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A MEAN VALUE THEOREM FOR THE SQUARE OF CLASS NUMBER TIMES REGULATOR OF QUADRATIC EXTENSIONS

by Takashi TANIGUCHI

Abstract. — Let k be a number field. In this paper, we give a formula for the mean value of the square of class number times regulator for certain families of quadratic extensions of k characterized by finitely many local conditions. We approach this by using the theory of the zeta function associated with the space of pairs of quaternion algebras. We also prove an asymptotic formula of the correlation coefficient for class number times regulator of certain families of quadratic extensions.

Résumé. — Soit k un corps de nombres. Dans cet article, nous donnons une formule pour la valeur moyenne du carré du nombre de classe multiplié par le régulateur pour certaines familles d'extensions quadratiques de k caractérisées par un nombre fini de conditions locales. Notre approche utilise la théorie de la fonction zêta associée à l'espace de paires d'algèbres de quaternions. Nous prouvons aussi une formule asymptotique pour le coefficient de corrélation du nombre de classe multiplié par le régulateur de certaines familles d'extensions quadratiques.

1. Introduction

We fix an algebraic number field k. Let $\mathfrak{M}$, $\mathfrak{M}_\infty$ and $\mathfrak{M}_{\mathrm{f}}$ denote respectively the set of all places of k, all infinite places and all finite places. For $v \in \mathfrak{M}$ let k_v denote the completion of k at v and if $v \in \mathfrak{M}_{\mathrm{f}}$ then let q_v denote the order of the residue field of k_v. We let Δ_k, r_1, r_2, and e_k be respectively the absolute discriminant, the number of real places, the number of complex places, and the number of roots of unity contained in k. We denote by $\zeta_k(s)$ the Dedekind zeta function of k.

Let $S \supset \mathfrak{M}_\infty$ be a finite set of places. We fix an S-tuple $L_S = (L_v)_{v\in S}$ where each L_v is a separable quadratic algebra of k_v, *i.e.*, either $k_v \times k_v$ or

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a quadratic extension of k_v. Let $\mathcal{Q}(L_S)$ be the following family of quadratic extensions of k;

$$\mathcal{Q}(L_S) := \{F \mid [F:k]=2, F\otimes k_v \cong L_v \text{ for all } v\in S\}.$$

Let h_F and R_F be the class number and the regulator of F, respectively. We would like to find the average value of $h_F^2 R_F^2$ for $F\in\mathcal{Q}(L_S)$. For $F\in\mathcal{Q}(L_S)$ we denote by $\Delta_{F/k}$ the relative discriminant of F/k and by $\mathcal{N}(\Delta_{F/k})$ its absolute norm. Let

$$\mathcal{Q}(L_S,X) := \{F\in\mathcal{Q}(L_S) \mid \mathcal{N}(\Delta_{F/k})\leqslant X\}.$$

The following is one of the main results of this paper.

Theorem 1.1 (Theorem 10.12). — *Let $L_S=(L_v)_{v\in S}$ be an S-tuple such that L_v is a field for at least two places of S. Then the limit*

$$\lim_{X\to\infty}\frac{1}{X^2}\sum_{F\in\mathcal{Q}(L_S,X)} h_F^2 R_F^2$$

exists, and its value is equal to

$$\frac{(\mathrm{Res}_{s=1}\zeta_k(s))^3\Delta_k^2 e_k^2\zeta_k(2)^2}{2^{r_1+r_2+1}2^{2r_1(L_S)}(2\pi)^{2r_2(L_S)}}\prod_{v\in S\cap\mathfrak{M}_{\mathrm{f}}}\mathfrak{e}_v(L_v)\prod_{v\in\mathfrak{M}_{\mathrm{f}}}(1-3q_v^{-3}+2q_v^{-4}+q_v^{-5}-q_v^{-6}).$$

Here we denote by $r_1(L_S)$ and $r_2(L_S)$ respectively the number of real and complex places of $F\in\mathcal{Q}(L_S,X)$ (these numbers do not depend on the choice of F) and also for $v\in\mathfrak{M}_{\mathrm{f}}$ we put

$$\mathfrak{e}_v(L_v)=\begin{cases}2^{-1}(1+q_v^{-1})(1-q_v^{-2}) & L_v\cong k_v\times k_v,\\ 2^{-1}(1-q_v^{-1})^3 & L_v \text{ is quadratic unramified},\\ 2^{-1}\mathcal{N}(\Delta_{L_v/k_v})^{-1}(1-q_v^{-1})(1-q_v^{-2})^2 & L_v \text{ is quadratic ramified.}\end{cases}$$

We discuss on the condition of L_S in Remark 10.13.

We explain one more theorem we prove in this paper. We fix a quadratic extension $\widetilde{k}$ of k. Let $\mathfrak{M}_{\mathrm{rm}}$, $\mathfrak{M}_{\mathrm{in}}$ and $\mathfrak{M}_{\mathrm{sp}}$ be the sets of finite places of k which are respectively ramified, inert and split on extension to $\widetilde{k}$. We assume $\mathfrak{M}_{\mathrm{rm}}$ does not contain places which divide 2. For any quadratic extension F of k other than $\widetilde{k}$, the compositum of F and $\widetilde{k}$ contains exactly three quadratic extensions of k. Let F^* denote the quadratic extension other than F and $\widetilde{k}$. Take any $F\in\mathcal{Q}(L_S)$ and put $L_v^*=F^*\otimes k_v$ for $v\in S$, which does not depend on the choice of F.

Theorem 1.2 (Theorem 11.2). — *Assume $S\supset\mathfrak{M}_{\mathrm{rm}}\cup\mathfrak{M}_\infty$. Let $L_S=(L_v)_{v\in S}$ be an S-tuple. Assume two of L_v's and two of L_v^*'s are fields. Then* *the limit*

$$\lim_{X\to\infty} \frac{\sum_{F\in\mathcal{Q}(L_S,X)} h_F R_F h_{F^*} R_{F^*}}{\left(\sum_{F\in\mathcal{Q}(L_S,X)} h_F^2 R_F^2\right)^{1/2} \left(\sum_{F\in\mathcal{Q}(L_S,X)} h_{F^*}^2 R_{F^*}^2\right)^{1/2}}$$

exists, and the value is equal to

$$\prod_{v\in\mathfrak{M}_{\mathrm{in}}\setminus S} \left(1 - \frac{2q_v^{-2}}{1+q_v^{-1}+q_v^{-2}-2q_v^{-3}+q_v^{-5}}\right).$$

It is an interesting phenomenon that the value is a product of factors indexed over $\mathfrak{M}_{\mathrm{in}}$. For example, if we take $\widetilde{k}$ such that $\widetilde{k}$ splits at all the small places of k, then $h_F R_F$ and $h_{F^*} R_{F^*}$ are strongly correlated.

We prove these density theorems by applying Tauberian methods to the *zeta functions* associated with *prehomogeneous vector spaces*. In the beautiful work of Wright and Yukie [18], they showed that 8 types of prehomogeneous vector space possess significant interest in arithmetic, and laid out a program to prove a series of density theorems. One advantage to using those zeta functions is that we can prove density theorems over general number field k rather than just $\mathbb{Q}$, as we stated above.

This paper is concerned with the representation

$$G' = \mathrm{GL}(2)\times\mathrm{GL}(2)\times\mathrm{GL}(2), \quad V' = k^2\otimes k^2\otimes k^2,$$

which is referred to as the D_4 case in [18]. It was found in [18] that the principal parts of the zeta function of this type are closely related to the asymptotic behavior of the mean value of $h_F^2 R_F^2$ of quadratic extensions F/k. However, the global theory of prehomogeneous vector spaces is difficult in general and more than ten meaningful cases including the case (G', V') are left open.

Our approach to work on this topic is to consider inner forms. Let $\mathcal{B}$ be a quaternion algebra of k and $\mathcal{B}^{\mathrm{op}}$ the opposite algebra. We regard $\mathcal{B}^\times$ and $(\mathcal{B}^{\mathrm{op}})^\times$ as algebraic groups over k. In this paper, we consider the representation

$$G = \mathcal{B}^\times \times (\mathcal{B}^{\mathrm{op}})^\times \times \mathrm{GL}(2), \quad V = \mathcal{B}\otimes k^2,$$

which is an inner form of (G, V). Note that if $\mathcal{B}$ splits then (G, V) is equivalent to (G', V'). We call (G, V) the *space of pairs of quaternion algebras*. As we saw in [15], the orbit space of V also carries a rich structure. The non-split case is useful because the global theory becomes much easier than the split case. In this paper we consider (G, V) when $\mathcal{B}$ is a division algebra over a number field k. For this case, we determined the principal parts of the global zeta function in [15] using Fourier analysis. The necessary result is quoted in Theorem 4.2.

As we will see in Proposition 10.3, the global zeta function is still an approximation of the counting function of $h_F^2 R_F^2$ for quadratic extensions, and we could not directly deduce Theorem 1.1 from the global theory [15]. The aim of this paper is to fill this gap by carrying out what is called the *filtering process* originally developed by Datskovsky and Wright [3] and Datskovsky [1]. This process requires a local theory in some detail. We consider the localizations of (G, V) at each place of k. We note that the localizations of (G, V) are equivalent to (G', V') all but finite number of places at which $\mathcal{B}$ ramifies. An outer form of the representation (G', V') is studied by Kable and Yukie [6, 7, 8] and some of their results are useful for us. After we prove Theorem 10.12, we study the correlation coefficient in the final section combined with the results of [6].

In the filtering process we change the order of limits and at present this hides information of the error term in Theorem 1.1. It is likely that the zeta function possesses much information on the error term also and the improvement of the filtering process is an important problem.

We note that there are several approach to studying the distributions of class numbers of quadratic extensions. In particular there is a good deal of works on moments of $h_F R_F$ for quadratic fields F over $\mathbb{Q}$. For example, Granville and Soundararajan [4] recently obtained the mean value of a general complex power of $h_F R_F$ for quadratic fields F with an estimate of the error term, which is much stronger than that given in Theorem 1.1 when $k = \mathbb{Q}$. Their method is based on the Pólya-Vinogradov inequality which is understood only over $\mathbb{Q}$, and it may be difficult to apply their method to other base fields. Some related results on relative quadratic extensions over a totally real field of class number one were obtained by Peter [10] using the space of binary quadratic forms. The approach using zeta functions of prehomogeneous vector spaces is completely independent of the base number field, and the study with non-principal characters will provide further arithmetic results as in [2], [5], or [12, 13]. All of these methods have different strengths and technical improvements will yield further arithmetic information in each approach.

For the remainder of this section, we will give the contents of the paper. In Section 2, we introduce the notations used throughout the paper. More specialized notations are introduced when required. In Section 3, we define the space of pairs of quaternion algebras, and recall from [15] its basic properties. In Section 4, we first define various invariant measures on the groups and the representation spaces. After that we introduce the global zeta function and review its analytic properties.

From Section 5 to Section 9, we consider the local theory. We establish the necessary local theory to obtain the density theorem in these sections. In Section 5, we define a measure on the stabilizer of semi-stable points, which is in some sense canonical. In Section 6, we define the local zeta function and the local density. Also we quote from [6] an estimate of the standard local zeta function, which we need in order to apply the filtering process in the proof of the mean value theorem in Section 10. In Sections 7, 8 and 9, we compute the local densities. Section 7 is for finite unramified places (the places $\mathcal{B}$ splits), Section 8 for finite ramified places, and Section 9 for infinite places. The unramified cases were almost done in [6, 7] and we essentially quote their result, but we will give a refinement for dyadic places by applying the method developed in [14]. After that we study the ramified cases.

In Section 10 we go back to the adelic situation. We first define some invariant measures and show that our zeta function is more or less the counting function of the unnormalized Tamagawa numbers of the stabilizers. After that we apply the filtering process to our case and find the mean value of the Tamagawa numbers. Then with an explicit computation, we give a formula for the mean value of the square of class numbers times regulators for a certain family of quadratic extensions. In Section 11, we define the correlation coefficient of class number times regulator of quadratic extensions. Then we explicitly compute the value in some cases by combining the results of [6] and this paper.

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2. Notation

In this section we collect basic notations used throughout in this paper.

If X is a finite set then $\#X$ will denote its cardinality. The standard symbols $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}$ will denote respectively the rational, real and complex numbers and the rational integers. The set of positive real numbers is denoted $\mathbb{R}_+$. For a complex number z, let $\Re(z), \Im(z)$ and $\bar{z}$ be the real part, the imaginary part, and the complex conjugate of z. If R is any ring then $R^\times$ is the set of invertible elements of R, and if V is a scheme defined over R and S is an R-algebra then V_S denotes its S-rational points. Let us denote by $\mathrm{M}(2,2)$ the set of 2×2 matrices.

We fix an algebraic number field k. Let $\mathfrak{M}$, $\mathfrak{M}_\infty$, $\mathfrak{M}_{\mathrm{f}}$, $\mathfrak{M}_{\mathrm{dy}}$, $\mathfrak{M}_{\mathbb{R}}$ and $\mathfrak{M}_{\mathbb{C}}$ denote respectively the set of all places of k, all infinite places, all finite places, all dyadic places (those dividing the place 2 of $\mathbb{Q}$), all real places and all complex places. Let $\mathcal{O}$ be the ring of integers of k. If $v \in \mathfrak{M}$ then k_v denotes the completion of k at v and $|\ |_v$ or $|\ |_{k_v}$ denotes the normalized absolute value on k_v. If $v \in \mathfrak{M}_{\mathrm{f}}$ then $\mathcal{O}_v$ denotes the ring of integers of k_v, $\mathfrak{p}_v$ the maximal ideal of $\mathcal{O}_v$ and q_v the cardinality of $\mathcal{O}_v/\mathfrak{p}_v$. For $t \in k_v^\times$, we define $\mathrm{ord}_v(t)$ so that $|t|_v = q_v^{-\mathrm{ord}_v(t)}$. For a practical purpose in Sections 7 and 8, we do *not* fix a uniformizer in $\mathcal{O}_v$ here. For any separable quadratic algebra L_v of k_v, let $\mathcal{O}_{L_v}$ denote the ring of integral elements of L_v. That is, if L_v is a quadratic extension then $\mathcal{O}_{L_v}$ is the integer ring of L_v and if $L_v = k_v \times k_v$ then $\mathcal{O}_{L_v} = \mathcal{O}_v \times \mathcal{O}_v$.

If k_1/k_2 is a finite extension of either local fields or number fields then we shall write Δ_{k_1/k_2} for the relative discriminant of the extension; it is an ideal in the ring of integers of k_2. For conventions, we let $\Delta_{k_2\times k_2/k_2}$ be the integer ring of k_2. If the extension k_1/k_2 is of number fields, let $\mathcal{N}(\Delta_{k_1/k_2})$ be the absolute norm of Δ_{k_1/k_2}. The symbol Δ_{k_1} will stand for $\mathcal{N}(\Delta_{k_1/\mathbb{Q}})$, the classical absolute discriminant of k_1 over $\mathbb{Q}$. We use the notation N_{k_1/k_2} for the norm in k_1/k_2.

Returning to k, we let r_1, r_2, h_k, R_k and e_k be respectively the number of real places, the number of complex places, the class number, the regulator and the number of roots of unity contained in k. It will be convenient to set

$$\mathfrak{C}_k = 2^{r_1}(2\pi)^{r_2} h_k R_k e_k^{-1}.$$

We refer to [17] as the basic reference for fundamental properties on adeles. The ring of adeles and the group of ideles are denoted by $\mathbb{A}$ and $\mathbb{A}^\times$, respectively. The adelic absolute value $|\ |$ on $\mathbb{A}^\times$ is normalized so that, for $t \in \mathbb{A}^\times$, $|t|$ is the module of multiplication by t with respect to any Haar measure dx on $\mathbb{A}$, *i.e.*, $|t| = d(tx)/dx$. Let $\mathbb{A}^0 = \{t \in \mathbb{A}^\times \mid |t| = 1\}$. Suppose $[k:\mathbb{Q}] = n$. For $\lambda \in \mathbb{R}_+$, $\underline{\lambda} \in \mathbb{A}^\times$ is the idele whose component at any infinite place is $\lambda^{1/n}$ and whose component at any finite place is 1. Then we have $|\underline{\lambda}| = \lambda$.

For a finite extension L/k, let $\mathbb{A}_L$ denote the adele ring of L. We define $\mathbb{A}_L^\times, \mathbb{A}_L^0, \mathfrak{C}_L$ etc., similarly. The adelic absolute value of L is denoted by

$|\ |_L$. There is a natural inclusion $\mathbb{A} \to \mathbb{A}_L$, under which an adele $(a_v)_v$ corresponds to the adele $(b_w)_w$ with $b_w = a_v$ if $w|v$. Using the identification $L \otimes_k \mathbb{A} \cong \mathbb{A}_L$, the norm map $\mathrm{N}_{L/k}$ can be extended to a map from $\mathbb{A}_L$ to $\mathbb{A}$. It is known (see p. 139 in [17]) that $|\mathrm{N}_{L/k}(t)| = |t|_L$ for $t \in \widetilde{\mathbb{A}}$. Suppose $[L:k] = m$. For $\lambda \in \mathbb{R}_+$, we denote by $\underline{\lambda}_L \in \mathbb{A}_L^\times$ the idele whose component at any infinite place is $\lambda^{1/mn}$ and whose component at any finite place is 1, so that $|\underline{\lambda}_L|_L = \lambda$. Clearly $\underline{\lambda} = \underline{\lambda}_L^m$ and hence $|\underline{\lambda}|_L = \lambda^m$. When we have to show the number field on which we consider $\underline{\lambda}$, we use the notation such as $\underline{\lambda}_k$.

If V is a vector space over k we write $V_\mathbb{A}$ for its adelization. Let $\mathcal{S}(V_\mathbb{A})$ and $\mathcal{S}(V_{k_v})$ be the spaces of Schwartz–Bruhat functions on each of the indicated domains.

For $v \in \mathfrak{M}_\mathrm{f}$, we choose a Haar measure dx_v on k_v to satisfy $\int_{\mathcal{O}_v} dx_v = 1$. We write dx_v for the ordinary Lebesgue measure if v is real, and for twice the Lebesgue measure if v is imaginary. We choose a Haar measure dx on $\mathbb{A}$ to satisfy $dx = \prod_{v\in\mathfrak{M}} dx_v$. Then $\int_{\mathbb{A}/k} dx = |\Delta_k|^{1/2}$ (see [17], p. 91).

For $v \in \mathfrak{M}_\mathrm{f}$, we normalize the Haar measure $d^\times t_v$ on $k_v^\times$ such that $\int_{\mathcal{O}_v^\times} d^\times t_v = 1$. Let $d^\times t_v(x) = |x|_v^{-1} dx_v$ if $v \in \mathfrak{M}_\infty$. We choose a Haar measure $d^\times t$ on $\mathbb{A}^\times$ so that $d^\times t = \prod_{v\in\mathfrak{M}} d^\times t_v$. Using this measure, we choose a Haar measure $d^\times t^0$ on $\mathbb{A}^0$ by

$$\int_{\mathbb{A}^\times} f(t)\, d^\times t = \int_0^\infty \int_{\mathbb{A}^0} f(\underline{\lambda} t^0)\, d^\times \lambda d^\times t^0,$$

where $d^\times\lambda = \lambda^{-1} d\lambda$. Then $\int_{\mathbb{A}^0/k^\times} d^\times t^0 = \mathfrak{C}_k$ (see [17], p. 95).

Let $\zeta_k(s)$ be the Dedekind zeta function of k. We define

$$Z_k(s) = |\Delta_k|^{s/2} \left(\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\right)^{r_1} \left((2\pi)^{1-s}\Gamma(s)\right)^{r_2} \zeta_k(s)\,.$$

This definition differs from that in [17], p.129 by the inclusion of the $|\Delta_k|^{s/2}$ factor. It is adopted here as the most convenient for our purposes. It is known ([17], p.129) that

$$\mathrm{Res}_{s=1}\, \zeta_k(s) = |\Delta_k|^{-\frac{1}{2}} \mathfrak{C}_k \quad \text{and so} \quad \mathrm{Res}_{s=1}\, Z_k(s) = \mathfrak{C}_k\,.$$

Let $\mathbb{H}$ denote the quaternion algebra of Hamiltonians over $\mathbb{R}$. We choose and fix an element $j \in \mathbb{H}$ so that $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$ as a left vector space over $\mathbb{C}$ and the multiplication law is given by $j^2 = -1$ and $j\alpha = \bar{\alpha} j$ for $\alpha \in \mathbb{C}$. Let us express elements of $\mathbb{H}$ as $x = x_1 + x_2 j$ where $x_1, x_2 \in \mathbb{C}$. We choose a Haar measure on $\mathbb{H}$ so that $dx = dx_1 dx_2$, where dx_1 and dx_2 are twice the Lebesgue measure on $\mathbb{C}$ as above. If we let $|x|_\mathbb{H} = |x_1|_\mathbb{C} + |x_2|_\mathbb{C}$, then $|x|_\mathbb{H}^{-2} dx$ defines a Haar measure on $\mathbb{H}^\times$. For practical purposes, we choose $d^\times t(x) = \pi^{-1}|x|_{\mathbb{H}}^{-2}dx$ as the normalized measure on $\mathbb{H}^\times$. We put $\mathbb{H}^0 = \{t \in \mathbb{H}^\times \mid |t|_{\mathbb{H}} = 1\}$.

