ANNALES

DE

L'INSTITUT FOURIER

Otto VAN KOERT

Open books on contact five-manifolds

Tome 58, nº 1 (2008), p. 139-157.

<http://aif.cedram.org/item?id=AIF_2008__58_1_139_0>

Ann. Inst. Fourier, Grenoble
58, 1 (2008) 139-157

OPEN BOOKS ON CONTACT FIVE-MANIFOLDS

by Otto VAN KOERT (*)

Abstract. — By using open book techniques we give an alternative proof of a theorem about the existence of contact structures on five-manifolds due to Geiges. The theorem asserts that simply-connected five-manifolds admit a contact structure in every homotopy class of almost contact structures.

Résumé. — En utilisant des techniques de livres ouverts, nous donnons une autre démonstration d'un théorème de Geiges sur l'existence de structures de contact sur des variétés de dimension cinq. Ce théorème affirme que les variétés simplement connexes de dimension cinq admettent une structure de contact dans toute classe d'homotopie de structures presque de contact.

1. Introduction

At the ICM of 2002 Giroux announced his results on the relation between contact manifolds and open book decompositions. The easy part of his results (and the part that we shall use) is a generalization of a construction due to Thurston and Winkelnkemper [10]; one can adapt certain open book decompositions to contact structures, thus giving a procedure to construct contact structures using open books. Roughly speaking Giroux's construction goes as follows. Take a compact Stein manifold P or more generally an exact symplectic manifold with boundary and a symplectomorphism ψ of P that is the identity near the boundary of P. The mapping torus of (P, ψ) can be shown to admit a natural contact structure. On the other hand, a neighborhood of the binding $\partial P \times D^2$ has a natural contact structure

Keywords: Contact topology, open books.
Math. classification: 53D35, 57R17.
(*) This paper is a part of my thesis, which I wrote under supervision of H. Geiges at the university of Cologne. I am indebted to him for his support and patience. I would also like to thank the anonymous referee for pointing out a nicer argument for a part of the construction. Currently I am supported by the F.N.R.S., Belgium.

that can be glued to the contact structure on the mapping torus, therefore giving rise to a closed contact manifold with an adapted contact structure.

In this paper, we will use Giroux's construction to reprove a theorem on the existence of contact structures on five-manifolds due to Geiges [3]. More precisely, we shall reprove the following theorem.

THEOREM 1.1 (Geiges). — *Let M be a simply-connected five-manifold. Then M admits a contact structure in every homotopy class of almost contact structures.*

The main idea of our alternative proof is very simple. Using the classification of simply-connected five-manifolds, we can reduce the problem to finding contact structures on certain model manifolds. We do this by explicit construction using Giroux's procedure. Although this is not necessary in the construction of Giroux, we will always take Stein surfaces as pages. Since the classification of simply-connected five-manifolds is determined by the homology groups and the second Stiefel-Whitney class, it suffices to track these topological invariants.

2. Preliminaries

We start by recalling Giroux's construction in a bit more detail. Let P be a compact Stein manifold of real dimension $2n$ and take a strictly plurisubharmonic function f. The function f defines an exact symplectic form $d\beta = -d(d^c f) = -d(df \circ J)$, where J is the complex structure on P. Let now $\psi : P \to P$ be a symplectomorphism that is the identity near the boundary of P. In general, ψ does not preserve β, which we would like to have. However, it turns out that the pull-back of β under ψ can be assumed to be exact due to the following lemma of Giroux [4].

LEMMA 2.1 (Giroux). — *The symplectomorphism ψ can be isotoped to a symplectomorphism ψ' that is the identity near the boundary and that satisfies*

$$\psi'^* \beta = \beta - dh.$$

Proof. — Let us denote the one-form $\psi^* \beta - \beta$ by μ. Since $d\beta$ is non-degenerate, we find a unique solution Y to the equation $i_Y d\beta = -\mu$. The flow of the vector field Y preserves $d\beta$, because μ is closed,

$$0 = -d\mu = d i_Y d\beta = \mathcal{L}_Y \beta.$$

Since ψ is the identity near the boundary, μ and hence Y vanish near the boundary. If we denote the time t flow of Y by φ_t, then we see that

$\psi' = \psi \circ \varphi_1$ is a symplectomorphism that is the identity near the boundary. Note that $\mathcal{L}_Y \mu = 0$, so $\varphi_t^* \mu = \mu$ for all t. We check that the difference of the pullback of β and β is indeed exact. We have

$$(\psi \circ \varphi_1)^* \beta - \beta = \varphi_1^*(\mu + \beta) - \beta = \mu + \varphi_1^* \beta - \beta.$$

On the other hand, we can express the difference $\varphi_1^* \beta - \beta$ as

$$\begin{aligned}\varphi_1^* \beta - \beta &= \int_0^1 \frac{d}{dt} \varphi_t^* \beta = \int_0^1 \varphi_t^* \mathcal{L}_Y \beta = \int_0^1 \varphi_t^* \left(i_Y d\beta + d(i_Y \beta) \right) \\ &= -\mu + \int_0^1 d\varphi_t^*(i_Y \beta).\end{aligned}$$

Moving μ to the left-hand-side, we see that $\mu + \varphi_1^* \beta - \beta$ is exact, which shows the claim of the lemma. □

Using this lemma we can make a mapping torus with a natural contact structure. The form

$$\alpha = d\varphi + \beta$$

is a contact form on $P \times \mathbb{R}$ that descends to the perturbed mapping torus

$$A := P \times \mathbb{R} / (x, \varphi) \sim \Psi(x, \varphi) = (\psi(x), \varphi + h(x)).$$

We see that α indeed gives A a well defined contact form, because

$$\Psi^* \alpha = d\varphi + dh + \psi^* \beta = d\varphi + dh + \beta - dh = \alpha.$$

The boundary of the page $K = \partial P$ inherits a natural contact form $\gamma = \beta|_{TK}$, since P is a compact Stein manifold. We use this to "complete" A into an open book. Glue $B := K \times D^2$ along its boundary to A. This can be done in a natural way, since ψ was assumed to be the identity near the boundary of P.

