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CONTRACTION OF EXCESS FIBRES BETWEEN THE MCKAY CORRESPONDENCES IN DIMENSIONS TWO AND THREE

by Samuel BOISSIÈRE & Alessandra SARTI (*)

ABSTRACT. — The quotient singularities of dimensions two and three obtained from polyhedral groups and the corresponding binary polyhedral groups admit natural resolutions of singularities as Hilbert schemes of regular orbits whose exceptional fibres over the origin reveal similar properties. We construct a morphism between these two resolutions, contracting exactly the excess part of the exceptional fibre. This construction is motivated by the study of some pencils of K3 surfaces appearing as minimal resolutions of quotients of nodal surfaces with high symmetries.

RÉSUMÉ. — Les singularités quotients de dimensions deux et trois obtenues par des groupes polyédraux et les groupes polyédraux binaires correspondants admettent des résolutions de singularités naturelles par les schémas de Hilbert d'orbites régulières, dont les fibres exceptionnelles au-dessus de l'origine révèlent des propriétés similaires. Nous construisons un morphisme entre ces deux résolutions, contractant exactement la partie excédentaire de la fibre exceptionnelle. Cette construction est motivée par l'étude de certains pinceaux de surfaces K3 apparaissant comme résolutions minimales de quotients de surfaces nodales à grandes symétries.

1. Introduction

Consider a binary polyhedral group $\tilde{G} \subset \mathrm{SU}(2)$ corresponding to a polyhedral group $G \subset \mathrm{SO}(3, \mathbb{R})$ through the double-covering $\mathrm{SU}(2) \rightarrow \mathrm{SO}(3, \mathbb{R})$. The group \tilde{G} acts freely on $\mathbb{C}^2 \setminus \{0\}$ and the quotient surface \mathbb{C}^2/\tilde{G} has an isolated singular point at the origin. The exceptional divisor of its minimal resolution of singularities $\mathcal{X} \rightarrow \mathbb{C}^2/\tilde{G}$ is a tree of smooth rational

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curves of self-intersection -2 , intersecting transversally, whose intersection graph is an A-D-E Dynkin diagram. The classical McKay correspondence ([15]) relates this intersection graph to the representations of the group \tilde{G} , associating bijectively each exceptional curve to a non trivial irreducible representation of the group: The correspondence in fact identifies the intersection graph with the McKay quiver of the action of \tilde{G} on \mathbb{C}^2 . Among these irreducible representations we find all irreducible representations of the group G : we call them *pure* and the remaining ones *binary*. Since $\tilde{G}/\{\pm 1\} \cong G$, one can produce a G -invariant cone $\mathbb{C}^2/\{\pm 1\} \xrightarrow{\sim} K \hookrightarrow \mathbb{C}^3$ whose quotient K/G is isomorphic to \mathbb{C}^2/\tilde{G} . In this note, we prove the following result, conjectured by W. P. Barth:

THEOREM 1.1. — *There exists a crepant resolution of singularities of \mathbb{C}^3/G containing a partial resolution $\mathcal{Y} \rightarrow K/G$ with the property that the intersection graph of its exceptional locus is precisely the McKay quiver of the action of G on \mathbb{C}^3 , together with a resolution map $\mathcal{X} \rightarrow \mathcal{Y}$ mapping isomorphically the exceptional curves corresponding to pure representations and contracting those associated with binary representations to ordinary nodes.*

We make this construction in the framework of the Hilbert schemes of regular orbits of Nakamura ([16]) providing, by the Bridgeland-King-Reid theorem ([3]), the natural candidates for the resolutions of singularities in dimensions two and three. We produce a morphism \mathcal{S} between these two resolutions of singularities, define our partial resolution \mathcal{Y} as the image of this map and study the effect of \mathcal{S} on the exceptional fibres:

$$\begin{array}{ccccc}
 \tilde{G}\text{-Hilb}(\mathbb{C}^2) & \xrightarrow{\mathcal{S}} & G\text{-Hilb}(\mathbb{C}^3) & & \\
 \downarrow \tilde{\pi} & \searrow \mathcal{S} & \downarrow \pi & & \\
 & \mathcal{Y} & & & \\
 \downarrow & \downarrow & \downarrow & & \\
 \mathbb{C}^2/\tilde{G} & \xrightarrow{\sim} & K/G & \longrightarrow & \mathbb{C}^3/G
 \end{array}$$

Although the exceptional fibres can be described very explicitly in all cases (see [12]), as a matter of principle our proof avoids any case-by-case analysis: The key consists in a systematic modular interpretation of the objects at issue.

In Sections 2 and 3 we introduce the notation and recall useful facts about clusters and Hilbert schemes of points and clusters. Sections 4-6 give a brief

survey on polyhedral, binary polyhedral and bipolyhedral groups, their representations and the classical McKay correspondences in dimensions two and three. In Section 7 we define and study the map \mathcal{S} (Lemma 7.1, Proposition 7.2). In Section 8 we show that the map \mathcal{S} contracts the curves corresponding to the binary representations and maps the curves corresponding to the pure representations isomorphically to the exceptional curves downstairs (Theorem 8.1) and get Theorem 1.1 as Corollary 8.5. In Section 9, as an example we describe in details the case when \tilde{G} is a cyclic group. Finally, Section 10 is devoted to an application to resolutions of pencils of K3 surfaces.

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2. Clusters

Let V be a n -dimensional complex vector space and \mathfrak{G} a finite subgroup of $\mathrm{SL}(V)$. We denote by $\mathcal{O}(V) := \mathbb{S}^*(V^\vee)$ the algebra of polynomial functions on V , with the induced left action $g \cdot f := f \circ g^{-1}$ for $f \in \mathcal{O}(V)$ and $g \in \mathfrak{G}$.

We choose a basis X_1, \dots, X_n of linear forms on V , denote the ring of polynomials in n indeterminates by $S := \mathbb{C}[X_1, \dots, X_n]$ and identify $\mathcal{O}(V) \cong S$. The ring S is given a grading by the total degree of a polynomial, where each indeterminate X_i has degree 1. In particular, the action of the group \mathfrak{G} on S preserves the degree.

Let $\mathfrak{m}_S := \langle X_1, \dots, X_n \rangle$ be the maximal ideal of S at the origin. We denote by $S^\mathfrak{G}$ the subring of \mathfrak{G} -invariant polynomials, by $\mathfrak{m}_{S^\mathfrak{G}}$ its maximal ideal at the origin and by $\mathfrak{n}_\mathfrak{G} := \mathfrak{m}_{S^\mathfrak{G}} \cdot S$ the ideal of S generated by the non constant \mathfrak{G} -invariant polynomials vanishing at the origin. The quotient ring of *coinvariants* is by definition $S_\mathfrak{G} := S/\mathfrak{n}_\mathfrak{G}$.

An ideal $\mathfrak{J} \subset S$ is called a \mathfrak{G} -cluster if it is globally invariant under the action of \mathfrak{G} and the quotient S/\mathfrak{J} is isomorphic as a \mathfrak{G} -module to the regular representation of \mathfrak{G} : $S/\mathfrak{J} \cong \mathbb{C}[\mathfrak{G}]$. A closed subscheme $Z \subset \mathbb{C}^n$ is called a \mathfrak{G} -cluster if its defining ideal $\mathfrak{J}(Z)$ is a \mathfrak{G} -cluster. Such a subscheme is then zero-dimensional, has length $|\mathfrak{G}|$ and contains only one orbit. For instance, a free \mathfrak{G} -orbit defines a \mathfrak{G} -cluster.