3. Review of the space of pairs of quaternion algebras

In this section, we define the prehomogeneous vector space of pairs of quaternion algebras which are at the heart of this work and reviewing their fundamental properties. Arithmetic plays no role here, so in this section we consider the representation over an arbitrary field K. We later use the result in this section both local and global situations.

Let $\mathcal{B}$ be a quaternion algebra over K. This algebra is isomorphic to either the algebra $\mathrm{M}(2,2)$ consisting of 2×2 matrices or a division algebra of dimension 4. Let $\mathcal{T}$ and $\mathcal{N}$ be the reduced trace and the reduced norm, respectively. We denote by $\mathcal{B}^{\mathrm{op}}$ the opposite algebra of $\mathcal{B}$. We introduce a group G_1 and its linear representation on $\mathcal{B}$ as follows. Let

$$G_{11} = \mathcal{B}^\times, \quad G_{12} = (\mathcal{B}^{\mathrm{op}})^\times, \quad \text{and} \quad G_1 = G_{11}\times G_{12}.$$

That is, G_1 is equal to $\mathcal{B}^\times\times\mathcal{B}^\times$ set theoretically and the multiplication law is given by $(g_{11}, g_{12})(h_{11}, h_{12}) = (g_{11}h_{11}, h_{12}g_{12})$. If there is no confusion, we drop 'op' and simply write $G_{12} = \mathcal{B}^\times$ instead. We regard G_1 as an algebraic group over K. The quaternion algebra $\mathcal{B}$ can be considered as a vector space over K. We define the action of G_1 on $\mathcal{B}$ as follows:

$$(g_1, w) \longmapsto g_{11}wg_{12}, \qquad g_1 = (g_{11}, g_{12}) \in G_1, w \in \mathcal{B}.$$

This defines a representation $\mathcal{B}$ of G_1. We consider the standard representation of $G_2 = \mathrm{GL}(2)$ on K^2. The group $G = G_1 \times G_2$ acts naturally on $V = \mathcal{B}\otimes K^2$. The representation (G, V) is the main object of this paper. This is a K-form of

$$(\mathrm{GL}(2)\times \mathrm{GL}(2)\times \mathrm{GL}(2), K^2\otimes K^2\otimes K^2), \tag{3.1}$$

and if $\mathcal{B}$ is split, (G, V) is equivalent to the above representation over K. The representation (3.1) was studied in [18] in some detail, and our review is a slight generalization [15] of that.

We describe the action more explicitly. Throughout this paper, we express elements of $V \cong \mathcal{B}\oplus\mathcal{B}$ as $x = (x_1, x_2)$. We identify $x = (x_1, x_2) \in V$ with the liner form $x(v) = v_1x_1 + v_2x_2$ in two variables $v = (v_1, v_2)$. Then the action of $g = (g_{11}, g_{12}, g_2) \in G$ on $x \in V$ is defined by

$$(gx)(v) = g_{11}x(vg_2)g_{12}.$$

We put $F_x(v) = \mathcal{N}(x(v))$. This is a binary quadratic form in variables $v = (v_1, v_2)$. We let $P(x)$ $(x \in V)$ be the discriminant of $F_x(v)$, which is a polynomial in V. That is, if we express $F_x(v) = a_0(x)v_1^2 + a_1(x)v_1v_2 + a_2(x)v_2^2$, then $P(x)$ is given by $P(x) = a_1(x)^2 - 4a_0(x)a_2(x)$. Let χ_i $(i = 1, 2)$ be the character of G_i defined by

$$\chi_1(g_1) = \mathcal{N}(g_{11})\mathcal{N}(g_{12}), \quad \chi_2(g_2) = \det g_2,$$

respectively. We define $\chi(g) = \chi_1(g_1)^2\chi_2(g_2)^2$. Then one can easily see that

$$P(gx) = \chi(g)P(x)$$

and hence $P(x)$ is a relative invariant polynomial with respect to the character χ. Let $V^{\mathrm{ss}} = \{x \in V \mid P(x) \neq 0\}$, and we call this the set of semi-stable points. That is, $x \in V$ is semi-stable if and only if $F_x(v)$ does not have a multiple root in $\mathbb{P}^1 = \{(v_1 : v_2)\}$.

Let $\widetilde{T} = \ker(G \to \mathrm{GL}(V))$. Then it is easy to see that

$$\widetilde{T} = \{(t_{11}, t_{12}, t_2) \mid t_{11}, t_{12}, t_2 \in \mathrm{GL}(1), t_{11}t_{12}t_2 = 1\},$$

which is contained in the center of G. Throughout this paper, we will identify $\widetilde{T}$ with $\mathrm{GL}(1)^2$ via the map

$$\widetilde{T} \longrightarrow \mathrm{GL}(1)^2, \qquad (t_{11}, t_{12}, (t_{11}t_{12})^{-1}) \longmapsto (t_{11}, t_{12}).$$

We are now ready to recall the description of the space of non-singular G_K-orbits in V_K.

Definition 3.1. — *For $x \in V_K^{\mathrm{ss}}$, we define*

$$Z_x = \operatorname{Proj} K[v_1, v_2]/(F_x(v)),$$
$$\widetilde{K}(x) = \Gamma(Z_x, \mathcal{O}_{Z_x}).$$

That is, $\widetilde{K}(x)$ is the global section of the scheme Z_x. Also we define $K(x)$ to be the splitting field of $F_x(v)$.

Note that $\widetilde{K}(x)$ may not be a field. Since V_K^{ss} is the set of x such that F_x does not have a multiple root, Z_x is a reduced scheme over K and $\widetilde{K}(x)$ is a separable commutative K-algebra of dimension 2. By definition, we immediately see that $\widetilde{K}(x) \cong K \times K$ if $F_x(v)$ has K-rational factors and $\widetilde{K}(x) \cong K(x)$ if $F_x(v)$ is irreducible over K.

The following lemma is useful for our practical purposes. For the proof, see [15, Lemma 3.3].

Lemma 3.2. — *Any G_K-orbit in V_K^{ss} contains an element of the form $w_u = (1, u)$ for some $u \in \mathcal{B}_K$.*

Note that for $u \in \mathcal{B}_K$, $w_u = (1, u)$ is semi-stable if and only if u is a separable quadratic element of $\mathcal{B}_K$. For $w_u \in V_K^{\mathrm{ss}}$, $F_{w_u}(v_1, -1) = \mathcal{N}(v_1 - u)$ is the characteristic polynomial of u and hence $\widetilde{K}(x)$ is isomorphic to $K[u] \subset \mathcal{B}_K$ as a K-algebra.

Definition 3.3. — *Let $\mathcal{A}_2(\mathcal{B}_K)$ be the set of isomorphism classes of separable commutative K-algebras of dimension 2 that are embeddable into $\mathcal{B}_K$.*

Note that if $\mathcal{B}_K$ is non-split, then any element of $\mathcal{A}_2(\mathcal{B}_K)$ is a quadratic extension of K. The following proposition is proved in [18], [15].

Proposition 3.4. — *The map $x \mapsto \widetilde{K}(x)$ gives a bijection between $G_K \backslash V_K^{\mathrm{ss}}$ and $\mathcal{A}_2(\mathcal{B}_K)$.*

For $x \in V_K^{\mathrm{ss}}$, let G_x be the stabilizer of x and G_x° its identity component. Both are algebraic groups defined over K. We have shown in [15] that $G_x^\circ \cong (\mathrm{GL}(1)_{\widetilde{K}(x)})^2$ as an algebraic group over K. We close this section with a detailed description of the K-rational points of the stabilizer G_{w_u}.

We first recall the isomorphism $G_{w_u\,K}^\circ \cong (K[u]^\times)^2$. Since $\{1, u\}$ is a basis of $K[u]$ as a K-vector space, for any $s_1, s_2 \in K[u]^\times$, $\{s_1 s_2, s_1 s_2 u\}$ is also a K-basis of $K[u]$. Hence there exists a unique element $g_{s_1 s_2} \in \mathrm{GL}(2)_K$ such that $g_{s_1 s_2}\, {}^t(s_1 s_2, s_1 s_2 u) = {}^t(1, u)$. Since $K[u]$ is a commutative algebra, $s_1 s_2 u = s_1 u s_2$. Therefore we have $(s_1, s_2, g_{s_1 s_2}) \in G_{w_u\,K}$. The following proposition is proved in [15, Lemma 3.4].

Proposition 3.5. — *The map*

$$\psi_u \colon (K[u]^\times)^2 \longrightarrow G_{w_u\,K}^\circ, \qquad (s_1, s_2) \longmapsto (s_1, s_2, g_{s_1 s_2})$$

gives an isomorphism of the two groups.

Finally we consider the structure of $G_{w_u\,K}/G_{w_u\,K}^\circ$. Let σ be the non-trivial K-automorphism of $K[u]$. Then there exists $\nu \in \mathcal{B}_K \setminus K[u]$ such that $\nu^2 \in K$, $\mathcal{B}_K = K[u] \oplus K[u]\nu$ as a $K[u]$-vector space, and the multiplication law is given by $\nu\alpha = \alpha^\sigma \nu$ for $\alpha \in K[u]$.

Proposition 3.6. — *Let $a = u + u^\sigma \in K$. We have $[G_{w_u\,K} : G_{w_u\,K}^\circ] = 2$ and $G_{w_u\,K}/G_{w_u\,K}^\circ$ is generated by the class of $\tau = \left(\nu^{-1}, \nu, \begin{pmatrix} 1 & 0 \\ a & -1 \end{pmatrix}\right)$.*

Proof. — A simple computation shows $\tau w_u = w_u$. On the other hand, by [18] we have $[G_{w_u\,\bar{K}} : G_{w_u\,\bar{K}}^\circ] = 2$ because (G, V) is a K-form of (3.1). Since $[G_{w_u\,K} : G_{w_u\,K}^\circ] \leqslant [G_{w_u\,\bar{K}} : G_{w_u\,\bar{K}}^\circ]$, the proposition follows. □

By Lemma 3.2, we have $[G_{x\,K} : G_{x\,K}^\circ] = 2$ for any $x \in V_K^{\mathrm{ss}}$.

4. Invariant measures and the global zeta function

For the rest of this paper, we fix a number field k and a non-split quaternion algebra $\mathcal{B}$ over k. In this section, we define various invariant measures in both local and adelic situations and summarize the necessary results. For the proof, see [16] for example. In this paper, we always choose the adelic measure as the product of local measures. After that we introduce the global zeta function of the prehomogeneous vector space (G, V) and recall from [15] its most basic analytic properties.

We define $\mathfrak{M}_{\mathcal{B}}$ to be the set of places v of k such that $\mathcal{B}$ is ramified at v. For $v \in \mathfrak{M}$, let $\mathcal{B}_v$ denote $\mathcal{B} \otimes_k k_v$. Then, by definition, $v \in \mathfrak{M}_{\mathcal{B}}$ if and only if $\mathcal{B}_v$ is a division algebra. It is well known that $\mathfrak{M}_{\mathcal{B}}$ is a finite set.

We give a normalization of invariant measure on G_{k_v} and V_{k_v}. First we consider the places $v \notin \mathfrak{M}_{\mathcal{B}}$. For each of these v, we fix once and for all a k_v-isomorphism $\mathcal{B}_v \cong \mathrm{M}(2,2)_{k_v}$ and identify these algebras. Then

$$G_{k_v} = \mathrm{GL}(2)_{k_v} \times \mathrm{GL}(2)_{k_v} \times \mathrm{GL}(2)_{k_v}, \qquad V_{k_v} = \mathrm{M}(2,2)_{k_v} \oplus \mathrm{M}(2,2)_{k_v}.$$

We choose a Haar measure dx_v on V_{k_v} so that

$$dx_v = dx_{1v}dx_{2v}, \qquad dx_{iv} = dx_{i11v}dx_{i12v}dx_{i21v}dx_{i22v} \quad (i = 1, 2)$$

for

$$x_v = (x_{1v}, x_{2v}), \qquad x_{iv} = \begin{pmatrix} x_{i11v} & x_{i12v} \\ x_{i21v} & x_{i22v} \end{pmatrix} \quad (i = 1, 2).$$

For $v \in \mathfrak{M}_{\mathrm{f}}$, we put $V_{\mathcal{O}_v} = \mathrm{M}(2,2)_{\mathcal{O}_v} \oplus \mathrm{M}(2,2)_{\mathcal{O}_v}$, which is a compact subgroup of V_{k_v}. We note that $\int_{V_{\mathcal{O}_v}} dx = 1$. If $v \in \mathfrak{M}_{\mathrm{f}}$, we put a maximal compact subgroup $\mathcal{K}_v$ of G_{k_v} as

$$\mathcal{K}_v = \mathrm{GL}(2)_{\mathcal{O}_v} \times \mathrm{GL}(2)_{\mathcal{O}_v} \times \mathrm{GL}(2)_{\mathcal{O}_v},$$

and normalize the measure dg_v on G_{k_v} so that the total volume of $\mathcal{K}_v$ is 1. For $v \in \mathfrak{M}_\infty$, we first give a measure for $\mathrm{GL}(2)_F$ where $F = \mathbb{R}$ or $\mathbb{C}$. As in Section 2, we shall take Lebesgue measure to be the standard measure on the real numbers and twice the Lebesgue measure to be the standard measure on the complex numbers. If $h_v = (h_{ijv})_{1 \leqslant i,j \leqslant 2}$, then $d\mu(h_v) = dh_{11v}dh_{12v}dh_{21v}dh_{22v}/|\det(h_v)|_F^2$ defines a Haar measure on $\mathrm{GL}(2)_F$. We put $dh_v = p_F d\mu(h_v)$ where $p_{\mathbb{R}} = \pi^{-1}$ and $p_{\mathbb{C}} = (2\pi)^{-1}$. Using this measure, we define dg_v for $v \in \mathfrak{M}_\infty$ as $dg_v = dg_{11v}dg_{12v}dg_{2v}$ where $g = (g_{11}, g_{12}, g_2) \in G_{k_v} = (\mathrm{GL}(2)_{k_v})^3$.

Next we consider the case $v \in \mathfrak{M}_{\mathcal{B}}$. For $v \in \mathfrak{M}_{\mathrm{f}}$, let $\mathcal{O}_{\mathcal{B}_v}$ be the ring consisting of integral elements of $\mathcal{B}_v$. We put $V_{\mathcal{O}_v} = \mathcal{O}_{\mathcal{B}_v} \oplus \mathcal{O}_{\mathcal{B}_v}$, which is a compact subgroup of V_{k_v}. We choose a Haar measure dx_v on $V_{k_v} = \mathcal{B}_v \oplus \mathcal{B}_v$

so that the volume of $\mathcal{O}_{\mathcal{B}_v} \oplus \mathcal{O}_{\mathcal{B}_v}$ is 1. Also we put a maximal compact subgroup $\mathcal{K}_v$ of G_{k_v} as

$$\mathcal{K}_v = \mathcal{O}_{\mathcal{B}_v}^{\times} \times \mathcal{O}_{\mathcal{B}_v}^{\times} \times \mathrm{GL}(2)_{\mathcal{O}_v},$$

and normalize the measure dg_v on G_{k_v} so that the total volume of $\mathcal{K}_v$ is 1.

Now the remaining case is for $v \in \mathfrak{M}_\infty \cap \mathfrak{M}_{\mathcal{B}}$, which is an element of $\mathfrak{M}_{\mathbb{R}}$. We fix an isomorphism $\mathcal{B}_v \cong \mathbb{H}$. Then

$$G_{k_v} = \mathbb{H}^{\times} \times \mathbb{H}^{\times} \times \mathrm{GL}(2)_{\mathbb{R}}, \qquad V_{k_v} = \mathbb{H} \oplus \mathbb{H}.$$

We set measures dg_v and dx_v on G_{k_v} and V_{k_v} as the product measures, where we consider the measures on $\mathbb{H}^{\times}, \mathbb{H}$ as in Section 2 and $\mathrm{GL}(2)_{\mathbb{R}}$ as above. For $v \in \mathfrak{M}_\infty$, we put

$$\mathcal{K}_v = \begin{cases} \mathrm{O}(2,\mathbb{R})^3 & v \in \mathfrak{M}_{\mathbb{R}} \setminus \mathfrak{M}_{\mathcal{B}}, \\ \mathrm{U}(2,\mathbb{C})^3 & v \in \mathfrak{M}_{\mathbb{C}} \setminus \mathfrak{M}_{\mathcal{B}}, \\ \mathbb{H}^0 \times \mathbb{H}^0 \times \mathrm{O}(2,\mathbb{R}) & v \in \mathfrak{M}_{\mathcal{B}}, \end{cases}$$

which is a maximal compact subgroup of G_{k_v}.

Using these local measures, we define the measures dg and dx on $G_{\mathbb{A}}$ and $V_{\mathbb{A}}$ by

$$dg = \prod_{v \in \mathfrak{M}} dg_v \qquad \text{and} \qquad dx = \prod_{v \in \mathfrak{M}} dx_v.$$

If we put

$$\Delta_{\mathcal{B}} = \Delta_k^4 \prod_{v \in \mathfrak{M}_{\mathcal{B}} \cap \mathfrak{M}_{\mathrm{f}}} q_v^2 \qquad \text{and} \qquad \Delta_V = \Delta_{\mathcal{B}}^2,$$

then it is well known that the volume of $V_{\mathbb{A}}/V_k$ with respect to the measure dx is $\Delta_V^{1/2}$. Hence our choice of measure dx on $V_{\mathbb{A}}$ in this paper is $\Delta_V^{1/2}$ times that of [15], in which we defined so that the volume of $V_{\mathbb{A}}/V_k$ is equal to 1.

Our definition of measure dg_v on G_{k_v} can naturally be considered as the product measure $dg_v = dg_{11v}dg_{12v}dg_{2v}$ for $g_v = (g_{11v}, g_{22v}, g_{2v})$ and we shall do so below. For example, if $v \in \mathfrak{M}_{\mathrm{f}} \cap \mathfrak{M}_{\mathcal{B}}$, the Haar measures dg_{11v}, dg_{12v} and dg_{2v} on G_{11k_v}, G_{12k_v} and G_{2k_v} are normalized so that

$$\int_{\mathcal{O}_{\mathcal{B}_v}^{\times}} dg_{11v} = \int_{\mathcal{O}_{\mathcal{B}_v}^{\times}} dg_{12v} = \int_{\mathrm{GL}(2)_{\mathcal{O}_v}} dg_{2v} = 1.$$

We define the measures dg_{11}, dg_{12} and dg_2 on $G_{11\mathbb{A}}, G_{12\mathbb{A}}$ and $G_{2\mathbb{A}}$ by

$$dg_{11} = \prod_{v \in \mathfrak{M}} dg_{11v}, \quad dg_{12} = \prod_{v \in \mathfrak{M}} dg_{12v} \quad \text{and} \quad dg_2 = \prod_{v \in \mathfrak{M}} dg_{2v}.$$

Clearly, we have $dg = dg_{11}dg_{12}dg_2$.

Since $\widetilde{T} \cong \mathrm{GL}(1) \times \mathrm{GL}(1)$ is a split torus, the first Galois cohomology set $H^1(k', \widetilde{T})$ is trivial for any field k' containing k. This implies that the set of k'-rational point of $\widetilde{G}$ coincides with $G_{k'}/\widetilde{T}_{k'}$. Therefore $(G/\widetilde{T})_{\mathbb{A}} = G_{\mathbb{A}}/\widetilde{T}_{\mathbb{A}}$ and $(G/\widetilde{T})_{\mathbb{A}}/(G/\widetilde{T})_k = G_{\mathbb{A}}/\widetilde{T}_{\mathbb{A}}G_k$. We put the measures $d^\times \widetilde{t}_v$ and $d^\times \widetilde{t}$ on $\widetilde{T}_v$ and $\widetilde{T}_{\mathbb{A}}$ respectively to satisfy $d^\times \widetilde{t}_v = d^\times t_{1v} d^\times t_{2v}$, $d^\times \widetilde{t} = d^\times t_1 d^\times t_2$ for $\widetilde{t}_v = (t_{1v}, t_{2v}, (t_{1v}t_{2v})^{-1}), \widetilde{t} = (t_1, t_2, (t_1t_2)^{-1})$. Using these, we normalize the invariant measure $d\widetilde{g}_v$ and $d\widetilde{g}$ on G_{k_v}/T_{k_v} and $G_{\mathbb{A}}/T_{\mathbb{A}}$ so that $dg_v = d\widetilde{g}_v d^\times \widetilde{t}_v, dg = d\widetilde{g} d^\times \widetilde{t}$. Note that $d\widetilde{g} = \prod_{v \in \mathfrak{M}} d\widetilde{g}_v$ since $dg = \prod_{v \in \mathfrak{M}} dg_v$ and $d^\times \widetilde{t} = \prod_{v \in \mathfrak{M}} d^\times \widetilde{t}_v$.