This construction involving a mapping torus is sometimes called an **abstract open book**. Note that one can put a contact form $\tilde{\alpha}$ on B that matches the contact form on A, thus giving rise to a closed contact manifold $X := A \cup_\partial B$. This contact form $\tilde{\alpha}$ has the form

$$\tilde{\alpha} = h_1(r)\gamma + h_2(r) d\varphi,$$

where (r, φ) are polar coordinates on D^2 and h_1 and h_2 are functions that are sketched in Figure 2.1. For the choice of functions indicated in Figure 2.1 the form $\tilde{\alpha}$ is in fact a contact form, since the contact condition can be rewritten as

$$\tilde{\alpha} \wedge d\tilde{\alpha}^n = h_1^{n-1} \frac{h_1 h_2' - h_2 h_1'}{r} \gamma \wedge d\gamma^{n-1} \wedge dr \wedge r d\vartheta.$$

This is a non-vanishing form, since $\frac{h_1 h_2' - h_2 h_1'}{r} \neq 0$ by our choice of functions h_1 and h_2. Also note that by choosing these functions suitably, we can ensure that the contact form $\tilde{\alpha}$ matches the contact form α near the boundary of A. Hence we get a well defined contact form on the entire abstract open book. We will call the abstract open book together with the contact form given by the above construction an **abstract contact open book**. In this procedure the contact structure is determined by the page P and the monodromy ψ up to isotopy.

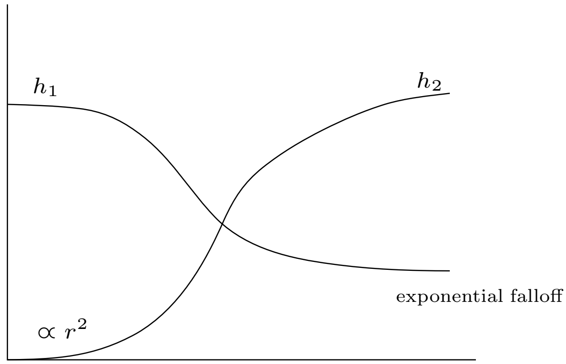

Figure 2.1. The functions h_1 and h_2

Remark 2.2. — If two manifolds, say M and N, are constructed via this procedure, then their connected sum $M\#N$ can also be constructed this way. Indeed, if M is an abstract contact open book coming from the pair (P_1, ψ_1) and N is constructed from (P_2, ψ_2), then we may consider the boundary connected sum $P_1 \natural P_2$, which is again a Stein manifold. Note that the symplectomorphisms ψ_1 and ψ_2 can be glued to a symplectomorphism $\psi_1 \natural \psi_2$ of $P_1 \natural P_2$, since both symplectomorphisms are the identity near the boundary. Then the abstract contact open book constructed from $(P_1 \natural P_2, \psi_1 \natural \psi_2)$ provides an open book decomposition for $M\#N$. This procedure is called a **book-connected sum**.

2.1. Classification of simply-connected five-manifolds

We now recall Barden's classification of simply-connected five-manifolds [2]. For a simply-connected manifold M we can regard the second Stiefel-Whitney class as a map $w_2(M): \; H_2(M) \to \mathbb{Z}_2$.

THEOREM 2.3 (Barden). — *Two simply-connected five-manifolds M_1 and M_2 are diffeomorphic if and only if there exists an isomorphism of groups $A: H_2(M_1) \to H_2(M_2)$ such that*

$$w_2(M_1) = w_2(M_2) \circ A.$$

Before we give a description of the decomposition of a simply-connected five-manifold into prime manifolds, we would like to point out that a necessary condition for the existence of a contact form is the existence of an almost contact structure. The existence of an almost contact structure is governed by purely topological considerations. For instance, a simply-connected five-manifold M admits an almost contact structure if and only if the third integral Stiefel-Whitney class $W_3(M) = 0$, see Lemma 7 from [3].

A simply-connected five-manifold can be uniquely decomposed into a connected sum of prime manifolds M_k for $1 \leqslant k \leqslant \infty$ with possibly one extra summand X_j with $j = -1$ or $1 \leqslant j \leqslant \infty$. The second Stiefel-Whitney class of X_j, the class $w_2(X_j)$, is always non-trivial.

The manifold M_k has homology group $H_2(M_k) \cong \mathbb{Z}_k \oplus \mathbb{Z}_k$ for $1 < k < \infty$. The manifold M_∞ can be identified with $S^2 \times S^3$. In the decomposition above we always take k to be a prime power if $k \neq \infty$. The manifold M_1 is S^5 and is only needed in a decomposition of M if $M \cong S^5$. These manifolds all carry an almost contact structure since $W_3(M_k) = 0$.

The manifold X_{-1} is known as the Wu-manifold and satisfies $H_2(X_{-1}) = \mathbb{Z}_2$. It does not carry an almost contact structure since $W_3(X_{-1}) \neq 0$. For $1 \leqslant j < \infty$ we have $H_2(X_j) = \mathbb{Z}_{2^j} \oplus \mathbb{Z}_{2^j}$. Again $W_3(X_j) \neq 0$, so we do not need to consider these manifolds because they cannot have a contact structure. Finally the manifold X_∞ can be identified with $S^2 \tilde{\times} S^3$, the non-trivial S^3-bundle over S^2 and has $H_2(X_\infty) \cong \mathbb{Z}$. Among the "$X$"-manifolds X_∞ is the only one with vanishing W_3, so we shall need to consider $S^2 \tilde{\times} S^3$.

Using this decomposition we see that it suffices to compute the second homology group and the second Stiefel-Whitney class in order to determine which contact five-manifold we have.

2.2. Some general arguments for computing the homology of open books

In our construction we will always use a simply-connected page. This implies that the abstract open book will also be a simply-connected manifold. Indeed, if we use P to denote the page of the open book and A to denote the mapping torus of P, we see that the homotopy exact sequence of a fibration implies that $\pi_1(A) = \mathbb{Z}$. Now consider the completed open book X, obtained by gluing $B := \partial P \times D^2$ to A along a collar neighborhood of its boundary. Note that the generator of the fundamental group of A gets killed in B; the curve $\{point\} \times S^1$ lying in the boundary of A represents the generator. In B, this curve bounds the disk $\{point\} \times D^2$. On the other hand, we can always choose a curve lying in $\partial P \times \{point\}$ to represent a generator of $\pi_1(B)$. However, such a curve will always be contractible in A, since it lies in a page. An application of the Seifert-Van Kampen theorem shows that the open book is simply-connected.

Since the classification of simply-connected five-manifolds is mainly controlled by homology, some general arguments to compute the homology of open books turn out to be useful. First of all, we shall stick to the notation introduced in Section 2, namely we shall denote the mapping torus of a compact Stein manifold P by A, the thickened binding by B and the closed manifold by $X := A \cup_\partial B$. We can, in fact, glue along a collar neighborhood of the boundary. Therefore, we can apply the Mayer-Vietoris sequence straight away to X and its "parts" A and B to compute the homology of X.

