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## OPTIMAL DESTABILIZING VECTORS IN SOME GAUGE THEORETICAL MODULI PROBLEMS

by Laurent BRUASSE

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ABSTRACT. — We prove that the well-known Harder-Narasimhan filtration theory for bundles over a complex curve and the theory of optimal destabilizing 1-parameter subgroups are the same thing when considered in the gauge theoretical framework.

Indeed, the classical concepts of the GIT theory are still effective in this context and the Harder-Narasimhan filtration can be viewed as a limit object for the action of the gauge group, in the direction of an optimal destabilizing vector. This vector appears as an extremal value of the so called “maximal weight function”. We give a complete description of these optimal destabilizing endomorphisms. Then we show how this principle may be applied to an other complex moduli problem: holomorphic pairs (i.e. holomorphic vector bundles coupled with morphisms with fixed source) over a complex curve. We get here a new version of the Harder-Narasimhan filtration theorem for the notion of  $\tau$ -stability. These results suggest that the principle holds in the whole gauge theoretical framework.

RÉSUMÉ. — Nous montrons que, du point de vue de la théorie de jauge, la filtration de Harder-Narasimhan d’un fibré vectoriel complexe au-dessus d’une courbe et la notion de sous-groupe déstabilisant optimal à un paramètre coïncident.

En utilisant l’approche de la GIT, la filtration de Harder-Narasimhan apparaît comme un objet limite pour l’action du groupe de jauge, dans la direction d’un vecteur déstabilisant optimal. Ce vecteur est un extremum de la “fonction de poids maximal”. Nous donnons une description complète de ces vecteurs déstabilisants optimaux. Nous montrons que le même principe s’applique à un autre problème de modules : celui des paires holomorphes (un fibré vectoriel complexe couplé avec un morphisme) sur une courbe complexe. On obtient dans ce contexte une nouvelle version du théorème de filtration de Harder-Narasimhan pour la notion de  $\tau$ -stabilité. Ces résultats suggèrent que le principe reste valable en toute généralité en théorie de jauge.

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*Keywords:* GIT, optimal 1-parameter subgroup, gauge theory, maximal weight map, complex moduli problem, stability, Harder-Narasimhan filtration, moment map.

*Math. classification:* 32M05, 53D20, 14L24, 14L30, 32L05, 32G13, 53C55.

## 1. Introduction

Harder and Narasimhan have proved [9] that any non semistable bundle on an algebraic curve admits a unique filtration by subsheaves such that the quotients are torsion free and semistable. This is now a classical result which was generalized for reflexive sheaves on projective manifolds [18], [13], and then to any reflexive sheaf on an arbitrary compact Hermitian manifold [4], [6].

The system of semistable quotients associated to a non semistable vector bundle by the Harder-Narasimhan theorem can be interpreted as a semistable object with respect to a new moduli problem: the moduli problem for  $G$ -bundles, where  $G$  is a product  $\prod_i \mathrm{GL}(r_i)$ . The motivation of this work is to find a general principle which applies to an arbitrary moduli problem: we want to associate in a *canonical way* to a non semistable object a new moduli problem and a semistable object for this new problem. One of the motivation is to find an analogous of the Harder-Narasimhan statement for other type of complex objects, for instance holomorphic bundles coupled with sections or endomorphisms.

On the other hand, the Geometric Invariant Theory (GIT) and its further developments (see [10], [14], [11], [17], [19], [7]) give powerful tools and intuitions, to deal with a stability condition. In [7], we discussed the relation between generalized GIT concepts and Harder-Narasimhan theory in the finite dimensional framework, for moduli problems associated with actions of a reductive group on a finite dimensional (*possibly non compact*) manifold. We look at a holomorphic action  $\alpha : G \times F \rightarrow F$  of a complex reductive Lie group  $G$  on a complex manifold  $F$ . We prove, that under a certain completeness assumption [19], the notion of stability *and of semistability* associated to the choice of a compact subgroup  $K \subset G$  and of a  $K$ -equivariant moment map  $\mu$  may be defined in term of a  $G$ -equivariant generalized maximal weight map  $\lambda : H(G) \times F \rightarrow \mathbb{R}$ , where  $H(G) \subset \mathfrak{g}$  is the union of all subspaces of the form  $i\xi$ . In the classical GIT, this map is known as the Hilbert numerical function (see [11]). We have proved the existence and unicity (up to equivalence) of an *optimal destabilizing vector*  $s$  in the Lie algebra of  $G$  associated to any non semistable point  $f \in F$ . The role of this optimal vector corresponds precisely to that of so-called adapted 1-parameter subgroups in classical GIT (see [10] or [17]). We have shown that the path  $t \mapsto e^{ts}f$  converges to a point  $f_0$  which is semistable with respect to a natural action of the centralizer  $Z(s)$  on a submanifold of  $F$ . The assignment  $f \mapsto f_0 = \lim_{t \rightarrow \infty} e^{ts}f$  is a finite dimensional “Harder-Narasimhan type” statement.

Therefore the general principle may be stated as follows: to get an analogue of the Harder-Narasimhan result for a given complex moduli problem, one has to give a gauge theoretical formulation of the problem (i.e. describe it in term of an action of a gauge group on certain complex variety) and then use the generalized GIT machinery to compute the maximal weight map and to study the optimal destabilizing vectors. Then the new semistable object is obtained as a limit in the direction of the optimal vector.

In gauge theory, there are already well-known links between the stability theory of vector bundles and the GIT concepts. The analogue of the maximal weight map in this setting has already been exploited by several authors, in particular to study the Kobayashi-Hitchin correspondance (see for example [15], [12]). In the case of plain bundles, the relation between the algebraic geometric notion of stability and the Morse theory associated to the Yang-Mills functional (see [1], [8]) is also well known. *But what appears to be unavailable in the literature, even in the case of plain bundles, is the very natural fact that the extremal value of the maximal weight map in the gauge theoretical framework corresponds exactly to the optimal direction which leads to the Harder-Narasimhan filtration (or precisely to the associated system of semistable quotients).*

The purpose of the present paper is to give a direct and complete proof of this fact in two classical gauge theoretical problems: holomorphic vector bundles and holomorphic pairs (i.e. vector bundles coupled with morphisms with fixed source) over a complex curve. These results were announced without proof in [12]. The case of a higher dimensional base will be studied in a forthcoming article.

We first focus on the moduli problem of complex vector bundles: we prove that one can associate to any non semistable bundle over a curve a maximal destabilizing element in the formal Lie algebra of the gauge group, then we give a complete description of this optimal vector. The main tools here are the notion of Harder-Narasimhan polygonal line (see [5]) and a well defined notion of energy for such lines. The system of quotients of the “classical” Harder-Narasimhan filtration appears as the limit in the direction of this optimal vector.

In a second part, we give an analogue description for the moduli problem associated to holomorphic pairs over a curve. Here the corresponding notion of stability, called  $\tau$ -stability (see [2]), depends on the choice of a real parameter. We prove, in this setting, a generalization of the Harder-Narasimhan filtration theorem for the  $\tau$ -stability (let us remark that a generalization of the Seshadri filtration in the context of  $\tau$ -stability was

given a few years ago in [3]). Using an explicit formula for the maximal weight function, we describe the optimal destabilizing vectors. Once again, the filtration is obtained as a limit in the direction of this extremal value of the maximal weight map.

It is a quite remarkable fact that when optimizing the maximal weight map we are led to study energy of piecewise affine plane curves. The Harder-Narasimhan polygonal line plays here a central role: It makes a link between the maximal weight map and a sort of “energy of the filtration”.

These two results and the recent work in [12] suggest that the principle holds in the infinite dimensional gauge theoretical framework: for moduli problems for principal  $G$ -bundles possibly coupled with linear or non-linear actions. The main tools for this generalization should be found in [12] and [7]: *the analogon of the Harder-Narasimhan filtration in this framework is a meromorphic reduction (see [12]) of the  $G$ -bundle structure to a parabolic subgroup  $L$  of  $G$*  (the group stabilizing the Harder-Narasimhan filtration in the present article). As in the finite dimensional case [7], this parabolic subgroup appears naturally when solving an optimization problem for the generalized maximal weight map (see [12]). Details will be studied in a forthcoming article.

