc. Math. France* **26** (1898), p. 238-248.

[2] B. Braaksma, “Transseries for a class of nonlinear difference equations”, *J. Differ. Equations Appl.* **7** (2001), p. 717-750.

[3] B. Braaksma & R. Kuik, “Resurgence relations for classes of differential and difference equations”, *Ann. Fac. Sci. Toulouse Math.* **13** (2004), p. 479-492.

[4] B. Candelpergher, J. C. Nosmas & F. Pham, *Approche de la résurgence*, Actualités Mathématiques, Hermann, 1993.

[5] ———, “Premiers pas en calcul étranger”, *Ann. Inst. Fourier (Grenoble)* **43** (1993), p. 201-224.

[6] O. Costin & R. Costin, “Rigorous WKB for finite-order linear recurrence relations with smooth coefficients”, *SIAM J. Math. Anal.* **27** (1996), p. 110-134.

[7] ———, “On Borel summation and Stokes phenomena for rank-1 nonlinear systems of ordinary differential equations”, *Duke Math. J.* **93** (1998), p. 289-344.

[8] ———, “Global reconstruction of analytic functions from local expansions”, preprint, 2006.

[9] O. Costin & S. Garoufalidis, “Resurgence of the Kontsevich-Zagier power series”, preprint, 2006, math.GT/0609619.

[10] ———, “Resurgence of 1-dimensional sums of sum-product type”, preprint, 2007.

[11] ———, “Resurgence of the fractional polylogarithms”, preprint, 2007, math.CA/0701743.

[12] E. Delabaere, “Introduction to the Écalle theory”, *Computer algebra and differential equations, London Math. Soc. Lecture Note Series* **193** (1994), p. 59-101.

[13] E. Delabaere & F. Pham, “Resurgent methods in semi-classical asymptotics”, *Ann. Inst. H. Poincaré Phys. Théor.* **71** (1999), p. 1-94.

[14] J. Écalle, *Resurgent functions, Vol. I–II*, Mathematical Publications of Orsay, **81**, 1981.

[15] ———, “Weighted products and parametric resurgence”, *Analyse algébrique des perturbations singulières, I (Marseille-Luminy) Travaux en Cours* **47** (1991), p. 7-49.

[16] S. Garoufalidis & J. Geronimo, “Asymptotics of q-difference equations”, *Contemporary Math. AMS* **416** (2006), p. 83-114.

[17] S. Garoufalidis, J. Geronimo & T. T. Q. Le, “Gevrey series in quantum topology”, J. Reine Angew. Math., in press.

[18] G. H. Hardy, *Divergent Series*, Clarendon Press, Oxford, 1949.

[19] R. Jungen, “Sur les séries de Taylor n’ayant que des singularités algébrico-logarithmiques sur leur cercle de convergence”, *Comment. Math. Helv.* **3** (1931), p. 266-306.

[20] B. Malgrange, “Introduction aux travaux de J. Écalle”, *Enseign. Math.* **31** (1985), p. 261-282.

[21] F. W. J. Olver, *Asymptotics and special functions*, Reprint. AKP Classics, A K Peters, Ltd., Wellesley, MA, 1997.

[22] D. Sauzin, “Resurgent functions and splitting problems”, preprint, 2006.

[23] D. Zagier, “Vassiliev invariants and a strange identity related to the Dedekind eta-function”, *Topology* **40** (2001), p. 945-960.

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