The homology of the mapping torus A, being a fiber bundle over S^1, can be determined from the Wang sequence [11], but see also [1]. This works as follows. Suppose P is a manifold and φ a diffeomorphism of P. If the mapping torus A is defined by

$$A := P \times [0,1]/(x,0) \sim (\varphi(x),1),$$

then we have the following long exact sequence in homology, called the Wang sequence,

$$\to H_3(A;\mathbb{Z}) \to H_2(P;\mathbb{Z}) \stackrel{\varphi_*-id}{\to} H_2(P;\mathbb{Z}) \stackrel{incl_*}{\to} H_2(A;\mathbb{Z}) \to .$$

The homology of B is simply the homology of the boundary of a page $K = \partial P$. Finally we have the homotopy equivalence $A \cap B \sim K \times S^1$, so the homology of $A \cap B$ can be determined using the Künneth formula for $K \times S^1$.

In order to simplify the sequences, we will use the following simple argument. If $\varphi : G \to G$ is a surjective homomorphism of finitely generated abelian groups, then φ is an isomorphism. This can be seen as follows. Write $G = \mathbb{Z}^k \oplus T$, where $\mathbb{Z}^k$ is a free abelian group of rank k and T is a torsion group. Write $\varphi = (f,g)$, where $f : G \to \mathbb{Z}^k$ and $g : G \to T$. Of course, f cannot depend on the torsion part of G, so f can be regarded as a surjective homomorphism from $\mathbb{Z}^k$ to $\mathbb{Z}^k$. This means f must be injective, since this would also be true if we extended f to a linear surjection from $\mathbb{Q}^k$ to $\mathbb{Q}^k$. This implies that if we restrict g to T, we get a surjective map from T to T. Since these are finite sets with an equal number of points, the map $g|_T$ must be injective as well, which in turn implies that φ is injective.

We will apply this for instance in the following situation. Consider the Mayer-Vietoris sequence of the pair (A, B) in X, where A and B are as above. Since we already saw that X is simply-connected, we also have that $H_1(X) = 0$, and hence a part of the Mayer-Vietoris sequence looks like

$$H_1(A \cap B) \xrightarrow{f} H_1(A) \oplus H_1(B) \to 0.$$

Note that by the Künneth formula $H_1(A \cap B) \cong H_1(A) \oplus H_1(B)$. Applying the above argument at this point shows that the map f is an isomorphism. This can also be seen in different ways, for instance using the fundamental groups of the involved spaces.

3. Contact open books for $S^2 \times S^3$ and $S^2 \tilde{\times} S^3$

Our construction starts by taking a simple Stein manifold $P := \Sigma_k$, the 2-disk-bundle over S^2 with Euler number $-k$ with $k \geqslant 2$. We remark that these manifolds carry often more than one Stein structure as can be seen in Figure 3.1. Here we use the Kirby diagram description of Stein surfaces due to Gompf [5]; by attaching two-handles in a suitable way to Legendrian knots, one can ensure that the resulting manifold carries a Stein structure, i.e. we choose the framing of a Legendrian knot K to be equal to the contact framing minus 1. First we will show that we get contact open books for

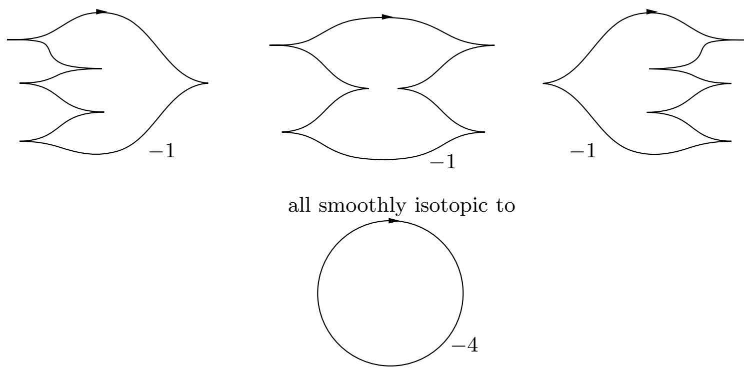

Figure 3.1. Different Stein structures on Σ_4

$S^2 \times S^3$ and $S^2 \tilde{\times} S^3$, then we will show that the different realizations from Figure 3.1 can give rise to different contact structures on $S^2 \times S^3$ and $S^2 \tilde{\times} S^3$.

Let S_k denote the contact boundary of Σ_k. It is well known that the manifold S_k can be identified with the circle bundle over S^2 with Euler number $-k$. We will use the identity as monodromy, so the mapping torus of the pair (Σ_k, id) is diffeomorphic to $A := \Sigma_k \times S^1$. A neighborhood of the binding will be written as $B := S_k \times D^2$. By gluing A and B in a collar neighborhood of their boundary we obtain a contact manifold $X := A \cup_\partial B$.

To see what manifold X is, consider the rank 4 disk bundle $\Sigma_k \times D^2$ over S^2. We can rewrite its boundary as

$$\partial(\Sigma_k \times D^2) = \Sigma_k \times S^1 \cup_\partial S_k \times D^2 = A \cup_\partial B = X.$$

In other words, the manifold X is a 3-sphere bundle over S^2. To see what sphere bundle it is, we look more closely at the vector bundle associated to the disk bundle Σ_k, which we shall denote by σ_k. If we denote the trivial bundle of rank 2 by ε^2, then $\Sigma_k \times D^2$ is the disk bundle associated to $\sigma_k \oplus \varepsilon^2$. Recall now that rank 4 vector bundles over S^2 are classified by their second Stiefel-Whitney class. In our case, this class is given by

$$w_2(\sigma_k \oplus \varepsilon^2) = w_2(\sigma_k) = k \mod 2.$$

So for k even the bundle $\sigma_k \oplus \varepsilon^2$ is trivial and for k odd the bundle $\sigma_k \oplus \varepsilon^2$ is the unique non-trivial bundle of rank 4 over S^2. As a result, we see that X is diffeomorphic to $S^2 \times S^3$ for k even. For k odd, the manifold X is diffeomorphic to $S^2 \tilde{\times} S^3 \cong X_\infty$.