We are particularly interested in \mathfrak{G} -clusters supported at the origin. Then $\mathfrak{J} \subset \mathfrak{m}_S$ and in fact this condition is enough to assert that the cluster is supported at the origin. Furthermore, one has automatically $\mathfrak{n}_\mathfrak{G} \subset \mathfrak{J}$, since

any non constant function $f \in \mathfrak{n}_{\mathfrak{G}}$ not contained in \mathfrak{J} would induce a new copy of the trivial representation in the quotient S/\mathfrak{J} , different from that given by the constant functions. Hence we wish to understand the structure of the \mathfrak{G} -clusters \mathfrak{J} such that $\mathfrak{n}_{\mathfrak{G}} \subset \mathfrak{J} \subset \mathfrak{m}_S$, equivalent to the study of the quotient ideals $\mathfrak{J}/\mathfrak{n}_{\mathfrak{G}} \subset \mathfrak{m}_S/\mathfrak{n}_{\mathfrak{G}} \subset S/\mathfrak{n}_{\mathfrak{G}} = S_{\mathfrak{G}}$, with the exact sequence:

$$(2.1) \quad 0 \longrightarrow \mathfrak{J}/\mathfrak{n}_{\mathfrak{G}} \longrightarrow S_{\mathfrak{G}} \longrightarrow S/\mathfrak{J} \longrightarrow 0.$$

From now on, we assume that there exists a complex reflection group $\mathfrak{R} \in \text{GL}(V)$ containing \mathfrak{G} such that $[\mathfrak{R} : \mathfrak{G}] = 2$. Then $\mathfrak{m}_S/\mathfrak{n}_{\mathfrak{G}}$ is a graded finite-dimensional algebra which as a \mathfrak{G} -module consists exactly of each non trivial representation ρ of \mathfrak{G} repeated $2 \dim \rho$ times: we denote the occurrences of each representation ρ by $V^{(1)}(\rho), \dots, V^{(2 \dim \rho)}(\rho)$ where each $V^{(i)}(\rho)$ is given by homogeneous polynomials modulo $\mathfrak{n}_{\mathfrak{G}}$ ([6, 7]).

By the exact sequence (2.1), giving a \mathfrak{G} -cluster supported at the origin consists in choosing $\dim \rho$ copies of ρ in $\mathfrak{m}_S/\mathfrak{n}_{\mathfrak{G}}$ for each non trivial representation ρ of \mathfrak{G} . This gives many choices since any linear combination of some $V^{(i)}(\rho)$ and $V^{(j)}(\rho)$ provides such a copy. The underlying idea is that one does not have to make all these choices to define \mathfrak{J} (see §9 for an explicit example).

For such an ideal \mathfrak{J} with $\mathfrak{n}_{\mathfrak{G}} \subset \mathfrak{J} \subset \mathfrak{m}_S$, we consider the finite-dimensional \mathfrak{G} -modules $W \subset S$ generating \mathfrak{J} in the sense that $\mathfrak{J} = W \cdot S + \mathfrak{n}_{\mathfrak{G}}$. Such modules exist by the preceding construction. Among these choices, we consider the *minimal* ones, such that no strict \mathfrak{G} -submodule of them generate \mathfrak{J} in the preceding sense.

If W is a generator in this sense, then:

$$\mathfrak{J} = W \cdot S + \mathfrak{n}_{\mathfrak{G}} = W + \mathfrak{m}_S \cdot W + \mathfrak{n}_{\mathfrak{G}} = W + \mathfrak{m}_S \cdot \mathfrak{J} + \mathfrak{n}_{\mathfrak{G}}.$$

This means that the \mathbb{C} -linear map $W \rightarrow \mathfrak{J}/(\mathfrak{m}_S \cdot \mathfrak{J} + \mathfrak{n}_{\mathfrak{G}})$ is surjective. Also, since W is a \mathfrak{G} -module and since $\mathfrak{m}_S \cdot \mathfrak{J} + \mathfrak{n}_{\mathfrak{G}}$ is \mathfrak{G} -stable, this map is \mathfrak{G} -linear. If W is a minimal set of generators, it satisfies in particular $W \cap (\mathfrak{m}_S \cdot \mathfrak{J} + \mathfrak{n}_{\mathfrak{G}}) = \{0\}$ since this intersection would provide a \mathfrak{G} -submodule whose complementary in W is a smaller \mathfrak{G} -submodule generating \mathfrak{J} . Hence, for W minimal one gets an isomorphism of \mathfrak{G} -modules $W \cong \mathfrak{J}/(\mathfrak{m}_S \cdot \mathfrak{J} + \mathfrak{n}_{\mathfrak{G}})$. We set then $V(\mathfrak{J}) := \mathfrak{J}/(\mathfrak{m}_S \cdot \mathfrak{J} + \mathfrak{n}_{\mathfrak{G}})$. The set of generators of $V(\mathfrak{J})$ may not be uniquely determined, but its structure as a \mathfrak{G} -module is unique. The important issue, that will be the core of the classification, will be to determine whether $V(\mathfrak{J})$ is irreducible or not, although it is a minimal set of generators.

Notation for the two- and three-dimensional cases. When applying the preceding constructions in dimensions two or three, we fix the following notation:

- For $n = 2$, the polynomial ring is denoted by $A := \mathbb{C}[x, y]$, the group by \tilde{G} and any ideal by I .
- For $n = 3$, the polynomial ring is denoted by $B := \mathbb{C}[a, b, c]$, the group by G and any ideal by J .

3. Moduli space of clusters

3.1. Hilbert scheme of points

Let $X \subset \mathbb{P}_{\mathbb{C}}^n$ be a quasi-projective scheme and N a positive integer. Consider the contravariant functor from the category of schemes to the category of sets $\mathcal{Hilb}_X^N: (\text{Schemes}) \rightarrow (\text{Sets})$ given by:

$$\mathcal{Hilb}_X^N(T) := \left\{ Z \subset T \times X \left| \begin{array}{l} (a) \quad Z \text{ is a closed subscheme} \\ (b) \quad \text{the morphism } Z \hookrightarrow T \times X \xrightarrow{p} T \text{ is flat} \\ (c) \quad \forall t \in T, Z_t \subset X \text{ is a closed subscheme} \\ \qquad \qquad \text{of dimension 0 and length } N \end{array} \right. \right\}$$

This functor is represented by a quasi-projective scheme $\text{Hilb}^N(X)$ coming with a *universal family* $\Xi_N^X \subset \text{Hilb}^N(X) \times X$. In the sequel, we always denote by p the projection to the moduli space (here $\text{Hilb}^N(X)$) and by q the projection to the base (here X). When X is projective, the scheme $\text{Hilb}^N(X)$ is projective and comes with a very ample line bundle (for $\ell \gg 0$):

$$\det \left(p_* \left(\mathcal{O}_{\Xi_N^X} \otimes q^* \mathcal{O}_X(\ell) \right) \right).$$

When $X = \mathbb{C}^n$, one gets an open immersion $\text{Hilb}^N(\mathbb{C}^n) \hookrightarrow \text{Hilb}^N(\mathbb{P}_{\mathbb{C}}^n)$ corresponding to the restriction of the universal family. The restriction of the determinant line bundle gives the very ample line bundle $\det \left(p_* \mathcal{O}_{\Xi_N^{\mathbb{C}^n}} \right)$ on $\text{Hilb}^N(\mathbb{C}^n)$.

There exists a natural projective morphism from $\text{Hilb}^N(X)$ to the symmetric product $S^N(X)$ sending a closed subscheme to the corresponding 0-cycle describing its support, called the *Hilbert-Chow* morphism:

$$\mathcal{H}: \text{Hilb}^N(X) \longrightarrow S^N(X).$$

By a theorem of Fogarty ([5]), the scheme $\text{Hilb}^N(X)$ is connected. For $\dim X = 2$, it is reduced, smooth and the morphism \mathcal{H} is a resolution of singularities.

