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# A LINEAR EXTENSION OPERATOR FOR WHITNEY FIELDS ON CLOSED O-MINIMAL SETS

by Wiesław PAWŁUCKI (*)

*Dedicated to my wife Jolanta*

Abstract. — A continuous linear extension operator, different from Whitney's, for $\mathcal{C}^p$-Whitney fields (p finite) on a closed o-minimal subset of $\mathbb{R}^n$ is constructed. The construction is based on special geometrical properties of o-minimal sets earlier studied by K. Kurdyka with the author.

Résumé. — On construit un opérateur d'extension linéaire et continu pour les champs de Whitney de classe $\mathcal{C}^p$ (p fini) sur un sous-ensemble fermé o-minimal de $\mathbb{R}^n$. La construction, différente de celle de Whitney, est basée sur des propriétés géométriques spéciales des ensembles o-minimaux, étudiées avant par K. Kurdyka et l'auteur.

## 1. Introduction

In 1997 K. Kurdyka and the author gave in [6] the following o-minimal version of the Whitney extension theorem:

Theorem 1.1 ([6]). — *Given any o-minimal structure on the ordered field of real numbers $\mathbb{R}$, a compact definable subset $E \subset \mathbb{R}^n$, a definable $\mathcal{C}^p$-Whitney field $F$ on $E$, where $p \in \mathbb{N} \setminus \{0\}$, then for any integer $q \geqslant p$, there exists a definable $\mathcal{C}^p$-extension $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ of $F$ which is $\mathcal{C}^q$ on $\mathbb{R}^n \setminus E$.*

However, the extension operator $F \mapsto f$ from [6] is not linear and it was not clear how the construction from [6] based on o-minimal geometry could be adapted to get an extension operator for *all* Whitney fields on

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any compact (or more generally closed) o-minimal subset $E$ of $\mathbb{R}^n$. The present paper is devoted to this question. The main goal here is to prove the following

Theorem 1.2. — *Let $E$ be a closed o-minimal subset of $\mathbb{R}^n$ and $p \in \mathbb{N}$. Let $\mathcal{E}^p(E)$ denote the Fréchet algebra of all $\mathcal{C}^p$-Whitney fields on $E$.*

*Then there exists a continuous linear extension operator $\mathcal{L} : \mathcal{E}^p(E) \longrightarrow \mathcal{C}^p(\mathbb{R}^n)$ which has the following properties*

1. *$\mathcal{L}$ is a finite composition of operators each of which either preserves definability or (only if $p > 0$) is an integration with respect to a parameter;*
2. *operators preserving definability in (1) are only of the following five types: substituting with a definable mapping; taking a linear combination with definable coefficients; differentiation; restriction to a definable subset and extending by zero;*
3. *there exists a constant $M > 0$ such that if $\omega$ is a modulus of continuity of a field $F$, then $M\omega$ is a modulus of continuity of $\mathcal{L}F$.*

Since $\mathcal{L}$ involves integration, it may not preserve definability in the initial o-minimal structure where $E$ is definable. For example, if $F$ is a (globally) subanalytic $\mathcal{C}^p$-Whitney field, then $\mathcal{L}F$ can *a priori* involve the function $t \mapsto t \log t$, not subanalytic at 0. By a result of Lion and Rolin [7], we get in this case the following

Corollary 1.3. — *Let $\mathcal{A}$ denote the algebra of real functions generated by (globally) subanalytic functions and their logarithms; i.e. $\mathcal{A}$ consists of all functions of the form $P(h_1, \dots, h_m, \log h_1, \dots, \log h_m)$, where $h_i : \mathbb{R}^n \longrightarrow \mathbb{R} \quad (i = 1, \dots, m)$ are subanalytic, $m \in \mathbb{N}\setminus\{0\}$, $P \in \mathbb{R}[Y_1, \dots, Y_{2m}]$, and where we adopt the convention: $\log t = 0$, for $t \leqslant 0$. Let $E$ be a closed subanalytic subset of $\mathbb{R}^n$ and $p \in \mathbb{N}$.*

*Then there exists a continuous linear extension operator $\mathcal{L} : \mathcal{E}^p(E) \longrightarrow \mathcal{C}^p(\mathbb{R}^n)$ which has the following properties:*

1. *if $F$ is a $\mathcal{C}^p$-Whitney field on $E$ all derivatives of which $F^\varkappa$ are (restrictions to $E$ of) functions in $\mathcal{A}$, then $\mathcal{L}F \in \mathcal{A}$;*
2. *there exists a constant $M > 0$ such that if $\omega$ is a modulus of continuity of a field $F$, then $M\omega$ is a modulus of continuity of $\mathcal{L}F$.*

The case $p = 0$ in Theorem 1.2, when integration is not used seems worth being stated separately

Corollary 1.4. — *Let $E$ be a closed o-minimal subset of $\mathbb{R}^n$ and let $\mathcal{C}(E)$ denote the Fréchet space of all real continuous functions on $E$*

*Then there exists a continuous linear extension operator $\mathcal{L} : \mathcal{C}(E) \longrightarrow \mathcal{C}(\mathbb{R}^n)$ preserving definability and such that there exists $M > 0$ such that, if $\omega$ is a modulus of continuity for $F \in \mathcal{C}(E)$, then $M\omega$ is a modulus of continuity for $\mathcal{L}F$.*

By an o-minimal subset of an Euclidean space $\mathbb{R}^n$ we mean a subset definable in any o-minimal structure on the ordered field of real numbers $\mathbb{R}$ (see [2, 3] for the definition and fundamental properties).

We refer the reader to [13], [4], [8], [11] or/and [12] for basic facts on Whitney fields. It will be convenient for us to adopt the following definition of a Whitney field.

Let $p \in \mathbb{N} \setminus \{0\}$ and let $A$ be a locally closed subset of $\mathbb{R}^n$; i.e. contained and closed in some open subset $G \subset \mathbb{R}^n$. A $\mathcal{C}^p$-*Whitney field* on $A$ is a polynomial

$$F(u, X) = \sum_{|\varkappa| \leqslant p} \frac{1}{\varkappa!} F^{\varkappa}(u) X^{\varkappa} \in \mathcal{C}(A)[X] = \mathcal{C}(A)[X_1, \dots, X_n],$$

which fulfills the following condition

$(*)$ for each $c \in A$ and each $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leqslant p$

$$D_X^{\alpha} F(a, 0) - D_X^{\alpha} F(b, a-b) = o(|a-b|^{p-|\alpha|}), \quad \text{when } A \ni a \to c,\ A \ni b \to c,$$

or equivalently (see [8], Chapter I, Theorem 2.2) - the condition

$(**)$ for each $c \in A$

$$F(a, x-a) - F(b, x-b) = o(|x-a|^p + |x-b|^p),$$

uniformly with respect to $x \in \mathbb{R}^n$, when $A \ni a \to c$, $A \ni b \to c$.

We will denote by $\mathcal{E}^p(A)$ the real algebra of all $\mathcal{C}^p$-Whitney fields on $A$. It is a Fréchet algebra with the topology defined by the following system of seminorms

$$||F||_p^K = |F|_p^K + \sup_{\substack{a,b \in K \\ a \neq b \\ |\alpha| \leqslant p}} \frac{|D_X^{\alpha} F(a, 0) - D_X^{\alpha} F(b, a-b)|}{|a-b|^{p-|\alpha|}},$$

where $K$ is a compact subset of $A$ and $| \,.\, |_p^K$ is a seminorm defined by

$$|F|_p^K = \sup_{\substack{a \in K \\ |\alpha| \leqslant p}} |F^{\alpha}(a)|.$$

Let $\mathcal{C}^p(G)$ denote the usual Fréchet algebra of real functions of class $\mathcal{C}^p$ ($\mathcal{C}^p$-functions) on $G$. Then we have the following homomorphism of Fréchet algebras

$$T : \mathcal{C}^p(G) \longrightarrow \mathcal{E}^p(A), \quad Tf(a,X) = T^p_a f(X) = \sum_{|\varkappa|\leqslant p} \frac{1}{\varkappa!} D^\varkappa f(a) X^\varkappa,$$

and the Whitney extension theorem [13] says that there exists a linear continuous mapping

$$W : \mathcal{E}^p(A) \longrightarrow \mathcal{C}^p(G) \qquad \text{such that} \qquad T \circ W = id_{\mathcal{E}^p(A)},$$

called an *extension operator*.

A subset $E$ of $\mathbb{R}^n$ is said to be *1-regular* (*with a constant* $C \geqslant 1$) if any two points $a, b$ of $E$ can be joined in $E$ by a rectifiable arc $\gamma : [0,1] \longrightarrow E$ of length $|\gamma| \leqslant C|a-b|$.

If $F \in \mathcal{E}^p(A)$ and $K$ is a compact 1-regular subset of $A$ with a constant $C$, then

$$|F|^K_p \leqslant ||F||^K_p \leqslant 2n^{\frac{p}{2}} C^p |F|^K_p \qquad \text{(See [12], p.76, (2.5.1)).}$$

Consequently, if every compact subset $L$ of $A$ is contained in a 1-regular compact subset $K$ of $A$, then the topology of $\mathcal{E}^p(A)$ is defined by the system of seminorms $|\ .\ |^K_p$.