We put

$$G^0_{1i\mathbb{A}} = \{g_{1i} \in G_{1i\mathbb{A}} \mid |\mathcal{N}(g_{1i})| = 1\} \quad (i = 1, 2),$$
$$G^0_{2\mathbb{A}} = \{g_2 \in G_{2\mathbb{A}} \mid |\det(g_2)| = 1\}.$$

Then the maps

$$\mathbb{R}_+ \times G^0_{1i\mathbb{A}} \longrightarrow G_{1i\mathbb{A}}, \quad (\lambda_{1i}, g^0_{1i}) \longmapsto \underline{\lambda_{1i}} g^0_{1i} \quad (i = 1, 2),$$
$$\mathbb{R}_+ \times G^0_{2\mathbb{A}} \longrightarrow G_{2\mathbb{A}}, \quad (\lambda_2, g^0_2) \longmapsto \underline{\lambda_2} g^0_2,$$

give isomorphisms of these groups. We choose Haar measures dg^0_{1i} and dg^0_2 on $G^0_{1i\mathbb{A}}$ and $G^0_{2\mathbb{A}}$ so that $dg_{1i} = 2d^\times \lambda_{1i} dg^0_{1i}, dg_2 = 2d^\times \lambda_2 dg^0_2$. Then it is known that

$$\int_{G^0_{1i\mathbb{A}}/G_{1ik}} dg^0_{1i} = \Delta_k^{1/2} \mathfrak{C}_k Z_k(2) \prod_{v \in \mathfrak{M}_{\mathcal{B}} \cap \mathfrak{M}_{\mathrm{f}}} (q_v - 1),$$
$$\int_{G^0_{2\mathbb{A}}/G_{2k}} dg^0_2 = \Delta_k^{1/2} \mathfrak{C}_k Z_k(2).$$

We now define the global zeta function.

Definition 4.1. — *For $\Phi \in \mathcal{S}(V_{\mathbb{A}})$ and a complex variable s, we define*

$$Z(\Phi, s) = \int_{G_{\mathbb{A}}/\widetilde{T}_{\mathbb{A}}G_k} |\chi(\widetilde{g})|^s \sum_{x \in V_k^{\mathrm{ss}}} \Phi(\widetilde{g}x)\, d\widetilde{g},$$

and call it the global zeta function.

It is known that the integral converges if $\Re(s)$ is sufficiently large and can be continued meromorphically to the whole complex plane. In [15], we described the principal parts of $Z(\Phi, s)$ by means of certain distributions. However, we used a slightly different formulation in [15], and we need some arguments to translate the results from that paper. Also, in this paper we only consider the rightmost pole of $Z(\Phi, s)$ because this is enough to deduce the density theorems.

We put $G^0_{1\mathbb{A}} = G^0_{11\mathbb{A}} \times G^0_{12\mathbb{A}}$. The domain of integration used in [15] is $\mathbb{R}_+ \times G^0_{\mathbb{A}}/G_k$, where $G^0_{\mathbb{A}} = G^0_{1\mathbb{A}} \times G^0_{2\mathbb{A}}$. Let $\widetilde{T}^0_{\mathbb{A}} = G^0_{\mathbb{A}} \cap \widetilde{T}_{\mathbb{A}}$. Then we have

$$(\mathbb{R}_+ \times G^0_{\mathbb{A}})/\widetilde{T}^0_{\mathbb{A}} \cong G_{\mathbb{A}}/\widetilde{T}_{\mathbb{A}}$$

via the map which sends the class of $(\lambda, g^0_{11}, g^0_{12}, g^0_2)$ to class of $(g^0_{11}, g^0_{12}, \underline{\lambda} g^0_2)$. In [15] $\mathbb{R}_+ \times G^0_{\mathbb{A}}$ is made to act on $V_{\mathbb{A}}$ by assuming that $(\lambda, 1)$ acts by multiplication by $\underline{\lambda}$, and the above isomorphism is compatible with the actions of the two groups on $V_{\mathbb{A}}$. We will compare the measure $d\widetilde{g}$ on $G_{\mathbb{A}}/\widetilde{T}_{\mathbb{A}}$ with the measure $d^\times\lambda dg^0$ on $\mathbb{R}_+ \times G^0_{\mathbb{A}}$ used in [15]. The argument in [15] is valid for any choice of measure on $G^0_{1\mathbb{A}}$ and we consider $dg^0_{11}dg^0_{12}$. We note that the measure dg^0_2 on $G^0_{2\mathbb{A}}$ in the present situation is $\Delta_k^{1/2}\mathfrak{C}_k^2$ times that of used in [15].

We have $G_{\mathbb{A}}/\widetilde{T}_{\mathbb{A}} \cong (\mathbb{R}^3_+ \times G^0_{\mathbb{A}})/(\mathbb{R}^2_+ \times \widetilde{T}^0_{\mathbb{A}})$ where $\mathbb{R}^2_+ \times \widetilde{T}^0_{\mathbb{A}}$ is included in $\mathbb{R}^3_+ \times G^0_{\mathbb{A}}$ via $(\lambda_1, \lambda_2, \widetilde{t}^0) \mapsto (\lambda_1, \lambda_2, \lambda_1^{-1}\lambda_2^{-1}, \widetilde{t}^0)$ and $\mathbb{R}^3_+ \times G^0_{\mathbb{A}}$ maps onto $G_{\mathbb{A}}/\widetilde{T}_{\mathbb{A}}$ via $(\lambda_1, \lambda_2, \lambda_3, g^0) \mapsto (\underline{\lambda_1}, \underline{\lambda_2}, \underline{\lambda_3})g^0$. With this identification we have chosen the measure $d\widetilde{g}$ to be compatible with the measure $8d^\times\lambda_1 d^\times\lambda_2 d^\times\lambda_3 dg^0$ on $\mathbb{R}^3_+ \times G^0_{\mathbb{A}}$ and $d^\times\lambda_1 d^\times\lambda_2 d^\times\widetilde{t}^0$ on $\mathbb{R}^2_+ \times \widetilde{T}^0_{\mathbb{A}}$, where the volume of $\widetilde{T}^0_{\mathbb{A}}/\widetilde{T}_k$ under $d^\times\widetilde{t}^0$ is $\mathfrak{C}_k^2$. Moreover, $|\chi(1, \underline{\lambda})| = \lambda^4$, and so if $Z^*(\Phi, s)$ denotes the zeta function studied in [15], then we have $Z(\Phi, s) = 8\Delta_k^{1/2} Z^*(\Phi, 4s)$. In [15], it is shown that $Z^*(\Phi, s)$ has a holomorphic continuation to the region $\Re(s) > 6$ except for a possible simple pole at $s = 8$ with residue

$$Z_k(2)\mathfrak{C}_k^{-1} \int_{G^0_{1\mathbb{A}}/G_{1k}} dg^0_{11} dg^0_{12} \cdot \int_{V_{\mathbb{A}}} \Phi(x)dx.$$

where the measure dx on $V_{\mathbb{A}}$ is $\Delta_V^{-1/2}$ times that of in this paper. Thus we arrive at:

Theorem 4.2. — *Assume that the Schwartz-Bruhat function $\Phi \in \mathcal{S}(V_{\mathbb{A}})$ has a product form $\Phi = \otimes_{v\in\mathfrak{M}}\Phi_v$ and each $\Phi_v \in \mathcal{S}(V_{k_v})$ is $\mathcal{K}_v$-invariant. The zeta function $Z(\Phi, s)$ has a meromorphic continuation to the region $\mathrm{Re}(s) > 3/2$ only with a possible simple pole at $s = 2$ with residue*

$$\mathcal{R}_1 \prod_{v\in\mathfrak{M}} \int_{V_{k_v}} \Phi_v(x_v)dx_v,$$

where we put

$$\mathcal{R}_1 = 2\Delta_k^{-5/2}\mathfrak{C}_k Z_k(2)^3 \prod_{v\in\mathfrak{M}_{\mathrm{f}}\cap\mathfrak{M}_{\mathcal{B}}} (1 - q_v^{-1})^2.$$

This completes our review of the analytic properties of the global zeta function. To arrive at the density theorem from this, we need various preparations from local theory. We do it in the next five sections.

5. The canonical measure on the stabilizer

In this section we shall define a measure on $G^{\circ}_{x\,k_v}$ for $x \in V^{\mathrm{ss}}_{k_v}$ which is canonical in the sense made precise by Proposition 5.1. Recall that there exists a unique division quaternion algebra $\mathcal{B}$ up to isomorphism over a local field F other than $\mathbb{C}$, and that for any separable quadratic extension L/F, there exists a injective homomorphism $L \to \mathcal{B}$ of F-algebras. Hence by Proposition 3.4, the set of rational orbits $G_k \backslash V^{\mathrm{ss}}_{k_v}$ corresponds to the set of all separable quadratic algebras of k_v if $v \notin \mathfrak{M}_{\mathcal{B}}$ and to the set of all separable quadratic extensions of k_v if $v \in \mathfrak{M}_{\mathcal{B}}$.

Following [6], we attach to each orbit in $V^{\mathrm{ss}}_{k_v}$ where $v \in \mathfrak{M}$, an index or type which records the arithmetic properties of v and the extension of k_v corresponding to the orbit. The orbit corresponding to $k_v \times k_v$ will have the index (ur sp). (This case does not occur when $v \in \mathfrak{M}_{\mathcal{B}}$.) The orbit corresponding to the unique unramified quadratic extension of k_v will have the index (rm ur) or (ur ur) according as v is in $\mathfrak{M}_{\mathcal{B}}$ or not. An orbit corresponding to a ramified quadratic extension of k_v will have the index (rm rm) if $v \in \mathfrak{M}_{\mathcal{B}}$ and (ur rm) if $v \notin \mathfrak{M}_{\mathcal{B}}$.

We first give a normalization of the measure on the stabilizer $G^{\circ}_{x\,k_v}$ for elements of $V^{\mathrm{ss}}_{k_v}$ of the form $w_u = (1, u)$. We recall that $k_v[u]$ is isomorphic to either $k_v \times k_v$ or a quadratic extension of k_v as a k_v-algebra. By using this isomorphism, we can construct an isomorphism of multiplicative group

$$k_v[u]^{\times} \cong \begin{cases} k_v^{\times} \times k_v^{\times} & w_u \text{ has type (ur sp)}, \\ L^{\times}_{v,w_u} & \text{otherwise}, \end{cases}$$

where L_{v,w_u} is the splitting field of $F_{w_u}(v)$ if this quadratic form is irreducible. Using the normalized measure of $k_v^{\times}$ and $L^{\times}_{v,w_u}$ in Section 2 (we consider the product measure on $k_v^{\times} \times k_v^{\times}$), we induce a measure $d^{\times}_u s$ on $k_v[u]^{\times}$ as the pullback measure via the above isomorphism. We note that this normalization does not depend on the choice of the isomorphism.

For an element of the form $w_u = (1, u)$, we constructed an isomorphism

$$\psi_u \colon k_v[u]^{\times} \times k_v[u]^{\times} \longrightarrow G^{\circ}_{w_u\,k_v}, \qquad (s_1, s_2) \longmapsto g''_{w_u,v} = (s_1, s_2, g_{s_1 s_2})$$

in Section 3. Using this isomorphism and the product measure $d^{\times}_u s_1 d^{\times}_u s_2$ on $k_v[u]^{\times} \times k_v[u]^{\times}$, we define a Haar measure $dg''_{w_u,v}$ on $G^{\circ}_{w_u\,k_v}$ by

$$dg''_{w_u,v} = (\psi_u)_*(d^{\times}_u s_1 d^{\times}_u s_2),$$

the pushout measure. For a general element $x \in V^{\mathrm{ss}}_{k_v}$ we choose an element $g \in G_{k_v}$ so that $x = g w_u$ for some $w_u \in V^{\mathrm{ss}}_{k_v}$, which is possible by

Lemma 3.2. Then

$$i_g\colon G^\circ_{w_u\,k_v} \longrightarrow G^\circ_{x\,k_v}, \qquad g''_{w_u,v} \longmapsto g''_{x,v} = g g''_{w_u,v} g^{-1}$$

gives an isomorphism of groups. We define the measure $dg''_{x,v}$ on $G^\circ_{x\,k_v}$ by

$$dg''_{x,v} = (i_g)_*(dg''_{w_u,v}).$$

We let $d\widetilde{g}''_{x,v}$ on $G^\circ_{x\,k_v}/\widetilde{T}_{k_v}$ such that $dg''_{x,v} = d\widetilde{g}''_{x,v} d^\times \widetilde{t}_v$. Note that we defined the measure $d^\times\widetilde{t}_v$ on $\widetilde{T}_{k_v}$ in Section 3.

We have to check that these normalizations are well-defined.

Proposition 5.1.

(1) *The above definition of $dg''_{x,v}$ does not depend on the choice of u and g.*

(2) *Moreover, suppose that $x, y \in V^{\rm ss}_{k_v}$ and that $y = g_{xy}x$ for some $g_{xy} \in G_{k_v}$. Let $i_{g_{xy}} : G^\circ_{y\,k_v} \to G^\circ_{x\,k_v}$ be the isomorphism $i_{g_{xy}}(g) = g_{xy}^{-1} g g_{xy}$. Then*

$$dg''_{y,v} = i^*_{g_{xy}}(dg''_{x,v}) \qquad \text{and} \qquad d\widetilde{g}''_{y,v} = i^*_{g_{xy}}(d\widetilde{g}''_{x,v})\,.$$

Proof. — By the construction of the measures, a formal consideration shows that it is enough to prove (2) for $x = (1, u_1), y = (1, u_2)$ where $u_1, u_2 \in \mathcal{B}_{k_v}$. We write

$$g_{xy} = (\alpha, \beta, g_2), \quad g_2 = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

Then, since $y = g_{xy}x$, we have

$$\beta = (p + qu_1)^{-1} \cdot \alpha^{-1}, \quad u_2 = \alpha \cdot \frac{r + su_1}{p + qu_1} \cdot \alpha^{-1}. \tag{5.1}$$

Therefore if we let

$$\eta\colon \mathcal{B}_v \longrightarrow \mathcal{B}_v, \qquad \eta(\theta) = \alpha^{-1} \cdot \theta \cdot \alpha,$$

we have $\eta(u_2) = (r + su_1)/(p + qu_1)^{-1} \in k_v[u_1]$ and hence η induces an isomorphism of k_v-algebras

$$\eta\colon k_v[u_2] \longrightarrow k_v[u_1] \tag{5.2}$$

and an isomorphism of groups

$$\eta\colon k_v[u_2]^\times \longrightarrow k_v[u_1]^\times. \tag{5.3}$$

Since (5.2) is an isomorphism of k_v-algebras, (5.3) is a measure preserving map.

Now we show that the diagram

$$
\begin{array}{ccc}
(k_v[u_2]^\times)^2 & \xrightarrow{\psi_{u_2}} & G^\circ_{y\,k_v} \\
{\scriptstyle(\eta,\eta)}\downarrow & & \downarrow{\scriptstyle i_{g_{xy}}} \\
(k_v[u_1]^\times)^2 & \xrightarrow{\psi_{u_1}} & G^\circ_{x\,k_v}
\end{array} \tag{5.4}
$$

is commutative. Let $s_1, s_2 \in k_v[u_2]^\times$. We compare

$$\psi_{u_1} \circ (\eta,\eta)(s_1,s_2) \qquad \text{and} \qquad i_{g_{xy}} \circ \psi_{u_2}(s_1,s_2). \tag{5.5}$$

Note that by Proposition 3.5, the G_2-part of an element of G°_x is uniquely determined by its G_1-part and hence to prove the above elements are same, it is enough to verify that their G_1-parts coincide. By the definition of the maps, we immediately see

$$
\begin{aligned}
\psi_{u_1} \circ (\eta,\eta)(s_1,s_2) &= (\alpha^{-1}s_1\alpha, \alpha^{-1}s_2\alpha, *), \\
i_{g_{xy}} \circ \psi_{u_2}(s_1,s_2) &= (\alpha,\beta,g_2)^{-1}(s_1,s_2,*)(\alpha,\beta,g_2) \\
&= (\alpha^{-1}s_1\alpha, \beta s_2 \beta^{-1}, *).
\end{aligned}
$$

Note that we defined G_{12} to be the multiplicative group of the opposite algebra of $\mathcal{B}$. We consider the G_{12}-part of the latter element. By (5.1), we have

$$\alpha\beta = \alpha(p+qu_1)^{-1}\alpha^{-1} = \eta^{-1}\left((p+qu_1)^{-1}\right) \in k_v[u_2]$$

and hence commute with $s_2 \in k_v[u_2]$. Therefore $\alpha\beta s_2 = s_2\alpha\beta$ and hence $\beta s_2 \beta^{-1} = \alpha^{-1}s_2\alpha$. This shows that the G_1-parts of (5.5) coincide and hence the diagram (5.4) is commutative. Since $(\eta,\eta)\colon (k_v[u_2]^\times)^2 \to (k_v[u_1]^\times)^2$ is measure preserving, the commutativity of the above diagram establishes the first claim of (2) and the second claim follows from the observation that $i_{g_{xy}}|_{\widetilde{T}_{k_v}}$ is the identity map. □

6. The local zeta function and the local density

In this section, we make a canonical choice of a measure on the stabilizer quotient $G_{k_v}/G^\circ_{x\,k_v}$ and define the local zeta function. We also choose a standard orbital representative for each G_{k_v}-orbit in $V^{\mathrm{ss}}_{k_v}$, and define the the local density E_v for $v \in \mathfrak{M}$ which will show up later in the Euler factor in the density theorem.

We choose a left invariant measure $dg'_{x,v}$ on $G_{k_v}/G^\circ_{x\,k_v}$ such that $dg_v = dg'_{x,v}dg''_{x,v}$. Recall that we defined invariant measures dg_v and $dg''_{x,v}$ on G_{k_v} and $G^\circ_{x\,k_v}$ in Sections 4 and 5, respectively. If $g_{xy} \in G_{k_v}$ satisfies $y = g_{xy}x$

and $i_{g_{xy}}$ is the inner automorphism $g \mapsto g_{xy}^{-1} g g_{xy}$ of G_{k_v} then $i_{g_{xy}}(G^\circ_{y\,k_v}) = G^\circ_{x\,k_v}$ and so $i_{g_{xy}}$ induces a homeomorphism $G_{k_v}/G^\circ_{y\,k_v} \to G_{k_v}/G^\circ_{x\,k_v}$, which we also write $i_{g_{xy}}$.

PROPOSITION 6.1. — *We have* $i^*_{g_{xy}}(dg'_{x,v}) = dg'_{y,v}$.

Proof. — Since the group G_{k_v} is unimodular, $i^*_{g_{xy}}(dg_v) = dg_v$. On the other hand, we have $i^*_{g_{xy}}(dg''_{x,v}) = dg''_{y,v}$ by Proposition 5.1. Hence,

$$\begin{aligned} dg'_{y,v} dg''_{y,v} &= dg_v = i^*_{g_{xy}}(dg_v) \\ &= i^*_{g_{xy}}(dg'_{x,v} dg''_{x,v}) = i^*_{g_{xy}}(dg'_{x,v}) i^*_{g_{xy}}(dg''_{x,v}) \\ &= i^*_{g_{xy}}(dg'_{x,v}) dg''_{y,v}. \end{aligned}$$

Therefore $i^*_{g_{xy}}(dg'_{x,v}) = dg'_{y,v}$. □

DEFINITION 6.2. — *For* $v \in \mathfrak{M}$ *and* $x \in V^{\mathrm{ss}}_{k_v}$ *we let* $b_{x,v} > 0$ *be the constant satisfying the following equation*

$$\int_{G_{k_v}/G^\circ_{x\,k_v}} f(g'_{x,v}x)\, dg'_{x,v} = b_{x,v} \int_{G_{k_v}x} f(y)|P(y)|_v^{-2}\, dy$$

for any function f *on* $G_{k_v}x \subset V_{k_v}$ *integrable with respect to* $dy/|P(y)|_v^2$.

This is possible because $dy/|P(y)|_v^2$ is a G_{k_v}-invariant measure on $V^{\mathrm{ss}}_{k_v}$ and each of the orbits $G_{k_v}x$ is an open set in $V^{\mathrm{ss}}_{k_v}$.

PROPOSITION 6.3. — *If* $x, y \in V^{\mathrm{ss}}_{k_v}$ *and* $G_{k_v}x = G_{k_v}y$ *then* $b_{x,v} = b_{y,v}$.