3.2. Hilbert scheme of regular orbits

We consider the sub-functor $\mathfrak{G}\text{-Hilb}_{\mathbb{C}^n}$ of $\text{Hilb}_{\mathbb{C}^n}^{|\mathfrak{G}|}$ given by

$$\mathfrak{G}\text{-Hilb}_{\mathbb{C}^n}(T) := \left\{ Z \in \text{Hilb}_{\mathbb{C}^n}^{|\mathfrak{G}|}(T) \mid \forall t \in T, Z_t \subset \mathbb{C}^n \text{ is a } \mathfrak{G}\text{-cluster} \right\}.$$

This functor is represented by a quasi-projective scheme $\mathfrak{G}\text{-Hilb}(\mathbb{C}^n)$ called the *Hilbert scheme of \mathfrak{G} -regular orbits*, which is a union of some connected components of the subscheme of \mathfrak{G} -fixed points $\left(\text{Hilb}^{|\mathfrak{G}|}(\mathbb{C}^n)\right)^{\mathfrak{G}}$. Furthermore, the quotient $\mathbb{C}^n/\mathfrak{G}$ can be identified with a closed subscheme of $\mathbb{S}^{|\mathfrak{G}|}(\mathbb{C}^n)$ and since the support of a \mathfrak{G} -cluster consists exactly in one orbit through \mathfrak{G} , the restriction of the Hilbert-Chow morphism factorizes through a projective morphism (see [3, 11, 18]):

$$\mathcal{H} : \mathfrak{G}\text{-Hilb}(\mathbb{C}^n) \longrightarrow \mathbb{C}^n/\mathfrak{G}.$$

There is a unique irreducible component of $\mathfrak{G}\text{-Hilb}(\mathbb{C}^n)$ containing the free \mathfrak{G} -orbits and mapping birationally onto $\mathbb{C}^n/\mathfrak{G}$. This component is taken as the definition of the Hilbert scheme of \mathfrak{G} -regular orbits in [16]. By the theorem of Bridgeland-King-Reid [3], if $n \leq 3$, then $\mathfrak{G}\text{-Hilb}(\mathbb{C}^n)$ is already irreducible, reduced, smooth and the morphism \mathcal{H} is a crepant resolution of singularities of the quotient $\mathbb{C}^n/\mathfrak{G}$. Moreover, \mathcal{H} is an isomorphism over the open subset of free \mathfrak{G} -orbits. As a byproduct, the two definitions coincide.

As before, the scheme $\mathfrak{G}\text{-Hilb}(\mathbb{C}^n)$ has a universal family $\mathcal{Z}_{\mathfrak{G}}$ which is the restriction of the universal family $\Xi_{|\mathfrak{G}|}^{\mathbb{C}^n}$ corresponding to the closed immersion $\mathfrak{G}\text{-Hilb}(\mathbb{C}^n) \hookrightarrow \text{Hilb}^{|\mathfrak{G}|}(\mathbb{C}^n)$. The restriction of the determinant line bundle gives the very ample line bundle $\det(p_*\mathcal{O}_{\mathcal{Z}_{\mathfrak{G}}})$ on $\mathfrak{G}\text{-Hilb}(\mathbb{C}^n)$ (see [10, §8.1]).

4. Rotation groups

Let $\text{SO}(3, \mathbb{R})$ be the group of rotations in \mathbb{R}^3 . Up to conjugation, there are five different types of finite subgroups of $\text{SO}(3, \mathbb{R})$, called *polyhedral groups*: The cyclic groups C_n , the dihedral groups D_n , the tetrahedral group \mathcal{T} , the octahedral group \mathcal{O} and the icosahedral group \mathcal{I} .

Consider the classical exact sequence:

$$0 \longrightarrow \{\pm 1\} \longrightarrow \text{SU}(2) \xrightarrow{\phi} \text{SO}(3, \mathbb{R}) \longrightarrow 0.$$

For any finite subgroup $G \subset \text{SO}(3, \mathbb{R})$, the inverse image $\tilde{G} := \phi^{-1}G$ is called a *binary polyhedral group*. It is a finite subgroup of $\text{SU}(2) \subset \text{SL}(2, \mathbb{C})$.

Consider the second exact sequence:

$$0 \longrightarrow \{\pm 1\} \longrightarrow \text{SU}(2) \times \text{SU}(2) \xrightarrow{\sigma} \text{SO}(4, \mathbb{R}) \longrightarrow 0.$$

For any binary polyhedral group \tilde{G} , the direct image $\sigma(\tilde{G} \times \tilde{G}) \subset \text{SO}(4, \mathbb{R})$ is called a *bipolyhedral group*. In §10, we make use of the following particular groups:

- $G_6 = \sigma(\tilde{\mathcal{T}} \times \tilde{\mathcal{T}})$ of order 288;
- $G_8 = \sigma(\tilde{\mathcal{O}} \times \tilde{\mathcal{O}})$ of order 1152;
- $G_{12} = \sigma(\tilde{\mathcal{I}} \times \tilde{\mathcal{I}})$ of order 7200.

Consider a binary polyhedral group \tilde{G} , the associated polyhedral group G and set $\tau := \{\pm 1\}$:

$$0 \longrightarrow \tau \longrightarrow \tilde{G} \xrightarrow{\phi} G \longrightarrow 0.$$

This exact sequence induces an injection of the set of irreducible representations of G in the set of irreducible representations of \tilde{G} : If $\rho: G \rightarrow \text{GL}(V)$ is an irreducible representation of G , it induces by composition a representation of \tilde{G} which is τ -invariant, *i.e.* such that $\rho(-g) = \rho(g)$ for all $g \in \tilde{G}$. If the representation ρ admits a non trivial \tilde{G} -submodule, it is also a non trivial G -submodule after going to the quotient $\tilde{G}/\tau \cong G$. This shows that the image of the injection:

$$\text{Irr}(G) \hookrightarrow \text{Irr}(\tilde{G})$$

consists precisely on the irreducible representations which are τ -invariant. These representations are called *pure* and the remaining representations are called *binary*. More precisely, if $\rho: \tilde{G} \rightarrow \text{GL}(V)$ is an irreducible representation of \tilde{G} , the subspace

$$V^\tau := \{v \in V \mid v = \rho(-1)v\}$$

is a \tilde{G} -submodule of V . Hence either $V^\tau = V$ and the representation ρ is pure, or ρ is binary and $V^\tau = \{0\}$.

For each type of binary polyhedral group, the binary representations are labelled by a “ \sim ” and the trivial representation is denoted by χ_0 .

5. Graph-theoretic intuition

If $\mathfrak{G} \subset \text{SL}(n, \mathbb{C})$ is a finite subgroup, it defines a natural faithful representation \mathcal{Q} of \mathfrak{G} . Let $\{V_0, \dots, V_k\}$ be a complete set of irreducible representations of \mathfrak{G} , where V_0 denotes the trivial one. For each such representation,

one may decompose the tensor products

$$\mathcal{Q} \otimes V_i \cong \bigoplus_{j=0}^k V_j^{\oplus a_{i,j}}$$

for some non negative integers $a_{i,j}$. When $a_{i,j} = a_{j,i}$ for all i, j , one defines the *McKay quiver* as the unoriented graph with vertices V_0, V_1, \dots, V_k and $a_{i,j}$ edges between the vertices V_i and V_j . In particular, this quiver may contain some loops. For our purpose, we only consider the *reduced* McKay quiver with vertices V_1, \dots, V_k and one edge between V_i and V_j if $i \neq j$ and $a_{i,j} \neq 0$: This means that we remove from the McKay quiver the vertex V_0 , all edges starting from it, all loops and all multiple edges. When there is an edge joining V_i and V_j , the vertices are called *adjacent*. All finite subgroups of $SU(2)$ and $SO(3, \mathbb{R})$ enter in this context.

For each binary polyhedral group $\tilde{G} \subset SU(2)$ and its corresponding polyhedral group $G \subset SO(3, \mathbb{R})$, we draw the reduced McKay quiver with our conventions. For the binary polyhedral groups, we denote by a white vertex the pure representations and by a black vertex the binary ones. We get (see for example [6, 7, 8]) the graphs of Figure 5.1 and 5.2.

In the sequel, we interpret these graphs as the intersection graphs of a family of smooth rational curves meeting transversally. One may then get the following intuition: Looking at the two-dimensional graphs, if one contracts the curves associated to a binary representation (black nodes), then one gets an intersection graph which is precisely the corresponding graph in dimension three!

Another property of the two-dimensional quivers is that no two pure representations and no two binary representations are adjacent. This means that the contraction contracts a disjoint union of -2 curves.

6. Exceptional fibres in dimensions two and three

Considering the Hilbert-Chow morphism $\mathcal{H}: \mathfrak{G}\text{-Hilb}(\mathbb{C}^n) \longrightarrow \mathbb{C}^n/\mathfrak{G}$, our purpose is to describe the exceptional fibre $\mathcal{H}^{-1}(O)$ over the origin $O \in \mathbb{C}^n/\mathfrak{G}$ in the two- and three-dimensional cases. Note that all finite subgroups of $SL(2, \mathbb{C})$ or $SO(3, \mathbb{R})$ enter in the context of §2 since they are subgroups of index 2 of a reflection group (see [7, §2.7]). Hence we may apply the general procedure for the study of the clusters supported at the origin.