As was shown by Glaeser [4] (see also [8], [12] or [11]) it is convenient to use a notion of a modulus of continuity in connection with Whitney fields. By a *modulus of continuity* we will understand any continuous, increasing and concave function $\omega : [0,+\infty) \longrightarrow [0,+\infty)$, vanishing at 0. By a modulus of continuity of a $\mathcal{C}^p$-Whitney field

$$F(u,X) = \sum_{|\varkappa|\leqslant p} \frac{1}{\varkappa!} F^\varkappa(u) X^\varkappa$$

on a subset $A$ of $\mathbb{R}^n$ we will understand such a modulus of continuity $\omega$ that

$$|D^\alpha_X(a,0) - D^\alpha_X(b,a-b)| \leqslant \omega(|a-b|)|a-b|^{p-|\alpha|},$$

whenever $|\alpha| \leqslant p$ and $a, b \in A$. For a $\mathcal{C}^p$-function $f \in \mathcal{C}^p(G)$ on an open subset $G$, by its modulus of continuity we will understand a modulus of continuity of the $\mathcal{C}^p$-Whitney field $Tf$ on $G$.

Every $\mathcal{C}^p$-Whitney field on a compact subset of $\mathbb{R}^n$ admits a modulus of continuity. If a $\mathcal{C}^p$-Whitney field $F$ on a subset $A$ has a modulus of continuity $\omega$, then it is easily seen that $F$ extends by uniform continuity to a $\mathcal{C}^p$-Whitney field on $\overline{A}$ with the same modulus of continuity. Whitney's extension operator [13] has the following property (see [4]):

There exists a constant $M$ depending only on $p$ and $n$ such that, for every $F \in \mathcal{E}^p(A)$ admitting a modulus of continuity $\omega$, $M\omega$ is a modulus of continuity for $WF$. (In fact a localization by a partition of unity is necessary.)

We have also the following

Proposition 1.5. — *Let $F$ be a $\mathcal{C}^p$-Whitney field on a (locally) closed 1-regular with constant $C$ subset $A$.*

1. *If $\omega$ is a modulus of continuity of $F$ on $A$, then $|F^\alpha(a) - F^\alpha(b)| \leqslant \omega(|a-b|)$, whenever $|\alpha| = p$, $a, b \in A$.*
2. *If $\omega$ is a modulus of continuity such that $|F^\alpha(a) - F^\alpha(b)| \leqslant \omega(|a-b|)$, whenever $|\alpha| = p$, $a, b \in A$, then $n^{\frac{p}{2}} C^p \omega$ is a modulus of continuity of $F$ on $A$.*

*Proof.* — (1) being trivial, for (2) see again [12], (2.5.1), p.76. □

Shortly, our construction of the extension operator $\mathcal{L}$ is as follows. First we show how to extend $\mathcal{C}^p$-Whitney fields from a linear subspace $\mathbb{R}^k \times 0$ of $\mathbb{R}^n$. Then we generalize the construction to the set of the form $\overline{\Omega} \times 0$, where $\Omega$ is open in $\mathbb{R}^k$ for fields flat on $\partial\Omega \times 0$, simply by Hestenes Lemma. Using induction on dimension of $A$, this gives an extension operator for $A = \overline{\Gamma}$, where $\Gamma = \Omega \times 0$ assuming we have it already built for the *boundary* $\partial\Gamma = \overline{\Gamma} \setminus \Gamma$ of $\Gamma$ which in this case is $\partial\Omega \times 0$. The next generalization is by taking $A = \overline{\Gamma}$, where $\Gamma$ is a $\Lambda_p$-*regular leaf* of dimension $k$ in the sense of [6], and again assuming the fields are flat on $\partial\Gamma$. Additionally, the extension can be chosen vanishing outside a *conical neighbourhood* of $\Gamma$; i.e. the set $\{x \in \Omega \times \mathbb{R}^{n-k} : d(x, \Gamma) < \varepsilon d(x, \partial\Gamma)\}$, where $\Omega$ is the orthogonal projection of $\Gamma$ to $\mathbb{R}^k \times 0$ and $\varepsilon$ is a positive arbitrary constant. The next generalization is to the closure of a *finite tower* of $\Lambda_p$-regular leaves lying over a common *open* $\Lambda_p$-*regular cell* in $\mathbb{R}^k$. To finish the construction we will prove that every closed definable $k$-dimensional subset $A$ admits a finite decomposition $A = M_0 \cup \cdots \cup M_s$ such that each $M_i$ is a finite tower of definable $\Lambda_p$-regular leaves in a suitable linear coordinate system and for any $i, j \in \{0, \ldots, s\}$, where $i \neq j$, $\overline{M}_i$ and $\overline{M}_j$ are *simply separated relative to* $\partial M_i$; i.e. $d(x, M_j) \geqslant C d(x, \partial M_i)$, for each $x \in M_i$, with some positive constant $C$. (The proof of this $\Lambda_p$-regular Decomposition Theorem is based on [6] and [10].)

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## 2. Extension operator for a linear subspace

Observe that if $\Omega$ is an open subset of $\mathbb{R}^k$ and $A = \Omega \times 0 \subset \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$, then the algebra $\mathcal{E}^p(A)$ can be identified with the algebra of polynomials

$$F(u, W) = \sum_{|\alpha| \leqslant p} \frac{1}{\alpha!} F^\alpha(u) W^\alpha = \sum_{|\alpha| \leqslant p} \frac{1}{\alpha!} F^\alpha(u) W_1^{\alpha_1} \dots W_l^{\alpha_l},$$

where $l = n - k$ and $F^\alpha \in \mathcal{C}^{p-|\alpha|}(\Omega)$, for each $\alpha \in \mathbb{N}^l$ such that $|\alpha| \leqslant p$ (cf. [4], Chap.III, (8.4)).

Let us now consider the case $k = n - 1$ and $A = \mathbb{R}^k \times 0$. Then the extension operator will be produced using regularization of functions $F^\alpha$ by convolution. Strictly, we have the following

Proposition 2.1. — *Let $\sigma \in \{0, \dots, p\}, g \in \mathcal{C}^{p-\sigma}(\mathbb{R}^k), \varphi \in \mathcal{C}^p(\mathbb{R}^k)$. Assume that supp$\varphi$ is compact and put*

$$\varphi_w(v) = \frac{1}{w^k} \varphi(\frac{v}{w})$$

$$\text{and} \quad G(u, w) = \frac{1}{\sigma!} (g \star \varphi_w)(u) w^\sigma = \frac{1}{\sigma!} \int_{\mathbb{R}^k} g(u - v) \varphi_w(v) w^\sigma dv,$$

*for $u \in \mathbb{R}^k$ and $w \in \mathbb{R}, w > 0$.*

*Then $G : \mathbb{R}^k \times (0, +\infty) \longrightarrow \mathbb{R}$ is a $\mathcal{C}^p$-function and for every $(\alpha, \beta) \in \mathbb{N}^k \times \mathbb{N}$ such that $|\alpha| + \beta \leqslant p$*

$$\lim_{w \to 0} D^{(\alpha,\beta)} G(u, w) = \begin{cases} 0\,, & \text{if } \beta < \sigma \\ D^\alpha g(u) \int \varphi\,, & \text{if } \beta = \sigma \\ \displaystyle\sum_{|\gamma| = \beta - \sigma} \omega_{\gamma\sigma} D^{\alpha+\gamma} g(u)\,, & \text{if } \beta > \sigma \end{cases}$$

*uniformly on compact subsets with respect to $u$, where $\omega_{\gamma\sigma}$ are some constants depending only on $\gamma$, $\sigma$ and $\varphi$.*

To prove Proposition 2.1 one needs the following

Lemma 2.2. — *For any $r \in \mathbb{R}$ and $\lambda \in \{0, \dots, p\}$*

$$\frac{\partial^\lambda}{\partial w^\lambda}\left[w^r \varphi\left(\frac{v}{w}\right)\right] = w^{r-\lambda}\varphi_\lambda(\frac{v}{w}),$$

*where $\varphi_\lambda$ is a $\mathcal{C}^{p-\lambda}$-function on $\mathbb{R}^k$ with a compact support and $\int \varphi_\lambda = (k+r)(k+r-1)\cdots(k+r-\lambda+1)\int \varphi$.*

*Proof of Lemma 2.2.* —

$$\frac{\partial}{\partial w}\left[w^r \varphi\left(\frac{v}{w}\right)\right] = rw^{r-1}\varphi\left(\frac{v}{w}\right)+w^r\sum_{i=1}^{k}\frac{\partial \varphi}{\partial v_i}\left(\frac{v}{w}\right)\left(-\frac{v_i}{w^2}\right) = w^{r-1}\varphi_1\left(\frac{v}{w}\right),$$

where $\varphi_1(v) = r\varphi(v) - \sum_{i=1}^{k} v_i \frac{\partial \varphi}{\partial v_i}(v)$. Moreover, integrating by parts,

$$\int \varphi_1 = r\int \varphi - \sum_{i=1}^{k}\int v_i\frac{\partial \varphi}{\partial v_i} = (r+k)\int \varphi$$

and Lemma 2.2 follows by induction. □

*Proof of Proposition 2.1.* — $G$ is of class $\mathcal{C}^p$ on $\mathbb{R}^k \times (0, +\infty)$, because

$$G(u,w) = \int g(v)\frac{1}{w^k}\varphi\left(\frac{u-v}{w}\right)w^\sigma dv.$$

(I) Assume first that $\beta \leqslant \sigma$ and $|\alpha| \leqslant p - \sigma$. Then

$$D^{(\alpha,\beta)}G(u,w) = \frac{1}{\sigma!}\int D^\alpha g(u-v)w^{\sigma-\beta-k}\varphi_\beta\left(\frac{v}{w}\right)dv =$$

$$\frac{1}{\sigma!}w^{\sigma-\beta}\int D^\alpha g(u-v)\frac{1}{w^k}\varphi_\beta\left(\frac{v}{w}\right)dv \longrightarrow \frac{1}{\sigma!}0^{\sigma-\beta}D^\alpha g(u)\int \varphi_\beta,$$

when $w \to 0$, the convergence being uniform on compact subsets with respect to $u$. Consequently, the limit is 0, if $\beta < \sigma$ and $D^\alpha g \int \varphi$, if $\beta = \sigma$.