Proof. — Let $f(y)$ be as in Definition 6.2 and $y = g_{xy}x$ for $g_{xy} \in G_{k_v}$. Then taking into Proposition 6.1 into account, we have

$$\begin{aligned} \int_{G_{k_v}x} f(y)|P(y)|_v^{-2}\, dy &= b_{x,v}^{-1} \int_{G_{k_v}/G^\circ_{x\,k_v}} f(g'_{x,v}x)\, dg'_{x,v} \\ &= b_{x,v}^{-1} \int_{G_{k_v}/G^\circ_{y\,k_v}} f(g_{xy}^{-1} g'_{y,v} y)\, dg'_{y,v} \\ &= b_{x,v}^{-1} b_{y,v} \int_{G_{k_v}x} f(y)|P(y)|_v^{-2}\, dy. \end{aligned}$$

Note that the last step is justified because $dg'_{y,v}$ is left G_{k_v}-invariant. Therefore $b_{x,v} = b_{y,v}$. □

DEFINITION 6.4. — *For* $\Phi \in \mathcal{S}(V_{k_v})$ *and* $s \in \mathbb{C}$ *we define*

$$Z_{x,v}(\Phi_v, s) = \int_{G_{k_v}/G^\circ_{x\,k_v}} |\chi(g'_{x,v})|_v^s \Phi_v(g'_{x,v}x)\, dg'_{x,v}$$

and call it the local zeta function.

By the definition of $b_{x,v}$ and the equation $P(g'_{x,v}x) = \chi(g'_{x,v})P(x)$, we have

$$Z_{x,v}(\Phi, s) = \frac{b_{x,v}}{|P(x)|_v^s} \int_{G_{k_v}x} |P(y)|_v^{s-2}\Phi(y)\, dy\,.$$

This integral converges absolutely at least when $\mathrm{Re}(s) > 2$. For $x, y \in V_{k_v}^{\mathrm{ss}}$ lying in the same orbit, by the above equation and Proposition 6.3, we obtain the following.

Proposition 6.5. — *If $x, y \in V_{k_v}^{\mathrm{ss}}$ and $G_{k_v}x = G_{k_v}y$ then*

$$Z_{x,v}(\Phi_v, s) = \frac{|P(y)|_v^s}{|P(x)|_v^s} Z_{y,v}(\Phi_v, s).$$

By this proposition, we see that the local zeta functions for the same G_{k_v}-orbit are related by a simple equation. In section 10, we define and consider certain Dirichlet series arising from the global zeta function. There, collecting the orbital zeta functions lying in the same G_{k_v}-orbit will be fundamental. For this purpose, we fix a representative element for each G_{k_v}-orbit in $V_{k_v}^{\mathrm{ss}}$, which also has some good arithmetic properties if $v \in \mathfrak{M}_{\mathrm{f}}$.

Definition 6.6. — *For each of G_{k_v}-orbits in $V_{k_v}^{\mathrm{ss}}$, we choose and fix an element x which satisfies the following condition.*

(1) *If $v \in \mathfrak{M}_{\mathrm{f}}$, then x is of the form $(1, u)$ and u generates $\mathcal{O}_{\widetilde{k}_v(x)}$ over $\mathcal{O}_v$ via the identification $k_v[u] \cong \widetilde{k}_v(x)$.*
(2) *If $v \in \mathfrak{M}_\infty$, then $|P(x)|_v = 1$.*

We call such fixed orbital representatives as the standard orbital representatives.

If $v \in \mathfrak{M}_{\mathrm{f}}$, for any standard representative $x = (1, u) \in V_{k_v}^{\mathrm{ss}}$, u is a root of $F_x(v_1, -1)$ and so the discriminant $P(x)$ of $F_x(v)$ generates the ideal $\Delta_{k_v(x)/k_v}$.

Definition 6.7. — *For any $v \in \mathfrak{M}_{\mathrm{f}}$, let $\Phi_{v,0}$ be the characteristic function of $V_{\mathcal{O}_v}$. Also we put*

$$Z_{x,v}(s) = Z_{x,v}(\Phi_{v,0}, s).$$

We call $Z_{x,v}(s)$ for any standard orbital representative x a standard local zeta function *of x.*

To describe estimates of Dirichlet series, we introduce the following notation.

Definition 6.8. — *Suppose that we have Dirichlet series $L_i(s) = \sum_{m=1}^{\infty} \ell_{i,m}m^{-s}$ for $i = 1, 2$. If $|\ell_{1,m}| \leqslant \ell_{2,m}$ for all $m \geqslant 1$ then we shall write $L_1(s) \preccurlyeq L_2(s)$.*

We set $S_0 = \mathfrak{M}_\infty \cup \mathfrak{M}_{\rm dy} \cup \mathfrak{M}_{\mathfrak{B}}$. To carry out the filtering process, we need a uniform estimate of the standard local zeta functions. The following proposition concerning the standard local zeta functions for $v \notin S_0$ is proved in [6, Corollary 8.24, Proposition 9.25]. Since S_0 is a finite set, the result is enough for our purposes.

Proposition 6.9. — *Let $v \notin S_0$ and $x \in V_{k_v}^{\rm ss}$ be one of the standard representatives. Then $Z_{x,v}(s)$ can be expressed as*

$$Z_{x,v}(s) = \sum_{n \geqslant 0} \frac{a_{x,v,n}}{q_v^{ns}}$$

with $a_{x,v,0} = 1$ and $a_{x,v,n} \geqslant 0$ for all n. Also let us define

$$L_v(s) = \frac{1 + 29q_v^{-2(s-1)} - 21q_v^{-4(s-1)} + 7q_v^{-6(s-1)}}{(1 - q_v^{-(2s-1)})(1 - q_v^{-2(s-1)})^4}.$$

Then $Z_{x,v}(s) \preccurlyeq L_v(s)$.

Now we define the local density.

Definition 6.10. — *Assume $x \in V_{k_v}^{\rm ss}$ is a standard orbital representative. We define*

$$\varepsilon_v(x) = \frac{|P(x)|_v^2}{b_{x,v}}.$$

Also we define the local density *at v by*

$$E_v = \sum_x \varepsilon_v(x)$$

where the sum is over all standard representatives for orbits in $G_{k_v} \backslash V_{k_v}^{\rm ss}$.

These values play essential roles in the density theorem. The purpose in the next three sections are to compute the local densities. To make the density theorem more precise, it is better to evaluate $\varepsilon_v(x)$ separately rather than the sum E_v. We compute for $v \in \mathfrak{M}_{\rm f}$ in Sections 7, 8 and for $v \in \mathfrak{M}_\infty$ in Section 9. For $v \notin \mathfrak{M}_{\mathfrak{B}}$, those were already almost carried out in [6, 7] and except for a refinement for dyadic places in Proposition 7.4, we quote their result.

Remark 6.11. — We briefly compare the definition of standard orbital representatives and the value $\varepsilon_v(x)$ in [6] and in this paper for $v \notin \mathfrak{M}_{\mathfrak{B}}$, to confirm that we can directly use their result. Let G'_{k_v} denote the group of the representation V_{k_v} used in [6]. Then one can easily see that the isomorphism $G'_{k_v} \to G_{k_v}$ given by $(g_1, g_2, g_3) \mapsto (g_1, {}^t g_2, g_3)$ is compatible with their actions on V_{k_v}. If we identify these groups using the isomorphism, we immediately see that our choice of measure on G_{k_v} coincides to that of in [6], and moreover, measures on $G^\circ_{x\,k_v}$ also. The latter claim holds because both papers used the isomorphism in Proposition 3.5 to normalize the measures on $G^\circ_{x\,k_v}$. The normalizations of the measures on $G_{k_v}/G^\circ_{x\,k_v}$ are slightly different, but from the definitions we can easily see that the constants $b_{x,v}$'s coincide. Although our choices of the standard orbital representatives x for $v \in \mathfrak{M}_{\mathrm{f}}$ are also slightly different, the values of $|P(x)|_v$ coincide since the standard orbital representatives x in [6] are also chosen so that $P(x)$ generate $\Delta_{k_v(x)/k_v}$. Since $\varepsilon_v(x)$ is determined only by $|P(x)|_v$ and $b_{x,v}$, this shows that our $\varepsilon_v(x)$'s coincide with those of [6].

7. Computation of the local densities at finite unramified places

In this and the next sections, we assume $v \in \mathfrak{M}_{\mathrm{f}}$. We first introduce some notations for these sections. For any $v \in \mathfrak{M}_{\mathrm{f}}$ we shall put $2\mathcal{O}_v = \mathfrak{p}_v^{m_v}$. Of course $m_v = 0$ unless $v \in \mathfrak{M}_{\mathrm{dy}}$. If $x \in V^{\mathrm{ss}}_{k_v}$ then let $\Delta_{k_v(x)/k_v} = \mathfrak{p}_v^{\delta_{x,v}}$. For $v \notin \mathfrak{M}_{\mathrm{dy}}$, $\delta_{x,v}$ is either 0 or 1. It is well-known that if v is dyadic then $\delta_{x,v}$ takes one of the values $0, 2, \dots, 2m_v, 2m_v+1$.

We now assume $v \notin \mathfrak{M}_{\mathfrak{B}}$. The following propositions are proved in [6, Lemma 7.3] and [7, Propositions 4.14, 4.15, 4.25].

Proposition 7.1. — *Assume $v \notin \mathfrak{M}_{\mathfrak{B}}$. Let $x \in V^{\mathrm{ss}}_{k_v}$ be one of the standard representatives.*

(1) *If x has type* (ur sp) *then* $\varepsilon_v(x) = 2^{-1}(1+q_v^{-1})(1-q_v^{-2})^2$.
(2) *If x has type* (ur ur) *then* $\varepsilon_v(x) = 2^{-1}(1-q_v^{-1})^3(1-q_v^{-2})$.

Proposition 7.2. — *Assume $v \notin \mathfrak{M}_{\mathfrak{B}}$. Let $x \in V^{\mathrm{ss}}_{k_v}$ be one of the standard representatives.*

(1) *If $v \notin \mathfrak{M}_{\mathrm{dy}}$ and x has type* (ur rm) *then* $\varepsilon_v(x) = 2^{-1}q_v^{-1}(1-q_v^{-1})(1-q_v^{-2})^3$.
(2) *If $v \in \mathfrak{M}_{\mathrm{dy}}$ then*

$$\sum_{2\leqslant \delta_{x,v}=2\ell\leqslant 2m_v} \varepsilon_v(x) = (1-q_v^{-1})^2(1-q_v^{-2})^3 q_v^{-\ell},$$

$$\sum_{\delta_{x,v}=2m_v+1} \varepsilon_v(x) = (1-q_v^{-1})(1-q_v^{-2})^3 q_v^{-(m_v+1)},$$

where x runs through all the standard representative with the given condition of discriminants.

Proposition 7.3. — *Let $v \notin \mathfrak{M}_{\mathrm{B}}$. Then*

$$E_v = (1 - q_v^{-2})(1 - 3q_v^{-3} + 2q_v^{-4} + q_v^{-5} - q_v^{-6}).$$

These results are already enough to prove our density theorems. However, if we know the each value of $\varepsilon_v(x)$ also for $v \in \mathfrak{M}_{\mathrm{dy}}$ in the Proposition 7.2, then the density theorems become finer. In this section we refine Proposition 7.2 to the following.

Proposition 7.4. — *Assume $v \notin \mathfrak{M}_{\mathrm{B}}$. Let $x \in V_{k_v}^{\mathrm{ss}}$ be a standard representative with the type* (ur rm). *Then*

$$\varepsilon_v(x) = 2^{-1}|\Delta_{k_v(x)/k_v}|_v^{-1}(1 - q_v^{-1})(1 - q_v^{-2})^3.$$

It is well known that there are $2q_v^{l-1}(q_v - 1)$ numbers of quadratic extensions of k_v with the absolute value of the relative discriminant q_v^{2l} if $1 \leqslant l \leqslant m_v$ and $2q_v^{m_v}$ numbers of quadratic extensions of k_v with the absolute value of the relative discriminant $q_v^{2m_v+1}$. Hence this is in fact a refinement of Proposition 7.2. We give the proof of this proposition after we prove Lemma 7.9.

Let L/k_v be a quadratic ramified extension, ϖ a uniformizer of L, and ϖ^τ the conjugate of ϖ with respect to L/k_v, henceforth fixed. We put $a_1 = \varpi + \varpi^\tau, a_2 = \varpi\varpi^\tau$. Following [6], we let

(7.1)

$$x = (x_1, x_2), \quad \text{where} \quad x_1 = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \text{ and } x_2 = \begin{pmatrix} 1 & a_1 \\ a_1 & a_1^2 - a_2 \end{pmatrix}.$$

Then $F_x(v) = -(v_1^2 + a_1 v_1 v_2 + a_2 v_2^2) = -(v_1 + \varpi v_2)(v_1 + \varpi^\tau v_2)$ and hence $L \cong k_v(x)$ and $P(x)$ generates the ideal $\Delta_{k_v(x)/k_v}$. Therefore we can replace the standard representative for the orbit corresponding to L to this x to compute $\varepsilon_v(x) = |P(x)|_v^2 b_{x,v}^{-1}$.

The following lemma is a consequence of [6, Lemma 7.3] and [7, Proposition 3.2].

Lemma 7.5. — *We have $\varepsilon_v(x) = \mathrm{vol}(\mathcal{K}_v x)$.*

We compute $\mathrm{vol}(\mathcal{K}_v x)$ with a slight modification of the method in [7], along the lines of [14]. To begin we introduce some notations, which we also use to consider similar problems in Section 8. We regard $\mathcal{K}_v$ as the set of $\mathcal{O}_v$-rational points $G_{\mathcal{O}_v}$ of a group scheme $G = \mathrm{GL}(2) \times \mathrm{GL}(2) \times \mathrm{GL}(2)$ defined over $\mathcal{O}_v$ acting on a module scheme $V = \mathrm{M}(2,2) \oplus \mathrm{M}(2,2)$ also defined over $\mathcal{O}_v$. Then, since x is an $\mathcal{O}_v$-rational point of V, we can consider the stabilizer of x as a group scheme also defined over $\mathcal{O}_v$, in the sense of [9]. Let G_x denote this group scheme. Note that this definition of G_x differs from [7, 8]. Let i be a positive integer. For an $\mathcal{O}_v$-scheme X, let $\mathfrak{r}_{X,i}$ denote the reduction map $X_{\mathcal{O}_v} \to X_{\mathcal{O}_v/\mathfrak{p}_v^i}$. If the situation is obvious we drop X and write $\mathfrak{r}_i$ instead. For rational points $y_1, y_2 \in X_{\mathcal{O}_v}$, we use the notation $y_1 \equiv y_2\ (\mathfrak{p}_v^i)$ if $\mathfrak{r}_i(y_1) = \mathfrak{r}_i(y_2)$. We also use the notation "y mod $\mathfrak{p}_v^i$" for $\mathfrak{r}_i(y)$.

For the element x of the form (7.1), let

$$A_x(c,d) = \begin{pmatrix} c & d \\ -a_2 d & c + a_1 d \end{pmatrix}.$$

Then if $A_x(c_i, d_i) \in \mathrm{GL}(2)_{\mathcal{O}_v}$ for $i = 1, 2$, by computation we see that the element

$$(A_x(c_1,d_1), A_x(c_2,d_2), A_x(c_1,d_1)^{-1} A_x(c_2,d_2)^{-1}) \in \mathcal{K}_v$$

stabilizes x. Let $N_{x\,\mathcal{O}_v}$ denote the subgroup of $\mathcal{K}_v$ consisting of elements of the form above. We naturally regard $N_{x\,\mathcal{O}_v}$ as the set of $\mathcal{O}_v$-rational points of a group scheme N_x, which is a subgroup of G_x, defined over $\mathcal{O}_v$.

Proposition 7.6. — *We have $N_x \cong (\mathcal{O}_{k_v(x)}^\times)^2$ as group schemes over $\mathcal{O}_v$.*

Proof. — Let R be any $\mathcal{O}_v$-algebra. Then we could see that the map

$$(A_x(c_1,d_1), A_x(c_2,d_2), A_x(c_1,d_1)^{-1} A_x(c_2,d_2)^{-1}) \mapsto (c_1 + \varpi d_1, c_2 + \varpi d_2)$$

gives an isomorphism between $N_{x\,R}$ and $\{(\mathcal{O}_{k_v(x)} \otimes_{\mathcal{O}_v} R)^\times\}^2$, and this map, denoted by $\psi_{x,R}$, satisfies the usual functorial property with respect to homomorphism of $\mathcal{O}_v$-algebras. This shows that there exists an isomorphism $\psi_x \colon N_x \to (\mathcal{O}_{k_v(x)}^\times)^2$ as groups schemes over $\mathcal{O}_v$ such that $\psi_{x,R}$ is the induced isomorphism for all R. □

We now consider the orbit $\mathcal{K}_v x$. The approach in [7] is to consider modulo $\mathfrak{p}_v$ congruence condition on $V_{\mathcal{O}_v}$ to compute the sum $\sum_x \mathrm{vol}(\mathcal{K}_v x)$ where x runs through all the standard representatives with the given relative discriminant. Let $n = \delta_{x,v} + 2m_v + 1$ as in [14]. Then, as we demonstrate below, deliberation of the congruence relation modulo $\mathfrak{p}_v^n$ allows us to treat the single orbit $\mathcal{K}_v x$. We note that the idea of considering modulo a certain high power of prime ideal is already presented in [7] and used to compute $\varepsilon_v(x)$ in some other cases.

Definition 7.7. — *We define $\mathcal{D} = \{y \in V_{\mathcal{O}_v} \mid y \equiv x\,(\mathfrak{p}_v^n)\}$.*

Lemma 7.8. — *We have $\mathcal{D} \subset \mathcal{K}_v x$.*

Proof. — Let $y \in \mathcal{D}$. First we show $y \in G_{k_v} x$. Since $P(y) \equiv P(x)\ (\mathfrak{p}_v^n)$ and $\mathrm{ord}_v(P(x)) = \delta_{x,v}$, we have $P(y)/P(x) \equiv 1\ (\mathfrak{p}_v^{2m_v+1})$. Then by Hensel's lemma, we have $P(y)/P(x) \in (k_v^\times)^2$. Therefore the splitting fields of $F_x(v)$ and $F_y(v)$ coincide and hence by Lemma 3.4, we have $y \in G_{k_v}x$. The rest of argument is exactly the same as that of [6, 7] and we omit it. □

Lemma 7.9. — *We have* $[G_{x\,\mathcal{O}_v/\mathfrak{p}_v^n} : N_{x\,\mathcal{O}_v/\mathfrak{p}_v^n}] = 2q_v^{\delta_{x,v}}$.

Proof. — The same argument as in the proof of [7, Proposition 4.15] shows that each right coset space of $N_{x\,\mathcal{O}_v/\mathfrak{p}_v^n}\backslash G_{x\,\mathcal{O}_v/\mathfrak{p}_v^n}$ contains exactly one element of the form $g = (g_1, g_2)$, $g_1 = (g_{11}, g_{12})$ with

$$g_{11} = \begin{pmatrix} 1 & 0 \\ u & s \end{pmatrix}, g_{22} = \begin{pmatrix} 1 & v \\ 0 & t \end{pmatrix}, g_2 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Hence we will consider when such an element actually lies in $G_{x\,\mathcal{O}_v/\mathfrak{p}_v^n}$. Suppose that g is in the form above and $gx = x$ in $V_{\mathcal{O}_v/\mathfrak{p}_v^n}$. We put $y = (y_1, y_2) = (g_1, 1)x$. Then by computation we have

$$y_1 = \begin{pmatrix} 0 & t \\ s & * \end{pmatrix}, \quad y_2 = \begin{pmatrix} 1 & v + a_1 t \\ u + a_1 s & * \end{pmatrix}.$$

Therefore, by comparing the (1,1), (1,2) and (2,1)-entries of x_1 and $\alpha y_1 + \beta y_2$, we have $\beta = 0$ and $s = t = \alpha^{-1}$. Also under the condition $s = t$, from $x_1 = \gamma y_1 + \delta y_2$ we have $\delta = 1$, $u = v$ and $\gamma = s^{-1}(a_1 - u - a_1 s)$. Hence

$$\alpha y_1 + \beta y_2 = \begin{pmatrix} 0 & 1 \\ 1 & 2u + a_1 s \end{pmatrix},$$
$$\gamma y_1 + \delta y_2 = \begin{pmatrix} 1 & a_1 \\ a_1 & -u^2 + (2u + a_1 s)a_1 - a_1 su - a_2 s^2 \end{pmatrix}$$

and therefore we see that $gx = x$ if and only if

$$2u + a_1 s = a_1 \quad \text{and} \quad u^2 + a_1 su + a_2 s^2 = a_2 \quad \text{in } \mathcal{O}_v/\mathfrak{p}_v^n.$$

This system is exactly as the one we considered in [14, Lemma 4.7] and it has $2q_v^{\delta_{x,v}}$ solutions in all. □

We are now ready to prove Proposition 7.4. Let $\mathfrak{r}_n$ be the reduction map $G_{\mathcal{O}_v} \to G_{\mathcal{O}_v/\mathfrak{p}_v^n}$. Then by Lemma 7.8, we have

$$\mathrm{vol}(\mathcal{K}_v x) = \mathrm{vol}(G_{\mathcal{O}_v} x) = \#(G_{\mathcal{O}_v}/\mathfrak{r}_n^{-1}(G_{x\,\mathcal{O}_v/\mathfrak{p}_v^n})) \cdot \mathrm{vol}(\mathcal{D}).$$

Since

$$G_{\mathcal{O}_v}/\mathfrak{r}_n^{-1}(G_{x\,\mathcal{O}_v/\mathfrak{p}_v^n}) \cong G_{\mathcal{O}_v/\mathfrak{p}_v^n}/G_{x\,\mathcal{O}_v/\mathfrak{p}_v^n},$$

by Lemma 7.9 we have

$$\begin{aligned}\mathrm{vol}(\mathcal{K}_v x) &= \mathrm{vol}(\mathcal{D}) \cdot \frac{\#(G_{\mathcal{O}_v/\mathfrak{p}_v^n})}{2q_v^{\delta_{x,v}} \cdot \#(N_{x\,\mathcal{O}_v/\mathfrak{p}_v^n})} \\ &= q_v^{-8n} \cdot \frac{\left\{q_v^{4n}(1-q_v^{-1})(1-q_v^{-2})\right\}^3}{2q_v^{\delta_{x,v}} \cdot \left\{q_v^{2n}(1-q_v^{-1})\right\}^2} \\ &= 2^{-1} q_v^{-\delta_{x,v}}(1-q_v^{-1})(1-q_v^{-2})^3.\end{aligned}$$

Since $|\Delta_{k(x)/k}|_v = q_v^{\delta_{x,v}}$, we obtained the desired result.