3.1. Chern classes of contact structures

Let us take a look at Figure 3.1. Legendrian unknots representing Σ_k have rotation numbers going from $-k+2, -k+4, \cdots, k-2$. Fix a Legendrian unknot representing Σ_k and denote its rotation number by r. Now Theorem 11.3.1 from the book of Gompf and Stipsicz [6] tells us how to compute the Chern class.

THEOREM 3.1 (Gompf). — *Suppose P is a Stein surface obtained by two-handle attachment along a Legendrian link L. Then $c_1(P)$ is represented by a cocycle whose value on each oriented two-handle h attached along a component K of L is given by $r(K)$.*

We have just a single Legendrian unknot, so application of this theorem shows that

$$c_1(\Sigma_k) = r \in \mathbb{Z} \cong H^2(\Sigma_k).$$

We now want to establish the relation between the Chern class of the contact structure corresponding to the open book decomposition we described and the Chern class of Σ_k, the page of the open book. We may regard the pull-back $p_1^* T\Sigma_k$ as a subbundle of TA. If we denote the symplectic form on Σ_k by ω, then we may write the contact form on A as $\alpha = dt + \beta$, where t is the local coordinate on $S^1 = \mathbb{R}/\mathbb{Z}$, and β satisfies $d\beta = p_1^*\omega$. We obtain a complex structure J for $p_1^* T\Sigma_k$ by pulling back the (almost) complex structure on Σ_k that is compatible with ω.

Next, we construct a vector bundle isomorphism from $p_1^* T\Sigma_k$ to the contact structure $\xi = \ker \alpha$. Define

$$\begin{aligned} \varphi : \ p_1^* T\Sigma_k &\to \xi \\ v &\mapsto v - \beta(v)\frac{\partial}{\partial t}. \end{aligned}$$

In the definition of this map, we regard both $p_1^* T\Sigma_k$ and ξ as subbundles of the tangent bundle. The vector field $\frac{\partial}{\partial t}$ generates the standard rotation in the S^1-direction.

The inverse of φ can be obtained as follows,

$$\varphi^{-1}(v) = H(Tp_1(v)),$$

where we use H to denote the obvious lift from $T\Sigma_k$ to TA. In other words, the inverse of φ projects out the $\frac{\partial}{\partial t}$-component of an element in $\xi \subset TA$. This map φ can be used to give ξ a complex structure. Put $\tilde{J} := \varphi \circ J \circ \varphi^{-1}$. This makes φ into a complex vector bundle isomorphism from $(p_1^* T\Sigma_k, J)$ to $(\xi, \tilde{J})$, because by construction $\tilde{J} \circ \varphi = \varphi \circ J$. We check now that $\tilde{J}$ is a complex structure for ξ compatible with $d\alpha = d\beta$. We set $\tilde{v} = \varphi(v)$ and $\tilde{w} = \varphi(w)$. Then

$$\begin{aligned} d\beta(\tilde{J}\tilde{v}, \tilde{J}\tilde{w}) &= d\beta(\varphi(Jv), \varphi(Jw)) = d\beta(Jv, Jw) = d\beta(v, w) \\ &= d\beta(\varphi(v), \varphi(w)) = d\beta(\tilde{v}, \tilde{w}) \end{aligned}$$

These steps hold true, because φ adds an S^1-component and $d\beta$ does not contain any dt part, so $d\beta(\varphi(\ldots), \varphi(\ldots)) = d\beta(\ldots, \ldots)$. Also, J is a complex structure on $(p_1^* T\Sigma_k, J)$ compatible with $d\beta$. For the same reasons, the following holds:

$$d\beta(\tilde{v}, \tilde{J}\tilde{v}) = d\beta(\varphi(v), \varphi(Jv)) = d\beta(v, Jv) > 0 \text{ if } \tilde{v} \neq 0.$$

This proves that $\tilde{J}$ is a complex structure compatible with the contact structure ξ. Since $(p_1^* T\Sigma_k, J)$ and $(\xi, \tilde{J})$ are isomorphic as complex vector bundles by φ (which covers the identity), their Chern classes are the same. We had already computed the Chern class of Σ_k, so we have proved that $c_1(\xi) = r \in \mathbb{Z} \cong H^2(A)$.

We resort to a Mayer-Vietoris argument to complete our computation of the Chern class of X. Consider the Mayer-Vietoris sequence for cohomology with integer coefficients. The part that is relevant to us looks like

$$0 \to \underbrace{H^1(A)}_{\cong\mathbb{Z}} \oplus \underbrace{H^1(B)}_{\cong 0} \xrightarrow{\alpha} \underbrace{H^1(A\cap B)}_{\cong\mathbb{Z}} \xrightarrow{f} \underbrace{H^2(X)}_{\cong\mathbb{Z}} \xrightarrow{(i^*,j^*)} \underbrace{H^2(A) \oplus H^2(B)}_{\cong\mathbb{Z}\oplus\mathbb{Z}_k}.$$

Since the map α is injective, it has to map 1 to some non-zero integer, say m. If m is not equal to ± 1, then we see that $f(m)=0$, but $f(1)\neq 0$ by exactness. However $H^2(X)$ has no torsion, so we see that $m=\pm 1$ and thus the map α is an isomorphism. Again, by exactness the map f has to be the zero homomorphism. So we see that the map (i^*,j^*) is injective. We can say a bit more, namely that i^* is injective. This can be seen by noting that $H^2(B)$ is torsion. We show that i^* is an isomorphism by looking at the sequence of the pair (X,A). The piece of the sequence that interests us, looks like

$$H^2(X) \xrightarrow{i^*} H^2(A) \to H^3(X,A).$$

By excision, we have $H^3(X,A)\cong H^3(B,\partial B)$. The latter group is seen to be isomorphic to $H_2(B)=0$ by Poincaré duality. This shows that i^* is surjective.

The restriction of the first Chern class of the contact structure ξ_X on X to A is given by $c_1(\xi_X)=r$. Since we just checked i^* to be an isomorphism, it follows that $c_1(\xi_X)=r\in\mathbb{Z}\cong H^2(X)$. There is an ambiguity in this notation, namely it depends on which generator of $H^2(X)$ we take.

These ambiguities do not matter for the point we want to make, which is showing that all possible Chern classes of ξ_X can be realized by our open books (i.e. both positive and negative elements in $H^2(X)$). Indeed, the isomorphism $i^*: H^2(X)\to H^2(A)$ only depends on the topological structure of Σ_k and S_k, and not on the Stein structure of Σ_k. Hence we can change the sign of the first Chern class of ξ_X without affecting the orientation of X, for example by replacing the Legendrian knot representing Σ_k by its mirror.