## 2. Holomorphic vector bundles

Here we will use the “GIT” intuition to describe the behavior of the following gauge theoretical moduli problem: classifying the holomorphic structures on a given complex vector bundle up to gauge equivalence. We will assume that the base manifold is a complex curve to avoid complications related to singular sheaves (and so the use of  $L^2_1$ -Sobolev sections). Besides, the completeness property used in [7] is *formally* satisfied for *linear* moduli problems on a complex curve, so this is natural to work first in this framework.

Let  $E$  be a complex vector bundle of rank  $r$  over the Hermitian curve  $(Y, g)$ . We denote by  $\mathcal{G}$  the complex gauge group

$$\mathcal{G} := \text{Aut}(E)$$

whose formal Lie algebra is  $A^0(\text{End}(E))$ . Let  $h$  be any Hermitian structure on  $E$  and let us denote by

$$\mathcal{K}_h := U(E, h) \subset \mathcal{G}$$

the real gauge group of unitary automorphisms of  $E$  with respect to  $h$ .

We will use here the terminology developed in [19] and [7]. An element  $s \in A^0(\text{End}(E))$  is said to be of *Hermitian type* if there exists a Hermitian metric  $h$  on  $E$ , such that  $s \in A^0(\text{Herm}(E, h))$ .

We identify a holomorphic structure  $\mathcal{E}$  on  $E$  with the corresponding integrable semiconnections  $\bar{\partial}_{\mathcal{E}}$  on  $E$  (see [12]). We are concerned with the stability theory for the action of  $\mathcal{G}$  on the space  $\mathcal{H}(E)$  of holomorphic structures. Fixing a Hermitian metric  $h$ , the moment map for the induced  $\mathcal{K}_h$ -action on  $\mathcal{H}(E)$  is given by

$$\mu(\mathcal{E}) = \Lambda_g(F_{\mathcal{E},h}) + \frac{2\pi i}{\text{vol}_g(Y)} m(\mathcal{E}) \text{id}_E$$

where  $F_{\mathcal{E},h}$  is the curvature of the Chern connection associated to  $\bar{\partial}_{\mathcal{E}}$  and  $h$ , and where the slope of  $\mathcal{E}$  is

$$m(\mathcal{E}) := \frac{\text{deg}(\mathcal{E})}{r}.$$

Let us recall that a holomorphic vector bundle  $\mathcal{E} \in \mathcal{H}(E)$  is semistable with respect to this moment map if and only if it is semistable in the sense of Mumford (see [14], [12]):

$$m(\mathcal{F}) := \frac{\text{deg}(\mathcal{F})}{\text{rank}(\mathcal{F})} \leq \frac{\text{deg}(\mathcal{E})}{\text{rank}(\mathcal{E})} := m(\mathcal{E})$$

for all reflexive subsheave  $\mathcal{F} \subset \mathcal{E}$  such that  $0 < \text{rank}(\mathcal{F}) < r$ .

One may give an analytic Hilbert type criterion for the stability theory associated to the moment map  $\mu$ . We will need the following notation: if  $f$  is an endomorphism of a vector space  $V$ , we will put for any  $a \in \mathbb{R}$

$$V_f(a) := \bigoplus_{a' \leq a} \text{Eig}(f, a').$$

We extend the notation for endomorphisms of  $E$  with constant eigenvalues in an obvious way.

Then, one has an explicit formula for the maximal weight map  $\lambda$  (see [16]): if  $\mathcal{E} \in \mathcal{H}(E)$  and  $s \in \text{Herm}(E, h)$  then

$$\lambda^s(\mathcal{E}) = \begin{cases} \lambda_k \text{deg}(\mathcal{E}) + \sum_{i=1}^{k-1} (\lambda_i - \lambda_{i+1}) \text{deg}(\mathcal{E}_i) - (\text{deg}(\mathcal{E})/r) \text{tr}(s) & \text{if the eigenvalues } \lambda_1 < \dots < \lambda_k \text{ of } s \text{ are constant} \\ \quad \text{and } \mathcal{E}_i := \mathcal{E}_s(\lambda_i) \text{ are holomorphic,} & \\ \infty & \text{if not.} \end{cases}$$

**STABILITY CRITERION.** — *A point  $\mathcal{E} \in \mathcal{H}(E)$  is semistable if and only if  $\lambda^s(\mathcal{E}) \geq 0$  for all  $s \in A^0(\text{Herm}(E, h))$ .*

Let us come to the definition of optimal destabilizing endomorphisms:

PROPOSITION 2.1. — *Let  $\mathcal{E}$  be a non semistable bundle. There exists a Hermitian endomorphism  $s_{\text{op}} \in A^0(\text{Herm}(E, h))$  such that*

$$\lambda^{s_{\text{op}}}(\mathcal{E}) = \inf_{\substack{s \in A^0(\text{Herm}(E, h)) \\ \|s\|=1}} \lambda^s(\mathcal{E}).$$

*Proof.* — It is sufficient to consider the  $s \in A^0(\text{Herm}(E, h))$  with constant eigenvalues  $\lambda_1 < \dots < \lambda_k$  and such that the  $\mathcal{E}_i$  are holomorphic. Condition  $\|s\| = 1$  implies that the eigenvalues  $\lambda_i$  are bounded. Moreover, we know that the degree of a subbundle of  $\mathcal{E}$  is bounded above (see [4], Prop. 2.2), so that, writing  $\lambda$  in the form

$$\lambda^s(\mathcal{E}) = \text{deg}(\mathcal{E}) \left[ \lambda_k - \frac{1}{r} \sum_{i=1}^k r_i \lambda_i \right] + \sum_{i=1}^{k-1} \text{deg}(\mathcal{E}_i) (\lambda_i - \lambda_{i+1}),$$

it becomes obvious that  $\lambda^s(\mathcal{E})$  is bounded from below on the sphere  $\|s\| = 1$ .

Now let  $(s_n)_n$  be a sequence of Hermitian endomorphisms such that

$$\lim_{n \rightarrow +\infty} \lambda^{s_n}(\mathcal{E}) = \inf_{\substack{s \in A^0(\text{Herm}(E, h)) \\ \|s\|=1}} \lambda^s(\mathcal{E}).$$

We always assume that the associated filtrations are holomorphic. Going to a subsequence if necessary, we may suppose that each  $s_n$  admits  $k$  distinct eigenvalues  $\lambda_1^n < \dots < \lambda_k^n$ .

Let us recall the following result which is a direct consequence of the convergence theorem for subsheaves proved in [6] (see also [4, Prop. 2.9]):

PROPOSITION 2.2. — *Let  $(\mathcal{F}_n)_n$  be a sequence of subsheaves of  $\mathcal{E}$  and assume that there exists a constant  $c \in \mathbb{R}$  such that for all  $n$   $\text{deg}(\mathcal{F}_n) \geq c$ . Then we may extract a subsequence  $(\mathcal{F}_m)_m$  which converges in the sense of weakly holomorphic subbundles to a subsheaf  $\mathcal{F}$  of  $E$ . In particular*

$$\limsup_{m \rightarrow +\infty} \text{deg}(\mathcal{F}_m) \leq \text{deg}(\mathcal{F}).$$

Using this proposition and the fact that the  $\lambda_i^n$  are bounded, one can easily extract from  $(s_n)_n$  a subsequence  $(s_m)_m$  such that:

- (i) *there exist indices  $0 = j_0 < j_1 < \dots < j_\ell = k$  and distinct values  $\lambda_{j_1} < \dots < \lambda_{j_\ell}$  such that  $\lambda_i^m \rightarrow \lambda_{j_{p+1}}$  for all  $i \in \{j_p, \dots, j_{p+1}\}$ ;*
- (ii) *there exists a filtration  $0 = \mathcal{E}_0 \subset \mathcal{E}_{j_1} \subset \dots \subset \mathcal{E}_{j_\ell} = \mathcal{E}$  such that each  $\mathcal{E}_{j_p}$  is the limit in the sense of weakly holomorphic subbundles of the  $\mathcal{E}_{j_p}^m$ , which implies that  $\text{deg}(\mathcal{E}_{j_p}) \geq \limsup_{m \rightarrow +\infty} \text{deg}(\mathcal{E}_{j_p}^m)$ .*

For the second point, we use the fact that inclusion of subsheaves is preserved when going through the limit in the sense of weakly holomorphic subbundles [4].