8. Computation of the local densities at finite ramified places

In this section we assume $v \in \mathfrak{M}_{\mathcal{B}}$ and so $\mathcal{B}_v$ is a non-split quaternion algebra of k_v. We briefly recall the algebraic structure of $\mathcal{B}_v$ and prepare the notations to begin with. We take a commutative subalgebra F_v of $\mathcal{B}_v$ such that F_v is a quadratic unramified extension of k_v and henceforth fixed in this section. Let σ denote the non-trivial element of $\mathrm{Gal}(F_v/k_v)$. Then for any prime element $\pi_v \in k_v$, $\mathcal{B}_v$ can be identified with $F_v \oplus F_v\sqrt{\pi_v}$ as a left vector space of F_v and the multiplication law is given by $\sqrt{\pi_v}\alpha = \alpha^\sigma\sqrt{\pi_v}$ for $\alpha \in F_v$. The involution of $\mathcal{B}_v$ is denoted by $a \mapsto a^*$. Then for $\alpha, \beta \in F_v$, $(\alpha + \beta\sqrt{\pi_v})^* = \alpha^\sigma - \beta\sqrt{\pi_v}$. Hence the reduced trace $\mathcal{T}$ and the reduced norm $\mathcal{N}$ of $\mathcal{B}_v$ are given by

$$\mathcal{T}(\alpha + \beta\sqrt{\pi_v}) = \alpha + \alpha^\sigma, \quad \mathcal{N}(\alpha + \beta\sqrt{\pi_v}) = \alpha\alpha^\sigma - \pi_v\beta\beta^\sigma,$$

for $\alpha, \beta \in F_v$. The map $u \mapsto \mathrm{ord}_v(\mathcal{N}(u))$ defines a discrete valuation of $\mathcal{B}_v$, and it is well known that $\mathcal{O}_{\mathcal{B}_v} = \{u \mid |\mathcal{N}(u)|_v \leqslant 1\}, \mathcal{O}_{\mathcal{B}_v}^\times = \{u \mid |\mathcal{N}(u)|_v = 1\}$. If we restrict the reduced norm to any quadratic subfield L_v, it coincides with the norm map N_{L_v/k_v} of the extension L_v/k_v. Hence $\mathcal{O}_{\mathcal{B}_v} \cap L_v = \mathcal{O}_{L_v}$ and $\mathcal{O}_{\mathcal{B}_v}^\times \cap L_v = \mathcal{O}_{L_v}^\times$. We fix an element $\theta \in \mathcal{O}_{F_v}$ so that $\mathcal{O}_{F_v} = \mathcal{O}_v[\theta]$. By computation we have the following.

Lemma 8.1. — *We have*

$$\begin{aligned}\mathcal{O}_{\mathcal{B}_v} &= \{\alpha + \beta\sqrt{\pi_v} \mid \alpha, \beta \in \mathcal{O}_{F_v}\}, \\ \mathcal{O}_{\mathcal{B}_v}^\times &= \{\alpha + \beta\sqrt{\pi_v} \mid \alpha \in \mathcal{O}_{F_v}^\times, \beta \in \mathcal{O}_{F_v}\}.\end{aligned}$$

For any quadratic ramified extension L_v of k_v contained in $\mathcal{B}_v$, we can also write

$$\mathcal{O}_{\mathcal{B}_v} = \{\alpha + \beta\theta \mid \alpha, \beta \in \mathcal{O}_{L_v}\},$$
$$\mathcal{O}_{\mathcal{B}_v}^{\times} = \{\alpha + \beta\theta \mid \alpha, \beta \in \mathcal{O}_{L_v}, \alpha \in \mathcal{O}_{L_v}^{\times} \text{ or } \beta \in \mathcal{O}_{L_v}^{\times}\},$$

and moreover, by changing θ if necessary, the multiplication law is given by $\theta\alpha = \alpha^{\tau}\theta$ for $\alpha \in L_v$ where τ denotes the non-trivial element of $\mathrm{Gal}(L_v/k_v)$.

As in Section 7, we can and shall regard $\mathcal{K}_v$ as the set of $\mathcal{O}_v$-rational points $G_{\mathcal{O}_v}$ of a group scheme G defined over $\mathcal{O}_v$ acting on a module scheme V also defined over $\mathcal{O}_v$. For example, the group $G_{\mathcal{O}_v} = \mathcal{O}_{\mathcal{B}_v}^{\times} \times (\mathcal{O}_{\mathcal{B}_v})^{\times} \times \mathrm{GL}(2)_{\mathcal{O}_v}$ acts on the module $V_{\mathcal{O}_v} = \mathcal{O}_{\mathcal{B}_v} \oplus \mathcal{O}_{\mathcal{B}_v}$. Then any standard orbital representative x is an element of $V_{\mathcal{O}_v}$ and as in Section 7, we regard the stabilizer G_x as a group scheme defined over $\mathcal{O}_v$. If $x \in V_{\mathcal{O}_v}$, then $F_x(v) \in \mathrm{Sym}^2\mathcal{O}_v^2$. We also regard $\mathrm{Sym}^2\mathcal{O}_v^2$ as a module scheme over $\mathcal{O}_v$ and the map $x \mapsto F_x(v)$ as a morphism of schemes. We continue to use the notation $\mathfrak{r}_i$ defined in Section 7. Further, for any quadratic extension L_v of k_v, we use the abbreviation $\mathcal{O}_{L_v}/\mathfrak{p}_v^i$ for $\mathfrak{r}_i(\mathcal{O}_{L_v})$, where we regard $\mathcal{O}_{L_v}$ as a scheme over $\mathcal{O}_v$. For example,

$$\mathcal{O}_{F_v}/\mathfrak{p}_v^i = (\mathcal{O}_v/\mathfrak{p}_v^i)[\theta] = \{\alpha + \beta\theta \mid \alpha, \beta \in \mathcal{O}_v/\mathfrak{p}_v^i\}.$$

In this section, we will express $g \in G_{k_v}$ as

$$g = (g_{11}, g_{12}, g_2), \quad g_2 = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

Proposition 8.2. — *Let x be one of the standard representatives. Then there exists an injective homomorphism $(\mathcal{O}_{\widetilde{k_v(x)}}^{\times})^2 \to G_x$ as a group scheme over $\mathcal{O}_v$.*

Proof. — Let $x = (1, u)$. We construct the injective homomorphism

$$\psi_{u\,R} \colon \{(\mathcal{O}_{\widetilde{k_v(x)}} \otimes R)^{\times}\}^2 \longrightarrow G_{x\,R}$$

for any commutative $\mathcal{O}_v$-algebra R.

We put $\widetilde{R}(x) = \mathcal{O}_{\widetilde{k_v(x)}} \otimes R$. Note that $\widetilde{R}(x) = R[u]$ is a subalgebra of $\mathcal{O}_{\mathcal{B}_v} \otimes R$ and is commutative. Since $\{1, u\}$ is a $\mathcal{O}_v$-basis of $\mathcal{O}_v[u]$, this is also an R-basis of $\widetilde{R}(x)$. Let $s_1, s_2 \in \widetilde{R}(x)^{\times}$. Then $\{s_1 s_2, s_1 s_2 u\}$ is also an R-basis of $\widetilde{R}(x)$, and so there exists a unique element $g = g_{s_1 s_2} \in \mathrm{GL}(2)_R$ such that $g\, {}^t(s_1 s_2, s_1 s_2 u) = {}^t(1, u)$. Hence

$$\psi_{u\,R} \colon (s_1, s_2) \longmapsto (s_1, s_2, g_{s_1 s_2})$$

gives an injective homomorphism from $(\widetilde{R}(x)^\times)^2$ to G_{xR}, and as in the proof of Proposition 7.6, we can regard this map as the induced one from the morphism of schemes. □

Let $N_x \subset G_x$ be the image of this homomorphism.

PROPOSITION 8.3. — *Let $x \in V_{k_v}^{\mathrm{ss}}$ be one of the standard representatives. Then*

$$\int_{\mathcal{K}_v \cap G^\circ_{x\,k_v}} dg''_{x,v} = 1.$$

Proof. — Let $x = (1, u)$ be a standard representative. We claim that $\psi_u^{-1}(\mathcal{K}_v \cap G^\circ_{x\,k_v}) = (\mathcal{O}_v[u]^\times)^2$ where ψ_u is defined in Section 5. The inclusion $\psi_u^{-1}(\mathcal{K}_v \cap G^\circ_{x\,k_v}) \subset (\mathcal{O}^\times_{k_v[u]})^2$ follows from $\mathcal{O}^\times_{\mathcal{B}_v} \cap k_v(x) = \mathcal{O}^\times_{k_v(x)}$. Let s_1, s_2 be elements of $\mathcal{O}^\times_{k_v[u]}$. Then since $\{s_1 s_2, s_1 s_2 u\}$ also forms an $\mathcal{O}_v$-basis of $\mathcal{O}_{k_v[u]}$, we have $g_{s_1 s_2} \in \mathrm{GL}(2)_{\mathcal{O}_v}$. This shows the reverse inclusion. Now the proposition follows from the definition of $dg''_{x,v}$. □

The following simple observation will be sometimes useful in the concrete calculations below. This easily follows from Proposition 3.5 and the properties of the norm map of the quadratic extension of local fields.

LEMMA 8.4. — *We define*

$$\varsigma : G_{k_v} \longrightarrow \mathbb{Z}^2, \quad \textit{as} \quad g \longmapsto (\mathrm{ord}_v(\mathcal{N}(g_{11})), \mathrm{ord}_v(\mathcal{N}(g_{12}))).$$

Then the image $\varsigma(G^\circ_{x\,k_v})$ is $(2\mathbb{Z})^2$ if x corresponds to the quadratic unramified extension and $\mathbb{Z}^2$ if x corresponds to a quadratic ramified extension.

From now on we consider the case $k_v(x)$ is unramified and ramified separately. We first consider the former case. Till Proposition 8.10, we assume x has type (rm ur). We note that in this case the polynomial $(F_x(v) \bmod \mathfrak{p}_v) \in \mathrm{Sym}^2(\mathcal{O}_v/\mathfrak{p}_v)^2$ is irreducible and especially $F_x(0,1) \in \mathcal{O}_v^\times$. By changing the choice of the included unramified extension F_v and the generator of the integer ring θ if necessary, we may assume $x = (1, \theta)$. Let us write $\theta = a + b\theta^\sigma, a \in \mathcal{O}_v, b \in \mathcal{O}_v^\times$ and set $\tau_\theta = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \in \mathrm{GL}(2)_{\mathcal{O}_v}$. We fix a prime element $\pi_v \in k_v$ and put $\tau_x = (\sqrt{\pi_v}^{-1}, \sqrt{\pi_v}, \tau_\theta)$, which then generates the non-trivial class of $G_{x\,k_v}/G^\circ_{x\,k_v}$.

LEMMA 8.5. — *Let x have type* (rm ur). *Then* $\mathcal{K}_v G_{x\,k_v} = \mathcal{K}_v G^\circ_{x\,k_v} \amalg \tau_x \mathcal{K}_v G^\circ_{x\,k_v}$.

Proof. — Since $\mathrm{ord}_v(\mathcal{N}(\sqrt{\pi_v}^{-1})) = \mathrm{ord}_v(\pi_v^{-1}) = -1$, we have $\tau_x \notin \mathcal{K}_v G^\circ_{x\,k_v}$ as a consequence of Lemma 8.4. Now the lemma follows since τ_x normalizes the group $\mathcal{K}_v$. □

Lemma 8.6. — *Let x have type* (rm ur). *Then $\varepsilon_v(x) = 2^{-1}\mathrm{vol}(\mathcal{K}_v x)$.*

Proof. — By the definition of $dg'_{x,v}$, Proposition 8.3 and Lemma 8.5,

$$\begin{aligned}1 = \int_{\mathcal{K}_v} dg_v &= \int_{\mathcal{K}_v G^\circ_{x\,k_v}/G^\circ_{x\,k_v}} dg'_{x,v} \cdot \int_{\mathcal{K}_v \cap G^\circ_{x\,k_v}} dg''_{x,v} \\ &= \int_{\mathcal{K}_v G^\circ_{x\,k_v}/G^\circ_{x\,k_v}} dg'_{x,v} \\ &= \frac{1}{2}\int_{\mathcal{K}_v G_{x\,k_v}/G^\circ_{x\,k_v}} dg'_{x,v}.\end{aligned}$$

Hence, if we let Φ_v be the characteristic function of $\mathcal{K}_v x$, by Definition 6.2 we have

$$\begin{aligned}2 = \int_{\mathcal{K}_v G_{x\,k_v}/G^\circ_{x\,k_v}} dg'_{x,v} &= \int_{G_{k_v}/G^\circ_{x\,k_v}} \Phi_v(g'_{x,v}x)\, dg'_{x,v} \\ &= b_{x,v}\int_{G_{k_v}x} \Phi_v(y)|P(y)|_v^{-2}\, dy \\ &= b_{x,v}\int_{\mathcal{K}_v x} |P(y)|_v^{-2}\, dy.\end{aligned}$$

Since $|P(y)|_v = |P(x)|_v$ for all $y \in \mathcal{K}_v x$, we have $\varepsilon_v(x) = |P(x)|_v^2 b_{x,v}^{-1} = 2^{-1}\mathrm{vol}(\mathcal{K}_v x)$. □

We will compute $\mathrm{vol}(\mathcal{K}_v x)$. In the case $k_v(x)$ is unramified extension, it is enough to consider the congruence relation modulo $\mathfrak{p}_v$.

Definition 8.7. — *We define $\mathcal{D} = \{y \in V_{\mathcal{O}_v} \mid y \equiv x\,(\mathfrak{p}_v)\}$.*

Lemma 8.8. — *We have $\mathcal{D} \subset \mathcal{K}_v x$.*

Proof. — Let $y \in \mathcal{D}$. Since $(F_y(v) \mod \mathfrak{p}_v) = (F_x(v) \mod \mathfrak{p}_v) \in \mathrm{Sym}^2 (\mathcal{O}_v/\mathfrak{p}_v)^2$, the splitting field of $F_y(v)$ is the quadratic unramified extension. Hence, $y \in G_{k_v}x$. Let $y = gx, g = (g_{11}, g_{12}, g_2) \in G_{k_v}$. Note that

$$|\chi(g)|_v = |\mathcal{N}(g_{11})\mathcal{N}(g_{12})\det(g_2)|_v^2 = 1$$

since $|P(y)|_v = |P(x)|_v$. We will show that $g \in \mathcal{K}_v G_{x\,k_v}$. By Lemma 8.4, multiplying an element of $G^\circ_{x\,k_v}$ and τ_x if necessary, we may assume that g satisfies either one of the following conditions.

(A) $|\mathcal{N}(g_{11})|_v = |\mathcal{N}(g_{12})|_v = 1$, (hence $|\det(g_2)|_v = 1$.)
(B) $|\mathcal{N}(g_{11})|_v = q_v, |\mathcal{N}(g_{12})|_v = 1$, (hence $|\det(g_2)|_v = q_v^{-1}$.)

From the definition of the representation we have

$$F_y(v) = \mathcal{N}(g_{11})\mathcal{N}(g_{12})F_x(vg_2)$$

and hence

$$F_y(1,0) = \mathcal{N}(g_{11})\mathcal{N}(g_{12})\mathrm{N}_{F_v/k_v}(p+q\theta),$$
$$F_y(0,1) = \mathcal{N}(g_{11})\mathcal{N}(g_{12})\mathrm{N}_{F_v/k_v}(r+s\theta).$$

On the other side, since $F_x(v) \equiv F_y(v)\ (\mathfrak{p}_v)$, both $F_y(1,0)$ and $F_y(0,1)$ are units of $\mathcal{O}_v$. If g satisfies the condition (B), then $\mathrm{ord}_v(\mathrm{N}_{F_v/k_v}(p+q\theta))$ must be 1. But this is a contradiction since F_v/k_v is the quadratic unramified extension. Hence we assume g satisfies the condition (A). Then both $\mathrm{N}_{F_v/k_v}(p+q\theta)$ and $\mathrm{N}_{F_v/k_v}(r+s\theta)$ are elements of $\mathcal{O}_v^\times$ and so $p,q,r,s \in \mathcal{O}_v$. Since $|\det(g_2)|_v = 1$, we conclude $g_2 \in \mathrm{GL}(2)_{\mathcal{O}_v}$. Thus $g \in \mathcal{K}_v$ and the lemma follows. □

Lemma 8.9. — *We have* $G_{x\,\mathcal{O}_v/\mathfrak{p}_v} = N_{x\,\mathcal{O}_v/\mathfrak{p}_v}$.

Proof. — In the proof of this lemma, if we have $y \equiv y'\ (\mathfrak{p}_v)$ for any two $\mathcal{O}_v$-rational points of an $\mathcal{O}_v$-scheme, we drop $(\mathfrak{p}_v)$ and simply write $y \equiv y'$ instead. Clearly $G_{x\,\mathcal{O}_v/\mathfrak{p}_v} \supset N_{x\,\mathcal{O}_v/\mathfrak{p}_v}$ and hence we prove the reverse inclusion. Let $g = (g_{11}, g_{12}, g_2) \in G_{x\,\mathcal{O}_v/\mathfrak{p}_v}$. We choose representatives of g_{11}, g_{12}, g_2 in $G_{11\mathcal{O}_v}, G_{12\mathcal{O}_v}, G_{2\mathcal{O}_v}$ and use the same notation for them. By Lemma 8.1 and Proposition 8.2, multiplying by an element of $N_{x\,\mathcal{O}_v/\mathfrak{p}_v}$ if necessary, we assume that

$$g_{11} = 1 + \alpha\sqrt{\pi_v}, \quad g_{12} = 1 + \beta\sqrt{\pi_v},$$

where $\alpha, \beta \in \mathcal{O}_{F_v}$. Put $y = (y_1, y_2) = (g_{11}, g_{12}, 1)x$. Then by computation we have

$$y_1 \equiv 1 + (\alpha+\beta)\sqrt{\pi_v}, \quad y_2 \equiv \theta + (\alpha\theta^\sigma + \beta\theta)\sqrt{\pi_v}.$$

Since ${}^t(y_1, y_2) \equiv g_2^{-1}\,{}^t(1,\theta)$ and $1, \theta \in \mathcal{O}_{F_v}$, we have $\mathfrak{r}_1(y_1), \mathfrak{r}_1(y_2) \in \mathcal{O}_{F_v}/\mathfrak{p}_v$. Hence $\alpha + \beta \equiv 0, \alpha\theta^\sigma + \beta\theta \equiv 0$. Then since $\theta - \theta^\sigma \in \mathcal{O}_{F_v}^\times$, we have $\alpha \equiv \beta \equiv 0$ and hence $g_{11} \equiv g_{12} \equiv 1$. Then $g_2 \equiv I_2$ and this shows $G_{x\,\mathcal{O}_v/\mathfrak{p}_v} \subset N_{x\,\mathcal{O}_v/\mathfrak{p}_v}$. □

Proposition 8.10. — *Let x has type* (rm ur)*. Then* $\varepsilon_v(x) = 2^{-1}(1 - q_v^{-2})(1 - q_v^{-1})$.