Notice that for $(X\cong S^2\times S^3,\xi_X)$ we can realize all even Chern classes and for $(X\cong S^2\tilde{\times}S^3,\xi_X)$ we can realize all odd Chern classes. Namely, observe that the rotation number r of the diagram in Figure 3.1 can attain any even value, provided that we have chosen k even and large enough for that purpose. The same argument works for odd rotation numbers.

4. Open books for prime manifolds

In this section we will construct open book decompositions of the remaining prime manifolds, i.e. those simply-connected five-manifolds with torsion H_2 and trivial Stiefel-Whitney class. In order to cover these remaining cases, we turn our attention to a well studied class of Stein manifolds, namely Brieskorn varieties. Note that for the cases we still need to cover, it is necessary to use a non-trivial monodromy.

4.1. Brieskorn varieties

Consider the polynomial

$$P_t(z) = \sum_{i=0}^{n} z_i^{a_i} - t$$

for $z = (z_0, \dots, z_n) \in \mathbb{C}^{n+1}$ and $t \in \mathbb{C}$. The zero set of this polynomial is a Stein manifold if $t \neq 0$. If $t = 0$, the zero set of P_t has a singularity at 0 if one of the exponents is larger than 1. We will denote the zero set of the polynomial P_t by Σ_a, where a indicates that this set depends on the exponents $a = (a_0, \dots, a_n)$. We will call the set Σ_a a **Brieskorn variety**. There is a group action of $\mathbb{Z}_{a_i}$ on Σ_a obtained by multiplying the i^{th} coordinate by a_i^{th} roots of unity for each $i = 0, \dots, n$. These Stein manifolds can be made into compact Stein manifolds by restricting Σ_a to a ball $B_R = \{z \in \mathbb{C}^{n+1} \mid |z| \leqslant R\}$ in $\mathbb{C}^{n+1}$ with sufficiently large radius. By abuse of notation, we will also denote this set by Σ_a. The boundary of this compact Stein manifold is a **Brieskorn manifold** with exponents a, provided that t is small enough. This property of Brieskorn manifolds can for instance be found in theorem 14.3 of [7].

We would like to use Brieskorn varieties as pages with the corresponding Brieskorn manifolds as binding in open books. In order to produce a non-trivial symplectomorphism, we consider the action of the generator of $\mathbb{Z}_{a_0}$ on Σ_a as monodromy, i.e. we use the "rotation" map

$$\begin{aligned} \varphi : \Sigma_a &\to \Sigma_a \\ (z_0, \dots, z_n) &\mapsto (\zeta_{a_0} z_0, z_1, \dots, z_n), \end{aligned}$$

where ζ_{a_0} is the a_0^{th} root of unity $e^{2\pi i/a_0}$. Since this is even a biholomorphism, we get a symplectomorphism of the page, but we still need to show that we can isotope this map symplectically to the identity near the boundary of the page. We will describe this in the following interlude.

4.1.1. The rotation maps φ are symplectically isotopic to the identity

Instead of considering the polynomial P, we take the function

$$g = \sum_{i=0}^{n} z_i^{a_i} - f(r),$$

where $r = \sqrt{\sum_{i=0}^{n} |z_i|^2}$ and the function f is a real valued function to be specified later. We denote the zero set of g by $\tilde{\Sigma}_a$. Note that this set is in general not a Stein manifold. We will, however, show that it is symplectic for suitable f, as one might suspect if f varies slowly. To be more precise, take a vector $X \in T\mathbb{C}^{n+1}|_{g^{-1}(0)}$. The condition that X be tangent to $\tilde{\Sigma}_a$ is

$$i_X dg = i_X \left(\sum_{i=0}^{n} a_i z_i^{a_i-1} dz_i - \frac{1}{2}\frac{\partial f}{\partial r} \sum_{i=0}^{n} (\frac{\bar{z}_i}{r} dz_i + \frac{z_i}{r} d\bar{z}_i) \right) = 0.$$

Let now ω_0 denote the standard symplectic form on $\mathbb{C}^{n+1}$ and suppose that $\omega_0|_{\tilde{\Sigma}_a}$ is degenerate for the vector X at some point of $\tilde{\Sigma}_a$. Then we have

$$i_X \omega_0 = (\lambda dg + \bar{\lambda} d\bar{g})$$

for some $\lambda \in \mathbb{C}$, because we know ω_0 is non-degenerate on $\mathbb{C}^{n+1}$. Using this relation, we deduce that

$$i_X dz_j = \frac{2}{i} \left(- \left(\frac{\partial f}{\partial r} \frac{z_j}{2r} \right) (\lambda + \bar{\lambda}) + a_j \bar{z}_j^{a_j-1} \bar{\lambda} \right).$$

Now we return to check the tangency condition of X. The previous relations now give us

$$0 = i_X dg = \frac{2}{i}\bar{\lambda} \left(\sum_j a_j^2 |z_j|^{2(a_j-1)} - \frac{\partial f}{\partial r} \sum_j \frac{a_j}{2r} (z_j^{a_j} + \bar{z}_j^{a_j}) \right)$$

The coefficient of $\bar{\lambda}$ has a term involving $a_i^2 |z_i|^{2(a_i-1)}$ in it. Now assume that the exponents are larger than 1 and that the derivative $\frac{\partial f}{\partial r} < 1 - \varepsilon$ for some positive ε. This means that the term with $a_i^2 |z_i|^{2(a_i-1)}$ will dominate for large r, i.e. the coefficient of $\bar{\lambda}$ will be non-zero and therefore $\bar{\lambda} = 0$. Since $|\bar{\lambda}| = |\lambda|$, it follows that λ must be zero, which in turn implies that X is zero. This last step shows that $\tilde{\Sigma}_a$ can be made symplectic for suitable f. To be more precise we choose f with the following properties.

1. The function f is constant 1 for $r \leqslant R_0$, where R_0 is chosen in such a way that the above mentioned term will indeed dominate.

2. For $r \geqslant R_1 > R_0 + 1$, the function f is constant 0. Note that this condition is not necessary for symplecticity. It will, however, be useful to make the rotation maps isotopic to the identity for large radii.
3. Between R_0 and R_1, the function f goes smoothly from 1 to 0, connecting smoothly to the already described parts of f. We will choose f such that its derivative is smaller than $1 - \varepsilon$.