Let  $s$  be the Hermitian endomorphism whose eigenvalues are the  $\lambda_{j_p}$  with corresponding filtration  $\{\mathcal{E}_{j_p}\}$  ( $s$  may have less than  $k$  distinct eigenvalues). Then we have

$$\lambda^s(\mathcal{E}) \leq \liminf_{m \rightarrow +\infty} \lambda^{s^m}(\mathcal{E}) = \inf_{\substack{s \in A^0(\text{Herm}(E, h)) \\ \|s\|=1}} \lambda^s(\mathcal{E})$$

so that  $s$  is an optimal destabilizing endomorphism of  $\mathcal{E}$ . □

Let us come to the proof of the result stated in [7]:

**THEOREM 2.3.** — *Let  $\mathcal{E} \in \mathcal{H}(E)$  be a non semistable bundle. Then it admits a unique optimal destabilizing element  $s_{\text{op}} \in A^0(\text{Herm}(E, h))$  which is given by*

$$s_{\text{op}} = \frac{\sum_{i=1}^k [\text{deg}(\mathcal{E})/r - \text{deg}(\mathcal{E}_i/\mathcal{E}_{i-1})/r_i]}{(\text{Vol}(Y))^{\frac{1}{2}} \left( \sum_{i=1}^k r_i [\text{deg}(\mathcal{E}_i/\mathcal{E}_{i-1})/r_i - \text{deg}(\mathcal{E})/r]^2 \right)^{\frac{1}{2}}} \text{id}_{F_i},$$

where

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \dots \subset \mathcal{E}_k = \mathcal{E}$$

is the Harder-Narasimhan filtration of  $\mathcal{E}$ ,  $F_i$  is the  $h$ -orthogonal complement of  $\mathcal{E}_{i-1}$  in  $\mathcal{E}_i$  and  $r_i := \text{rank}(\mathcal{E}_i/\mathcal{E}_{i-1})$ .

*Proof.* — Let  $s \in A^0(\text{Herm}(E, h))$  such that  $s$  has constant eigenvalues  $\lambda_1 < \lambda_2 < \dots < \lambda_k$  and the filtration given by the  $\mathcal{E}_i = \mathcal{E}_s(\lambda_i)$  is holomorphic. Let us denote by

$$m_i := \frac{\text{deg}(\mathcal{E}_i/\mathcal{E}_{i-1})}{r_i}, \quad 1 \leq i \leq k, \quad m := m(\mathcal{E}),$$

the slopes of the associated quotient sequence. The expression of the maximal weight map becomes

$$\lambda^s(\mathcal{E}) = \sum_{i=1}^k \lambda_i r_i (m_i - m).$$

We want to minimize this expression with respect to  $s$  under the assumption

$$\|s\| = \left( \int_Y \text{tr}(ss^*) \text{vol}_g \right)^{\frac{1}{2}} = (\text{Vol}(Y))^{\frac{1}{2}} \left( \sum_{i=1}^k r_i \lambda_i^2 \right)^{\frac{1}{2}} = 1.$$

Assume first that the filtration  $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \dots \subset \mathcal{E}_k = \mathcal{E}$  is fixed and hence that the  $r_i$  and the  $m_i$  are fixed. We have to minimize the map

$$g_{\{\mathcal{E}_i\}} : (\lambda_1, \dots, \lambda_k) \mapsto \sum_{i=1}^k \lambda_i r_i (m_i - m)$$

over the smooth ellipsoid

$$S = \left\{ (\lambda_1, \dots, \lambda_k) ; \sum_{i=1}^k r_i \lambda_i^2 = 1 / \text{Vol}(Y) \right\}$$

with the additional open condition

$$(\star) \quad \lambda_1 < \lambda_2 < \dots < \lambda_k.$$

Resolving the Lagrange problem for  $g|_S$  we see that there are two critical points of  $g$  over  $S$  which are obtained for

$$\lambda_i = \varepsilon(m_i - m), \quad 1 \leq i \leq k \quad \text{and} \quad \varepsilon = \frac{\pm 1}{(\text{Vol}(Y))^{\frac{1}{2}} \left( \sum_{i=1}^k r_i (m_i - m)^2 \right)^{\frac{1}{2}}}.$$

Let us remark that  $g_{\{\mathcal{E}_i\}}$  is negative only for  $\varepsilon < 0$ .

We need the following

LEMMA 2.4. — *The filtration  $\{\mathcal{E}_i\}_{1 \leq i \leq k}$  associated to any optimal destabilizing element  $s_{\text{op}}$  satisfies  $m_1 > m_2 > \dots > m_k$ .*

*Proof.* — If the sequence  $(m_i)_{1 \leq i \leq k}$  is not decreasing, the critical point of  $g_{\{\mathcal{E}_i\}}$  which may correspond to a minimum does not satisfy condition  $(\star)$ . So let  $s_{\text{op}}$  be an optimal destabilizing element and  $\lambda_1 < \dots < \lambda_k$  its eigenvalues. Assume that for the corresponding filtration the sequence  $(m_i)_{1 \leq i \leq k}$  is not decreasing, then  $\nu = (\lambda_1, \dots, \lambda_k)$  is not a critical point of  $g = g_{\{\mathcal{E}_i\}}$ , so that the gradient  $\text{grad}_\nu(g|_S)$  is non zero. Therefore, moving slightly the point  $\nu$  in the opposite direction, we may get a new point  $\nu' \in S$  which still satisfies the open condition  $(\star)$  and with  $g(\nu') < g(\nu)$ . Thus, the corresponding Hermitian endomorphism  $s'$  satisfies  $\lambda^{s'}(\mathcal{E}) < \lambda^{s_{\text{op}}}(\mathcal{E})$  which is a contradiction. □

Keeping in mind this result, it is sufficient to consider endomorphisms whose associated filtration  $\{\mathcal{E}_i\}_{1 \leq i \leq k}$  satisfies  $m_1 > m_2 > \dots > m_k$ . We will call such a filtration an *admissible filtration*.

For admissible filtrations, the map  $g_{\{\mathcal{E}_i\}}$  reaches its minimum for

$$(\lambda_1, \dots, \lambda_k) = \frac{1}{(\text{Vol}(Y))^{\frac{1}{2}} \left( \sum_{i=1}^k r_i (m_i - m)^2 \right)^{\frac{1}{2}}} (m - m_1, \dots, m - m_k)$$

and

$$g_{\{\mathcal{E}_i\}}(\lambda_1, \dots, \lambda_k) = -\frac{1}{(\text{Vol}(Y))^{\frac{1}{2}}} \left( \sum_{i=1}^k r_i (m_i - m)^2 \right)^{\frac{1}{2}}.$$

Thus, we have to maximize  $\sum_{i=1}^k r_i (m_i - m)^2$  among all admissible filtrations.