Proof. — By Lemma 8.8, we have

$$\mathrm{vol}(\mathcal{K}_v x) = \mathrm{vol}(G_{\mathcal{O}_v} x) = \#(G_{\mathcal{O}_v}/\mathfrak{r}_1^{-1}(G_{x\,\mathcal{O}_v/\mathfrak{p}_v})) \cdot \mathrm{vol}(\mathcal{D}).$$

Since

$$G_{\mathcal{O}_v}/\mathfrak{r}_1^{-1}(G_{x\,\mathcal{O}_v/\mathfrak{p}_v}) \cong G_{\mathcal{O}_v/\mathfrak{p}_v}/G_{x\,\mathcal{O}_v/\mathfrak{p}_v},$$

by Lemma 8.9 we have

$$\begin{aligned}\mathrm{vol}(\mathcal{K}_v x) &= \mathrm{vol}(\mathcal{D}) \cdot \frac{\#(G_{\mathcal{O}_v/\mathfrak{p}_v})}{\#(N_{x\,\mathcal{O}_v/\mathfrak{p}_v})} \\ &= q_v^{-8} \cdot \frac{(q_v^2(q_v^2-1))^2 \cdot (q_v^2-1)(q_v^2-q_v)}{(q_v^2-1)^2} \\ &= (1-q_v^{-2})(1-q_v^{-1}).\end{aligned}$$

Now the proposition follows from Lemma 8.6. □

Next we consider orbits corresponding to quadratic ramified extensions. From now on to Proposition 8.15, we assume x has type (rm rm). Let $x = (1, \varpi)$. Then ϖ is a prime element of $L_v = k_v(\varpi) \cong k_v(x)$. Let τ denote the non-trivial element of $\mathrm{Gal}(L_v/k_v)$. Then $F_x(v_1, v_2) = (v_1 + \varpi v_2)(v_1 + \varpi^\tau v_2)$ is an Eisenstein polynomial and $(\varpi - \varpi^\tau)^2 \in \mathcal{O}_v$ generates the relative discriminant $\Delta_{k_v(x)/k_v} = \mathfrak{p}_v^{\delta_{x,v}}$.

Lemma 8.11. — *Let x have type* (rm rm). *Then* $\varepsilon_v(x) = \mathrm{vol}(\mathcal{K}_v x)$.

Proof. — We can prove this lemma exactly the same as Lemma 8.6. The only difference is that we can take the generator τ of $G_{x\,k_v}/G^\circ_{x\,k_v}$ in Proposition 3.6 from $\mathcal{K}_v$ and hence $\mathcal{K}_v G_{x\,k_v} = \mathcal{K}_v G^\circ_{x\,k_v}$. □

We put $n = \delta_{x,v} + 2m_v + 1$. As in the case x has type (ur rm) in Section 7, we consider the congruence relation of modulo $\mathfrak{p}_v^n$ to compute $\mathrm{vol}(\mathcal{K}_v x)$.

Definition 8.12. — *We define* $\mathcal{D} = \{y \in V_{\mathcal{O}_v} \mid y \equiv x\,(\mathfrak{p}_v^n)\}$.

Lemma 8.13. — *We have* $\mathcal{D} \subset \mathcal{K}_v x$.

Proof. — Let $y \in \mathcal{D}$. Then as in the proof of Lemma 7.8, we have $y \in G_{k_v} x$. The rest of argument is similar to that of Lemma 8.8 and we shall be brief. Let $y = gx, g \in G_{k_v}$. By Lemma 8.4, multiplying by an element of $G^\circ_{x\,k_v}$ if necessary, we may assume that

$$|\mathcal{N}(g_{11})|_v = |\mathcal{N}(g_{12})|_v = 1, \quad \text{and hence} \quad |\det(g_2)|_v = 1.$$

Since $F_y(v) \in \mathrm{Sym}^2 \mathcal{O}_v^2$, we have

$$\begin{aligned}F_y(1,0) &= \mathcal{N}(g_{11})\mathcal{N}(g_{12})\mathrm{N}_{k_v(x)/k_v}(p + q\varpi) \in \mathcal{O}_v, \\ F_y(0,1) &= \mathcal{N}(g_{11})\mathcal{N}(g_{12})\mathrm{N}_{k_v(x)/k_v}(r + s\varpi) \in \mathcal{O}_v.\end{aligned}$$

Hence both $\mathrm{N}_{k_v(x)/k_v}(p + q\varpi)$ and $\mathrm{N}_{k_v(x)/k_v}(r + s\varpi)$ are elements of $\mathcal{O}_v$ and so $p, q, r, s \in \mathcal{O}_v$. Since $|\det(g_2)|_v = 1$, we conclude $g_2 \in \mathrm{GL}(2)_{\mathcal{O}_v}$. Hence $g \in \mathcal{K}_v$ and the lemma follows. □

Lemma 8.14. — *We have* $[G_{x\,\mathcal{O}_v/\mathfrak{p}_v^n} : N_{x\,\mathcal{O}_v/\mathfrak{p}_v^n}] = 2q_v^{\delta_{x,v}}$.

Proof. — We shall count the number of elements of the right coset space $N_{x\,\mathcal{O}_v/\mathfrak{p}_v^n}\backslash G_{x\,\mathcal{O}_v/\mathfrak{p}_v^n}$. Let $g' \in G_{x\,\mathcal{O}_v/\mathfrak{p}_v^n}$. By Lemma 8.1 and Proposition 8.2, the right coset $N_{x\,\mathcal{O}_v/\mathfrak{p}_v^n}g'$ contains an element $g = (g_1, g_2)$ with $g_1 = (g_{11}, g_{12})$ of one of the following forms

(A) $g_{11} = 1 + \alpha\theta, g_{12} = 1 + \beta\theta$,
(B) $g_{11} = 1 + \alpha\theta, g_{12} = \beta + \theta$,
(C) $g_{11} = \alpha + \theta, g_{12} = 1 + \beta\theta$,
(D) $g_{11} = \alpha + \theta, g_{12} = \beta + \theta$,

where $\alpha, \beta \in \mathcal{O}_{L_v}/\mathfrak{p}_v^n$, and also they are determined by the coset $N_{x\,\mathcal{O}_v/\mathfrak{p}_v^n}g'$ only. We will count the possibilities for g for each of the above cases. We choose representatives of α, β in $\mathcal{O}_{L_v}$ and use the same notation.

From now on we consider the case $v \notin \mathfrak{M}_{\mathrm{dy}}$ and $v \in \mathfrak{M}_{\mathrm{dy}}$ separately. We first consider the case $v \notin \mathfrak{M}_{\mathrm{dy}}$. In this case $\delta_{x,v} = 1$ and $n = 2$. Also since $2 \in \mathcal{O}_v^\times$, by changing θ and $x = (1, \varpi)$ if necessary, we may assume that $\theta^2 \in \mathcal{O}_v^\times$ and $\varpi^\tau = -\varpi$. Let $y = (y_1, y_2) = (g_1, 1)x$.

First consider the case (A). By computation we have

$$y_1 = 1 + \alpha\beta^\tau\theta^2 + (\alpha + \beta)\theta, \quad y_2 = \varpi(1 - \alpha\beta\theta^2) + \varpi(\beta - \alpha)\theta.$$

Since ${}^t(y_1, y_2) \equiv g_2^{-1}\,{}^t(1, \varpi)\ (\mathfrak{p}_v^2)$, we have $\mathfrak{r}_2(y_1), \mathfrak{r}_2(y_2) \in \mathcal{O}_{L_v}/\mathfrak{p}_v^2$. Hence

$$\alpha + \beta \equiv 0\ (\mathfrak{p}_v^2) \qquad \text{and} \qquad \varpi(\beta - \alpha) \equiv 0\ (\mathfrak{p}_v^2).$$

It is easy to see that there are q_v possibilities for pairs of (α, β) modulo $\mathfrak{p}_v^2$ satisfying the above condition. On the other hand, for each of these pairs, we have $y_1 \equiv 1\ (\mathfrak{p}_v^2)$ and $y_2 \equiv \varpi\ (\mathfrak{p}_v^2)$, and hence $(1, g_2)y \equiv x\ (\mathfrak{p}_v^2)$ if and only if $g_2 \equiv I_2\ (\mathfrak{p}_v^2)$.

Next we consider the case (B). In this case, we have

$$y_1 = \beta + \alpha\theta^2 + (1 + \alpha\beta^\tau)\theta, \quad y_2 = \varpi(\beta - \alpha\theta^2) + \varpi(1 - \alpha\beta^\tau)\theta.$$

Again since ${}^t(y_1, y_2) \equiv g_2^{-1}\,{}^t(1, \varpi)$, we have $\mathfrak{r}_2(y_1), \mathfrak{r}_2(y_2) \in \mathcal{O}_{L_v}/\mathfrak{p}_v^2$. Hence

$$1 + \alpha\beta^\tau \equiv 0\ (\mathfrak{p}_v^2) \qquad \text{and} \qquad \varpi(1 - \alpha\beta^\tau) \equiv 0\ (\mathfrak{p}_v^2),$$

but this is impossible since $2 \in \mathcal{O}_v^\times$. Hence any right coset of $G_{x\,\mathcal{O}_v/\mathfrak{p}_v^2}$ does not contain elements of the form (B).

The remaining two cases are similar. We can see that there are no possibilities for g of the form (C) and q_v possibilities for g of the form (D). These give the desired description for $v \notin \mathfrak{M}_{\mathrm{dy}}$.

We next consider the case $v \in \mathfrak{M}_{\mathrm{dy}}$. In this case we may choose θ so that $\theta^2 = \theta + c$ for some $c \in \mathcal{O}_v^\times$. Again we let $y = (y_1, y_2) = (g_1, 1)x$.

Let us consider the case when g is of the form (A). By computation we have

$$y_1 = 1 + c\alpha\beta^\tau + (\alpha + \beta + \alpha\beta^\tau)\theta,$$
$$y_2 = (\varpi + c\alpha\beta^\tau\varpi^\tau) + (\alpha\varpi^\tau + \beta\varpi + \alpha\beta^\tau\varpi^\tau)\theta.$$

Hence as before, we need

$$\beta + \alpha + \alpha\beta^\tau \equiv 0 \ (\mathfrak{p}_v^n) \qquad \text{and} \qquad \beta\varpi + \varpi^\tau(\alpha + \alpha\beta^\tau) \equiv 0 \ (\mathfrak{p}_v^n).$$

Under the first equation, the second equation is equivalent to $\beta(\varpi - \varpi^\tau) \equiv 0 \ (\mathfrak{p}_v^n)$. If we write $\beta = \beta_1 + \beta_2\varpi$ where $\beta_1, \beta_2 \in \mathcal{O}_v/\mathfrak{p}_v^n$, this equation holds if and only if

$$\begin{cases} \beta_1, \beta_2 \in \mathfrak{p}_v^{n-\delta_{x,v}/2}/\mathfrak{p}_v^n & 2 \leqslant \delta_{x,v} \leqslant 2m_v, \\ \beta_1 \in \mathfrak{p}_v^{n-m_v}/\mathfrak{p}_v^n, \ \ \beta_2 \in \mathfrak{p}_v^{n-m_v-1}/\mathfrak{p}_v^n & \delta_{x,v} = 2m_v + 1. \end{cases}$$

Hence there are $q_v^{\delta_{x,v}}$ possibilities for β. Also for each of these β, $1 + \beta^\tau$ is invertible and so α is uniquely determined by the first equation.

Then for each of these pairs (α, β), we have $y_1 \equiv 1 \ (\mathfrak{p}_v^n)$ and $y_2 \equiv \varpi \ (\mathfrak{p}_v^n)$, and hence $(1, g_2)y \equiv x \ (\mathfrak{p}_v^n)$ is equivalent to $g_2 \equiv I_2 \ (\mathfrak{p}_v^n)$. Hence there are $q_v^{\delta_{x,v}}$ choices of g of the form (A) in all.

The remaining three cases are similar. There are no possibilities for g of the form (B) and (C), and $q_v^{\delta_{x,v}}$ choice of g of the form (D). We have thus proved the lemma. □

Proposition 8.15. — *Suppose the standard representative x has type (rm ur). Then* $\varepsilon_v(x) = 2^{-1}|\Delta_{k_v(x)/k_v}|_v^{-1}(1 + q_v^{-1})(1 - q_v^{-2})^2$.

Proof. — By Lemma 8.13, $\mathrm{vol}(\mathcal{K}_v x) = \#(G_{\mathcal{O}_v}/\mathfrak{r}_n^{-1}(G_{x\,\mathcal{O}_v/\mathfrak{p}_v^n})) \cdot \mathrm{vol}(\mathcal{D})$. Since

$$G_{\mathcal{O}_v}/\mathfrak{r}_n^{-1}(G_{x\,\mathcal{O}_v/\mathfrak{p}_v^n}) \cong G_{\mathcal{O}_v/\mathfrak{p}_v^n}/G_{x\,\mathcal{O}_v/\mathfrak{p}_v^n},$$

by Lemma 8.14 we have

$$\begin{aligned} \mathrm{vol}(\mathcal{K}_v x) &= \mathrm{vol}(\mathcal{D}) \cdot \frac{\#(G_{\mathcal{O}_v/\mathfrak{p}_v^n})}{2q_v^{\delta_{x,v}} \cdot \#(N_{x\,\mathcal{O}_v/\mathfrak{p}_v^n})} \\ &= q_v^{-8n} \cdot \frac{\left\{q_v^{4n}(1 - q_v^{-2})\right\}^2 \cdot q_v^{4n}(1 - q_v^{-1})(1 - q_v^{-2})}{2q_v^{\delta_{x,v}} \cdot \left\{q_v^{2n}(1 - q_v^{-1})\right\}^2} \\ &= 2^{-1}q_v^{-\delta_{x,v}}(1 + q_v^{-1})(1 - q_v^{-2})^2. \end{aligned}$$

Now the proposition follows from Lemma 8.11. □

9. Computation of the local densities at infinite places

In this section, we compute $\varepsilon_v(x)$ at infinite places. We assume $v \in \mathfrak{M}_\infty$ in this section. For the unramified places, the values were already computed in [7], and the remaining case is for places $v \in \mathfrak{M}_\mathcal{B} \cap \mathfrak{M}_\infty$. Note that this case does not occur if $v \in \mathfrak{M}_\mathbb{C}$ and that for these places $V_{k_v}^{\mathrm{ss}}$ is the single G_{k_v}-orbit. In the computation we need to know the 8×8 Jacobian determinant associated with the map $g \mapsto gx$ in some coordinate system. This calculation is carried out using the MAPLE computer algebra package [11].

PROPOSITION 9.1. — *Let $v \in \mathfrak{M}_\mathcal{B} \cap \mathfrak{M}_\infty$. Then $\varepsilon_v(x) = \pi^3/2$.*

Proof. — Since $|P(x)|_v = 1$ if x is a standard representative, $\varepsilon_v(x) = b_{x,v}^{-1}$. We proved in Proposition 6.3 that if $y \in G_{k_v}x$ then $b_{y,v} = b_{x,v}$. Therefore we will compute $b_{x,v}$ for $x = (1, \sqrt{-1})$ instead of the standard representative.

We define

$$\iota\colon \mathbb{C}^\times \longrightarrow \mathrm{GL}(2)_\mathbb{R} \quad \text{as} \quad t \longmapsto \begin{pmatrix} \Re(t) & \Im(t) \\ -\Im(t) & \Re(t) \end{pmatrix},$$

which is an injective homomorphism. Then the isomorphism in Proposition 3.5 can be expressed as

$$\psi_{\sqrt{-1}}\colon (\mathbb{C}^\times)^2 \longrightarrow G^\circ_{x\,\mathbb{R}}, \qquad (s_1, s_2) \longmapsto (s_1, s_2, \iota(s_1s_2)^{-1}).$$

Recall that the measure $dg''_{x,v}$ on $G^\circ_{x\,\mathbb{R}}$ was defined as the pushout measure of $d^\times s_1 d^\times s_2$.

Let $\mathbb{H}' = \{(u+j)s \mid u \in \mathbb{C}, s \in \mathbb{C}^\times\}$ and $\mathcal{D} = \mathbb{H}' \times \mathbb{H}' \times \mathrm{GL}(2)_\mathbb{R}$. Then $\mathcal{D}$ is an open dense subset of G_{k_v} and we will compare the measures on this set. Any element g of $\mathcal{D}$ can be written uniquely as $g = g'_{x,v}g''_{x,v}$ where

$$g'_{x,v} = (u_1 + j, u_2 + j, g_3), \quad g''_{x,v} = (s_1, s_2, \iota(s_1s_2)^{-1})$$

with

$$u_1, u_2 \in \mathbb{C}, \quad g_3 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{GL}(2)_\mathbb{R}, \quad \text{and} \quad s_1, s_2 \in \mathbb{C}^\times,$$

and when $g'_{x,v}$ is written in this form, $\Re(u_i), \Im(u_i)$ and a_{ij} for $i, j = 1, 2$ may be regarded as coordinates on $G_\mathbb{R}/G^\circ_{x\,\mathbb{R}}$. An easy computation shows that

$$dg'_{x,v} = \frac{1}{\pi^3} \cdot \frac{du_1}{(|u_1|_\mathbb{C} + 1)^2} \cdot \frac{du_2}{(|u_2|_\mathbb{C} + 1)^2} \cdot d\mu(g_3)$$

with respect to these coordinates. Note that we are setting du_i twice the Lebesgue measure on $\mathbb{C}$ as usual and that we defined $d\mu(g_3)$ to be $da_{11}da_{12}da_{21}da_{22}/|\det(g_3)|_{\mathbb{R}}^2$.

We consider the Jacobian determinant of the map

$$G_{\mathbb{R}}/G^{\circ}_{x\,\mathbb{R}} \to G_{\mathbb{R}}x, \qquad g'_{x,v} \mapsto g'_{x,v}x.$$

To do this, we choose their respective $\mathbb{R}$-coordinates. For $G_{\mathbb{R}}/G^{\circ}_{x\,\mathbb{R}}$, we regarded $\Re(u_i)$, $\Im(u_i)$ and a_{ij} for $i,j=1,2$ as its $\mathbb{R}$-coordinates. For $G_{\mathbb{R}}x$, which is an open subset of $V_{\mathbb{R}} = \mathbb{H}\oplus\mathbb{H}$, by expressing elements of $G_{\mathbb{R}}x$ as

$$y = (y_{11}+y_{12}j, y_{21}+y_{22}j), \qquad y_{ij}\in\mathbb{C}\ (i,j=1,2),$$

we regard $\Re(y_{ij}), \Im(y_{ij})$ for $i,j=1,2$ as $\mathbb{R}$-coordinates of $G_{\mathbb{R}}x$. Then with respect to the coordinate systems above, the Jacobian determinant of the map is found to be $4(|u_1|_{\mathbb{C}}+1)^2(|u_2|_{\mathbb{C}}+1)^2|\det(g_3)|_{\mathbb{R}}^2$ by using MAPLE [11]. Note that this map is a double cover since $[G_{x\,\mathbb{R}} : G^{\circ}_{x\,\mathbb{R}}] = 2$. As $P(g'_{x,v}x) = \chi(g'_{x,v})P(x)$ and

$$|\chi(g'_{x,v})|_{\mathbb{R}} = (|u_1|_{\mathbb{C}}+1)^2(|u_2|_{\mathbb{C}}+1)^2|\det(g_3)|_{\mathbb{R}}^2, \quad |P(x)|_{\mathbb{R}} = 4,$$

it follows that the pullback measure of $dy/|P(y)|_{\mathbb{R}}^2$ to $G_{\mathbb{R}}/G^{\circ}_{x\,\mathbb{R}}$ is

$$\frac{1}{2}\cdot\frac{du_1}{(|u_1|_{\mathbb{C}}+1)^2}\cdot\frac{du_2}{(|u_2|_{\mathbb{C}}+1)^2}\cdot d\mu(g_3).$$

We note that we chose the measure dy on $V_{\mathbb{R}}$ to be 2^4 times to that of product of Lebesgue measures $\prod_{i,j}[d\{\Re(y_{ij})\}d\{\Im(y_{ij})\}]$. Comparing this measure and $dg'_{x,v}$, we have $b_{x,v} = 2/\pi^3$ and hence the proposition follows. □

Combining [7, Propositions 5.2, 5.4] and the above proposition, we obtain the following.

PROPOSITION 9.2. — *Assume $v\in\mathfrak{M}_\infty$. Let $x\in V^{\rm ss}_{k_v}$ be one of the standard representatives.*

1. *If $v\in\mathfrak{M}_{\mathbb{R}}$ then $\varepsilon_v(x) = \pi^3/2$ for any type of the standard representative.*
2. *If $v\in\mathfrak{M}_{\mathbb{C}}$ then $\varepsilon_v(x) = 4\pi^3$.*

All of these finish the necessary preparations from local theory and we are now ready to go back to the adelic situation.

10. The mean value theorem

In this section, we will deduce our mean value theorem by putting together the results we have obtained before. We will see in Proposition 10.3 that the global zeta function is approximately the Dirichlet generating series for the sequence $\mathfrak{C}_L^2$ for quadratic extensions L of k which are embeddable into $\mathcal{B}$. If it were exactly this generating series, the Tauberian theorem would allow us to extract the mean value of the coefficients from the analytic behavior of this series. However, our global zeta function contains an additional factor in each term. We will surmount this difficulty by using the technique called the filtering process, which was originally formulated by Datskovsky and Wright [3].