(II) Now assume that $\beta \leqslant \sigma$ and $|\alpha| > p - \sigma$. Then $\alpha = \gamma + \delta$, where $|\gamma| = p - \sigma$ and $\delta \neq 0$.

$$D^{(\gamma,\beta)}G(u,w) = \frac{1}{\sigma!}\int D^\gamma g(u-v)w^{\sigma-\beta-k}\varphi_\beta\left(\frac{v}{w}\right)dv =$$

$$\frac{1}{\sigma!}\int D^\gamma g(v)w^{\sigma-\beta-k}\varphi_\beta\left(\frac{u-v}{w}\right)dv.$$

$$D^{(\alpha,\beta)}G(u,w) = \frac{1}{\sigma!}\int D^\gamma g(v)w^{\sigma-\beta-k}w^{-|\delta|}D^\delta\varphi_\beta\left(\frac{u-v}{w}\right)dv =$$

$$\frac{1}{\sigma!}w^{\sigma-\beta-|\delta|}\int D^\gamma g(u-wv)D^\delta\varphi_\beta(v)dv.$$

Notice that $\sigma - \beta - |\delta| = p - |\alpha| - \beta \geqslant 0$ and $\int D^\delta \varphi_\beta(v)dv = 0$, since $\varphi_\beta$ has a compact support. Consequently, $D^{(\alpha,\beta)}G(u,w) \longrightarrow 0$, when $w \to 0$.

(III) Finally, let $\beta > \sigma$. Then $|\alpha| \leqslant p - \beta < p - \sigma$ and $\beta = \sigma + \rho$, where $\rho > 0$. By the case (I),

$$D^{(\alpha,\sigma)}G(u,w) = \frac{1}{\sigma!}\int D^\alpha g(u - vw)\varphi_\sigma(v)dv.$$

$D^\alpha g$ being of class $p - \sigma - |\alpha| \geqslant \rho$, one obtains

$$\begin{aligned} D^{(\alpha,\beta)}G(u,w) =& D^{(0,\rho)}(D^{(\alpha,\sigma)}G)(u,w) \\ =& \frac{1}{\sigma!}\sum_{|\mu|=\rho}\int D^{\alpha+\mu}g(u - vw)(-v)^\mu \varphi_\sigma(v)dv, \end{aligned}$$

which tends to $\sum_{|\mu|=\rho} \omega_{\mu\sigma} D^{\alpha+\mu}g(u)$ uniformly on compact subsets with respect to $u$, when $w \to 0$, where $\omega_{\mu\sigma} = \dfrac{1}{\sigma!}\displaystyle\int (-v)^\mu \varphi_\sigma(v)dv$. □

Proposition 2.3. — *Let $\varphi \in \mathcal{C}^p(\mathbb{R}^k)$ be with compact support and such that $\int \varphi = 1$. Then the formula*

$$\begin{aligned} L(gW^\sigma)(u,w) = \left(\frac{w}{|w|}\right)^\sigma \Bigg[&\frac{1}{\sigma!}(g \star \varphi_{|w|})(u)|w|^\sigma \\ &- \sum_{0<|\gamma|\leqslant p-\sigma} \frac{1}{(\sigma + |\gamma|)!}\omega_{\gamma\sigma} L(D^\gamma g W^{\sigma+|\gamma|})(u,|w|)\Bigg], \end{aligned}$$

*for $\sigma \in \{1,\ldots,p\}$, $g \in \mathcal{C}^{p-\sigma}(\mathbb{R}^k)$, $u \in \mathbb{R}^k$ and $w \in \mathbb{R} \setminus \{0\}$, completed by putting*

$$L(gW^\sigma)(u,0) = 0, \quad \text{and} \quad L(gW^0) = L(g) = g, \quad \text{for } g \in \mathcal{C}^p(\mathbb{R}^k),$$

*defines (inductively) a continuous linear extension operator $L = L_p : \mathcal{E}^p(\mathbb{R}^k \times 0) \longrightarrow \mathcal{C}^p(\mathbb{R}^{k+1})$.*

*Moreover, there exists a constant $M > 0$ (depending only on $k, p$ and $\varphi$) such that if $\omega$ is a modulus of continuity of a field $F \in \mathcal{E}^p(\mathbb{R}^k \times 0)$, then $M\omega$ is a modulus of continuity of the $\mathcal{C}^p$-function $LF$.*

*Proof.* — This follows immediately from Proposition 2.1. □

Now we generalize our extension operator to any linear subspace of $\mathbb{R}^n$.

Proposition 2.4. — *Let $\mathbb{R}^k \times 0 \subset \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l$, where $l > 1$. Then the formula*

$$L_p(gW_1^{\alpha_1}\cdots W_l^{\alpha_l}) = L_p(L_{p-\alpha_l}(gW_1^{\alpha_1}\cdots W_{l-1}^{\alpha_{l-1}})W_l^{\alpha_l}),$$

*where* $\alpha = (\alpha_1, \dots, \alpha_l) \in \mathbb{N}^l, |\alpha| \leqslant p$ *and* $g \in \mathcal{C}^{p-|\alpha|}(\mathbb{R}^k)$, *defines by induction on* $l$ *a linear continuous extension operator* $L = L_p : \mathcal{E}^p(\mathbb{R}^k \times 0) \longrightarrow \mathcal{C}^p(\mathbb{R}^n)$.

*Moreover, there is a constant* $M > 0$ *such that if* $\omega$ *is a modulus of continuity for* $F \in \mathcal{E}^p(\mathbb{R}^k \times 0)$, *then* $M\omega$ *is a modulus of continuity for* $LF$.

*Proof.* — This follows easily by induction from Proposition 2.3. □

## 3. A generalization to the ideal of $\mathcal{C}^p$-Whitney fields on $\overline{\Omega} \times 0$ $p$-flat on $\partial\Omega \times 0$ ($\Omega$- an open $\Lambda_p$-regular cell in $\mathbb{R}^k = \mathbb{R}^k \times 0 \subset \mathbb{R}^k \times \mathbb{R}^l$)

If $A$ is any locally closed subset of $\mathbb{R}^n$ and $B$ any closed subset of $A$, $\mathcal{E}^p(A, B)$ will denote the ideal of all $\mathcal{C}^p$-Whitney fields F on $A$ *p-flat* on $B$; i.e. $F^\alpha(u) = 0$, when $|\alpha| \leqslant p$ and $u \in B$. It is closed in $\mathcal{E}^p(A)$.

Let first $\Omega$ be any open subset of $\mathbb{R}^k$. By the Hestenes Lemma (see [12], Lemma 4.3, p.80)

$$\begin{aligned}\mathcal{E}^p(\overline{\Omega} \times 0, \partial\Omega \times 0) =& \{F = \sum_{|\alpha| \leqslant p} \frac{1}{\alpha!} F^\alpha W^\alpha : F^\alpha \in \mathcal{C}^{p-|\alpha|}(\Omega), \\ & \lim_{u \to a} D^\beta F^\alpha(u) = 0, \text{ if } a \in \partial\Omega, |\beta| \leqslant p - |\alpha|\},\end{aligned}$$

and putting

$$\widetilde{F}^\alpha(u) = \begin{cases} F^\alpha(u), & \text{if } u \in \Omega \\ 0, & \text{if } u \in \mathbb{R}^k \setminus \Omega \end{cases} \qquad \text{and} \quad \widetilde{F} = \sum_\alpha \frac{1}{\alpha!} \widetilde{F}^\alpha W^\alpha,$$

one obtains a linear continuous extension operator

$$\mathcal{E}^p(\overline{\Omega} \times 0, \partial\Omega \times 0) \ni F \longrightarrow \widetilde{F} \in \mathcal{E}^p(\mathbb{R}^k \times 0, (\mathbb{R}^k \setminus \Omega) \times 0)$$

preserving modulus of continuity.