Let us remind some fundamental property of the Harder-Narasimhan filtration (see [5], [4] for details):

PROPOSITION 2.5. — *The Harder-Narasimhan filtration is the unique filtration  $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \dots \subset \mathcal{E}_\ell = \mathcal{E}$  such that*

- (i) *each quotient  $\mathcal{E}_i/\mathcal{E}_{i-1}$  is semistable for  $1 \leq i \leq \ell$ ;*
- (ii) *the slope sequence satisfies*

$$m(\mathcal{E}_i/\mathcal{E}_{i-1}) < m(\mathcal{E}_{i+1}/\mathcal{E}_i) \quad \text{for } 1 \leq i \leq \ell - 1.$$

We may associate to any filtration  $\mathcal{D} = (0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \dots \subset \mathcal{E}_\ell = \mathcal{E})$  a polygonal line  $\mathcal{P}(\mathcal{D})$  in  $\mathbb{R}^2$  defined by the sequence of points

$$p_i = (\text{rank } \mathcal{E}_i, \text{deg } \mathcal{E}_i)$$

for  $0 \leq i \leq \ell$ . The line associated to the Harder-Narasimhan filtration is called *the Harder-Narasimhan polygonal line* (see Fig. 2.1).

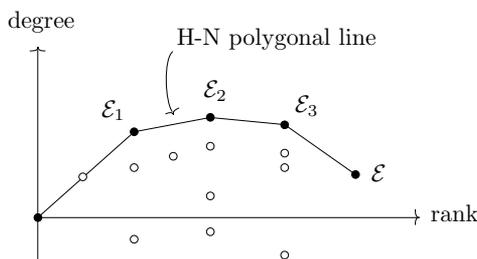


Figure 2.1. *The Harder-Narasimhan polygonal line*

For any filtration  $\mathcal{D} = \{\mathcal{E}_i\}_{1 \leq i \leq k}$ , the condition  $m_1 > \dots > m_k$  is equivalent to the fact that the line  $\mathcal{P}(\mathcal{D})$  is concave and let us remark that it is satisfied by the Harder-Narasimhan filtration. Now, the expression  $\sum_{i=1}^k r_i (m_i - m)^2$  can be interpreted as an energy of the corresponding polygonal line. Indeed, let us denote by  $f_{\mathcal{D}} \in C^0([0, r], \mathbb{R})$  the map whose graph is the polygonal line  $\mathcal{P}(\mathcal{D})$ ; this is a piecewise affine function and so

let  $a_0 = 0 < a_1 < \dots < a_\ell = r$  be the corresponding partition of  $[0, r]$ . We define

$$E(f_{\mathcal{D}}) := \sum_{i=1}^k r_i (m_i - m)^2 = \sum_{i=0}^{\ell-1} \int_{a_i}^{a_{i+1}} (f'_{\mathcal{D}}(x) - m)^2 dx$$

We will delay the proof of the following fundamental result, which asserts that the Harder-Narasimhan filtration is energy-maximizing among all admissible filtrations:

PROPOSITION 2.6. — *Let  $\mathfrak{D}$  be the Harder-Narasimhan filtration of  $\mathcal{E}$  and  $\mathcal{D}$  be any other admissible filtration, then  $E(f_{\mathfrak{D}}) > E(f_{\mathcal{D}})$ . The maximum of the energy among all concave polygonal lines associated to a filtration of  $\mathcal{E}$  is obtained for the Harder-Narasimhan polygonal line.*

Using this proposition we see that the maximal weight map  $\lambda^s(\mathcal{E})$  reaches its minimum for a Hermitian element  $s \in A^0(\text{Herm}(E, h))$  whose associated filtration is the Harder-Narasimhan filtration of  $\mathcal{E}$  and whose eigenvalues are  $\lambda_i = (m - m_i) / ((\text{Vol}(Y))^{\frac{1}{2}} (\sum_{i=1}^k r_i (m_i - m)^2)^{\frac{1}{2}})$ . Moreover it follows from the proof that this is a strict minimum.  $\square$

To prove proposition 2.6, we need the following lemma about the energy of concave piecewise affine maps.

LEMMA 2.7. — *Let  $f, g : [0, r] \rightarrow \mathbb{R}$  be two continuous piecewise affine concave functions. Assume that  $g(0) = f(0)$  and  $f(r) = g(r)$ . If for all  $x \in [0, r]$ ,  $f(x) \geq g(x)$  then  $E(f) \geq E(g)$ . Equality may occur if and only if  $f = g$ .*

*Proof.* — Let  $a_0 = 0 < a_1 < \dots < a_\ell = r$  be a partition of  $[0, r]$  common to  $f$  and  $g$ . Let  $F(t, x) = tf(x) + (1-t)g(x)$  then  $x \rightarrow F(t, x)$  is continuous, concave, and affine on each segment  $[a_i, a_{i+1}]$ . So we may define for each  $t$  the energy  $E(F(t, x))$  of the map  $x \mapsto F(t, x)$ . One has

$$\begin{aligned} \frac{d}{dt} E(F_t) &= \sum_{i=0}^{\ell-1} 2 \int_{a_i}^{a_{i+1}} \frac{d^2}{dt dx} F(t, x) \left( \frac{d}{dx} F(t, x) - m \right) dx \\ &= \sum_{i=0}^{\ell-1} \left[ \frac{d}{dt} F(t, x) \left( \frac{d}{dx} F(t, x) - m \right) \right]_{a_i}^{a_{i+1}} \\ &\quad - 2 \sum_{i=0}^{\ell-1} \int_{a_i}^{a_{i+1}} \frac{d}{dt} F(t, x) \frac{d^2}{dx^2} F(t, x) dx \end{aligned}$$

$$\geq \sum_{i=1}^{\ell-1} [(f(a_i) - g(a_i)) \times (t((f)'_g(a_i) - (f)'_d(a_i)) + (1-t)((g)'_g(a_i) - (g)'_d(a_i)))] > 0.$$

Indeed, the concavity of each polygonal line implies that its derivative is decreasing, that  $d^2F(t, x)/dx^2 \leq 0$  and we have  $(f(a_i) - g(a_i)) \geq 0$  of course. Let us remark that if  $f \neq g$  the strict inequality  $E(f) > E(g)$  obviously occurs.  $\square$

*Proof of proposition 2.6.* — Let us denote by  $\mathfrak{D}$  the Harder-Narasimhan filtration of  $\mathcal{E}$  and let  $\mathcal{D}$  be any other filtration whose polygonal line is concave.

PROPOSITION 2.8 (see [5]). — *For any subsheaf  $\mathcal{F}$  of  $\mathcal{E}$ , the point*

$$(\text{rank } \mathcal{F}, \text{deg } \mathcal{F})$$

*is located below the Harder-Narasimhan polygonal line  $\mathcal{P}(\mathfrak{D})$ . As a consequence, any polygonal line  $\mathcal{P}(\mathcal{D})$  associated to a filtration  $\mathcal{D}$  of  $\mathcal{E}$  is located below the Harder-Narasimhan polygonal line  $\mathcal{P}(\mathfrak{D})$ .*

Then, by Proposition 2.8, the polygonal line  $\mathcal{P}(\mathcal{D})$  lies below  $\mathcal{P}(\mathfrak{D})$ , that is for any  $x \in [0, r]$ ,  $f_{\mathcal{D}}(x) \leq f_{\mathfrak{D}}(x)$  with strict inequality on an open subset (see Fig. 2.2). We use Lemma 2.7 to conclude.  $\square$

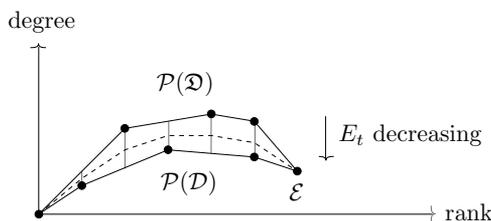


Figure 2.2. Illustration of Proposition 2.6

Using the identification of  $\mathcal{H}(E)$  with the space of integrable semiconnections, it is not difficult (see for instance [15, Lemma 2.3.2]) to show that

$$\lim_{t \rightarrow +\infty} (e^{t s_{\text{op}}} \bar{\partial}_{\mathcal{E}}) = \bar{\partial}_{F_1} \oplus \cdots \oplus \bar{\partial}_{F_k}.$$

In other words the holomorphic structure  $e^{t s_{\text{op}}} \mathcal{E}$  converges to the direct sum holomorphic structure  $\bigoplus_{i=1}^k \mathcal{E}_i / \mathcal{E}_{i-1}$  as  $t \rightarrow +\infty$ .