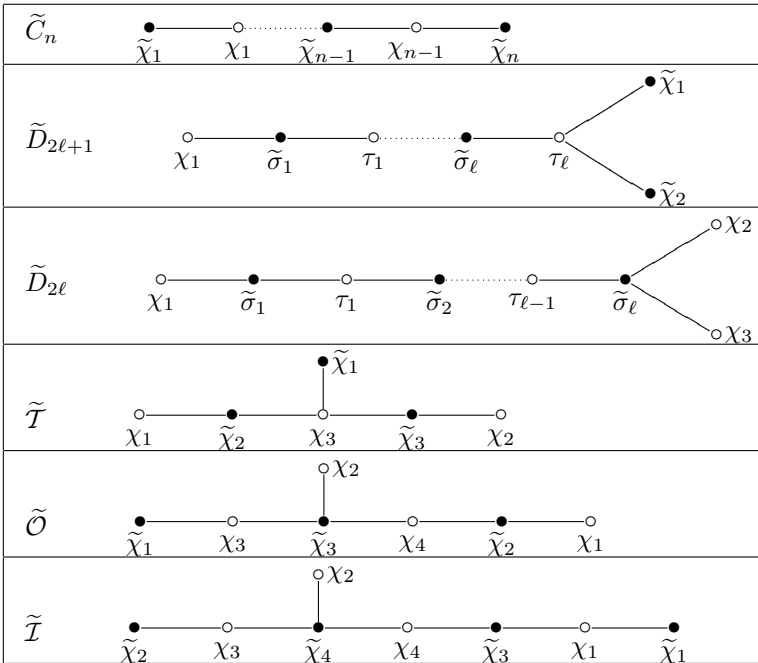


Figure 5.1. Reduced McKay quivers in dimension 2

The understanding of the exceptional fibre in these cases was achieved by Ito-Nakamura [12, 13] in dimension two and by Gomi-Nakamura-Shinoda [6, 7] in dimension three, by a case-by-case analysis. For the two-dimensional case, there is another proof by Crawley-Boevey [4] avoiding this case-by-case analysis. For any finite group \mathfrak{G} , $\text{Irr}^*(\mathfrak{G})$ denotes the set of irreducible representations but the trivial one.

Exceptional fibre in dimension two. Let $\tilde{G} \subset \text{SL}(2, \mathbb{C})$ be a binary polyhedral group and denote by $\tilde{\pi}: \tilde{G}\text{-Hilb}(\mathbb{C}^2) \rightarrow \mathbb{C}^2/\tilde{G}$ the Hilbert-Chow morphism. For each non trivial irreducible representation ρ of \tilde{G} , set:

$$E(\rho) := \{I \in \tilde{\pi}^{-1}(O)_{\text{red}} \mid V(I) \supset \rho\}.$$

THEOREM 6.1. — ([12, Theorem 3.1])

- Each $E(\rho)$ is a smooth rational curve of self-intersection -2 .
- $\tilde{\pi}^{-1}(O)_{\text{red}} = \bigcup_{\rho} E(\rho)$ and $\tilde{\pi}^{-1}(O) = \sum_{\rho} \dim \rho \cdot E(\rho)$ as a Cartier divisor (with $\rho \in \text{Irr}^*(\tilde{G})$).
- If $I \in E(\rho)$ and $I \notin E(\rho')$ for all $\rho' \neq \rho$, then $V(I) \cong \rho$.

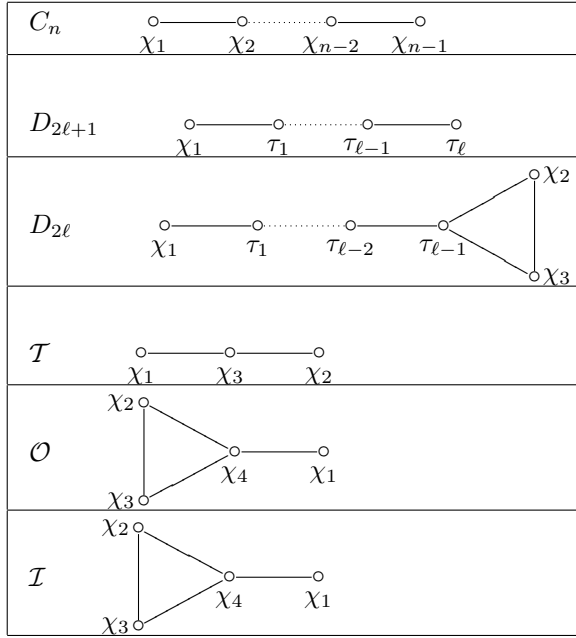


Figure 5.2. Reduced McKay quivers in dimension 3

- If $I \subset E(\rho) \cap E(\rho')$, then $V(I) \cong \rho \oplus \rho'$ and the curves $E(\rho)$ and $E(\rho')$ intersect transversally at I .
- The intersection graph of these curves is the reduced McKay quiver of the group \tilde{G} .

In particular, a generator $V(I)$ does not contain more than one copy of any irreducible representation, and $E(\rho) \cap E(\rho') \neq \emptyset$ if and only if the representations ρ and ρ' are adjacent.

Exceptional fibre in dimension three. Let $G \subset \text{SO}(3, \mathbb{R})$ be a polyhedral group. Denote by $\pi: G\text{-Hilb}(\mathbb{C}^3) \rightarrow \mathbb{C}^3/G$ the Hilbert-Chow morphism. For each non trivial irreducible representation ρ of G , set:

$$C(\rho) := \{J \in \pi^{-1}(O)_{\text{red}} \mid V(J) \supset \rho\}.$$

THEOREM 6.2. — ([7, Theorem 3.1])

- Each $C(\rho)$ is a smooth rational curve.
- $\pi^{-1}(O)_{\text{red}} = \bigcup_{\rho} C(\rho)$ (with $\rho \in \text{Irr}^*(G)$).
- If $J \in C(\rho)$ and $J \notin C(\rho')$ for all $\rho' \neq \rho$, then $V(J) \cong \rho$.
- The intersection graph of these curves is the reduced McKay quiver of the group G .

6.1. Explicit parameterizations

We explain briefly the explicit parameterizations of the exceptional curves. This description holds both in dimensions two and three so we do it with our general notation. The example of the cyclic group is treated in §9. As we explained in §2,

$$\mathfrak{m}_S/\mathfrak{n}_{\mathfrak{G}} \cong \bigoplus_{\substack{\rho \in \text{Irr}(\mathfrak{G}) \\ \rho \neq \rho_0}} \bigoplus_{i=1}^{2 \dim \rho} V^{(i)}(\rho)$$

where ρ_0 denotes the trivial representation. With the exact sequence:

$$0 \longrightarrow \mathfrak{J}/\mathfrak{n}_{\mathfrak{G}} \longrightarrow \mathfrak{m}_S/\mathfrak{n}_{\mathfrak{G}} \longrightarrow \mathfrak{m}_S/\mathfrak{J} \longrightarrow 0,$$

if one wants to parameterize a (flat) family of clusters over $\mathbb{P}_{\mathbb{C}}^1$, one has to choose, in the trivial sheaf:

$$\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1} \otimes \bigoplus_{\substack{\rho \in \text{Irr}(\mathfrak{G}) \\ \rho \neq \rho_0}} \bigoplus_{i=1}^{2 \dim \rho} V^{(i)}(\rho),$$

a locally free \mathfrak{G} -equivariant sheaf which is a copy of the regular representation on each fibre whose quotient is also locally free. The parameterizations are produced as follows: One chooses *one* non trivial subbundle

$$\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}(-1) \otimes \rho \hookrightarrow \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1} \otimes (V^{(i)}(\rho) \oplus V^{(j)}(\rho))$$

for some appropriate choice of the indices, and shows that this gives the required family whose points \mathfrak{J} are characterized by their generator:

$$V(\mathfrak{J}) \subset \mathbb{P}(V^{(i)}(\rho) \oplus V^{(j)}(\rho)).$$

That is: Once one choice has been made, the other choices are automatic, and they always correspond to a trivial subbundle (see §8.4).