Now we will consider the case when $\Omega$ is an open $\Lambda_p$-regular cell in $\mathbb{R}^k$ (cf. [6]). We will first recall the notion of $\Lambda_p$-regular mapping. Let $\psi : D \longrightarrow \mathbb{R}^m$ be a mapping on an open subset $D \subset \mathbb{R}^n$. We say that $\psi$ is $\Lambda_p$-*regular* (on $D$) if it is of class $\mathcal{C}^p$ and there is a constant $C \geqslant 0$ such that

$$|D^\varkappa \psi(x)| \leqslant C/d(x, \partial D)^{|\varkappa|-1}, \quad \text{whenever} \quad 1 \leqslant |\varkappa| \leqslant p \quad \text{and} \quad x \in D.$$

*Remark 3.1.* — Let $\psi$ be $\Lambda_p$-regular on $D$. Then

(1) it is $\Lambda_p$-regular on every open $D' \subset D$;

(2) if $A \subset \Omega$ is a 1-regular subset, then the restriction $\psi|A$ is Lipschitz and thus it has a continuous extension $\overline{\psi|A}$ to $\overline{A}$.

We shall say (after [6]) that $S$ is an *open* $\Lambda_p$*-regular* (*definable in a given o-minimal structure*) *cell* in $\mathbb{R}^n$ iff

(1) $S$ is an open interval in $\mathbb{R}$, when $n = 1$;
(2) $S = \{(x', x_n) : x' \in T, \quad \psi_1(x') < x_n < \psi_2(x')\}$, where $T$ is an open $\Lambda_p$-regular (definable) cell in $\mathbb{R}^{n-1}$ and each $\psi_i$ $(i = 1, 2)$ is a function on $T$ being either real $\Lambda_p$-regular (definable) function on $T$, or identically equal to $-\infty$, or identically equal to $+\infty$, and $\psi_1(x') < \psi_2(x')$, for all $x' \in T$, when $n > 1$.

*Remark 3.2.* — Such a cell $S$ is 1-regular and if $\psi_i$ is finite it is Lipschitz on $T$, thus it admits a continuous extension $\overline{\psi}_i$ to $\overline{T}$.

For any open (definable) $\Lambda_p$-regular cell in $\mathbb{R}^n$, one defines, by induction on $n$, a sequence $\rho_j : \overline{S} \longrightarrow \mathbb{R} \cup \{+\infty\}$ $(j = 1, \dots, 2n)$ of the *functions associated with the cell* $S$:

(1) When $n = 1$ and $S = (a_1, a_2)$, we put

$$\rho_1(x) = \begin{cases} x - a_1\,, & \text{if } a_1 \in \mathbb{R} \\ +\infty\,, & \text{if } a_1 = -\infty \end{cases} \quad \text{and} \quad \begin{cases} \rho_2(x) = a_2 - x\,, & \text{if } a_2 \in \mathbb{R} \\ +\infty\,, & \text{if } a_2 = +\infty. \end{cases}$$

(2) When $n > 1$ and $S = \{(x', x_n) : x' \in T \quad \psi_1(x') < x_n < \psi_2(x')\}$, let $\sigma_j$ $(j = 1, \dots, 2n-2)$ be the functions associated with $T$. We put, for any $x = (x', x_n) \in \overline{S}$, $\rho_j(x) = \sigma_j(x')$ for $j = 1, \dots, 2n-2$ and

$$\rho_{2n-1}(x) = \begin{cases} x_n - \overline{\psi}_1(x')\,, & \text{if } \psi_1 : T \to \mathbb{R} \\ +\infty\,, & \text{if } \psi_1 \equiv -\infty \end{cases} \quad \text{and}$$

$$\rho_{2n}(x) = \begin{cases} \overline{\psi}_2(x') - x_n\,, & \text{if } \psi_2 : T \to \mathbb{R} \\ +\infty\,, & \text{if } \psi_2 \equiv +\infty. \end{cases}$$

*Remark 3.3* ([6], *Lemma 3).* — There exists a constant $\Theta > 0$ such that

$$\Theta \min_j \rho_j(x) \leqslant d(x, \partial S) \leqslant \min_j \rho_j(x), \qquad \text{for} \quad x \in \overline{S}.$$

(We adopt the convention: $d(x, \emptyset) = +\infty$.)

*Remark 3.4* ([6], *Lemma 4).* — The functions $\rho_j$ which are finite are $\Lambda_p$-regular on $S$, Lipschitz on $\overline{S}$ and definable, if $S$ is so.

Lemma 3.5 (cf. [6], Lemma 5). — *Let* $\varphi_\nu : \Omega \longrightarrow \mathbb{R}$ $(\nu = 1, \dots, m)$ *be* $\Lambda_p$*-regular functions on an open subset* $\Omega \subset \mathbb{R}^k$*. Assume that* $r(u) :=$

$(\sum_{\nu=1}^{m}\varphi_\nu^2(u))^{\frac{1}{2}} \neq 0$ *for each* $u \in \Omega$. *Then there exists a constant* $\widetilde{C} > 0$ *such that for each* $u \in \Omega$

$$\left|D^\alpha\left(\frac{1}{r}\right)(u)\right| \leqslant \frac{\widetilde{C}}{r(u)\min(r(u), d(u,\partial\Omega))^{|\alpha|}}, \quad \textit{where } 0 \leqslant |\alpha| \leqslant p\,;$$

$$\textit{consequently} \quad \left|D^\alpha\left(\frac{1}{r}\right)(u)\right| \leqslant \frac{\widetilde{C}}{\min(r(u), d(u,\partial\Omega))^{|\alpha|+1}}.$$

*Proof*. — Induction on $|\alpha|$. □

PROPOSITION 3.6 (cf. [6], Lemmas 6-7). — *Let* $\Omega$ *be an open subset of* $\mathbb{R}^k$, *let* $f \in \mathcal{C}^p(\Omega \times \mathbb{R}^l)$ *and* $r \in \mathcal{C}^p(\Omega)$, *and let* $t : \Omega \longrightarrow (0, +\infty)$ *be any positive function such that* $t(u) \leqslant d(u, \partial\Omega)$ *for any* $u \in \Omega$. *Let* $\varepsilon > 0$ *and put*

$$\Delta_\varepsilon := \{(u,w) \in \Omega \times \mathbb{R}^l : |w| < \varepsilon t(u)\}.$$

*Assume that there exists a constant* $\widetilde{C} > 0$ *such that* $|D^\alpha(\frac{1}{r})| \leqslant \frac{\widetilde{C}}{t^{|\alpha|+1}}$, *when* $\alpha \in \mathbb{N}^k$, *and for each* $c \in \partial\Omega$, $D^\varkappa f(u,w) = o(t(u)^{p-|\varkappa|})$, *when* $\Delta_\varepsilon \ni (u,w) \to (c,0)$ *and* $\varkappa \in \mathbb{N}^k \times \mathbb{N}^l$, $|\varkappa| \leqslant p$.

*Let* $\xi : \mathbb{R} \longrightarrow \mathbb{R}$ *be any* $\mathcal{C}^p$*-function. Fix* $i \in \{1, \dots, l\}$ *and put*

$$g(u,w) := \xi\Big(\frac{w_i}{r(u)}\Big) f(u,w), \quad \textit{for } (u,w) \in \Omega \times \mathbb{R}^l.$$

*Then for each* $c \in \partial\Omega$, $D^\varkappa g(u,w) = o(t(u)^{p-|\varkappa|})$, *when* $\Delta_\varepsilon \ni (u,w) \to (c,0)$ *and* $\varkappa \in \mathbb{N}^k \times \mathbb{N}^l, |\varkappa| \leqslant p$.

*Proof*. — Put $h(u,w) = \xi\big(\frac{w_i}{r(u)}\big)$. By the Leibniz formula $D^\varkappa g = \sum_{\lambda \leqslant \varkappa} \binom{\varkappa}{\lambda} D^\lambda h D^{\varkappa-\lambda} f$, so it suffices to check that there exists a constant $C'_\varepsilon > 0$ such that $|D^\lambda h(u,w)| \leqslant C'_\varepsilon t(u)^{-|\lambda|}$, when $(u,w) \in \Delta_\varepsilon$ and $|\lambda| \leqslant p$. First, it is easy to see this for $h_0(u,w) := \dfrac{w_i}{r(u)}$ using Lemma 3.5. Then for $h = \xi \circ h_0$ we have

$$\frac{\partial h}{\partial x_j} = (\xi' \circ h_0)\frac{\partial h_0}{\partial x_j}, \quad \text{where } (x_1, \dots, x_n) = (u_1, \dots, u_k, w_1, \dots, w_l)$$

and $D^\lambda\left(\dfrac{\partial h}{\partial x_j}\right) = \sum_{\mu \leqslant \lambda} \binom{\lambda}{\mu} D^\mu(\xi' \circ h_0) D^{\lambda-\mu}\left(\dfrac{\partial h_0}{\partial x_j}\right)$, if $|\lambda| \leqslant p-1$, so we conclude by induction. □

*Remark 3.7.* — Suppose that $f$ is a $\mathcal{C}^p$-function on the whole space $\mathbb{R}^k \times \mathbb{R}^l$ and such that for each $c \in \partial\Omega$, $D^\varkappa f(u,0) = o(t(u)^{p-|\varkappa|})$, when $\Omega \ni u \to c$ and $\varkappa \in \mathbb{N}^k \times \mathbb{N}^l, |\varkappa| \leqslant p$.