This illustrates our principle that the Harder-Narasimhan filtration is a limit object precisely in the direction given by the extremal value of the maximal weight map. The direct sum holomorphic structure is of course semistable with respect to the smaller gauge group  $\prod_{i=1}^k \text{Aut}(E_i/E_{i-1})$  (which corresponds to the centralizer  $Z(s)$  of the optimal destabilizing vector  $s$  in the finite dimensional case (see [7]).

### 3. Holomorphic pairs

Here we will give an analogous result for a different moduli problem associated to holomorphic pairs.

Let  $\mathcal{F}_0$  be a fixed holomorphic vector bundle of rank  $r_0$  with a fixed Hermitian metric  $h_0$  and  $E$  a complex vector bundle of rank  $r$  on the Hermitian curve  $(Y, g)$ . We are interested in the following moduli problem: classifying the holomorphic pairs  $(\mathcal{E}, \varphi)$  where  $\mathcal{E}$  is a holomorphic structure on  $E$  and  $\varphi$  is a holomorphic morphism  $\varphi : \mathcal{F}_0 \rightarrow \mathcal{E}$ . Such a pair will be called a holomorphic pair of type  $(E, \mathcal{F}_0)$  and we will denote by  $\mathcal{H}(E, \mathcal{F}_0)$  the space of such pairs. Here the complex gauge group is once again the group  $\mathcal{G} := \text{Aut}(E)$ .

Let us fix a Hermitian metric  $h$  on  $E$ , and let us denote by  $\mathcal{K}_h := U(E, h)$  the group of unitary automorphisms. The moment map for the  $\mathcal{K}_h$  action on  $\mathcal{H}(E, \mathcal{F}_0)$  has the form:

$$\mu(\mathcal{E}, \varphi) = \Lambda_g F_{\mathcal{E}, h} - \frac{1}{2}i \varphi \circ \varphi^* + \frac{1}{2}i t \text{id}_E.$$

In the sequel we will assume that the topological condition  $\mu(\mathcal{E}) \geq \tau$  holds.

Then we have the following characterisation of semistable pairs  $(\mathcal{E}, \varphi)$  (see [2]): let  $\tau := (1/4\pi)t \text{Vol}_g(Y)$ , then  $(\mathcal{E}, \varphi)$  is semistable with respect to the moment map  $\mu$  if and only if it is  $\tau$ -semistable in the following sense:

- (i)  $m(\mathcal{F}) := \text{deg}(\mathcal{F})/\text{rk}(\mathcal{F}) \leq \tau$  for all reflexive subsheaves  $\mathcal{F} \subset \mathcal{E}$  with  $0 < \text{rk}(\mathcal{F}) < r$ ;
- (ii)  $m(\mathcal{E}/\mathcal{F}) := \text{deg}(\mathcal{E}/\mathcal{F})/\text{rk}(\mathcal{E}/\mathcal{F}) \geq \tau$  for all reflexive subsheaves  $\mathcal{F} \subset \mathcal{E}$  with  $0 < \text{rk}(\mathcal{F}) < r$  and  $\varphi \in H^0(\text{Hom}(\mathcal{F}_0, \mathcal{F}))$ .

Now we may give an analogue of the Harder-Narasimhan theorem for this notion of stability:

**THEOREM 3.1.** — *Let  $(\mathcal{E}, \varphi)$  be a non  $\tau$ -semistable holomorphic pair of type  $(E, \mathcal{F}_0)$ . Then there exists a unique holomorphic filtration with torsion free quotients*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \dots \subset \mathcal{E}_m \subset \mathcal{E}_{m+1} \subset \dots \subset \mathcal{E}_k = \mathcal{E}$$

of  $\mathcal{E}$  such that:

(i) The slopes sequence satisfies:

$$m(\mathcal{E}_1/\mathcal{E}_0) > \cdots > m(\mathcal{E}_m/\mathcal{E}_{m-1}) > \tau \\ > m(\mathcal{E}_{m+2}/\mathcal{E}_{m+1}) > \cdots > m(\mathcal{E}_k/\mathcal{E}_{k-1})$$

and the additional condition  $\tau \geq m(\mathcal{E}_{m+1}/\mathcal{E}_m)$ .

(ii) The quotients  $\mathcal{E}_{i+1}/\mathcal{E}_i$  are semistable for  $i \neq m$ .

(iii) One of the following properties holds:

(a)  $\text{Im}(\varphi) \not\subset \mathcal{E}_m$ ,  $\tau > \deg(\mathcal{E}_{m+1}/\mathcal{E}_m)/\text{rk}(\mathcal{E}_{m+1}/\mathcal{E}_m)$  and the pair  $(\mathcal{E}_{m+1}/\mathcal{E}_m, \bar{\varphi})$  is  $\tau$ -semistable, where  $\bar{\varphi}$  is the  $\mathcal{E}_{m+1}/\mathcal{E}_m$ -valued morphism induced by  $\varphi$ .

(b)  $\text{Im}(\varphi) \not\subset \mathcal{E}_m$ ,  $\tau = \deg(\mathcal{E}_{m+1}/\mathcal{E}_m)/\text{rk}(\mathcal{E}_{m+1}/\mathcal{E}_m)$  and  $\mathcal{E}_{m+1}/\mathcal{E}_m$  is semistable of slope  $\tau$ . This implies that the pair  $(\mathcal{E}_{m+1}/\mathcal{E}_m, \bar{\varphi})$  is  $\tau$ -semistable.

(c)  $\text{Im}(\varphi) \subset \mathcal{E}_m$  and  $\mathcal{E}_{m+1}/\mathcal{E}_m$  is semistable.

Moreover, in cases (b) and (c) the obtained filtration coincides with the “classical” Harder-Narasimhan filtration of  $\mathcal{E}$  and the additional condition  $m(\mathcal{E}_{m+1}/\mathcal{E}_m) > m(\mathcal{E}_{m+2}/\mathcal{E}_{m+1})$  holds.

*Proof.* — We will use the results and methods of Theorem 3.2 in [4]. In order to build the filtration, let us first consider type (i) destabilizing subsheaves of  $\mathcal{E}$ , i.e. subsheaves  $\mathcal{F}$  which satisfy  $m(\mathcal{F}) > \tau$ . If there exists such a subsheaf, we let  $\mathcal{E}_1$  be the maximal destabilizing type (i) subsheaf: it is nothing but the first element of the Harder-Narasimhan filtration of  $\mathcal{E}$ . If  $\text{Im}(\varphi) \subset \mathcal{E}_1$  then we follow with the classical Harder-Narasimhan filtration (case (c)), if not we consider the pair  $(\mathcal{E}/\mathcal{E}_1, \varphi_1)$  where  $\varphi_1$  is induced by  $\varphi$  and we follow the same principle until there is no more type (i) destabilizing subsheaves: we get a sequence

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_m$$

which coincides with the first terms of the usual Harder-Narasimhan filtration and such that  $\mathcal{E}/\mathcal{E}_m$  has no type (i) destabilizing subsheaf.

Then we consider type (ii) destabilizing subsheaves and we take the subsheaf  $\mathcal{F}_1$  containing  $\text{Im}(\varphi)$  and minimizing the slope  $m(\mathcal{E}/\mathcal{F}_1)$  (we take  $\mathcal{F}_1$  of maximal rank with this property, it exists and it is unique by the arguments developed in [4]). This will be the last term of our filtration. By definition, we have  $m(\mathcal{E}/\mathcal{F}_1) < \tau$ . Moreover  $\mathcal{F}_1$  contains  $\mathcal{E}_m$  by the following lemma.