7. Geometric construction

Let \tilde{G} be a binary polyhedral group acting on $A = \mathbb{C}[x, y]$. Set as before $\tau := \langle \pm 1 \rangle \subset \tilde{G}$ and $G := \tilde{G}/\tau$ the associated polyhedral group. We aim to define a morphism

$$\mathcal{S}: \tilde{G}\text{-Hilb}(\mathbb{C}^2) \longrightarrow G\text{-Hilb}(\mathbb{C}^3)$$

inducing a morphism between the exceptional fibres over the origin.

Since $A^\tau = \mathbb{C}[x^2, y^2, xy]$, we consider the following composite of ring morphisms, with $B = \mathbb{C}[a, b, c]$:

$$(7.1) \quad \sigma: B \twoheadrightarrow B / \langle ab - c^2 \rangle \xrightarrow{\sim} A^\tau \longrightarrow A$$

where the identification is defined by $a = x^2, b = y^2, c = xy$. The action of \tilde{G} on A induces an action of G on A^τ . Using the identification, we can define an action of G on the coordinates a, b, c , inducing an action on B with the property that the cone $K = \langle ab - c^2 \rangle$ is G -invariant.

Let I be an ideal of A and $J := \sigma^{-1}(I)$ the corresponding ideal of B . Observe the following property of the map σ :

LEMMA 7.1. — *If I is a \tilde{G} -cluster in A , then J is a G -cluster in B . Furthermore, if I is supported at the origin, then so is J .*

Proof. — If I is a \tilde{G} -cluster, then $A/I \cong \mathbb{C}[\tilde{G}]$. Since the group τ is finite, we have isomorphisms:

$$B/J \cong A^\tau/I^\tau \cong (A/I)^\tau \cong \mathbb{C}[\tilde{G}]^\tau \cong \mathbb{C}[G],$$

hence J is a G -cluster in B . Furthermore, note that $\sigma^{-1}\mathfrak{m}_A = \mathfrak{m}_B$ hence if I is a \tilde{G} -cluster supported at the origin, one has $I \subset \mathfrak{m}_A$ and then $J \subset \mathfrak{m}_B$, which implies that J is also supported at the origin (see §2). \square

This construction defines set-theoretically a map between the two moduli spaces of clusters $\mathcal{S}: \tilde{G}\text{-Hilb}(\mathbb{C}^2) \rightarrow G\text{-Hilb}(\mathbb{C}^3)$ by $\mathcal{S}(I) =_{\text{def}} J$.

PROPOSITION 7.2. — *The map \mathcal{S} is regular, projective, and induces a morphism between the exceptional fibres.*

Proof. —

Step 1. To get that the map \mathcal{S} is regular, we show that it is induced by a natural transformation between the two functors of points

$$\tilde{G}\text{-Hilb}_{\mathbb{C}^2}(\cdot) \implies G\text{-Hilb}_{\mathbb{C}^3}(\cdot).$$

Let T be a scheme and $Z \in \tilde{G}\text{-Hilb}_{\mathbb{C}^2}(T)$. Then $Z \subset T \times \mathbb{C}^2$ is a flat family of \tilde{G} -clusters over T and the map $Z \hookrightarrow T \times \mathbb{C}^2$ is τ -equivariant (for a trivial action on T). It induces a family

$$Z/\tau \hookrightarrow T \times (\mathbb{C}^2/\tau) \hookrightarrow T \times \mathbb{C}^3$$

where the quotient \mathbb{C}^2/τ is considered as the cone $\langle ab - c^2 \rangle$ in \mathbb{C}^3 . If T is a point, this is precisely our set-theoretic construction: if Z is given by an ideal I , Z/τ is given by the ideal I^τ .

To show that $Z/\tau \in G\text{-Hilb}_{\mathbb{C}^3}(T)$, we have to prove that this family is flat over T . Since this problem is local in T , we may assume that T is an affine

scheme, say $T = \text{Spec } R$. Then the family Z is given by a τ -equivariant quotient $R \otimes A \twoheadrightarrow Q$ so that the composition $R \hookrightarrow R \otimes_{\mathbb{C}} A \twoheadrightarrow Q$ makes Q a flat R -module. The family Z/τ is then given by the quotient

$$R \hookrightarrow R \otimes_{\mathbb{C}} B \twoheadrightarrow R \otimes_{\mathbb{C}} A^{\tau} \twoheadrightarrow Q^{\tau},$$

where the quotient $R \otimes_{\mathbb{C}} B \twoheadrightarrow R \otimes_{\mathbb{C}} A^{\tau}$ is induced by tensorization of the quotient $B \twoheadrightarrow A^{\tau}$. We have to show that this makes Q^{τ} a flat R -module. By hypothesis, the functor $Q \otimes_R -$ in the category of R -modules is exact. Since τ is finite, the functor $(-)^{\tau}$ is also exact in this category, and we note that the functor $Q^{\tau} \otimes_R -$ is the composition of this two functors since $Q^{\tau} \otimes_R N = (Q \otimes_R N)^{\tau}$ for any R -module N . Hence the functor $Q^{\tau} \otimes_R -$ is exact, which means that the family is flat.

Step 2. The composite of ring morphisms (7.1) gives an equivariant ring morphism

$$\begin{array}{ccc} \mathbb{C}[a, b, c] & \xrightarrow{\sigma} & \mathbb{C}[x, y] \\ \underbrace{(\quad)}_G & & \underbrace{(\quad)}_{\tilde{G}} \end{array}$$

inducing a surjective map at the level of the invariants: $\mathbb{C}[a, b, c]^G \twoheadrightarrow \mathbb{C}[x, y]^{\tilde{G}}$, hence a closed immersion:

$$\eta: \mathbb{C}^2/\tilde{G} = \text{Spec } \mathbb{C}[x, y]^{\tilde{G}} \longrightarrow \text{Spec } \mathbb{C}[a, b, c]^G = \mathbb{C}^3/G.$$

Taking more care of the cone $K = \mathbb{C}^2/\tau$, the equivariant morphism

$$\begin{array}{ccccc} \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2/\tau & \longrightarrow & \mathbb{C}^3 \\ \underbrace{(\quad)}_{\tilde{G}} & & \underbrace{(\quad)}_G & & \underbrace{(\quad)}_G \end{array}$$

induces the morphism η between the quotients:

$$\eta: \mathbb{C}^2/\tilde{G} \xrightarrow{\sim} (\mathbb{C}^2/\tau)/G \longrightarrow \mathbb{C}^3/G$$

sending the origin $O \in \mathbb{C}^2/\tilde{G}$ to the origin $O \in \mathbb{C}^3/G$ and by definition of \mathcal{S} the following diagram is commutative:

$$\begin{array}{ccc} \tilde{G}\text{-Hilb}(\mathbb{C}^2) & \xrightarrow{\mathcal{S}} & G\text{-Hilb}(\mathbb{C}^3) \\ \tilde{\pi} \downarrow & & \downarrow \pi \\ \mathbb{C}^2/\tilde{G} & \xrightarrow{\eta} & \mathbb{C}^3/G \end{array}$$

This implies that \mathcal{S} induces a morphism between the exceptional fibres:

$$\mathcal{S}: \tilde{\pi}^{-1}(O) \rightarrow \pi^{-1}(O).$$

Step 3. We prove that the morphism \mathcal{S} is proper by applying the valuative criterion of properness. Let K be any field over \mathbb{C} and $R \subset K$ any valuation ring with quotient field K . Consider a commutative diagram:

$$\begin{array}{ccc}
 \text{Spec } K & \xrightarrow{\phi} & \tilde{G}\text{-Hilb}(\mathbb{C}^2) \\
 \downarrow i & & \downarrow \mathcal{S} \\
 \text{Spec } R & \xrightarrow{\psi} & G\text{-Hilb}(\mathbb{C}^3)
 \end{array}$$

We have to show that there exists a unique factorization

$$\begin{array}{ccc}
 \text{Spec } K & \xrightarrow{\phi} & \tilde{G}\text{-Hilb}(\mathbb{C}^2) \\
 \downarrow i & \nearrow \tilde{\phi} & \downarrow \mathcal{S} \\
 \text{Spec } R & \xrightarrow{\psi} & G\text{-Hilb}(\mathbb{C}^3)
 \end{array}$$

making the whole diagram commute.