Then for each $c \in \partial\Omega$, $D^\varkappa f(u,w) = o(t(u)^{p-|\varkappa|})$, when $\Delta_\varepsilon \ni (u,w) \to (c,0)$ and $\varkappa \in \mathbb{N}^k \times \mathbb{N}^l, |\varkappa| \leqslant p$. This follows immediately from the Taylor formula

$$D^\varkappa f(u,w) = \sum_{|\lambda| \leqslant p-|\varkappa|} \frac{1}{\lambda!} D^{\varkappa+(0,\lambda)} f(u,0) w^\lambda + o(|w|^{p-|\varkappa|}),$$

when $u \to c, w \to 0$.

Let now $\Omega$ be an open $\Lambda_p$-regular cell in $\mathbb{R}^k$ and $\rho_j$ $(j = 1, \dots, 2k)$ - the functions associated with $\Omega$. We define an extension operator

$$\mathcal{L} : \mathcal{E}^p(\overline{\Omega} \times 0, \partial\Omega \times 0) \longrightarrow \mathcal{C}^p(\mathbb{R}^n), \quad \text{where} \quad \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l,$$

by the following formula

$$\mathcal{L}F(u,w) = \begin{cases} \prod_{i=1}^{l} \prod_{j=1}^{2k} \xi\left(Q \dfrac{w_i}{\rho_j(u)}\right)(L\widetilde{F})(u,w), & \text{if } u \in \Omega \\ 0, & \text{if } u \in \mathbb{R}^k \setminus \Omega, \end{cases}$$

where $Q$ is any real number $> \sqrt{l}\Theta^{-1}$, $\Theta$ is a constant from Remark 3.3 and $\xi : \mathbb{R} \longrightarrow \mathbb{R}$ is a (definable, if we wish) $\mathcal{C}^p$-function equal to 1 in a neighborhood of 0, and equal to 0 outside the open interval $(-1,1)$.

To check that $\mathcal{L}F \in \mathcal{C}^p(\mathbb{R}^n)$ we use repeatedly Proposition 3.6 with $r = \rho_j \not\equiv +\infty$ and $t(u) = d(u, \partial\Omega)$(at the beginning we take $f = L\widetilde{F}$ as in Remark 3.7) and the Hestenes Lemma. The factors involving $\rho_j \equiv +\infty$ being obviously 1 can be omitted in the above formula.

Observe that if $\varepsilon$ is any constant from $(0,1)$, we can choose $Q$ in such a way that $\mathcal{L}F$ is $p$-flat outside the set

$$\begin{aligned} \Delta_\varepsilon(\Omega \times 0) :=& \{x \in \mathbb{R}^n : d(x, \overline{\Omega} \times 0) < \varepsilon d(x, \partial\Omega \times 0)\} \\ =& \{(u,w) \in \Omega \times \mathbb{R}^l : |w| < \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} d(u, \partial\Omega)\}. \end{aligned}$$

*Remark 3.8.* — If $r$ and $t$ are as in Proposition 3.6 and $F \in \mathcal{E}^p(\overline{\Omega} \times 0, \partial\Omega \times 0)$ is such that, for each $c \in \partial\Omega$, $F^\varkappa(u,0) = o(t(u)^{p-|\varkappa|})$, when $\Omega \ni u \to c$ and $|\varkappa| \leqslant p$, the above formula for an extension of $F$ can be modified by putting

$$\mathcal{L}'F(u,w) = \begin{cases} \prod_{i=1}^{l} \xi\left(\sqrt{l} \dfrac{w_i}{r(u)}\right) \mathcal{L}F(u,w), & \text{if } u \in \Omega \\ 0, & \text{if } u \in \mathbb{R}^k \setminus \Omega. \end{cases}$$

Then $\mathcal{L}'F$ is $p$-flat, outside the neighborhood $\{(u,w)\in\Omega\times\mathbb{R}^l : |w|<r(u)\}$ of $\Omega\times 0$ and outside $\Delta_\varepsilon(\Omega\times 0)$.

In order that $\mathcal{L}F$ (or $\mathcal{L}'F$) and $F$ have the same (up to a multiplicative constant) modulus of continuity we will prove the following

PROPOSITION 3.9. — *Under the assumptions of Proposition 3.6 assume additionally that $\Omega$ is 1-regular, $r\in\mathcal{C}^{p+1}(\Omega)$ such that*

$$|D^\alpha(\frac{1}{r})|\leqslant\frac{\widetilde{c}}{t^{|\alpha|+1}},\quad \textit{when}\quad \alpha\in\mathbb{N}^k, |\alpha|\leqslant p+1$$

*and $t$ is Lipschitz. Then there exists a constant $M>0$ such that if $\omega$ is a modulus of continuity for $f$ on $\Delta_\varepsilon$ satisfying*

$$|D^\varkappa f(u,w)|\leqslant\omega(t(u))t(u)^{p-|\varkappa|},$$

*when $(u,w)\in\Delta_\varepsilon$ and $|\varkappa|\leqslant p$, then $M\omega$ is a modulus of continuity for $g$ on $\Delta_\varepsilon$ satisfying*

$$|D^\varkappa g(u,w)|\leqslant M\omega(t(u))t(u)^{p-|\varkappa|},$$

*when $(u,w)\in\Delta_\varepsilon$ and $|\varkappa|\leqslant p$.*

*Proof.* — In view of the proof of Proposition 3.6, it suffices to check that, for a constant $M>0$, $M\omega$ is a modulus of continuity for $g$ on $\Delta_\varepsilon$. First observe that $\Delta_\varepsilon$ is 1-regular, because $\Omega$ is so and the function $t$ is Lipschitz. There exists a constant $C\geqslant 1$ such that $|t(u_1)-t(u_2)|\leqslant C|u_1-u_2|$, for any $u_1,u_2\in\Omega$.

Fix any $\varkappa\in\mathbb{N}^{k+l}$ such that $|\varkappa|=p$, any $\lambda\leqslant\varkappa$ and any two points $x_i=(u_i,w_i)\in\Delta_\varepsilon$ $(i=1,2)$. We have to estimate

$$|D^\lambda h(x_1)D^{\varkappa-\lambda}f(x_1)-D^\lambda h(x_2)D^{\varkappa-\lambda}f(x_2)|.$$

*Case I* : $\quad t(u_i)\leqslant 2C|x_1-x_2|$ $(i=1,2)$.

Then $|D^\lambda h(x_i)D^{\varkappa-\lambda}f(x_i)|\leqslant C'_\varepsilon t(u_i)^{-|\lambda|}\omega(t(u_i))t(u_i)^{p-|\varkappa-\lambda|}$

$$\leqslant C'_\varepsilon\omega(2C|x_1-x_2|)\leqslant 2CC'_\varepsilon\omega(|x_1-x_2|).$$

*Case II* : $\quad t(u_1)>2C|x_1-x_2|$.

Then $|u_1-u_2|\leqslant C|x_1-x_2|<\frac{1}{2}t(u_1)\leqslant\frac{1}{2}d(u_1,\Omega)$; thus $[x_1,x_2]\subset\Omega\times\mathbb{R}^l$.

We have $|D^\lambda h(x_1)[D^{\varkappa-\lambda}f(x_1)-D^{\varkappa-\lambda}f(x_2)]|\leqslant|D^\lambda h(x_1)|\times$

$$\Big[\sum_{1\leqslant|\mu|\leqslant p-|\varkappa-\lambda|}\frac{1}{\mu!}|D^{\varkappa-\lambda+\mu}f(x_1)||x_1-x_2|^{|\mu|}+\omega(|x_1-x_2|)|x_1-x_2|^{p-|\varkappa-\lambda|}\Big]\leqslant$$

$$M_1\omega(t(u_1))t(u_1)^{-1}|x_1-x_2|+M_2\omega(|x_1-x_2|)\leqslant M'\omega(|x_1-x_2|),$$

where $M_1, M_2$ and $M'$ are positive constants and we use: $\omega(s)t \leqslant \omega(t)s$ if $t \leqslant s$.

On the other hand $|[D^\lambda h(x_1) - D^\lambda h(x_2)]D^{\varkappa-\lambda} f(x_2)| \leqslant$

$$\sup_{x\in[x_1,x_2]} \sum_{j=1}^{k+l} |D^{\lambda+(j)} h(x)||x_1 - x_2||D^{\varkappa-\lambda} f(x_2)|.$$

For any $x = (u, w) \in [x_1, x_2]$, $\quad 2|t(u_1) - t(u)| \leqslant 2C|u_1 - u| \leqslant 2C|x_1 - x_2| < t(u_1)$ and $2|w_1 - w| \leqslant 2C|x_1 - x_2| < t(u_1)$; thus $\frac{1}{2}t(u_1) < t(u) < \frac{3}{2}t(u_1)$ and $|w| \leqslant |w_1| + |w_1 - w| < \varepsilon t(u_1) + t(u) \leqslant (2\varepsilon + 1)t(u)$.