Let $x \in V_k^{\rm ss}$. We define measures dg''_x and $d\widetilde{g}''_x$ on $G^\circ_{x\mathbb{A}}$ and $G^\circ_{x\mathbb{A}}/\widetilde{T}_\mathbb{A}$ to be $dg''_x = \prod_{v\in\mathfrak{M}} dg''_{x,v}$ and $d\widetilde{g}''_x = \prod_{v\in\mathfrak{M}} d\widetilde{g}''_{x,v}$, where we defined $dg''_{x,v}$ and $d\widetilde{g}''_{x,v}$ in Section 5. We choose a left invariant measure on $G_\mathbb{A}/G^\circ_{x\,\mathbb{A}}$. Since G°_x is isomorphic to $(\mathrm{GL}(1)_{\widetilde{k(x)}})^2$ as an algebraic group over k, the first Galois cohomology set $H^1(k', G^\circ_x)$ is trivial for any field k' containing k. This implies that the set of k'-rational points of $G_{k'}/G^\circ_{x\,k'}$ coincides with $(G/G^\circ_x)_{k'}$. Therefore $G_\mathbb{A}/G^\circ_{x\,\mathbb{A}} = (G/G^\circ_x)_\mathbb{A}$. Hence if we let $dg'_x = \prod_v dg'_{x,v}$ (we defined $dg'_{x,v}$ in Section 6), then this defines an invariant measure on $G_\mathbb{A}/G^\circ_{x\,\mathbb{A}}$. We have $dg = dg'_x dg''_x$ since $dg_v = dg'_{x,v} dg''_{x,v}$ for all v, and hence $d\widetilde{g} = dg'_x d\widetilde{g}''_x$.

We first determine the volume of $G^\circ_{x\,\mathbb{A}}/\widetilde{T}_\mathbb{A} G^\circ_{x\,k}$ under $d\widetilde{g}''_x$, which is the weighting factor of the Dirichlet series arising from our global zeta function.

Proposition 10.1. — *Suppose $x \in V_k^{\rm ss}$. Then the volume of $G^\circ_{x\mathbb{A}}/\widetilde{T}_\mathbb{A} G^\circ_{xk}$ with respect to the measure $d\widetilde{g}''_x$ is $(2\mathfrak{C}_{k(x)}/\mathfrak{C}_k)^2$.*

Proof. — Identifying $\widetilde{T}$ with $(\mathrm{GL}(1)_k)^2$ and G°_x with $(\mathrm{GL}(1)_{k(x)})^2$, we define $\widetilde{T}^0_\mathbb{A} = (\mathbb{A}^0)^2$ and $G^{\circ 0}_{x\,\mathbb{A}} = (\mathbb{A}^0_{k(x)})^2$. Let $d^\times \widetilde{t}^0$ and dg''^0_x be the measures on $\widetilde{T}^0_\mathbb{A}$ and $G^{\circ 0}_{x\,\mathbb{A}}$, such that $dg''_x = d^\times\lambda_1 d^\times\lambda_2 dg''^0_x$, $d^\times\widetilde{t} = d^\times\lambda_1 d^\times\lambda_2 d^\times\widetilde{t}^0$ for

$$g''_x = (\underline{\lambda_1}_{k(x)}, \underline{\lambda_2}_{k(x)}) g''^0_x, \quad \widetilde{t} = (\underline{\lambda_1}_k, \underline{\lambda_2}_k)\widetilde{t}^0.$$

Note that if $\lambda \in \mathbb{R}_+$ then the absolute value of $\underline{\lambda}_k$ as an idele of $k(x)$ is λ^2. Therefore, $dg''_x = 2^2 d^\times\lambda_1 d^\times\lambda_2 dg''^0_x$ for $g''_x = (\underline{\lambda_1}_k, \underline{\lambda_2}_k) g''^0_x$. Since $dg''_x = d^\times\widetilde{t} d\widetilde{g}''_x$ this implies that $2^2 dg''^0_x = d^\times\widetilde{t}^0 d\widetilde{g}''_x$. Therefore

$$\begin{aligned} 2^2 \int_{G^{\circ 0}_{x\,\mathbb{A}}/G^\circ_{x\,k}} dg''^0_x &= \int_{G^{\circ 0}_{x\,\mathbb{A}}/G^\circ_{x\,k}\widetilde{T}^0_\mathbb{A}} d\widetilde{g}''_x \int_{\widetilde{T}^0_\mathbb{A}/\widetilde{T}_k} d^\times\widetilde{t}^0 \\ &= \mathrm{vol}(G^{\circ 0}_{x\,\mathbb{A}}/\widetilde{T}^0_\mathbb{A} G^\circ_{x\,k}) \int_{\widetilde{T}^0_\mathbb{A}/\widetilde{T}_k} d^\times\widetilde{t}^0\,. \end{aligned}$$

Since

$$\int_{G_{x\,\mathbb{A}}^{\circ 0}/G_{x\,k}^{\circ}} dg_x''^0 = \mathfrak{C}_{k(x)}^2 \qquad \text{and} \qquad \int_{\widetilde{T}_{\mathbb{A}}^0/\widetilde{T}_k} d^\times \widetilde{t}^0 = \mathfrak{C}_k^2,$$

this proves the proposition. □

For $x \in V_k^{\mathrm{ss}}$ and $\Phi = \otimes\Phi_v \in \mathcal{S}(V_{\mathbb{A}})$ we define the *orbital zeta function* of x to be $Z_x(\Phi, s) = \prod_{v\in\mathfrak{M}} Z_{x,v}(\Phi_v, s)$. Note that we defined $Z_{x,v}(\Phi_v, s)$ in Section 6. If x lies in the orbit of the standard representative $\omega_{v,x}$ in $V_{k_v}^{\mathrm{ss}}$ then we shall write $\Xi_{x,v}(\Phi_v, s) = Z_{\omega_{v,x},v}(\Phi_v, s)$ and $\Xi_x(\Phi, s) = \prod_{v\in\mathfrak{M}} \Xi_{x,v}(\Phi_v, s)$. We call $\Xi_x(\Phi, s)$ the *standard orbital zeta function.*

Proposition 10.2. — *For $x \in V_k^{\mathrm{ss}}$ and $\Phi = \otimes\Phi_v \in \mathcal{S}(V_{\mathbb{A}})$ we have*

$$Z_x(\Phi, s) = \mathcal{N}(\Delta_{k(x)/k})^{-s}\Xi_x(\Phi, s)\,.$$

Proof. — By Proposition 6.5, we have

$$Z_x(\Phi, s) = \left(\prod_{v\in\mathfrak{M}} \frac{|P(\omega_{v,x})|_v}{|P(x)|_v}\right)^s \Xi_x(\Phi, s).$$

Since $P(x) \in k^\times$, we have $\prod_{v\in\mathfrak{M}} |P(x)|_v = 1$ by the Artin product formula. Also since $P(\omega_{v,x})$ generate the local discriminant $\Delta_{k_v(x)/k_v}$ of $k_v(x)$ if $v \in \mathfrak{M}_{\mathrm{f}}$ and $|P(\omega_{v,x})|_v = 1$ if $v \in \mathfrak{M}_\infty$, we have

$$\prod_{v\in\mathfrak{M}} |P(\omega_{v,x})|_v = \prod_{v\in\mathfrak{M}_{\mathrm{f}}} |P(\omega_{v,x})|_v = \prod_{v\in\mathfrak{M}_{\mathrm{f}}} |\Delta_{k_v(x)/k_v}|_v = \mathcal{N}(\Delta_{k(x)/k})^{-1}.$$

Thus we have the proposition. □

Proposition 10.3. — *If $\Phi = \otimes\Phi_v \in \mathcal{S}(V_{\mathbb{A}})$ then we have*

$$Z(\Phi, s) = \frac{2}{\mathfrak{C}_k^2} \sum_{x\in G_k\backslash V_k^{\mathrm{ss}}} \frac{\mathfrak{C}_{k(x)}^2}{\mathcal{N}(\Delta_{k(x)/k})^s}\Xi_x(\Phi, s).$$

Proof. — By the usual manipulation, we have

$$Z(\Phi, s) = \sum_{x\in G_k\backslash V_k^{\mathrm{ss}}} \frac{\int_{G_{x\,\mathbb{A}}^{\circ}/\widetilde{T}_{\mathbb{A}}G_{x\,k}^{\circ}} d\tilde{g}_x''}{[G_{x\,k} : G_{x\,k}^{\circ}]} \int_{G_{\mathbb{A}}/G_{x\,\mathbb{A}}^{\circ}} |\chi(g_x')|^s \Phi(g_x' x) dg_x'.$$

For each $x \in G_k\backslash V_k^{\mathrm{ss}}$, the last integral in the above equation is equal to $Z_x(\Phi, s)$ since $\Phi = \otimes_v\Phi_v$ and $dg_x' = \prod_v dg_{x,v}'$. Now the proposition follows from $[G_{x\,k} : G_{x\,k}^{\circ}] = 2$ and Propositions 10.1, 10.2. □

We are now ready to describe the filtering process. This process was originally used in [3] and was also used in [6]. Since our situation is quite similar to [6], we follow this reference.

We fix a finite set $S \supseteq S_0$ of places of k. Let T denote any finite subset $T \supseteq S$ of $\mathfrak{M}$. Let $\Xi_{x,v}(s) = \Xi_{x,v}(\Phi_{v,0}, s)$. (We defined S_0 and $\Phi_{v,0}$ in Section 6.)

Definition 10.4. — *For each finite subset $T \supseteq S$ of $\mathfrak{M}$, we define*

$$\Xi_{x,T}(s) = \prod_{v \notin T} \Xi_{x,v}(s) \qquad \text{and} \qquad L_T(s) = \prod_{v \notin T} L_v(s),$$

where $L_v(s)$ is as in Proposition 6.9.

By Proposition 6.9, we have the following.

Proposition 10.5. — *Both $\Xi_{x,T}(s)$ and $L_T(s)$ are Dirichlet series. We let*

$$\Xi_{x,T}(s) = \sum_{m=1}^{\infty} \frac{a^*_{x,T,m}}{m^s} \qquad \text{and} \qquad L_T(s) = \sum_{m=1}^{\infty} \frac{l^*_{T,m}}{m^s}.$$

Then $0 \leqslant a^*_{x,T,m} \leqslant l^*_{T,m}$ for all m and $a^*_{x,T,1} = 1$. Also $L_T(s)$ converges absolutely and locally uniformly in the region $\mathrm{Re}(s) > 3/2$.

We consider T-tuples $\omega_T = (\omega_v)_{v \in T}$ where each ω_v is one of the standard orbital representatives for the orbits in $V^{\mathrm{ss}}_{k_v}$. If $x \in V^{\mathrm{ss}}_k$ and $x \in G_{k_v}\omega_v$ then we write $x \approx \omega_v$ and if $x \approx \omega_v$ for all $v \in T$ then we write $x \approx \omega_T$.

Definition 10.6. — *We define*

$$\xi_{\omega_T}(s) = \sum_{x \in G_k \backslash V^{\mathrm{ss}}_k, x \approx \omega_T} \frac{\mathfrak{C}^2_{k(x)}}{\mathcal{N}(\Delta_{k(x)/k})^s} \Xi_{x,T}(s)$$

and

$$\xi_{\omega_S,T}(s) = \sum_{x \in G_k \backslash V^{\mathrm{ss}}_k, x \approx \omega_S} \frac{\mathfrak{C}^2_{k(x)}}{\mathcal{N}(\Delta_{k(x)/k})^s} \Xi_{x,T}(s),$$

which is the sum of $\xi_{\omega_T}(s)$ over all $\omega_T = (\omega_v)_{v \in T}$ which extend the fixed S-tuple ω_S.

The following lemma is exactly the same as [6, Lemma 6.17] and we omit the proof.

Lemma 10.7. — *Let $v \in \mathfrak{M}$, $x \in V^{\mathrm{ss}}_{k_v}$ and $r \in \mathbb{C}$. Then there exists a $\mathcal{K}_v$-invariant Schwartz-Bruhat function Φ_v such that the support of Φ_v is contained in $G_{k_v}x$, $Z_{x,v}(\Phi_v, s)$ is an entire function and $Z_{x,v}(\Phi_v, r) \neq 0$.*

Proposition 10.8. — *Let $T \supseteq S$ be a finite set of places of k and ω_T be a T-tuple, as above. The Dirichlet series $\xi_{\omega_T}(s)$ has a meromorphic* *continuation to the region* $\mathrm{Re}(s) > 3/2$*. Its only possible singularity in this region is a simple pole at* $s = 2$ *with residue*

$$\mathcal{R}_2 \prod_{v \in \mathfrak{M}_{\mathcal{B}} \cap \mathfrak{M}_{\mathrm{f}}} (1 - q_v^{-1})^2 \prod_{v \in T} \varepsilon_v(\omega_v),$$

where

$$\mathcal{R}_2 = \Delta_k^{-5/2} \mathfrak{C}_k^3 Z_k(2)^3.$$

Proof. — For each $v \in T$ we choose $\mathcal{K}_v$-invariant Schwartz-Bruhat function Φ_v such that $\mathrm{supp}(\Phi_v) \subseteq G_{k_v}\omega_v$. Let $\Phi = \bigotimes_{v \in T} \Phi_v \otimes \bigotimes_{v \notin T} \Phi_{v,0} \in \mathcal{S}(V_{\mathbb{A}})$. For $v \in T$ we have $\Xi_{x,v}(\Phi_v, s) = 0$ unless $x \approx \omega_v$ and hence by Proposition 10.3 we have

$$\begin{aligned} Z(\Phi, s) &= \frac{2}{\mathfrak{C}_k^2} \sum_{x \in G_k \backslash V_k^{\mathrm{ss}}, x \approx \omega_T} \frac{\mathfrak{C}_{k(x)}^2}{\mathcal{N}(\Delta_{k(x)/k})^s} \left(\prod_{v \in T} \Xi_{x,v}(\Phi_v, s) \right) \Xi_{x,T}(s) \\ &= \frac{2}{\mathfrak{C}_k^2} \left(\prod_{v \in T} Z_{\omega_v, v}(\Phi_v, s) \right) \xi_{\omega_T}(s). \end{aligned}$$

Using Lemma 10.7 and Theorem 4.2, this formula implies the first statement. Also by the equation just before Proposition 6.5, we have

$$Z_{\omega_v, v}(\Phi, 2) = \varepsilon_v(\omega_v)^{-1} \int_{V_{k_v}} \Phi_v(x) dx_v.$$

Since $\int_{V_{k_v}} \Phi_{v,0}(x)\, dx_v = 1$ for $v \notin T$, by Theorem 4.2 we have the residue of $\xi_{\omega_T}(s)$. □

As a corollary to this proposition, we obtain the following.

Corollary 10.9. — *The Dirichlet series* $\xi_{\omega_S, T}(s)$ *has a meromorphic continuation to the region* $\mathrm{Re}(s) > 3/2$*. Its only possible singularity in this region is a simple pole at* $s = 2$ *with residue*

$$\mathcal{R}_2 \prod_{v \in \mathfrak{M}_{\mathcal{B}} \cap \mathfrak{M}_{\mathrm{f}}} (1 - q_v^{-1})^2 \prod_{v \in S} \varepsilon_v(\omega_v) \cdot \prod_{v \in T \setminus S} E_v.$$

We are now ready to prove a preliminary version of the density theorem. Since the proof is exactly same as that of [6, Theorem 6.22], we omit it. Note that by Proposition 7.3, the product $\prod_{v \in \mathfrak{M}} E_v$ converges to a positive number.

Theorem 10.10. — *Let* $S \supset S_0$ *be a finite set of places of* k *and* ω_S *an* S*-tuple of standard orbital representatives. Then*

$$\lim_{X \to \infty} \frac{1}{X^2} \sum_{\substack{x \in G_k \backslash V_k^{\mathrm{ss}}, x \approx \omega_S \\ \mathcal{N}(\Delta_{k(x)/k}) \leqslant X}} \mathfrak{C}_{k(x)}^2 = \frac{1}{2} \mathcal{R}_2 \prod_{v \in \mathfrak{M}_{\mathcal{B}} \cap \mathfrak{M}_{\mathrm{f}}} (1 - q_v^{-1})^2 \prod_{v \in S} \varepsilon_v(\omega_v) \cdot \prod_{v \notin S} E_v .$$

We will rewrite Theorem 10.10 as a mean value theorem for the square of class number times regulator of quadratic extensions. Let $S \supset \mathfrak{M}_\infty$ be a finite set of places. As in Section 1, we let $L_S = (L_v)_{v\in S}$ be an S-tuple where each L_v is a separable quadratic algebra of k_v, and put

$$\mathcal{Q}(L_S) = \{F \mid [F:k]=2,\ F\otimes k_v \cong L_v \text{ for all } \ v\in S\},$$
$$\mathcal{Q}(L_S,X) = \{F\in\mathcal{Q}(L_S) \mid \mathcal{N}(\Delta_{F/k}) \leqslant X\}$$

where X is a positive number.

To state our main theorem, we define the constants as follows.

Definition 10.11.

(1) *For $v\in\mathfrak{M}_{\mathrm{f}}$ and L_v a separable quadratic algebra over k_v, we put*

$$\mathfrak{e}_v(L_v) = \begin{cases} 2^{-1}(1+q_v^{-1})(1-q_v^{-2}) & L_v\cong k_v\times k_v, \\ 2^{-1}(1-q_v^{-1})^3 & L_v \text{ is quadratic unramified}, \\ 2^{-1}|\Delta_{L_v/k_v}|_v^{-1}(1-q_v^{-1})(1-q_v^{-2})^2 & L_v \text{ is quadratic ramified.}\end{cases}$$

(2) *Let $S\supset\mathfrak{M}_\infty$. For an S-tuple $L_S=(L_v)_{v\in S}$ of separable quadratic algebras, we define*

$$\mathfrak{e}_\infty(L_S) = 2^{-r_1(L_s)}(2\pi)^{-r_2(L_s)}$$

where we put

$$r_1(L_S) = \#\{v\in\mathfrak{M}_\mathbb{R} \mid L_v\cong\mathbb{R}\times\mathbb{R}\}\times 2,$$
$$r_2(L_S) = \#\{v\in\mathfrak{M}_\mathbb{R} \mid L_v\not\cong\mathbb{R}\times\mathbb{R}\}+2r_2.$$

(3) *For $v\in\mathfrak{M}_{\mathrm{f}}$, we put*

$$\mathfrak{E}_v = 1-3q_v^{-3}+2q_v^{-4}+q_v^{-5}-q_v^{-6}.$$

Also we define

$$\mathcal{R}_k = 2^{-(r_1+r_2+1)}e_k^2\mathfrak{C}_k^3.$$

Note that the constants $\mathfrak{e}_v(L_v)$ are $(1-q_v^{-2})^{-1}$ times those we have listed in Propositions 7.1, 7.4 and $(1-q_v^{-1})^2(1-q_v^{-2})^{-1}$times those we have evaluated in Propositions 8.10, 8.15.

The following theorem is one of the main results of this paper.

Theorem 10.12. — *Let $S\supset\mathfrak{M}_\infty$ and $L_S=(L_v)_{v\in S}$ an S-tuple. Assume there are at least 2 places v such that L_v are fields. Then we have*

$$\lim_{X\to\infty}\frac{1}{X^2}\sum_{F\in\mathcal{Q}(L_S,X)} h_F^2R_F^2 = \mathcal{R}_k\Delta_k^{1/2}\zeta_k(2)^2\mathfrak{e}_\infty(L_S)^2\prod_{v\in S\cap\mathfrak{M}_{\mathrm{f}}}\mathfrak{e}_v(L_v)\prod_{v\notin S}\mathfrak{E}_v.$$

Proof. — We choose $v_1, v_2 \in S$ so that L_{v_1}, L_{v_2} are fields. We take the quaternion algebra $\mathcal{B}$ of k so that $\mathfrak{M}_{\mathcal{B}} = \{v_1, v_2\}$, which is possible by the Hasse principle. We consider the prehomogeneous vector space (G, V) for this $\mathcal{B}$. Since the set of k_v-rational orbits $G_{k_v}\backslash V_{k_v}^{\mathrm{ss}}$ corresponds to the set of all quadratic extensions of k_v if $v \in \mathfrak{M}_{\mathcal{B}}$ and to the set of all separable quadratic algebras of k_v if $v \notin \mathfrak{M}_{\mathcal{B}}$, we can take an S-tuple $\omega_S = (\omega_v)_{v\in S}$ of standard orbital representatives so that each ω_v corresponds to L_v. We claim that if a quadratic extension F of k satisfies $F \in \mathcal{Q}(L_S)$ then there exists $x \in V_k^{\mathrm{ss}}$ so that $F \cong k(x)$. In fact, if $F \in \mathcal{Q}(L_S)$ then $F \otimes k_{v_i} \cong L_{v_i}$ is embeddable into $\mathcal{B}_{v_i}$ for $i = 1, 2$. Since $\mathcal{B}_v \cong \mathrm{M}(2,2)_{k_v}$ for $v \notin \mathfrak{M}_{\mathcal{B}}$, this shows that $F \otimes \mathcal{B}_v \cong \mathrm{M}(2,2)_{F\otimes k_v}$ for all v and by the Hasse principle we have $F \otimes \mathcal{B} \cong \mathrm{M}(2,2)_F$. Hence F is embeddable into $\mathcal{B}$ and so by Proposition 3.4, there exists $x \in V_k^{\mathrm{ss}}$ such that $F \cong k(x)$.