LEMMA 3.2. — Let  $\mathcal{G} \subset \mathcal{E}$  be a maximal destabilizing subsheaf of type (i) and let  $\mathcal{F} \subset \mathcal{E}$  be a type (ii) maximal destabilizing subsheaf, then  $\mathcal{G} \subset \mathcal{F}$ .

*Proof.* — Assume first that  $\mathcal{G} \cap \mathcal{F} = 0$ , then  $(\mathcal{F} + \mathcal{G})/\mathcal{F} \simeq \mathcal{G}$  and of course  $\text{Im}(\varphi) \subset \mathcal{G} + \mathcal{F}$ . Then we have

$$m((\mathcal{F} + \mathcal{G})/\mathcal{F}) = m(\mathcal{G}) > \tau > m(\mathcal{E}/\mathcal{F}),$$

$$0 \rightarrow (\mathcal{F} + \mathcal{G})/\mathcal{F} \rightarrow \mathcal{E}/\mathcal{F} \rightarrow \mathcal{E}/(\mathcal{F} + \mathcal{G}) \rightarrow 0.$$

We get  $m(\mathcal{E}/(\mathcal{F} + \mathcal{G})) < m(\mathcal{E}/\mathcal{F})$  which is a contradiction. Assume now that  $\mathcal{F} \cap \mathcal{G}$  is a non trivial subsheaf of  $\mathcal{G}$  then using the following exact sequence

$$0 \rightarrow \mathcal{G} \cap \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{G}/\mathcal{G} \cap \mathcal{F} \rightarrow 0$$

and the isomorphism  $(\mathcal{G} + \mathcal{F})/\mathcal{F} \simeq \mathcal{G}/\mathcal{G} \cap \mathcal{F}$ , we get

$$m((\mathcal{F} + \mathcal{G})/\mathcal{F}) = m(\mathcal{G}/\mathcal{G} \cap \mathcal{F}) < m(\mathcal{E}/\mathcal{F}) < \tau < m(\mathcal{G})$$

and  $m(\mathcal{G} \cap \mathcal{F}) > m(\mathcal{G})$  which is a contradiction. So we get  $\mathcal{G} \cap \mathcal{F} = \mathcal{G}$  so that  $\mathcal{G} \subset \mathcal{F}$ . □

*Remark 3.3.* — This lemma simply states that one can always build a Harder-Narasimhan filtration starting either from its first term or from its last term.

Following the process, we get a sequence  $\mathcal{F}_\ell \subset \dots \subset \mathcal{F}_1 \subset \mathcal{E}$  where  $\mathcal{F}_\ell$  does contain  $\mathcal{E}_m$  and admits no type (ii) destabilizing subsheaf. It is quite easy to prove that each quotient  $\mathcal{F}_i/\mathcal{F}_{i+1}$  is semistable. Moreover, it is clear that  $\mathcal{F}_\ell/\mathcal{E}_m$  has no type (i) destabilizing subsheaf. So, putting things together, we get a filtration

$$0 \subset \mathcal{E}_0 \subset \dots \subset \dots \mathcal{E}_m \subset \mathcal{E}_{m+1}(= \mathcal{F}_\ell) \subset \dots \subset \mathcal{E}_{k-1} = (\mathcal{F}_1) \subset \mathcal{E}_k = \mathcal{E}$$

where  $\text{Im}(\varphi) \subset \mathcal{E}_{m+1}$ .

Also we have

$$m(\mathcal{E}_1/\mathcal{E}_0) > \dots > m(\mathcal{E}_m/\mathcal{E}_{m-1}) > \tau \quad \text{and}$$

$$m(\mathcal{E}_{m+2}/\mathcal{E}_{m+1}) > \dots > m(\mathcal{E}_k/\mathcal{E}_{k-1}).$$

If  $\text{Im}(\varphi) \subset \mathcal{E}_m$ , then clearly the filtration coincides with the classical Harder-Narasimhan filtration and we are in case (c) of the theorem. Else the pair  $(\mathcal{E}_{m+1}/\mathcal{E}_m, \bar{\varphi})$  is by construction  $\tau$ -semistable with  $m(\mathcal{E}_{m+1}/\mathcal{E}_m) \leq \tau$ . Let us remark to conclude that if  $m(\mathcal{E}_{m+1}/\mathcal{E}_m) = \tau$ , then the notion of  $\tau$ -semistability and semistability coincide for  $\mathcal{E}_{m+1}/\mathcal{E}_m$  (case (b) of the theorem).

Unicity is proved in the same way as in the algebraic “classical” case (see [4] or [5]). □

*Remark 3.4.* — Theorem 3.1 clearly holds for any bundle over a compact Hermitian manifold  $(X, g)$  where  $g$  is a Gauduchon metric (see [4]). Indeed, we did not use the fact that  $Y$  is one dimensional in this proof.

Coming back to our gauge moduli problem, one can once again give a formula for the maximal weight function:

$$\lambda^s(\mathcal{E}) = \begin{cases} \lambda_k \deg(\mathcal{E}) + \sum_{i=1}^{k-1} (\lambda_i - \lambda_{i+1}) \deg(\mathcal{E}_i) - \tau \operatorname{tr}(s) & \text{if the eigenvalues } \lambda_1 < \dots < \lambda_k \text{ of } s \text{ are constant,} \\ \mathcal{E}_i := \mathcal{E}_s(\lambda_i) \text{ are holomorphic, and } \varphi \in H^0(\operatorname{Hom}(\mathcal{F}_0, \mathcal{E}_s(0))); & \\ \infty \text{ if not.} & \end{cases}$$

**CRITERION.** — A holomorphic pair  $(\mathcal{E}, \varphi)$  is semistable with respect to the moment map  $\mu$  if and only if  $\lambda^s(\mathcal{E}) \geq 0$  for all  $s \in A^0(\operatorname{Herm}(E, h))$ .

Put again  $r_i := \operatorname{rk}(\mathcal{E}_i/\mathcal{E}_{i-1})$  and  $m_i = m(\mathcal{E}_i/\mathcal{E}_{i-1})$ . We have the following result:

**THEOREM 3.5.** — For any Hermitian metric  $h$  on  $E$  and any non semistable holomorphic pair  $(\mathcal{E}, \varphi)$ , there exists a unique normalised Hermitian endomorphism  $s_{\text{op}} \in A^0(\operatorname{Herm}(E, h))$  which satisfies

$$\lambda^{s_{\text{op}}}(\mathcal{E}) = \inf_{\substack{s \in A^0(\operatorname{Herm}(E, h)) \\ \|s\|=1}} \lambda^s(\mathcal{E}).$$

It is given by

(i) If  $\operatorname{Im}(\varphi) \subset \mathcal{E}_m$  then

$$s_{\text{op}} = \frac{\sum_{i=1}^k [\tau - m_i] \operatorname{id}_{F_i}}{(\operatorname{Vol}(Y))^{\frac{1}{2}} \left( \sum_{i=1}^k r_i [m_i - \tau]^2 \right)^{\frac{1}{2}}}.$$

(ii) If  $\operatorname{Im}(\varphi) \not\subset \mathcal{E}_m$

$$s_{\text{op}} = \frac{\sum_{\substack{i=1 \\ i \neq m+1}}^k [\tau - m_i] \operatorname{id}_{F_i}}{(\operatorname{Vol}(Y))^{\frac{1}{2}} \left( \sum_{\substack{i=1 \\ i \neq m+1}}^k r_i [m_i - \tau]^2 \right)^{\frac{1}{2}}},$$

where  $F_i$  is the  $h$ -orthogonal complement of  $\mathcal{E}_{i-1}$  in  $\mathcal{E}_i$ .

*Proof.* — The existence of an optimal destabilizing element  $s_{\text{op}}$  is proved in the same way as in proposition 2.1: we simply use the fact that inclusion of subsheaves is preserved when going through the limit in the sense

of weakly holomorphic subbundles to deal with the additional condition  $\text{Im}(\varphi) \subset \mathcal{E}_s(0)$ .