Consequently $x \in \Delta_{2\varepsilon+1}$ and

$$|D^{\lambda+(j)} h(x)| \leqslant C'_{2\varepsilon+1} t(u)^{-|\lambda|-1} \leqslant 2^{|\lambda|+1} C'_{2\varepsilon+1} t(u_1)^{-|\lambda|-1}$$

and

$$|D^{\varkappa-\lambda} f(x_2)| \leqslant \omega(t(u_2))t(u_2)^{|\lambda|} \leqslant \Big(\frac{3}{2}\Big)^{|\lambda|+1} \omega(t(u_1)))t(u_1)^{|\lambda|}.$$

The needed inequality follows. □

*Remark 3.10.* — Suppose that $f$ is a $\mathcal{C}^p$-function on the whole space $\mathbb{R}^k \times \mathbb{R}^l$ and $\omega$ is its modulus of continuity such that

$$|D^\varkappa f(u, 0)| \leqslant \omega(t(u))t(u)^{p-|\varkappa|},$$

when $u \in \Omega$ and $\varkappa \in \mathbb{N}^{k+l}, |\varkappa| \leqslant p$.

Then there exists a constant $M'' > 0$ such that

$$|D^\varkappa f(u, w)| \leqslant M''\omega(t(u))t(u)^{p-|\varkappa|},$$

when $(u, w) \in \Delta_\varepsilon$, and $\varkappa \in \mathbb{N}^{k+l},\ |\varkappa| \leqslant p$.

Indeed, this follows immediately from

$$|D^\varkappa f(u, w) - \sum_{|\lambda|\leqslant p-|\varkappa|} \frac{1}{\lambda!} D^{\varkappa+(0,\lambda)} f(u, 0)w^\lambda| \leqslant \omega(|w|)|w|^{p-|\varkappa|}.$$

*Remark 3.11.* — If $\Omega$ is an open $\Lambda_{p+1}$-regular cell in $\mathbb{R}^k$ and $\xi$ is a $\mathcal{C}^{p+1}$-function, then there exists a positive constant $M$, such that, for any $F \in \mathcal{E}^p(\overline{\Omega} \times 0, \partial\Omega \times 0)$ (respectively, fulfilling additional conditions: $|F^\varkappa(u, 0)| \leqslant \omega(r(u))r(u)^{p-|\varkappa|}$, when $u \in \Omega,\ \varkappa \in \mathbb{N}^{k+l},\ |\varkappa| \leqslant p$) if $\omega$ is a modulus of continuity for $F$, then $M\omega$ is a modulus of continuity for $\mathcal{L}F$ (respectively, for $\mathcal{L}'F$).

## 4. A generalization to the ideal of $\mathcal{C}^p$-Whitney fields on the closure of a $\Lambda_p$-regular leaf $p$-flat on its boundary

Now we will transpose the extension operator $\mathcal{L}$ to the closure of any $\Lambda_p$-regular leaf. A subset $E \subset R^n$ is called a (definable) $\Lambda_p$-*regular leaf* of dimension $k$ in $\mathbb{R}^n$ if it is the graph $E = \{(u, \varphi(u)) : u \in \Omega\}$ of a (definable) $\Lambda_p$-regular mapping $\varphi : \Omega \longrightarrow \mathbb{R}^l$ defined on an open (definable) $\Lambda_p$-regular cell $\Omega$ in $\mathbb{R}^k$. A reduction of this case to the previous one will be by the following Lipschitz automorphism

$$\overline{\Omega} \times \mathbb{R}^l \ni (u, w) \longmapsto (u, w + \overline{\varphi}(u)) \in \overline{\Omega} \times \mathbb{R}^l$$

and the following

Proposition 4.1 (cf. [6], Proposition 3). — *Let $\varphi : \Omega \longrightarrow \mathbb{R}^l$ be a $\Lambda_p$-regular mapping defined on an open subset $\Omega \subset \mathbb{R}^k$. Let $t : \Omega \longrightarrow (0, +\infty)$ be any function such that $t(u) \leqslant d(u, \partial\Omega)$, for each $u \in \Omega$. Let $E$ be any closed subset of $\Omega \times \mathbb{R}^l$ and*

$$F(u, w; U, W) = \sum_{|\alpha|+|\beta|\leqslant p} \frac{1}{\alpha!\beta!} F^{(\alpha,\beta)}(u, w) U^\alpha W^\beta \quad \left\{ \begin{array}{l} U = (U_1, \dots, U_k), \\ W = (W_1, \dots, W_l) \end{array} \right.$$

*a $\mathcal{C}^p$-Whitney field on $E$ such that, for any $c \in \partial\Omega$*
*$F^{(\alpha,\beta)}(u, w) = o(t(u)^{p-|\alpha|-|\beta|})$, when $u \to c$ and $|\alpha| + |\beta| \leqslant p$.*

*Let $F_\varphi(u, v; U, V)$ be a polynomial in $(U, V)$ of degree $\leqslant p$ such that*

$$F_\varphi(u, v; U, V) = \sum_{|\alpha|+|\beta|\leqslant p} \frac{1}{\alpha!\beta!} F^{(\alpha,\beta)}(u, v + \varphi(u)) U^\alpha \Big(V + \sum_{1\leqslant|\varkappa|\leqslant p} \frac{1}{\varkappa!} D^\varkappa \varphi(u) U^\varkappa\Big)^\beta \; mod (U, V)^{p+1}$$

*defined for $(u, v) \in E_\varphi$, where $E_\varphi = \{(u, v) \in \Omega \times \mathbb{R}^l : (u, v + \varphi(u)) \in E\}$.*

*Then $F_\varphi$ is a $\mathcal{C}^p$-Whitney field on $E_\varphi$ such that, for any $c \in \partial\Omega$*
*$F_\varphi^{(\alpha,\beta)}(u, v) = o(t(u)^{p-|\alpha|-|\beta|})$, when $u \to c$ and $|\alpha| + |\beta| \leqslant p$.*

*Proof.* — It is easy to check that $F_\varphi$ fulfills the condition $(**)$ from Introduction, thus it is a $\mathcal{C}^p$-Whitney field on $E_\varphi$. Besides

$$F_\varphi(u, v; U, V) = \sum_{|\alpha|+|\beta|\leqslant p} \frac{1}{\alpha!\beta!} F^{(\alpha,\beta)}(u, v + \varphi(u)) U^\alpha \times$$

$$\sum_{\gamma+\sum_\varkappa \delta_\varkappa=\beta} \frac{\beta!}{\gamma! \prod \delta_\varkappa!} V^\gamma \prod_\varkappa \left[ \frac{1}{\varkappa!^{|\delta_\varkappa|}} U^{|\delta_\varkappa|\varkappa} (D^\varkappa \varphi(u))^{\delta_\varkappa} \right] \quad mod (U, V)^{p+1},$$

thus

$$F_\varphi^{(\sigma,\gamma)}(u,v) = \sum_{\alpha+\sum_\varkappa |\delta_\varkappa|\varkappa=\sigma} [\,.\,]F^{(\alpha,\gamma+\sum_\varkappa \delta_\varkappa)}(u,v+\varphi(u))\prod_\varkappa (D^\varkappa\varphi(u))^{\delta_\varkappa},$$

where $[\,.\,]$ denotes constants. To conclude notice that

$$F^{(\alpha,\gamma+\sum_\varkappa \delta_\varkappa)}(u,v+\varphi(u))\prod_\varkappa (D^\varkappa\varphi(u))^{\delta_\varkappa} =$$

$$o(1)t(u)^{p-|\alpha|-|\gamma|-\sum_\varkappa|\delta_\varkappa|}C\prod_\varkappa d(u,\partial\Omega)^{-|\delta_\varkappa||\varkappa|+|\delta_\varkappa|} =$$

$$o\big(t(u)^{p-|\sigma|-|\gamma|}\big).$$

□

*Remark 4.2.* — If $E = \{(u,\varphi(u)) : u \in \Omega\}$ (resp. $E = \Omega\times\mathbb{R}^l$), then $F_\varphi$ extends to a $\mathcal{C}^p$-Whitney field on $\overline{E_\varphi} = \overline{\Omega}\times 0$ (resp. $\overline{E_\varphi} = \overline{\Omega}\times\mathbb{R}^l$) $p$-flat on $\partial E_\varphi = \partial\Omega\times 0$ (resp. $\partial E_\varphi = \partial\Omega\times\mathbb{R}^l$).

*Proof.* — The both cases follow from the Hestenes Lemma. □

Proposition 4.3. — *Under the assumptions of Proposition 4.1, assume additionally that the mapping $\varphi$ is $\Lambda_{p+1}$-regular, $E$ and $\Omega$ are both 1-regular and $\overline{E}$ and $\partial\Omega\times\mathbb{R}^l$ are simply separated* (*). *Then there exists a constant $M>0$ such that, for each $F\in\mathcal{E}^p(\overline{E},\partial E)$, if $\omega$ is a modulus of continuity of $F$, then $M\omega$ is a modulus of continuity of $F_\varphi$.*

*Moreover, if $|F^\varkappa(u,w)| \leqslant \omega(t(u))t(u)^{p-|\varkappa|}$, when $(u,w)\in E$ and $|\varkappa|\leqslant p$, then $|F_\varphi^\varkappa(u,v)| \leqslant M\omega(t(u))t(u)^{p-|\varkappa|}$, when $(u,v)\in E_\varphi$ and $|\varkappa|\leqslant p$.*

*Proof.* — Observe that $E_\varphi$ is 1-regular. Let $\sigma\in\mathbb{N}^k$, $\gamma\in\mathbb{N}^l$ be such that $|\sigma|+|\gamma| = p$ and let $(u_i,v_i)\in E_\varphi, (i=1,2)$. We have to estimate

$$|F_\varphi^{(\sigma,\gamma)}(u_1,v_1) - F_\varphi^{(\sigma,\gamma)}(u_2,v_2)| \leqslant$$

$$\sum_{\alpha+\sum_\varkappa |\delta_\varkappa|\varkappa=\sigma} [\,.\,]|F^{(\alpha,\gamma+\sum_\varkappa \delta_\varkappa)}(u_1,v_1+\varphi(u_1))\prod_\varkappa (D^\varkappa\varphi(u_1))^{\delta_\varkappa} -$$

$$F^{(\alpha,\gamma+\sum_\varkappa \delta_\varkappa)}(u_2,v_2+\varphi(u_2))\prod_\varkappa (D^\varkappa\varphi(u_2))^{\delta_\varkappa}|.$$

Fix $\lambda = (\alpha,\gamma+\sum_\varkappa\delta_\varkappa)$ and put $x_i = (u_i, v_i+\varphi(u_i))$ and

$$\theta(u) = \prod_\varkappa (D^\varkappa\varphi(u))^{\delta_\varkappa}.$$

(*) See the beginning of Section 5 for the definition of simple separation.