By modular interpretation, the data of the morphism ϕ consists of an ideal $I \subset K[x, y]$ such that $K[x, y]/I \cong \mathbb{C}[\tilde{G}] \otimes_{\mathbb{C}} K$ and $K[x, y]/I$ is K -flat (it is here trivial since K is a field). Similarly, the data of the morphism ψ consists in an ideal $J \subset R[a, b, c]$ such that $R[a, b, c]/J \cong \mathbb{C}[G] \otimes_{\mathbb{C}} R$ and $R[a, b, c]/J$ is R -flat. The commutativity $\mathcal{S} \circ \phi = \psi \circ i$ means the following. Consider the diagram of ring morphisms induced by natural extension of scalars and base-change from the map σ :

$$\begin{array}{ccc}
 R[a, b, c] & \xrightarrow{\sigma_R} & R[x, y] \\
 \downarrow & & \downarrow \\
 K[a, b, c] & \xrightarrow{\sigma_K} & K[x, y]
 \end{array}$$

Then the commutativity condition means that $\sigma_K^{-1}(I) = J \cdot K[a, b, c]$.

We are looking for a morphism $\tilde{\phi}$ such that $\tilde{\phi} \circ i = \phi$ and $\mathcal{S} \circ \tilde{\phi} = \psi$, *i.e.* for an ideal $\tilde{I} \subset R[x, y]$ such that $R[x, y]/\tilde{I} \cong \mathbb{C}[\tilde{G}] \otimes_{\mathbb{C}} R$ and $R[x, y]/\tilde{I}$ is R -flat, satisfying the conditions $\tilde{I} \cdot K[x, y] = I$ and $\sigma_R^{-1}(\tilde{I}) = J$. A natural candidate is $\tilde{I} =_{\text{def}} I \cap R[x, y]$. We have to prove that it satisfies all the conditions and that it is unique for these properties. Denote by $\nu: K - \{0\} \rightarrow H$ the valuation with values in a totally ordered group H , satisfying the properties:

$$\nu(x \cdot y) = \nu(x) + \nu(y) \text{ and } \nu(x + y) \geq \min(\nu(x), \nu(y)) \quad \text{for } x, y \in K \setminus \{0\}$$

and such that $R = \{x \in K \mid \nu(x) \geq 0\} \cup \{0\}$. Recall that R is by definition integral and that a R -module is flat if and only if it is torsion-free (see for instance [1, 9]).

i. It is already clear that $\tilde{I} \cdot K[x, y] \subset I$. Let $P = \sum_{i,j} p_{i,j} x^i y^j \in I$ and $p \in \{p_{i,j}\}$ an element of minimal valuation. If $\nu(p) \geq 0$, then $P \in \tilde{I}$. Otherwise all coefficients of $p^{-1}P$ have positive valuation and so $p^{-1}P \in \tilde{I}$. So $P = p \cdot (p^{-1}P) \in \tilde{I} \cdot K[x, y]$, hence the equality.

ii. By commutativity of the above diagram,

$$\begin{aligned} \sigma_R^{-1}(\tilde{I}) &= \sigma_R^{-1}(I \cap R[x, y]) \\ &= \sigma_K^{-1}(I) \cap R[a, b, c] \\ &= (J \cdot K[a, b, c]) \cap R[a, b, c]. \end{aligned}$$

It is clear that $J \subset (J \cdot K[a, b, c]) \cap R[a, b, c]$. Let $P \in (J \cdot K[a, b, c]) \cap R[a, b, c]$, decomposed as $P = \sum_{\ell} U_{\ell} \cdot V_{\ell}$ with $U_{\ell} \in J$ and $V_{\ell} \in K[a, b, c]$. As before, there exists a coefficient q in all V_{ℓ} of minimal valuation, and we assume $\nu(q) < 0$ (otherwise there is no problem). Then $q^{-1}P \in J$. By assumption, the R -module $R[a, b, c]/J$ is torsion-free, so the multiplication by $q^{-1} \in R$ is injective. This means that $P \in J$.

iii. By definition, we have an R -linear inclusion $R[x, y]/\tilde{I} \hookrightarrow K[x, y]/I$, which shows that $R[x, y]/\tilde{I}$ is torsion-free, hence flat. It inherits an action of \tilde{G} and since $K[x, y]/I \cong \mathbb{C}[\tilde{G}] \otimes_{\mathbb{C}} K$, there exists a subrepresentation V of $\mathbb{C}[\tilde{G}]$ such that $R[x, y]/\tilde{I} \cong V \otimes_{\mathbb{C}} R$ (this uses the flatness, see [13, Lemma 9.4]). By the isomorphism of R -modules $R[x, y]/\tilde{I} \otimes_R K \cong K[x, y]/I$, the representation V is such that $V \otimes_R K = \mathbb{C}[\tilde{G}] \otimes_{\mathbb{C}} K$, which forces $V \cong \mathbb{C}[\tilde{G}]$.

iv. The uniqueness follows from the condition $\tilde{I} \cdot K[x, y] = I$ since we already noted that $I \cap R[x, y] = (\tilde{I} \cdot K[x, y]) \cap R[x, y] = \tilde{I}$, so our natural candidate is the only possibility.

Step 4. To conclude, remark that any proper morphism between two quasi-projective varieties is automatically projective. □

8. Contracted versus non contracted fibres

THEOREM 8.1. — *Consider the restriction of the morphism \mathcal{S} to a reduced curve $E(\rho)$. Then:*

- (1) If the representation ρ is pure, then \mathcal{S} maps isomorphically the curve $E(\rho)$ onto the curve $C(\rho)$.
- (2) If the representation ρ is binary, then \mathcal{S} contracts the curve $E(\rho)$ to a point.

Proof. — Let $E(\rho)$ be any exceptional curve. Since the morphism \mathcal{S} sends this curve to the tree of curves $\pi^{-1}(O)$, the image lies in some irreducible component C and the restricted morphism $\mathcal{S} : E(\rho) \rightarrow C$ is a proper morphism. We prove that:

- If the representation ρ is binary, then the morphism $\mathcal{S} : E(\rho) \rightarrow C$ contracts the curve to a point;
- If the representation ρ is pure, then $C = C(\rho)$ and the restricted morphism $\mathcal{S} : E(\rho) \rightarrow C(\rho)$ is an isomorphism.

The parameterizations of the two curves $E(\rho)$ and C define a proper morphism f whose properties reflect those of the restriction of \mathcal{S} :

$$\begin{array}{ccc}
 \mathbb{P}_{\mathbb{C}}^1 & \xrightarrow[\phi]{\sim} & E(\rho) \subset \tilde{G}\text{-Hilb}(\mathbb{C}^2) \\
 \downarrow f & & \downarrow \mathcal{S} \\
 \mathbb{P}_{\mathbb{C}}^1 & \xrightarrow[\psi]{\sim} & C \subset G\text{-Hilb}(\mathbb{C}^3)
 \end{array}$$

We know (see [9, II.6.8,II.6.9]) that either the morphism f contracts the curve to a point, or it is a finite surjective morphism. The basic idea to determine which case occurs is to take an ample line bundle $\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}(a)$ on the target (with $a > 0$): If the morphism f contracts the curve to a point, then $f^*\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}(a)$ is trivial, otherwise $f^*\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}(a) \cong \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}(\deg(f) \cdot a)$ is ample.

The natural candidate for an ample line bundle over the curve C is the determinant $\det(p_*\mathcal{O}_{Z(C)})$ obtained by restriction of the universal family $Z(C) := \mathcal{Z}_G|_C$.