Now, let  $s \in A^0(\text{Herm}(E, h))$  with constant eigenvalues  $\lambda_1 < \dots < \lambda_k$ , such that the filtration given by the  $\mathcal{E}_i = \mathcal{E}_s(\lambda_i)$  is holomorphic and  $\text{Im}(\varphi) \subset \mathcal{E}_s(0)$ . Using the previous notations the expression of the minimal weight map becomes

$$\lambda^s(\mathcal{E}) = \sum_{i=1}^k \lambda_i r_i (m_i - \tau).$$

As in the classical situation, we want to minimize this expression with respect to  $s$  under the assumption

$$\|s\| = (\text{Vol}(Y))^{\frac{1}{2}} \left( \sum_{i=1}^k r_i \lambda_i^2 \right)^{\frac{1}{2}} = 1$$

We use the same idea as in theorem 2.3. Assume first that the filtration by eigenspaces is fixed; we have to minimize the map

$$g_{\{\mathcal{E}_i\}} : (\lambda_1, \dots, \lambda_k) \longmapsto \sum_{i=1}^k \lambda_i r_i (m_i - \tau)$$

over the smooth ellipsoid

$$S = \left\{ (\lambda_1, \dots, \lambda_k); \sum_{i=1}^k r_i \lambda_i^2 = 1/\text{Vol}(Y) \right\}$$

with the two additional conditions

$$(\star_1) : \lambda_1 < \lambda_2 < \dots < \lambda_k. \quad \text{and} \quad (\star_2) : \text{Im}(\varphi) \subset \mathcal{E}_s(0).$$

Once again, resolving the Lagrange problem for  $g_{\{\mathcal{E}_i\}}|_S$ , we see that there are two critical points of  $g_{\{\mathcal{E}_i\}}$  over  $S$  which are obtained for

$$\lambda_i = \varepsilon(m_i - \tau), \quad 1 \leq i \leq k \quad \text{and} \quad \varepsilon = \frac{\pm 1}{(\text{Vol}(Y))^{\frac{1}{2}} \left( \sum_{i=1}^k r_i (m_i - \tau)^2 \right)^{\frac{1}{2}}}.$$

Let us remark that  $g_{\{\mathcal{E}_i\}}$  is negative only for  $\varepsilon < 0$ .

We have the following lemma:

LEMMA 3.6. — *Assume  $s$  is an optimal destabilizing Hermitian endomorphism and let  $\{\mathcal{E}_i\}_{1 \leq i \leq k}$  its associated filtration. Then there exists  $\ell$  such that*

- (i)  $m_1 > \dots > m_\ell > \tau > m_{\ell+2} > \dots > m_k = m(\mathcal{E});$
- (ii)  $\tau \geq m_{\ell+1}$  and the property  $\text{Im}(\varphi) \subset \mathcal{E}_{\ell+1}$  holds;

- (iii) if moreover  $\text{Im}(\varphi) \subset \mathcal{E}_\ell$ , then the additional condition  $m_{\ell+1} > m_{\ell+2}$  holds.

A filtration which satisfies these conditions will be called an admissible filtration.

*Proof.* — These are once again gradient arguments. Assume  $s$  is optimal and let  $\ell$  such that  $\lambda_1 < \dots < \lambda_\ell < 0 \leq \lambda_{\ell+1} < \dots < \lambda_k$ .

If  $\text{Im}(\varphi) \subset \mathcal{E}_\ell$ , since  $\lambda_\ell < 0$ , then the filtration must satisfy  $m_1 > \dots > m_\ell > \tau \geq m_{\ell+1} > m_{\ell+2} > \dots > m_k = m(\mathcal{E})$ , otherwise the same gradient argument as in lemma 2.4 contradicts the optimality of  $s$ .

If  $\text{Im}(\varphi) \not\subset \mathcal{E}_\ell$ , then obviously  $\text{Im}(\varphi) \subset \mathcal{E}_{\ell+1}$  and  $\lambda_{\ell+1} = 0$ . Then restricting  $g_{\{\mathcal{E}_i\}}$  to  $\{\lambda \mid \lambda_{\ell+1} = 0\}$ , a similar argument shows that the associated filtration satisfies i). For the second point, an explicit computation of the gradient of  $g_{\{\mathcal{E}_i\}}$ , shows that if  $m_{\ell+1} > \tau$ , we may move the point  $\lambda$  such that  $g_{\{\mathcal{E}_i\}}$  decreases and  $\lambda_{\ell+1} \leq 0$ , which contradicts the optimality of  $s$ . Thus  $\tau \geq m_{\ell+1}$ .  $\square$

So it is sufficient for our problem to minimize  $g_{\{\mathcal{E}_i\}}$  where  $\{\mathcal{E}_i\}$  is an admissible filtration. Under this assumption we get:

- If  $\text{Im}(\varphi) \subset \mathcal{E}_\ell$ , the minimal value of  $g_{\{\mathcal{E}_i\}}|_S$  is obtained for

$$(\lambda_1, \dots, \lambda_k) = \frac{1}{(\text{Vol}(Y))^{\frac{1}{2}} \left( \sum_{i=1}^k r_i (m_i - \tau)^2 \right)^{\frac{1}{2}}} (\tau - m_1, \dots, \tau - m_k)$$

and

$$g_{\{\mathcal{E}_i\}}(\lambda_1, \dots, \lambda_k) = -\frac{1}{(\text{Vol}(Y))^{\frac{1}{2}}} \left( \sum_{i=1}^k r_i (m_i - \tau)^2 \right)^{\frac{1}{2}}.$$

- If  $\text{Im}(\varphi) \not\subset \mathcal{E}_\ell$ , then  $g_{\{\mathcal{E}_i\}}|_S$  reaches its minimum for

$$(\lambda_1, \dots, \lambda_k) = \frac{1}{\|\lambda\|} (\tau - m_1, \dots, \tau - m_\ell, 0, \tau - m_{\ell+2}, \dots, \tau - m_k)$$

and

$$g_{\{\mathcal{E}_i\}}(\lambda_1, \dots, \lambda_k) = -\frac{1}{(\text{Vol}(Y))^{\frac{1}{2}}} \left( \sum_{\substack{i=1 \\ i \neq \ell+1}}^k r_i (m_i - \tau)^2 \right)^{\frac{1}{2}}.$$

Now we want to minimize these expressions among all admissible filtrations. As in section 2, we will work on the polygonal line associated to any admissible filtration. We use the same notations as in section 2, and we use

the following definition for the energy  $E(f)$  of a polygonal line  $f$ :

$$E(f) := \sum_{i=1}^k r_i(m_i - \tau)^2 = \sum_{i=0}^{\ell-1} \int_{a_i}^{a_{i+1}} (f'_{\mathcal{D}}(x) - \tau)^2 dx$$

The difficult point here is that the polygonal line associated to an admissible filtration *may no longer be concave*. We will in fact compare the energy of the different concave parts.

Let us postpone the proof of the following technical lemma:

LEMMA 3.7. — *Let  $f : [0, r] \rightarrow \mathbb{R}$  and  $g : [0, s] \rightarrow \mathbb{R}$  two distinct concave piecewise affine maps. Assume that*

- (i)  $f(0) = g(0)$ ;
- (ii)  $f(x) \geq g(x)$  for all  $x \in [0, \min(r, s)]$ ;
- (iii)  $f'(x) \geq \tau$  and  $g'(x) \geq \tau$  where they are defined;
- (iv)  $[f(r) - g(s)] / (r - s) \leq \tau$ .

Then  $E(f) > E(g)$ .