*Case I*: $|x_1 - x_2| \geqslant \frac{1}{2} d(u_i, \partial\Omega)$ for $i = 1, 2$.

$$|F^\lambda(x_i)\theta(u_i)| \leqslant \omega(d(x_i, \partial E)) d(x_i, \partial E)^{p-|\lambda|} |\theta(u_i)| \leqslant$$
$$\omega(C d(u_i, \partial\Omega))[C d(u_i, \partial\Omega)]^{p-|\lambda|} |\theta(u_i)| \leqslant$$
$$\omega(2C|x_1 - x_2|)[C d(u_i, \partial\Omega)]^{p-|\lambda|} \prod_{\varkappa} d(u_i, \partial\Omega)^{-|\delta_\varkappa||\varkappa| + |\delta_\varkappa|} \leqslant M\omega(|x_1 - x_2|).$$

*Case II*: $|x_1 - x_2| \leqslant \frac{1}{2} d(u_1, \partial\Omega)$.

$$|F^\lambda(x_1)\theta(u_1) - F^\lambda(x_2)\theta(u_2)| \leqslant$$
$$|F^\lambda(x_1) - F^\lambda(x_2)||\theta(u_2)| + |F^\lambda(x_1)||\theta(u_1) - \theta(u_2)| \leqslant$$
$$\Bigg[\sum_{1 \leqslant |\mu| \leqslant p-|\lambda|} \frac{1}{\mu!} |F^{\lambda+\mu}(x_1)||x_2 - x_1|^{|\mu|} + \omega(|x_1 - x_2|)|x_1 - x_2|^{p-|\lambda|}\Bigg] |\theta(u_2)| +$$
$$|F^\lambda(x_1)| \sup_{z \in [u_1, u_2]} \sum_{j=1}^{k} |D^{(j)}\theta(z)||u_1 - u_2| \leqslant$$
$$\Bigg[\sum_{1 \leqslant |\mu| \leqslant p-|\lambda|} \frac{1}{\mu!} \omega(d(x_1, \partial E)) d(x_1, \partial E)^{p-|\lambda|-|\mu|} |x_1 - x_2| d(u_1, \partial\Omega)^{|\mu|-1} +$$
$$\omega(|x_1 - x_2|) d(u_1, \partial\Omega)^{p-|\lambda|}\Bigg] |\theta(u_2)| +$$
$$\omega(d(x_1, \partial\Omega))|x_1 - x_2| \sup_{z \in [u_1, u_2]} \sum_{j=1}^{k} |D^{(j)}\theta(z)|$$
$$\Bigg[C_1 \omega(d(u_1, \partial\Omega))|x_1 - x_2| d(u_1, \partial\Omega)|^{p-|\lambda|-1} +$$
$$\omega(|x_1 - x_2|) d(u_1, \partial\Omega)^{p-|\lambda|}\Bigg] |\theta(u_2)| +$$
$$C_2 \omega(d(u_1, \partial\Omega))|x_1 - x_2| \sup_{z \in [u_1, u_2]} \prod_{\varkappa} d(z, \partial\Omega)^{-|\delta_\varkappa||\varkappa| + |\delta_\varkappa| - 1}.$$

Now it suffices to observe that $\omega(d(u_1, \partial\Omega))|x_1 - x_2| \leqslant \omega(|x_1 - x_2|) d(u_1, \partial\Omega)$ and $d(z, \partial\Omega) \geqslant d(u_1, \partial\Omega) - |z - u_1| \geqslant d(u_1, \partial\Omega) - |x_1 - x_2| \geqslant \frac{1}{2} d(u_1, \partial\Omega)$, if $z \in [u_1, u_2]$. □

Assume now that $E = \{(u, \varphi(u)) : u \in \Omega\}$ is a $\Lambda_p$-regular leaf of dimension $k$ in $\mathbb{R}^n$. We define an extension operator $\mathcal{L} : \mathcal{E}^p(\overline{E}, \partial E) \longrightarrow \mathcal{C}^p(\mathbb{R}^n)$ by the formula

$$\mathcal{L}F = \begin{cases} (\mathcal{L}F_\varphi)_{-\varphi}, & \text{on } \Omega \times \mathbb{R}^l \\ 0, & \text{on } (\mathbb{R}^k \setminus \Omega) \times \mathbb{R}^l. \end{cases}$$

For any constant $\varepsilon > 0$, we can specify this operator in such a way that for each $F \in \mathcal{E}^p(\overline{E}, \partial E)$, $\mathcal{L}F$ is flat outside the neighborhood $\Delta_\varepsilon(E) := \{x \in \mathbb{R}^n : d(x,E) < \varepsilon d(x, \partial E)\}$.

## 5. A generalization to a finite tower of $\Lambda_p$-regular leaves

Here we will generalize the extension operator $\mathcal{L}$ to the ideal $\mathcal{E}^p(\overline{E}, \partial E)$, where $E$ is a finite disjoint union $E = E_1 \cup \cdots \cup E_s$ of graphs of $\Lambda_p$-regular mappings $\varphi_\sigma : \Omega \longrightarrow \mathbb{R}^l$ $(\sigma = 1, \ldots, s)$ defined on a common open $\Lambda_p$-regular cell $\Omega \subset \mathbb{R}^k$. Put $r_\sigma(u) := |\varphi_\sigma(u) - \varphi_s(u)|$ for $\sigma = 1, \ldots, s-1$ and $u \in \Omega$.

We first define $\mathcal{L}F$ for any $F \in \mathcal{E}^p(\overline{E}, \overline{E}_1 \cup \cdots \cup \overline{E}_{s-1} \cup \partial E_s)$.

Then we put

$$\mathcal{L}F = \begin{cases} \left[ \prod_{\sigma=1}^{s-1} \prod_{i=1}^{l} \xi\left(\sqrt{l}\, \frac{w_i}{r_\sigma(u)}\right) \mathcal{L}\big((F|\overline{E}_s)_{\varphi_s}\big)\right]_{-\varphi_s}, & \text{on } \Omega \times \mathbb{R}^l \\ 0, & \text{on } (\mathbb{R}^k \setminus \Omega) \times \mathbb{R}^l, \end{cases}$$

which gives an extension operator according to Proposition 3.6 (used repeatedly with $t(u) := \min\big(\{r_\sigma(u)\}, d(u, \partial\Omega)\big)$), Remark 3.8 and Proposition 4.1.

Let now consider a general case where $F$ is any element of $\mathcal{E}^p(\overline{E}, \partial E)$. Proceeding by induction, assume that $\mathcal{L}(F|\overline{E}_1 \cup \cdots \cup \overline{E}_{s-1})$ has already been defined. Then $H := F - T\mathcal{L}(F|\overline{E}_1 \cup \cdots \cup \overline{E}_{s-1})|\overline{E} \in \mathcal{E}^p(\overline{E}, \overline{E}_1 \cup \cdots \cup \overline{E}_{s-1} \cup \partial E_s)$ and we put

$$\mathcal{L}F = \mathcal{L}H + \mathcal{L}(F|\overline{E}_1 \cup \cdots \cup \overline{E}_{s-1}).$$

For any $\varepsilon > 0$, we can specify this operator in such a way that $\mathcal{L}F$ is $p$-flat outside the set $\Delta_\varepsilon(E) := \{x \in \mathbb{R}^n : d(x,E) < \varepsilon d(x, \partial E)\}$.