Now that we know that $\tilde{\Sigma}_a$ is symplectic, we want to see that the corresponding rotation map can be isotoped to the identity. First define the map $\varphi : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$, sending $(z_0, \dots, z_n) \mapsto (\zeta_{a_0} z_0, z_1, \dots, z_n)$. Now choose the following Hamiltonian function on $\mathbb{C}^{n+1}$;

$$H = \sum_{i=0}^{n} \frac{\pi}{a_i} |z_i|^2.$$

The time t flow of the Hamiltonian vector field associated to H induces the map

$$\psi_t : (z_0, \dots, z_n) \mapsto (e^{2\pi i \frac{t}{a_0}} z_0, \dots, e^{2\pi i \frac{t}{a_n}} z_n).$$

Note that this map sends $\tilde{\Sigma}_a$ to $\tilde{\Sigma}_a$ for $r > R_1$. Choose a function h that is constant 0 for $0 \leqslant r \leqslant R_1$ and that increases to 1 at $r = R_2 > R_1$, after which it is constant 1. Let $\tilde{\psi}_t$ denote the time t flow of the Hamiltonian vector field associated to $\tilde{H} = hH$. The map $\tilde{\psi}_t$ sends $\tilde{\Sigma}_a$ to $\tilde{\Sigma}_a$ for all radii. By choosing $t_0 \in \mathbb{Z}$ such that $t_0 = -1 \bmod a_0$ and $t_0 = 0 \bmod a_i$ for $i = 1, \dots, n$, we undo the rotation in the first coordinate for large radii and hence we see that $\tilde{\psi}_t$ is the identity near the boundary. Note this choice is not always possible, but if a_0 is relatively prime to a_i for $i = 1, \dots, n$, it is. Altogether, we have the map

$$\tilde{\varphi} = \tilde{\psi}_{t_0} \circ \varphi : \ \tilde{\Sigma}_a \to \tilde{\Sigma}_a,$$

which is the identity near the boundary of $\tilde{\Sigma}_a$. Also note that the choice of t_0 is not unique.

4.1.2. Homomorphism on homology induced by the rotation map

We shall take this isotoped rotation map as the monodromy for the page $\tilde{\Sigma}_a$. In order to invoke Barden's classification result, we need to know what map the monodromy induces on the homology of $\tilde{\Sigma}_a$. The Wang sequence we discussed in Section 2.2 gives the homology of the mapping torus.

First, we observe that φ and $\tilde{\varphi}$ are isotopic, so they induce the same maps on homology. And we may, in fact, work with the non-deformed Stein manifold Σ_a and the rotation map defined there (which we will also refer to as φ), because $\tilde{\Sigma}_a$ and Σ_a coincide in ball of radius R_0 around the origin as subsets of $\mathbb{C}^{n+1}$.

These Stein manifolds Σ_a have been studied carefully in the past (see for instance [7]) and many results about their properties, including their homology, are known. We will give a short summary of some of the results that we will use. The results that we are listing are from Hirzebruch-Mayer, [7], but date back to Pham, see [8].

In the following we will use the group action on Σ_a induced by multiplication by roots of unity. To that end, we introduce some notation. The group of a_j^{th}-roots of unity will be written as $G_{a_j} \cong \mathbb{Z}_{a_j}$ when we consider it as an abstract group, and we will denote a generator of G_{a_j} by w_j. As a subgroup of $\mathbb{C}^*$, we shall write $\tilde{G}_{a_j}$. The roots of unity will be indicated by ζ_j. We will write $G_a = G_{a_1} \oplus G_{a_2} \oplus \cdots \oplus G_{a_n}$. Let us now consider the deformation retract of the Stein manifolds Σ_a indicated in the following theorem.

Theorem 4.1 (Pham, see [7] and [8]). — *The set $U_a = \{z \in \Sigma_a \mid z_j^{a_j} \geqslant 0$ for all $j\}$ is a deformation retract of Σ_a. This deformation is compatible with the group action mentioned above.*

We can parametrize the set U_a in the following way,

$$U_a = \{(\zeta_0 t_0, \ldots, \zeta_n t_n) \in \mathbb{C}^{n+1} \mid \zeta_j \in \tilde{G}_{a_j},\ t_j \geqslant 0 \text{ and } \sum_{i=0}^{n} t_j^{a_j} = 1\}.$$

On the other hand, note that the join $G_{a_0} * \cdots * G_{a_n}$ may be written as

$$\tilde{G}_{a_0} * \cdots * \tilde{G}_{a_n} = \{(\zeta_0 t_0, \ldots, \zeta_n t_n) \in \mathbb{C}^{n+1} \mid \zeta_j \in \tilde{G}_{a_j},\ t_j \geqslant 0 \text{ and } \sum_{i=0}^{n} t_j = 1\}.$$

These sets can be identified if we rescale the t_j's. Notice that this identification is compatible with the group action, because G_a acts only on the roots of unity.

General theory gives us that the join $G_{a_0} * \cdots * G_{a_n}$ is an n-dimensional simplicial complex with an n-simplex for each element in G_a. This is again compatible with the group action in the following sense. Let e denote the simplex corresponding to $1 \in G_a$. The other simplices are obtained by letting G_a act. In other words, the simplicial chain complex in degree n can be written as

$$C_n(U_a) = \mathbb{Z}(G_a)e,$$

where $\mathbb{Z}(G_a)$ denotes the group ring of G_a.

Now define

$$h := (1 - w_0)(1 - w_1) \ldots (1 - w_n)e.$$

In the lecture notes of Hirzebruch and Mayer [7] it is shown that h is a cycle. In fact, one can establish an isomorphism (see [7] for more details)

$$\tilde{H}_n(U_a) \cong \mathbb{Z}(G_a)h$$

coming from the homomorphism

$$\begin{array}{rcl}\Phi:\ C_n(U_a)\cong \mathbb{Z}(G_a) & \rightarrow & \mathbb{Z}(G_a)h \\ w & \mapsto & wh.\end{array}$$

The kernel of Φ is the ideal I_a generated by

$$1 + w_j + w_j^2 + \cdots + w_j^{a_j - 1} \text{ for } j = 0, \dots, n.$$

Let us consider the basis of $\tilde{H}_n(U_a)$ represented by elements in $C_n(U_a)$ of the form

$$w_0^{k_0} w_1^{k_1} \cdot \ldots \cdot w_n^{k_n} \text{ with } 0 \leqslant k_j \leqslant a_j - 2 \text{ for } j = 0, \dots, n. \tag{4.1}$$

With respect to this basis, we can give a matrix representation of φ_*, the isomorphism on homology induced by the rotation map φ. The "rotation" map φ corresponds to multiplication by w_0 on $C_n(U_a)$. That is to say that φ shifts the basis in Formula 4.1 by w_0. For the induced map on homology, we use the ideal I_a to simplify the results if necessary, for instance

$$\begin{array}{rcl} w_0 & \mapsto & w_0^2 \\ w_0^{a_0-2} & \mapsto & w_0^{a_0-1} \equiv -1 - w_0 - \ldots - w_0^{a_0-2} \bmod I_a. \end{array}$$

Hence the matrix representation of φ_* consists of $(a_1 - 1) \cdot \ldots \cdot (a_n - 1)$ blocks on the diagonal that look like the $(a_0 - 1) \times (a_0 - 1)$-matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \\ 0 & \cdots & 0 & 1 & -1 \end{pmatrix}$$

if we order the basis by its degree in w_1, then by its degree in w_2 and so on.