Therefore, applying Theorem 10.10 for ω_S, we obtain

(10.1)
$$\lim_{X\to\infty} \frac{1}{X^2} \sum_{F\in\mathcal{Q}(L_S,X)} \mathfrak{C}_F^2 = \frac{1}{2}\mathcal{R}_2 \prod_{v\in\mathfrak{M}_{\mathcal{B}}\cap\mathfrak{M}_{\mathrm{f}}} (1-q_v^{-1})^2 \prod_{v\in S} \varepsilon_v(\omega_v) \cdot \prod_{v\notin S} E_v \,.$$

We consider the value $\mathfrak{C}_F^2$. Let $r_1(F)$ and $r_2(F)$ be the number of set of real places and complex places, respectively. Then if $F \in \mathcal{Q}(L_S)$ we immediately see $r_i(F) = r_i(L_S)$ for $i = 1, 2$. Also one can easily see that $e_F = e_k$ all but finitely-many quadratic extensions F of k. This finite exceptions may be ignored in the limit, and we have

$$\mathfrak{C}_F^2 = \mathfrak{e}_\infty(L_S)^{-2} e_k^{-2} h_F^2 R_F^2$$

for almost all $F \in \mathcal{Q}(L_S)$. Let us consider the right hand side of (10.1). By Proposition 9.2 and the definition of $Z_k(s)$, we have

$$\begin{aligned}\frac{1}{2}\mathcal{R}_2 \prod_{v\in\mathfrak{M}_\infty} \varepsilon_v(\omega_v) &= \frac{\mathfrak{C}_k^3}{2\Delta_k^{5/2}} \left(\frac{\Delta_k}{\pi^{r_1}(2\pi)^{r_2}}\zeta_k(2)\right)^3 (\frac{\pi^3}{2})^{r_1}(4\pi^3)^{r_2} \\ &= \frac{1}{2^{r_1+r_2+1}} \Delta_k^{1/2}\mathfrak{C}_k^3\zeta_k(2)^3 = e_k^{-2}\mathcal{R}_k\Delta_k^{1/2}\zeta_k(2)^3.\end{aligned}$$

Since $\zeta_k(2) = \prod_{v\in\mathfrak{M}_{\mathrm{f}}}(1-q_v^{-2})^{-1}$ and $E_v = (1-q_v^{-2})\mathfrak{E}_v$, from (10.1) we have

$$\begin{aligned}(10.2)\quad \lim_{X\to\infty} \frac{1}{X^2} \sum_{F\in\mathcal{Q}(L_S,X)} h_F^2 R_F^2 &= \Delta_k^{1/2}\zeta_k(2)^2\mathcal{R}_k\mathfrak{e}_\infty(L_S) \\ &\times \prod_{v\in(S\cap\mathfrak{M}_{\mathrm{f}})\backslash\mathfrak{M}_{\mathcal{B}}} \frac{\varepsilon_v(\omega_v)}{1-q_v^{-2}} \prod_{v\in S\cap\mathfrak{M}_{\mathrm{f}}\cap\mathfrak{M}_{\mathcal{B}}} \frac{(1-q_v^{-1})^2\varepsilon_v(\omega_v)}{1-q_v^{-2}} \prod_{v\notin S} \mathfrak{E}_v.\end{aligned}$$

As in the observation after Definition 10.11, one can see that $\varepsilon_v(\omega_v)/(1-q_v^{-2}) = \mathfrak{e}_v(L_v)$ for $v \in \mathfrak{M}_{\mathrm{f}} \setminus \mathfrak{M}_{\mathcal{B}}$ and $(1-q_v^{-1})^2\varepsilon_v(\omega_v)/(1-q_v^{-2}) = \mathfrak{e}_v(L_v)$ for $v \in \mathfrak{M}_{\mathrm{f}} \cap \mathfrak{M}_{\mathcal{B}}$. Hence we obtain the desired description. □

Remark 10.13. — Let $S \supset \mathfrak{M}_\infty$ and $L_S = (L_v)_{v\in S}$ any S-tuple of separable quadratic algebras. For a finite set T of places of k, let $\mathcal{Q}_T$ be the set of quadratic extensions L of k such that L does not split at least two places of T. Then by Theorem 10.12, for any T so that $T \cap S = \emptyset$, we can see that

$$\lim_{X\to\infty} \frac{1}{X^2} \sum_{F\in\mathcal{Q}_T, F\in\mathcal{Q}(L_S,X)} h_F^2 R_F^2 = \mathcal{R}_k \Delta_k^{1/2} \zeta_k(2)^2 \mathfrak{e}_\infty(L_S)^2 \prod_{v\in S\cap\mathfrak{M}_{\mathrm{f}}} \mathfrak{e}_v(L_v) \prod_{v\notin(S\cup T)} \mathfrak{E}_v \times \left(\prod_{v\in T} \mathfrak{E}_v - \prod_{v\in T} \frac{(1+q_v^{-1})(1-q_v^{-2})}{2} \right)$$

and hence

$$\lim_{T\nearrow(\mathfrak{M}\setminus S)} \lim_{X\to\infty} \frac{1}{X^2} \sum_{F\in\mathcal{Q}_T, F\in\mathcal{Q}(L_S,X)} h_F^2 R_F^2 = \mathcal{R}_k \Delta_k^{1/2} \zeta_k(2)^2 \mathfrak{e}_\infty(L_S)^2 \prod_{v\in S\cap\mathfrak{M}_{\mathrm{f}}} \mathfrak{e}_v(L_v) \prod_{v\notin S} \mathfrak{E}_v .$$

If we could change the order of limits in the left hand side of the above formula, we obtain the statement of Theorem 10.12 for unconditional S-tuples also. But to assert the statement is true, we probably have to know the principal part at the rightmost pole of the global zeta function for $\mathcal{B} = \mathrm{M}(2,2)$, which is an open problem. We conclude this section with this conjecture.

CONJECTURE 10.14. — *The statement of Theorem 10.12 also holds for any unconditional S-tuple L_S.*

11. The correlation coefficient

In this section, we define the correlation coefficient of class number times regulator of certain families of quadratic extensions, and give the value in some cases. The author would like to thank A. Yukie, who suggested working on this topic.

We fix a quadratic extension $\widetilde{k}$ of k. For any quadratic extension F of k other than $\widetilde{k}$, the compositum of F and $\widetilde{k}$ contains exactly three quadratic extensions of k. Let F^* denote the quadratic extension other than F and $\widetilde{k}$. Note that if we write $\widetilde{k} = k[x]/(x^2-\alpha)$ and $F = k[x]/(x^2-\beta)$ where $\alpha, \beta \in k$ then $F^* = k[x]/(x^2-\alpha\beta)$. If $F = \widetilde{k}$, we put $h_{F^*}R_{F^*} = 0$. As in Section 10, let S always denote the finite set of places of k containing $\mathfrak{M}_\infty$ and $L_S = (L_v)_{v\in S}$ an S-tuple of separable quadratic algebras L_v of k_v.

Definition 11.1. — *We define*

$$\mathrm{Cor}(L_S) = \lim_{X\to\infty} \frac{\sum_{F\in\mathfrak{Q}(L_S,X)} h_F R_F h_{F^*} R_{F^*}}{\left(\sum_{F\in\mathfrak{Q}(L_S,X)} h_F^2 R_F^2\right)^{1/2} \left(\sum_{F\in\mathfrak{Q}(L_S,X)} h_{F^*}^2 R_{F^*}^2\right)^{1/2}}$$

if the limit of the right hand side exists and call it the correlation coefficient.

The asymptotic behavior of the numerator as $X \to \infty$ was investigated by [6, 7, 8], while the denominator is considered in this paper. Hence we find the correlation coefficients for certain types of $\widetilde{k}$ and L_S. Let $\mathfrak{M}_{\mathrm{rm}}$, $\mathfrak{M}_{\mathrm{in}}$ and $\mathfrak{M}_{\mathrm{sp}}$ be the sets of finite places of k which are respectively ramified, inert and split on extension to $\widetilde{k}$. Take any $F \in \mathfrak{Q}(L_S)$ to put $L_v^* = F^* \otimes k_v$ and $L_S^* = (L_v^*)_{v\in S}$, which does not depend on the choice of F. In this section we prove the following theorem.

Theorem 11.2. — *Assume $\mathfrak{M}_{\mathrm{rm}} \cap \mathfrak{M}_{\mathrm{dy}} = \emptyset$ and $S \supset \mathfrak{M}_{\mathrm{rm}}$. Let $L_S = (L_v)_{v\in S}$ is an S-tuple of separable quadratic algebras such that there are at least two places v with L_v are fields. Further assume that there are at least two places v with L_v^* are fields. Then we have*

$$\mathrm{Cor}(L_S) = \prod_{v\in\mathfrak{M}_{\mathrm{in}}\setminus S} \left(1 - \frac{2q_v^{-2}}{1+q_v^{-1}+q_v^{-2}-2q_v^{-3}+q_v^{-5}}\right).$$

We first recall from [6] the asymptotic behavior of the function

$$\sum_{F\in\mathfrak{Q}(L_S,X)} h_F R_F h_{F^*} R_{F^*}$$

as $X \to \infty$. We define the constants as follows.

Definition 11.3.

(1) *Let $v \in \mathfrak{M}_{\mathrm{f}} \setminus \mathfrak{M}_{\mathrm{rm}}$ and L_v a separable quadratic algebra over k_v. We define $\mathfrak{f}_v(L_v)$ as follows.*

(i) *If $v \in \mathfrak{M}_{\mathrm{sp}}$, then we put $\mathfrak{f}_v(L_v) = \mathfrak{e}_v(L_v)$.*

(ii) *If $v \in \mathfrak{M}_{\mathrm{in}}$, then we define*

$$\mathfrak{f}_v(L_v) = \begin{cases} 2^{-1}(1-q_v^{-1})(1+q_v^{-2}) & \begin{array}{l} L_v \cong k_v \times k_v \textit{ or} \\ L_v \textit{ is quadratic unramified,} \end{array} \\ 2^{-1}|\Delta_{L_v/k_v}|_v^{-1}(1-q_v^{-1})(1-q_v^{-4}) & L_v \textit{ is quadratic ramified.} \end{cases}$$

(iii) *If $v \in \mathfrak{M}_{\mathrm{rm}} \setminus \mathfrak{M}_{\mathrm{dy}}$, then we define*

$$\mathfrak{f}_v(L_v) = \begin{cases} 2^{-1}(1-q_v^{-2}) & L_v \cong k_v \times k_v, \\ 2^{-1}(1-q_v^{-1})^2 & L_v \textit{ is quadratic unramified,} \\ 2^{-1}q_v^{-2}(1-q_v^{-2}) & L_v \cong \widetilde{k}_v, \\ 2^{-1}q_v^{-2}(1-q_v^{-1})^2 & L_v \textit{ is quadratic ramified and } L_v \not\cong \widetilde{k}_v. \end{cases}$$

(2) *For an S-tuple $L_S = (L_v)_{v\in S}$ we define $\mathfrak{f}_\infty(L_S) = \mathfrak{e}_\infty(L_S)$.*

(3) *For $v \in \mathfrak{M}_{\mathrm{f}} \setminus \mathfrak{M}_{\mathrm{rm}}$, we put*

$$\mathfrak{F}_v = \begin{cases} \mathfrak{E}_v & v \in \mathfrak{M}_{\mathrm{sp}}, \\ (1+q_v^{-2})(1-q_v^{-2}-q_v^{-3}+q_v^{-4}) & v \in \mathfrak{M}_{\mathrm{in}}. \end{cases}$$

Then the following is a refinement of [6, Corollary 7.17] in case of $\mathfrak{M}_{\mathrm{dy}} \cap \mathfrak{M}_{\mathrm{rm}} = \emptyset$.

Proposition 11.4. — *Assume $\mathfrak{M}_{\mathrm{dy}} \cap \mathfrak{M}_{\mathrm{rm}} = \emptyset$ and $S \supset \mathfrak{M}_{\mathrm{rm}}$. Then the limit*

$$\lim_{X\to\infty} \frac{1}{X^2} \sum_{F \in \mathfrak{Q}(L_S, X)} h_F R_F h_{F^*} R_{F^*}$$

exists and the value is equal to

$$\mathcal{R}_k \zeta_{\widetilde{k}}(2) \Delta_{\widetilde{k}}^{1/2} \Delta_k^{-1/2} \mathfrak{f}_\infty(L_S) \mathfrak{f}_\infty(L_S^*) \prod_{v \in S \cap \mathfrak{M}_{\mathrm{f}}} \mathfrak{f}_v(L_v) \prod_{v \notin S} \mathfrak{F}_v.$$

Proof. — The only new part is that we determine the constant $\mathfrak{f}_v(L_v)$ for $v \in \mathfrak{M}_{\mathrm{dy}}$ and L_v a quadratic ramified extension, whereas in [6] only the sum of $\mathfrak{f}_v(L_v)$ for L_v's with the same relative discriminants were given. We consider the constants $\mathfrak{f}_v(L_v)$ for these cases. For $v \in \mathfrak{M}_{\mathrm{sp}}$, we see from [6] that Proposition 7.4 gives not only $\mathfrak{e}_v(L_v)$ but also the value $\mathfrak{f}_v(L_v)$. Let $v \in \mathfrak{M}_{\mathrm{in}}$. Then a similar argument from Lemma 7.5 to Lemma 7.9 again leads us to the problem to count the number of the system of congruence equations considered in [14, Lemma 4.7], and the result follows. Since the argument is much the same as the case of $v \in \mathfrak{M}_{\mathrm{sp}}$, we choose not to include the details here. □

We next consider the second term in the denominator.

We compare $\Delta_{L_v^*/k_v}$ and Δ_{L_v/k_v}. For $v \in \mathfrak{M}_{\mathrm{rm}}$, we put $\mathrm{sgn}(L_v) = -1$ if L_v is a quadratic ramified extension and $\mathrm{sgn}(L_v) = 1$ otherwise. Then in the case $v \notin \mathfrak{M}_{\mathrm{rm}} \cap \mathfrak{M}_{\mathrm{dy}}$, the results are described as follows.

Lemma 11.5. — *We have $\Delta_{L_v^*/k_v} = \mathfrak{p}_v^{\mathrm{sgn}(L_v)} \Delta_{L_v/k_v}$ if $v \in \mathfrak{M}_{\mathrm{rm}} \setminus \mathfrak{M}_{\mathrm{dy}}$, while $\Delta_{L_v^*/k_v} = \Delta_{L_v/k_v}$ if $v \in \mathfrak{M}_{\mathrm{sp}} \cup \mathfrak{M}_{\mathrm{in}}$.*

Proof. — For $v \in \mathfrak{M}_{\mathrm{sp}}$, since $L_v^* = L_v$ we have $\Delta_{L_v^*/k_v} = \Delta_{L_v/k_v}$. If $v \in \mathfrak{M}_{\mathrm{rm}} \setminus \mathfrak{M}_{\mathrm{dy}}$, then L_v is quadratic ramified if and only if L_v^* is not. Also Δ_{L_v/k_v} is $\mathfrak{p}_v$ if L_v quadratic ramified and is $\mathcal{O}_v$ otherwise, and the result follows. We consider the case $v \in \mathfrak{M}_{\mathrm{in}}$. If L_v is not quadratic ramified, then one of L_v and L_v^* is the quadratic unramified extension and the other is $k_v \times k_v$. Hence their relative discriminants are concurrent. Assume L_v is quadratic ramified. If $v \notin \mathfrak{M}_{\mathrm{dy}}$ then L_v and L_v^* are the distinct quadratic ramified extensions of k_v with relative discriminants $\mathfrak{p}_v$, and therefore the result follows. If $v \in \mathfrak{M}_{\mathrm{dy}}$ then $\Delta_{L_v^*/k_v} = \Delta_{L_v/k_v}$ is a corollary of [8, Proposition 3.10]. Thus we obtained the desired description. □

For an S-tuple L_S, we define

$$\Delta_{L_S} = \prod_{v \in \mathfrak{M}_{\mathrm{rm}}} q_v^{\mathrm{sgn}(L_v)}.$$

Proposition 11.6. — *Assume $\mathfrak{M}_{\mathrm{rm}} \cap \mathfrak{M}_{\mathrm{dy}} = \emptyset$ and $S \supset \mathfrak{M}_{\mathrm{rm}}$. Let $L_S = (L_v)_{v \in S}$ be an S-tuple such that there are at least two places v with L_v^* fields. Then we have*

$$\lim_{X \to \infty} \frac{1}{X^2} \sum_{F \in \mathcal{Q}(L_S, X)} h_{F^*}^2 R_{F^*}^2 = \Delta_{L_S}^2 \mathcal{R}_k \Delta_k^{1/2} \zeta_k(2)^2 \mathfrak{e}_\infty(L_S^*)^2 \prod_{v \in S \cap \mathfrak{M}_{\mathrm{f}}} \mathfrak{e}_v(L_v^*) \prod_{v \notin S} \mathfrak{E}_v.$$

Proof. — By Lemma 11.5 we have $\mathcal{N}(\Delta_{F^*/k}) = \Delta_{L_S} \mathcal{N}(\Delta_{F/k})$. Also by definition, $F \in \mathcal{Q}(L_S)$ if and only if $F^* \in \mathcal{Q}(L_S^*)$. Hence, $F \in \mathcal{Q}(L_S, X)$ if and only if $F^* \in \mathcal{Q}(L_S^*, \Delta_{L_S} X)$. Therefore by applying L_S^* to Theorem 10.12, we have the proposition. □

All of these establish the necessary preparations and now we go back to the proof of Theorem 11.2. By Theorem 10.12 and Propositions 11.4, 11.6, we have

$$\mathrm{Cor}(L_S) = \mathcal{N}(\Delta_{\widetilde{k}/k})^{1/2} \Delta_{L_S}^{-1} \frac{\zeta_{\widetilde{k}}(2)}{\zeta_k(2)^2} \prod_{v \in S \cap \mathfrak{M}_{\mathrm{f}}} \frac{\mathfrak{f}_v(L_v)}{\{\mathfrak{e}_v(L_v)\mathfrak{e}_v(L_v^*)\}^{1/2}} \prod_{v \notin S} \frac{\mathfrak{C}_v}{\mathfrak{E}_v}.$$

Note that we used the relation $\mathcal{N}(\Delta_{\widetilde{k}/k}) = \Delta_{\widetilde{k}}/\Delta_k^2$. We naturally regard the right hand side of the equation above as the Euler product of finite places $\prod_{v\in\mathfrak{M}_{\mathrm{f}}} \alpha_v$. Then we immediately see $\alpha_v = 1$ for $v \in \mathfrak{M}_{\mathrm{sp}}$ and

$$\alpha_v = \frac{(1-q_v^{-2})^2}{1-q_v^{-4}} \cdot \frac{\mathfrak{C}_v}{\mathfrak{E}_v} = 1 - \frac{2q_v^{-2}}{1+q_v^{-1}+q_v^{-2}-2q_v^{-3}+q_v^{-5}}$$

for $v \in \mathfrak{M}_{\mathrm{in}} \setminus S$. Now the remaining task is to verify $\alpha_v = 1$ for $v \in S \setminus \mathfrak{M}_{\mathrm{sp}}$ and this is easily carried out by one by one calculation. For example, if $v \in S \cap \mathfrak{M}_{\mathrm{in}}$ and L_v is quadratic ramified, then we have

$$\alpha_v = \frac{(1-q_v^{-2})^2}{1-q_v^{-4}} \cdot \frac{2^{-1}|\Delta_{L_v/k_v}|_v^{-1}(1-q_v^{-1})(1-q_v^{-4})}{2^{-1}|\Delta_{L_v/k_v}|_v^{-1}(1-q_v^{-1})(1-q_v^{-2})^2} = 1,$$

and if $v \in S \cap \mathfrak{M}_{\mathrm{rm}}$ (note that by the assumption $\mathfrak{M}_{\mathrm{rm}} \cap \mathfrak{M}_{\mathrm{dy}} = \emptyset$, we have $v \notin \mathfrak{M}_{\mathrm{dy}}$ in this case) and L_v is quadratic unramified, then we have

$$\alpha_v = q_v^{(1/2)-1}(1-q_v^{-2}) \cdot \frac{2^{-1}(1-q_v^{-1})^2}{2^{-1}q_v^{-1/2}(1-q_v^{-1})^2(1-q_v^{-2})} = 1.$$

The other cases are similar and we omit the routine computation here.

Remark 11.7. — The purpose of the condition on the S-tuple L_S in Theorem 11.2 is to make the use of Theorem 10.12 for L_S and L_S^* possible. If Conjecture 10.14 is true, then we obtain Theorem 11.2 for unconditional L_S also.

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