## 6. Extension operator for a closed definable subset of $\mathbb{R}^n$

Definition 6.1 (cf. [10]). — *Let $A, B, Z \subset \mathbb{R}^n$. We say that $A$ and $B$ are simply $Z$-separated if one of the following equivalent conditions holds*

(1) $\exists M > 0 \forall x \in A, \quad d(x,B) \geqslant M d(x,Z)$;

(2) $\exists C > 0 \forall x \in \mathbb{R}^n, \quad d(x,A) + d(x,B) \geqslant C d(x,Z)$. *(If (1) holds, one can take $C = M/(M+1)$.)*

*We say that $A$ and $B$ are simply separated if they are simply $A \cap B$-separated.*

Proposition 6.2. — *Let $E_i \supset E'_i \quad (i = 1, \dots, s)$ be closed subsets of $\mathbb{R}^n$ and let $C > 0$ be a constant such that, for any $i, j \in \{1, \dots, s\}, i \neq j$ and any $x \in \mathbb{R}^n$*

$$d(x, E_i) + d(x, E_j) \geqslant C d(x, E'_i).$$

*Let $\varepsilon \in (0, C/2]$. Put $\Gamma_\varepsilon(E_i, E'_i) := \{x \in \mathbb{R}^n : d(x, E_i) < \varepsilon d(x, E'_i)\}$. Suppose that, for each $i = 1, \dots, s$*

$$\mathcal{L}_i : \mathcal{E}^p(E_i, E'_i) \longrightarrow \mathcal{C}^p(\mathbb{R}^n)$$

*is an extension operator such that $\mathcal{L}_i F$ is $p$-flat outside $\Gamma_\varepsilon(E_i, E'_i)$, for any $F \in \mathcal{E}^p(E_i, E'_i)$.*

*Then the formula*

$$\mathcal{L}F = \sum_{i=1}^{s} \mathcal{L}_i(F|E_i)$$

*defines an extension operator $\mathcal{L} : \mathcal{E}^p(\bigcup_i E_i, \bigcup_i E'_i) \longrightarrow \mathcal{C}^p(\mathbb{R}^n)$. Moreover, if each $\mathcal{L}_i$ preserves (up to a multiplicative constant) a modulus of continuity, then $\mathcal{L}$ has the same property.*

*Proof.* — It suffices to check that $\Gamma_\varepsilon(E_i, E'_i) \cap \Gamma_\varepsilon(E_j, E'_j) = \emptyset$, if $i \neq j$. If there were $x \in \Gamma_\varepsilon(E_i, E'_i) \cap \Gamma_\varepsilon(E_j, E'_j)$, then

$$2\varepsilon[d(x, E'_i) + d(x, E'_j)] > 2[d(x, E_i) + d(x, E_j)] \geqslant C[d(x, E'_i) + d(x, E'_j)],$$

a contradiction. □

A proof of the following theorem will be given in the next section.

$\Lambda_p$-regular Decomposition Theorem 6.3. — *Let $E$ be a closed subset of $\mathbb{R}^n$ definable in some fixed o-minimal structure on the ordered field of the real numbers $\mathbb{R}$. Let $k = \dim E$. Let $Z$ be any definable subset of $E$ of dimension $< k$.*

*Then there exists a finite decomposition*

$$E = M_1 \cup \dots \cup M_s \cup A$$

*such that each $M_i$ is a finite tower of $\Lambda_p$-regular $k$-dimensional definable leaves in an appropriate linear coordinate system, $A$ is a closed definable subset of $\dim < k$ containing $Z$ and, for any $i, j \in \{1, \dots, s\}$ $(i \neq j)$, $\overline{M}_i$ and $\overline{M}_j$ are simply $\partial M_i$-separated and, for any $i$, $\overline{M}_i$ and $A$ are simply $\partial M_i$-separated.*

In order to define an extension operator for any closed definable subset $E \subset \mathbb{R}^n$ we will use induction on $\dim E$. By the induction hypothesis we have an extension operator

$$\mathcal{L}_0 : \mathcal{E}^p(\cup_{i=1}^s \partial M_i \cup A) \longrightarrow \mathcal{C}^p(\mathbb{R}^n),$$

and by Section 5 combined with Proposition 6.2 we have an extension operator

$$\mathcal{L}_1 : \mathcal{E}^p(E, \cup_{i=1}^s \partial M_i \cup A) \longrightarrow \mathcal{C}^p(\mathbb{R}^n).$$

Now an extension operator for $E$ is defined by the formula

$$\mathcal{L}F = \mathcal{L}_1[F - T\mathcal{L}_0(F| \cup_i \partial M_i \cup A)|E] + \mathcal{L}_0(F| \cup_i \partial M_i \cup A).$$

## 7. Proof of $\Lambda_p$-regular Decomposition Theorem

Let $P \subset \mathbb{R}^n$ be any definable subset and $V$ - a linear subspace of $\mathbb{R}^n$ of dimension $n-k$. Following [10], we will say that $P$ is *perfectly situated relative to* $V$ if, for a/any linear complement $W$ of $V$ in $\mathbb{R}^n$, $P$ can be represented as a disjoint union

$$P = \bigcup \{\hat{\varphi} : \varphi \in \mathcal{F}\}$$

of a finite family $\mathcal{F}$ of definable $\mathcal{C}^1$-mappings $\varphi : \Delta_\varphi \longrightarrow V$ defined on connected $\mathcal{C}^1$-submanifolds $\Delta_\varphi \subset W$ and with bounded derivatives ($\hat{\varphi}$ stands here for the graph $\{u + \varphi(u) : u \in \Delta_\varphi\}$ of $\varphi$).

We will use the following

Theorem 7.1 (cf. [10], Theorem 0). — *Let $\Sigma = \{\sigma \subset \{1, \ldots, n\} : \operatorname{card} \sigma = n-k\} = \{\sigma_1, \ldots, \sigma_q\}$, where $q = \binom{n}{k}$.*

*Let $V_i = \bigoplus_{\nu \in \sigma_i} \mathbb{R} e_\nu$ $(i = 1, \ldots, q)$, where $e_1, \ldots, e_n$ is the canonical basis in $\mathbb{R}^n$.*

*Any definable closed subset $E \subset \mathbb{R}^n$ of dimension $k$ is a union $E = \bigcup_{i=1}^q E_i$ of definable closed subsets $E_i$ such that, for each $i$, $E_i$ is perfectly situated relative to $V_i$ and, for each $j \neq i$, $E_i$ and $E_j$ are simply separated and* $\dim(E_i \cap E_j) < k$.

From the last theorem and easy properties of simply separated sets (see [10], Proposition 2; (1) and (3)), it follows that it suffices to prove $\Lambda_p$-regular Decomposition Theorem for each $E_i$ and $Z_i = (Z \cap E_i) \cup (\bigcup_{j \neq i} E_i \cap E_j)$ separately, therefore - up to a permutation of variables - it suffices to prove it assuming that $E$ is perfectly situated relative to $0 \times \mathbb{R}^l$, where $l = n-k$. The proof in this case is based on the following two propositions.

Proposition 7.2 ([6], Proposition 2). — *If $\varphi : \Omega \longrightarrow \mathbb{R}^l$ is a definable $\Lambda_1$-regular mapping defined on an open $\Omega \subset \mathbb{R}^k$, then there exists a closed definable subset $Z$ of $\Omega$ such that $\dim Z < k$ and $\varphi|\Omega \setminus Z$ is $\Lambda_p$-regular mapping on $\Omega \setminus Z$.*

Proposition 7.3 ([6], Proposition 4). — *For any definable open subset $\Omega \subset \mathbb{R}^k$, there exists a finite family $\mathcal{S}$ of disjoint subsets of $\Omega$ such that $\dim(\Omega \setminus \bigcup \mathcal{S}) < k$ and each $S \in \mathcal{S}$ is an open definable $\Lambda_p$-regular cell in an appropriate linear system of coordinates in $\mathbb{R}^k$.*

*Proof of Proposition 7.3.* — See [6], Proposition 4, where the set is assumed bounded, but this assumption is not essential. Alternatively, first one can apply [10]; Theorem 1, $(B_k)$ to get the case $p = 1$ of Proposition 7.3, which is the theorem of Kurdyka [5] and Parusiński [9], and then by induction on $k$ one gets the case of any $p \geqslant 1$, applying Proposition 7.2. □

To finish the proof of the theorem, first represent $E$ as union of graphs with bounded derivatives:

$$E = \bigcup\{\hat{\varphi} : \varphi \in \mathcal{F}\},$$

as in the beginning of the section. Adding to $Z$ all the graphs with $\dim \Delta_\varphi < k$, one can assume that

$$E = Z \cup \bigcup\{\hat{\varphi} : \varphi \in \mathcal{F}_*\},$$

where $\mathcal{F}_* = \{\varphi \in \mathcal{F} : \Delta_\varphi \quad \text{non-empty open in} \quad \mathbb{R}^k\}$. By Proposition 7.2, for each $\varphi \in \mathcal{F}_*$ there exists a closed definable subset $K_\varphi$ of $\Delta_\varphi$ of dim $< k$ such that $\varphi|\Delta_\varphi \setminus K_\varphi$ is $\Lambda_p$-regular. Let

$$\boldsymbol{\Theta} := \overline{\pi(Z)} \cup \bigcup\{\partial\Delta_\varphi \cup K_\varphi : \varphi \in \mathcal{F}_*\},$$

where $\pi : \mathbb{R}^k \times \mathbb{R}^l \longrightarrow \mathbb{R}^k$ is the canonical projection. Take a family $\mathcal{S}$ as in Proposition 7.3 for the open subset

$$\Omega := \bigcup\{\Delta_\varphi : \varphi \in \mathcal{F}_*\} \setminus \boldsymbol{\Theta}.$$

Now it suffices to define, for each $S \in \mathcal{S}$

$$M_S := E \cup \pi^{-1}(S) \qquad \text{and} \qquad A := E \setminus \bigcup\{M_S : S \in \mathcal{S}\}.$$

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