The parameterization $\mathbb{P}_{\mathbb{C}}^1 \xrightarrow{\phi} \tilde{G}\text{-Hilb}(\mathbb{C}^2)$ of the curve $E(\rho)$ corresponds to a flat family $Z_{\tilde{G}}(\rho) \subset \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{C}^2$ which is the restriction to $E(\rho)$ of the universal family $\mathcal{Z}_{\tilde{G}}$ over $\tilde{G}\text{-Hilb}(\mathbb{C}^2)$. The direct image $p_*\mathcal{O}_{Z_{\tilde{G}}(\rho)}$ is a vector bundle of rank $|\tilde{G}|$ over $\mathbb{P}_{\mathbb{C}}^1$ equipped with an action of \tilde{G} inducing the regular representation on each fibre. It admits an isotypical decomposition over the irreducible representations of \tilde{G} and we recall the well-known explicit decomposition:

LEMMA 8.2. —

$$p_*\mathcal{O}_{Z_G(\rho)} \cong \left(\mathcal{O}_{\mathbb{P}_c^1}(1) \oplus \mathcal{O}_{\mathbb{P}_c^1}^{\oplus \dim \rho - 1} \right) \otimes \rho \oplus \bigoplus_{\substack{\rho' \in \text{Irr}(\tilde{G}) \\ \rho' \neq \rho}} \mathcal{O}_{\mathbb{P}_c^1}^{\oplus \dim \rho'} \otimes \rho'$$

Proof of the lemma. — This is an equivalent form of [14, §2.1] or [11, Proposition 6.2(3)]. We recall briefly the argument. Since this bundle is a quotient of $\mathcal{O}_{\mathbb{P}_c^1} \otimes A$ (see §6.1), it is generated by its global sections, hence it is a sum of line bundles $\mathcal{O}_{\mathbb{P}_c^1}(a)$ for $a \geq 0$. Since $\deg(p_*\mathcal{O}_{Z_G(\rho)}) = 1$ (see [8]), all line bundles are trivial but one, of degree one. \square

In particular, note that $\det(p_*\mathcal{O}_{Z_G(\rho)}) \cong \mathcal{O}_{\mathbb{P}_c^1}(\dim \rho)$ is the ample determinant line bundle in dimension two. By the functorial definition of the morphism \mathcal{S} , the composite $\mathbb{P}_c^1 \xrightarrow{\phi} \tilde{G}\text{-Hilb}(\mathbb{C}^2) \xrightarrow{\mathcal{S}} G\text{-Hilb}(\mathbb{C}^3)$ parameterizes the flat family $Z_{\tilde{G}}(\rho)/\tau$ with structural sheaf $\mathcal{O}_{Z_{\tilde{G}}(\rho)/\tau} = \left(\mathcal{O}_{Z_{\tilde{G}}(\rho)} \right)^\tau$ and one gets:

$$f^*(\det(p_*\mathcal{O}_{Z(C)})) = \det \left((p_*\mathcal{O}_{Z_G(\rho)})^\tau \right).$$

Now, as we noticed in §4, taking the invariants under τ keeps invariant the pure representations and kills the binary ones. Hence:

- If the representation ρ is binary, then:

$$\left(p_*\mathcal{O}_{Z_G(\rho)} \right)^\tau \cong \bigoplus_{\rho' \in \text{Irr}(G)} \mathcal{O}_{\mathbb{P}_c^1}^{\oplus \dim \rho'} \otimes \rho'$$

hence $\det(p_*\mathcal{O}_{Z_G(\rho)})^\tau \cong \mathcal{O}_{\mathbb{P}_c^1}$ is trivial;

- If the representation ρ is pure, then:

$$\left(p_*\mathcal{O}_{Z_G(\rho)} \right)^\tau \cong \left(\mathcal{O}_{\mathbb{P}_c^1}(1) \oplus \mathcal{O}_{\mathbb{P}_c^1}^{\oplus \dim \rho - 1} \right) \otimes \rho \oplus \bigoplus_{\substack{\rho' \in \text{Irr}(G) \\ \rho' \neq \rho}} \mathcal{O}_{\mathbb{P}_c^1}^{\oplus \dim \rho'} \otimes \rho'$$

hence $\det(p_*\mathcal{O}_{Z_G(\rho)})^\tau \cong \mathcal{O}_{\mathbb{P}_c^1}(\dim \rho)$ is ample.

This achieves the first part of the proof. It remains to show that in the case of a pure representation ρ , the target curve is $C = C(\rho)$ and that the finite surjective morphism f is an isomorphism. We do it by hand. A point $I \in E(\rho)$ is characterized by the choice of $V(I)$ and generically $V(I) \cong \rho$. For a pure representation ρ , the polynomials defining $V(I)$ are even so:

$$V(I^\tau) = V((A \cdot V(I) + \mathbf{n}_A)^\tau) \supset V(I)$$

so generically $V(I^\tau) = V(I)$ (only changed by $a = x^2, b = y^2, c = xy$). This means that $C = C(\rho)$ and if $I \neq J \in E(\rho)$, then $V(I) \neq V(J)$ hence

the images are also different, so the morphism is generically injective. This concludes the proof. \square

As a byproduct of our argument, we get the following equivalent in dimension three of Lemma 8.2 which, to our knowledge, does not appear explicitly in the literature:

COROLLARY 8.3. — *For any finite subgroup $G \subset \text{SO}(3, \mathbb{R})$ and any non trivial representation ρ of G , the restriction of the tautological bundle to the exceptional curve $C(\rho)$ decomposes as:*

$$p_*\mathcal{O}_{Z_G(\rho)} \cong \left(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}^{\oplus \dim \rho - 1} \right) \otimes \rho \oplus \bigoplus_{\substack{\rho' \in \text{Irr}(G) \\ \rho' \neq \rho}} \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}^{\oplus \dim \rho'} \otimes \rho'.$$

Proof. — The same argument as in the proof of Lemma 8.2 shows that this bundle is generated by its global sections. The bijectivity of the morphism f on the curves associated to pure representations (in the notation of the proof of Theorem 8.1) implies that $\det(p_*\mathcal{O}_{Z_G(\rho)}) \cong \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(\dim \rho)$, hence in the isotypical decomposition there is only one non trivial line bundle, of degree one, and we already know by the explicit parameterizations that the isotypical component corresponding to ρ is not trivial. \square

Remark 8.4. — In the decomposition of Lemma 8.2, the presence of a unique $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$ corresponds to the choice of the line $V(I)$ in a projective space $\mathbb{P}(\rho \oplus \rho)$ as explicitly described in §6.1. The fact that no other ample bundle occurs reflects the property that once one choice has been made, the other generators of the ideal do not involve the choice any more, as one can easily notice from the explicit computations of [13, §13, §14] (see §9 in this paper for an example). In the three-dimensional case, the same situation occurs by Corollary 8.3.

We get now Theorem 1.1, presented in the introduction, as a consequence of Theorem 8.1:

COROLLARY 8.5. — *The image $\mathcal{Y} := \mathcal{S}(\tilde{G}\text{-Hilb}(\mathbb{C}^2))$ projects onto the quotient K/G , inducing a partial resolution of singularities containing only the exceptional curves corresponding to pure representations. The morphism $\mathcal{S}: \tilde{G}\text{-Hilb}(\mathbb{C}^2) \rightarrow \mathcal{Y}$ is a resolution of singularities contracting the excess exceptional curves to ordinary nodes.*

Proof. — The projection $\pi: \mathcal{Y} \rightarrow \mathbb{C}^3/G$ factors through K/G by construction of \mathcal{Y} . The other assertions result from Theorem 8.1. The excess curves contract to ordinary nodes since, as one checks with Figures 5.1 and 5.2, each excess (-2) -curve is contracted to a different point. \square

Remark 8.6. — It is not difficult to see that in fact $\mathcal{Y} = G\text{-Hilb}(K)$.

9. Example: The cyclic group case

Let the cyclic group $\tilde{C}_n \cong \mathbb{Z}/(2n)\mathbb{Z}$ act on \mathbb{C}^2 with generator:

$$\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \quad \text{with } \xi = e^{\frac{2\pi i}{2n}}.$$

The choice of coordinates made in §7 implies that the group $C_n \cong \mathbb{Z}/n\mathbb{Z}$ acts on \mathbb{C}^3 with generator:

$$\begin{pmatrix} \xi^2 & 0 & 0 \\ 0 & \xi^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The irreducible representations of the cyclic group \tilde{C}_n are given by the matrices (ξ^i) , $i = 0, \dots, 2n - 1$. For i even, they are also the irreducible representations of C_n . There are then n pure and n binary representations. We set $\chi_i := \rho_{2i}$ and $\tilde{\chi}_i = \rho_{2i+1}$ for $i = 0, \dots, n - 1$. By Theorem 8.1, the exceptional curves on $\tilde{C}_n\text{-Hilb}(\mathbb{C}^2)$ corresponding to the binary representations are contracted by \mathcal{S} to a node on $\mathcal{S}(\tilde{C}_n\text{-Hilb}(\mathbb{C}^2))$ whereas the curves corresponding to the pure representations are in 1 : 1 correspondence with the exceptional curves downstairs (see Figure 9.1). In this section, we check this by a direct computation.