The above representation of φ_* can be used to compute the homology of the mapping torus

$$A' := \Sigma_a \times I/\sim, \text{ where } (x, 0) \sim (\varphi(x), 1).$$

This is done using the Wang sequence. We use the facts that $H_3(\Sigma_a) = 0$ and that $\pi_1(\Sigma_a) = 0$ (and hence also $H_1(\Sigma_a) = 0$). The piece that is relevant to us looks like

$$0 \to H_3(A') \to H_2(\Sigma_a) \overset{\varphi_*-id}{\to} H_2(\Sigma_a) \to H_2(A') \to 0.$$

Using the above matrix representation of φ_* we see that $\varphi_* - id$ is injective, because the determinant of the associated matrix is non-zero. Hence we conclude that $H_3(A') = 0$ and that $H_2(A') \cong \mathrm{coker}(\varphi_* - id)$. We have

$$H_2(A') \cong \mathrm{coker}(\varphi_* - id) \cong \underbrace{\mathbb{Z}_{a_0} \oplus \cdots \oplus \mathbb{Z}_{a_0}}_{(a_1-1)\cdot\ldots\cdot(a_n-1)\text{times}}.$$

Indeed, each block of the matrix representation of $\varphi_* - id$ corresponding to the above block has a cokernel isomorphic to $\mathbb{Z}_{a_0}$, which can be seen by performing Gauss elimination. Together with the discussion at the beginning of this section this gives us the homology of the mapping torus of $\tilde{\Sigma}_a$ with monodromy $\tilde{\varphi}$. Let A denote this mapping torus,

$$A = \tilde{\Sigma}_a \times I/\sim, \text{ where } (x, 0) \sim (\tilde{\varphi}(x), 1).$$

Then we have

$$H_2(A) \cong \underbrace{\mathbb{Z}_{a_0} \oplus \cdots \oplus \mathbb{Z}_{a_0}}_{(a_1-1)\cdot\ldots\cdot(a_n-1)\text{times}}. \tag{4.2}$$

The homotopy exact sequence of the fibration $A \to S^1$ shows that $\pi_1(A) \cong \mathbb{Z}$, so we see that $H_1(A) \cong \mathbb{Z}$ as well. All higher homology groups (grade larger than two) are zero.

4.1.3. Homology of the open book

Now we choose suitable exponents for the Brieskorn varieties and use them to give the remaining prime manifolds contact open books.

First of all, we consider the Brieskorn variety $\tilde{\Sigma}_a$ with exponents $a_0 = p^k$, $a_1 = 3$ and $a_2 = 2$, where p is a prime different from 2 and 3, and k some positive integer. Notice that the associated Brieskorn manifold K is then a homology sphere, i.e. $H_1(K) = 0$. The set A denotes the mapping torus of $\tilde{\Sigma}_a$ with monodromy $\tilde{\varphi}$ as in the previous section. As is our convention, we define $B := K \times D^2$ and set $X := A \cup_\partial B$.

The arguments from Section 2.2 show that X is simply-connected. By Poincaré duality we see that $H_4(X) = 0$, and since K is a homology sphere we also have $H_2(A \cap B) = 0$. Consider the following piece of the Mayer-Vietoris sequence,

$$0 \to \underbrace{H_2(A)}_{\cong \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}} \oplus \underbrace{H_2(B)}_{\cong 0} \to H_2(X) \to 0.$$

Here we have used the arguments from Section 2.2 to split off a part of the sequence. We see directly that $H_2(X) \cong \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$. In particular, the rank of $H_2(X)$ is zero, so $H_3(X) = 0$ as well by Poincaré duality and the universal coefficient theorem. This shows that the prime manifolds M with $H_2(M) \cong \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$ admit contact open books for $p \neq 2, 3$. The binding is a Brieskorn homology sphere of the form $\Sigma(p^k, 3, 2)$, and the page is the Brieskorn variety $\tilde{\Sigma}_a$. Together with our earlier results, this covers all prime manifolds except those with 2- or 3-torsion in their second homology group. To get them, we consider Brieskorn varieties with different exponents.

First we shall tackle the case of 2-torsion in homology. Consider the Brieskorn varieties $\tilde{\Sigma}_a$ with exponents $a_0 = 2^k$, $a_1 = 3$ and $a_2 = 3$. Since the exponents are not relatively prime, we cannot conclude that K is a homology sphere. We can, however, compute the homology of K by using the algorithm of Randell [9]. We get $H_1(K) \cong \mathbb{Z}_{2^k} \oplus \mathbb{Z}_{2^k}$.

Let $X := A \cup_\partial B$ be the open book as before, but now with the new exponents. If we consider the Mayer-Vietoris sequence for (A, B) in X with rational coefficients, we easily see that the rank of $H_3(X)$ is zero. Together with the arguments from Section 2.2 this reduces the remaining part of the Mayer-Vietoris sequence for (A, B) with integer coefficients to

$$0 \to \underset{\cong \mathbb{Z}^2_{2^k}}{H_2(A \cap B)} \overset{i \oplus j}{\to} \underset{\cong \mathbb{Z}^4_{2^k}}{H_2(A)} \oplus \underset{=0}{H_2(B)} \to H_2(X) \to 0.$$

We have used to Künneth formula to determine $H_2(A \cap B)$. The rank of $H_1(K)$ is zero, so by Poincaré duality $H_2(K) = 0$ and hence we also have $H_2(B) = 0$. Formula (4.2) gives the homology of A. Injectivity of the map $i \oplus j$ means that we can represent this map by a (4×2) matrix which has a (2×2) subdeterminant that is a unit in $\mathbb{Z}_{2^k}$. Hence we see that we can extend this matrix to form a basis of $\mathbb{Z}^4_{2^k}$. So after applying a basis transformation on $\mathbb{Z}^4_{2^k}$, we see that

$$\operatorname{im} i \oplus j = \mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k} \times \{0\} \times \{0\}.$$

Hence by exactness, we obtain $H_2(X) \cong \mathbb{Z}^2_{2^k}$.

The arguments for the 3-torsion case are almost completely the same. The exponents for Σ_a are different, of course. We take $a_0 = 3^k$, $a_1 = 4$ and $a_2 = 2$. As before we use the algorithm of Randell [9] to compute the homology of the Brieskorn manifold K. This time we get $H_1(K) \cong \mathbb{Z}_{3^k}$. Formula (4.2) shows that $H_2(A) = \mathbb{Z}^3_{3^k}$. Again, using the arguments from Section 2.2 we can split off a part of the Mayer-Vietoris sequence. By tensoring with $\mathbb{Q}$ we see that the rank of $H_2(X)$ is zero, and hence

$H_3(X) = 0$. This reduces the sequence to

$$0 \to \underbrace{H_2(A \cap B)}_{\cong \mathbb{Z}_{3^k}} \stackrel{i \oplus j}{\to} \underbrace{H_2(A)}_{\cong \mathbb{Z}^3_{3^k}} \oplus \underbrace{H_2(B)}_{\cong 0} \to H_2(X) \to 0.$$

The map $i \oplus j$ is injective, so $i \oplus j(\bar{1}) = (a, b, c)$ is an element of order 3^k. This means that one of the elements a, b, c is a unit in $\mathbb{Z}_{3^k}$. Therefore we can include the vector (a, b, c) into a basis of $\mathbb{Z}^3_{3^k}$. With respect to this basis we have $i \oplus j(\bar{1}) = (1, 0, 0)$. By exactness, we see directly that $H_2(X) \cong \mathbb{Z}_{3^k} \oplus \mathbb{Z}_{3^k}$.

Remark 4.2. — An easier way to see that these prime manifolds admit contact structures is by considering Brieskorn manifolds. Namely, we have

$$H_2(\Sigma(p^k, 3, 3, 3)) \cong \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k} \text{ for } p \text{ not divisible by } 3$$

and

$$H_2(\Sigma(p^k, 2, 4, 4)) \cong \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k} \text{ for } p \text{ not divisible by } 2.$$

This can be shown by applying Randell's algorithm [9]. Of course, we do not obtain the abstract open books in this way.

5. Conclusion and discussion

In Section 3 and Section 4 we constructed abstract contact open books for the prime manifolds in Barden's classification. Note that we can easily obtain an abstract contact open book for S^5. Simply take D^4 with standard symplectic structure as page and use the identity as monodromy. In view of Remark 2.2, this gives abstract contact open books for all simply-connected five-manifolds that admit an almost contact structure. Moreover, we can realize a contact structure with any admissible Chern class, since a non-zero Chern class can only come from an $S^2 \times S^3$- or an $S^2 \tilde{\times} S^3$-factor. For the latter two manifolds we have shown that we can realize all possible Chern classes. In [3] Lemma 7, it is shown that, for an oriented five-manifold, the almost contact structure is completely determined by the first Chern class.

We can change the orientation by replacing a contact form α by $-\alpha$. Hence we get a contact open book for every homotopy class of almost contact structures on a simply-connected five-manifold. This completes our alternative proof of Theorem 1.1.

Remark 5.1. — In our construction there is still a lot of freedom left, even though we took very explicit cases. For $S^2 \times S^3$ and $S^2 \tilde{\times} S^3$ we can, for instance, vary the page of the abstract open book but keep the Chern class fixed. This can for instance be done by adding two stabilizations to the Legendrian unknot used for the handle attachment; by adding one stabilization on the left side and one on the right side of the Legendrian unknot, we fix the rotation number, but decrease the framing $(tb-1)$ by 2. The resulting abstract contact open books have the same Chern class, but are they contactomorphic?

For the other prime manifolds, we can vary the monodromy in the following way. The parameter t_0 we used in isotoping the "rotation" map to the identity near the boundary in Section 4.1.1 is not unique. The obvious question is, whether different choices can lead to different contact structures on the same manifold. Here one should note that although we used a Hamilton vector field for the isotopy, we did not use one with compact support. Hence we could get different maps that are not symplectically isotopic relative to boundary.

BIBLIOGRAPHY

[1] N. A'Campo, "Feuilletages de codimension 1 sur des variétés de dimension 5", *C. R. Acad. Sci. Paris Sér. A-B* **273** (1971), p. A603-A604.

[2] D. Barden, "Simply connected five-manifolds", *Ann. of Math. (2)* **82** (1965), p. 365-385.

[3] H. Geiges, "Contact structures on 1-connected 5-manifolds", *Mathematika* **38** (1991), no. 2, p. 303-311.

[4] E. Giroux & J. Mohsen, "Contact structures and symplectic fibrations over the circle", lecture notes.

[5] R. Gompf, "Handlebody construction of Stein surfaces", *Ann. of Math. (2)* **148** (1998), no. 2, p. 619-693.

[6] R. Gompf & A. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics, vol. 20, American Mathematical Society, Providence, RI, 1999.

[7] F. Hirzebruch & K. Mayer, *O(n)-Mannigfaltigkeiten, exotische Sphären und Singularitäten*, Lecture Notes in Mathematics, No. 57, Springer-Verlag, Berlin, 1968.

[8] F. Pham, "Formules de Picard-Lefschetz généralisées et ramification des intégrales", *Bull. Soc. Math. France* **93** (1965), p. 333-367.

[9] R. Randell, "The homology of generalized Brieskorn manifolds", *Topology* **14** (1975), no. 4, p. 347-355.

[10] W. Thurston & H. Winkelnkemper, "On the existence of contact forms", *Proc. Amer. Math. Soc.* **52** (1975), p. 345-347.

[11] H.-C. Wang, "Homology of fibre bundles Duke", *Math Journal* **16** (1949), p. 33-38.

Manuscrit reçu le 13 avril 2006,
accepté le 15 mars 2007.

Otto VAN KOERT
Université Libre de Bruxelles
Département de Mathématiques - CP 218
Boulevard du Triomphe
1050 Bruxelles (Belgique)
ovkoert@ulb.ac.be