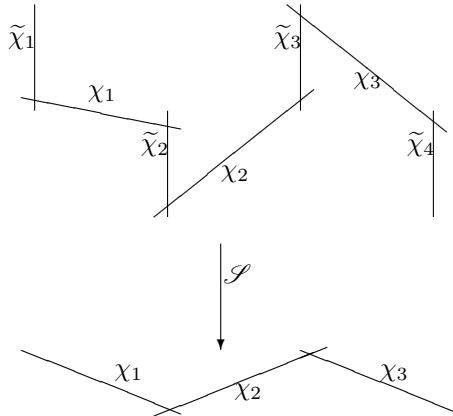


Figure 9.1. Contracted fibres for \tilde{C}_4

The ring of invariants $\mathbb{C}[x, y]^{\tilde{C}_n}$ is generated by x^{2n}, y^{2n}, xy and $\mathbb{C}[a, b, c]^{C_n}$ is generated by c, a^n, b^n, ab . We recall the description of the exceptional curves of \tilde{C}_n -Hilb(\mathbb{C}^2) following [13, Theorem 12.3]. We sort the basis of the algebra of coinvariants with respect to each irreducible representation:

$$\{1\}, \{x, y^{2n-1}\}, \dots, \{x^i, y^{2n-i}\}, \dots, \{x^{2n-1}, y\}.$$

To choose a cluster I/\mathfrak{n}_A supported at the origin amounts in choosing one copy of each non trivial representation, *i.e.* for all $i = 1, \dots, 2n - 1$ a point $(p_i : q_i) \in \mathbb{P}^1_{\mathbb{C}}$ defining the ideal by the generators:

$$\langle p_1x - q_1y^{2n-1}, \dots, p_ix^i - q_iy^{2n-i}, \dots, p_{2n-1}x^{2n-1} - q_{2n-1}y \rangle.$$

But the point is that one only needs *one* choice. Assume that there exists an index i such that $p_iq_i \neq 0$, and take the smallest i with this property. Set $p = p_i, q = q_i$ and $v = px^i - qy^{2n-i}$. Then since xy is invariant, $x^{i+1}, \dots, x^{2n-1} \in I/\mathfrak{n}_A$ and $y^{2n-i+1}, \dots, y^{2n-1} \in I/\mathfrak{n}_A$ so all our other choices were trivial, and $V(I) = \mathbb{C} \cdot v$. More formally, we parameterized the exceptional curve $E(\rho_i)$ by a subbundle:

$$\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1) \otimes \rho_i \oplus \bigoplus_{j \neq i} \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}} \otimes \rho_j \hookrightarrow \bigoplus_j (\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}} \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}) \otimes \rho_j.$$

If there is no such index, let x^i be the minimal power of x in the choice: To find once each non trivial representation one has to choose y^{2n-i+1} and the minimal set of generators $V(I) = \mathbb{C} \cdot x^i \oplus \mathbb{C} \cdot y^{2n-i+1}$ contains two adjacent representations.

Otherwise stated, a \tilde{C}_n -cluster at the origin takes the form:

$$I_j(p : q) := \langle px^j - qy^{2n-j}, xy, x^{j+1}, y^{2n-j+1} \rangle$$

for $1 \leq j \leq 2n - 1, (p : q) \in \mathbb{P}^1_{\mathbb{C}}$ and $E(\rho_j) = \{I_j(p : q)\}$.

By the same method, a C_n -cluster at the origin takes the form:

$$J_k(s : t) := \langle sa^k - tb^{n-k}, c, a^{k+1}, b^{n-k+1}, ab \rangle$$

for $1 \leq k \leq n - 1, (s : t) \in \mathbb{P}^1_{\mathbb{C}}$ and $C(\chi_k) = \{J_k(s : t)\}$.

With the construction (7.1) we have to compute $\sigma^{-1}(I_j(p : q))$. Setting $\bar{\sigma} : B / \langle ab - c^2 \rangle \rightarrow A$, it is equivalent to compute $\bar{\sigma}^{-1}(I_j(p : q))$. First we compute $I_j(p : q)^\tau \in A^\tau$. We distinguish two cases:

- j even, *i.e.* $j = 2j', j' = 1, \dots, n - 1$. We have in $A^\tau = \mathbb{C}[x^2, y^2, xy]$:

$$I_j(p : q)^\tau = I_j(p : q) = \langle p(x^2)^{j'} - q(y^2)^{n-j'}, xy, (x^2)^{j'+1}, (y^2)^{n-j'+1} \rangle$$

$$\bar{\sigma}^{-1}(I_j(p : q)) = \langle pa^{j'} - qb^{n-j'}, c, a^{j'+1}, b^{n-j'+1} \rangle = J_{j'}(p : q).$$

- j odd, i.e. $j = 2j' + 1, j' = 0, \dots, n - 1$. Note that $xy \in I_j(p : q)^\tau$ and $(x^2)^{j'+1}, y^{n-j'} \in I_j(p : q)^\tau$, but $px^{2j'+1} - qy^{2n-2j'-1} \notin I_j(p : q)^\tau$. So $\bar{\sigma}^{-1}(I_j(p : q)) = \langle a^{j'+1}, b^{n-j'}, c \rangle$. Since:

$$\bar{\sigma}^{-1}(I_j(p : q)) = J_{j'}(0 : 1) = J_{j'+1}(1 : 0),$$

one has $\bar{\sigma}^{-1}(I_j(p : q)) \in C(\rho_{j'}) \cap C(\rho_{j'+1})$.

The curves $E(\rho_j)$ with j even correspond to the pure representations and are not contracted by \mathcal{S} as the previous computation shows. The curves with j odd correspond to the binary representations and are contracted by \mathcal{S} .

10. Application: Pencils of symmetric surfaces

The polynomial invariants of the bipolyhedral groups G_6, G_8 and G_{12} are studied by the second author in [17]: The first non trivial invariant other than a power of the quadric $Q: x^2 + y^2 + z^2 + t^2 = 0$ is a homogeneous polynomial S_n of degree n . Consider then the following pencil of G_n -symmetric surfaces in $\mathbb{P}^3_{\mathbb{C}}$:

$$X_n(\lambda) = \{S_n + \lambda Q^{n/2} = 0\}, \quad \lambda \in \mathbb{C}.$$

The general surface $X_n(\lambda)$ is smooth and for each n there are precisely four singular surfaces in the corresponding pencil. The singularities of these surfaces are ordinary nodes forming one orbit through G_n (see [17]).

Consider now the pencil of quotient surfaces in $\mathbb{P}^3_{\mathbb{C}}/G_n$:

$$\{X_n(\lambda)/G_n\}, \quad \lambda \in \mathbb{C}.$$

These quotient surfaces have only A-D-E singularities and the minimal resolutions of singularities $Y_n(\lambda) \rightarrow X_n(\lambda)/G_n$ are K3 surfaces with Picard number greater than 19 (see [2]). For the four nodal surfaces in each pencil, a careful study of the stabilizers of the nodes shows that, if X denotes one of these nodal surfaces, the image of the node on $X/G_n \subset \mathbb{P}^3_{\mathbb{C}}/G_n$ is a particular quotient singularity locally isomorphic to $\mathbb{C}^2/\tilde{G} \subset \mathbb{C}^3/G$ for some polyhedral group G explicitly computed (see [2, §3, Proposition 3.1]):

- for $n = 6$: C_3, \mathcal{T} ;
- for $n = 8$: $D_2, D_3, D_4, \mathcal{O}$;
- for $n = 12$: $D_3, D_5, \mathcal{T}, \mathcal{I}$.

Therefore, Theorem 1.1 gives locally a group-theoretic interpretation of the exceptional curves of the K3 surfaces $Y_n(\lambda)$ over the particular singularities of the nodal surfaces.

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