Let  $\mathcal{D} = (0 = \mathcal{E}_0 \subset \dots \subset \mathcal{E}_m \subset \mathcal{E}_{m+1} \subset \dots \subset \mathcal{E}_k = \mathcal{E})$  be the generalized Harder-Narasimhan filtration given by Theorem 3.1 and  $f_{\mathcal{D}} \in C^0([0, r], \mathbb{R})$  the corresponding piecewise affine map. Put  $m_0 = m$  if  $\text{Im}(\varphi) \subset \mathcal{E}_m$ ,  $m_0 = m + 1$  otherwise. Then  $f_{\mathcal{D}}$  is not a concave map but admits two concave parts corresponding to  $(0 = \mathcal{E}_0 \subset \dots \subset \mathcal{E}_m)$  and  $(\mathcal{E}_{m_0} \subset \dots \subset \mathcal{E}_k = \mathcal{E})$ . Keep in mind that the first  $m$  subsheaves of this filtration are just those of the classical Harder-Narasimhan filtration. Let  $D = (0 = \mathcal{E}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_p = \mathcal{E})$  be any admissible filtration. Let us denote by

$$r'_i := \text{rank}(\mathcal{F}_i / \mathcal{F}_{i-1}) \quad \text{and} \quad m'_i := m(\mathcal{F}_i / \mathcal{F}_{i-1})$$

and assume  $m'_j > \tau \geq m'_{j+1}$ .

Assume that  $\text{rank}(\mathcal{F}_j) > \text{rank}(\mathcal{E}_m)$ . It follows by the definition of the filtration  $\mathcal{D}$  that  $(\text{deg}(\mathcal{E}_m) - \text{deg}(\mathcal{F}_j)) / (\text{rank}(\mathcal{E}_m) - \text{rank}(\mathcal{F}_j)) < \tau$ : it is an obvious consequence of the fact that the point corresponding to  $\mathcal{F}_j$  is located below the classical Harder-Narasimhan polygonal line and the fact that  $\mathcal{E}_m$  admits by definition no more type (i) destabilizing subsheaf. Hence applying Lemma 3.7 to the first concave part of each filtration, we get

$$\sum_{i=1}^j r'_i(\tau - m'_i)^2 \leq \sum_{i=1}^m r_i(\tau - m_i)^2.$$

If  $\text{rank}(\mathcal{F}_j) \leq \text{rank}(\mathcal{E}_m)$ , then the points corresponding to  $\mathcal{F}_i, 1 \leq i \leq j$  are located below the polygonal line  $\mathcal{P}(\mathcal{D})$  and using again Lemma 3.7 we get the same result.

We can apply exactly the same argument to the second part of the filtration. Let  $j_0 = j$  if  $\text{Im}(\varphi) \subset \mathcal{F}_j$  and  $j_0 = j + 1$  in the other case. Then, one can prove that the points corresponding to  $(\mathcal{F}_i)_{j_0 \leq i \leq p}$  satisfy the following conditions:

- if  $\text{rank}(\mathcal{F}_i) > \text{rank}(\mathcal{E}_{m_0})$  for any  $i \in \{j_0, \dots, p\}$ , then  $\mathcal{F}_i$  is located below the second part of the generalized Harder-Narasimhan polygonal line (i.e. the points corresponding to  $\mathcal{E}_{m_0} \subset \dots \subset \mathcal{E}$ );
- if  $\text{rank}(\mathcal{F}_{j_0}) < \text{rank}(\mathcal{E}_{m_0})$ , the following relation holds:  

$$(\deg(\mathcal{F}_{j_0}) - \deg(\mathcal{E}_{m_0})) / (\text{rank}(\mathcal{F}_{j_0}) - \text{rank}(\mathcal{E}_{m_0})) \geq \tau;$$
- the part of the associated polygonal line corresponding to the indices  $\{j_0, \dots, p\}$  is concave.

The first and the second point can be seen as a Harder-Narasimhan property for subsheaves containing  $\text{Im}(\varphi)$ . Then using an analogue of Lemma 3.7 (in fact a symmetric proposition), we get

$$\sum_{u=j_0}^p r'_u (\tau - m'_u)^2 \leq \sum_{u=m_0}^k r_u (\tau - m_u)^2.$$

This is exactly that we wanted since in the case where  $\text{Im}(\varphi) \not\subset \mathcal{F}_j$ , then the expression of  $\lambda(\mathcal{F})$  does not contain the  $j$ -th term (the eigenvalue of the endomorphism must vanish).

In Figure 3.1, the energy of the left and right parts of the filtrations are respectively compared to the energy of the corresponding parts of the generalized Harder-Narasimhan filtration. The segments drawn with dotted lines give no contribution for the energy.

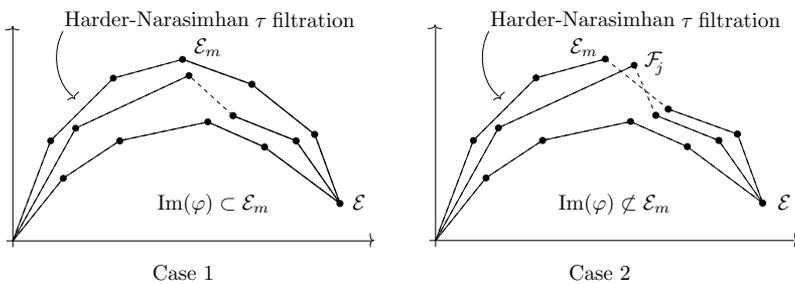


Figure 3.1. Illustration of Lemma 3.7

In either case, we have proved that the minimum of  $\lambda(\mathcal{E})$  is achieved for the Hermitian element  $s_{\text{op}}$  whose associated filtration is the generalized

Harder-Narasimhan filtration with the corresponding eigenvalues described in the theorem.  $\square$

*Proof of the technical Lemma 3.7.* — Let us fix  $f : [0, r] \rightarrow \mathbb{R}$ ; let  $a_0 = 0 < a_1 < \dots < a_\ell = s$  be the partition of  $[0, s]$  corresponding to  $g : [0, s] \rightarrow \mathbb{R}$ . We will do an induction on the length  $\ell$  of the partition.

For the case  $\ell = 1$ , assume first that  $s \leq r$  then the result is a consequence of Lemma 2.7 applied to  $f|_{[0,s]}$  and  $g$ : using the additional condition (iii), the conclusion of the lemma is still correct even if  $g(s) \neq f(s)$ .

If  $s > r$ , let us denote by  $h : [0, r] \rightarrow \mathbb{R}$  the straight line defined by  $h(0) = 0$  and  $h(r) = f(r)$ , then by Lemma 2.7,  $E(f) \geq E(h)$ . Denote by  $h_1$  the slope of the line  $h$  and  $g_1$  those of the line  $g$ . By hypothesis

$$\frac{h(r) - g(s)}{r - s} \leq \tau$$

such that

$$h(r) - r\tau \geq g(s) - s\tau$$

and using conditions (ii) and (iii) we get  $(h(r) - r\tau)^2/r \geq (g(s) - s\tau)^2/s$ , so that

$$r(h_1 - \tau)^2 \geq s(g_1 - \tau)^2.$$

Thus  $E(f) \geq E(h) \geq E(g)$ , equality can only occur if  $f = g$ .

Now assume that the result is proved for a polygonal line of length less or equal than  $\ell$  and let  $g$  be a polygonal line of length  $\ell + 1$ . Once again, if  $s \leq r$ , we use Lemma 2.7 to conclude. Else, we define a new polygonal line  $h$  as follows:

- $h|_{[0,a_\ell]} = g|_{[0,a_\ell]}$ ;
- if adding the segment  $[g(a_\ell), f(r)]$  preserves the concavity of  $h$  we do so (case 1 below), if not we extend the last segment  $[g(a_{\ell-1}), g(a_\ell)]$  in order that it reaches the line of slope  $\tau$  going through  $f(r)$  (case 2 in Figure 3.2).

Using the same argument as in the first step of the induction we get easily  $E(h) > E(g)$ . Then we use Lemma 2.7 in the first case and the induction hypothesis applied to  $f$  and  $h$  in the second case to get  $E(f) \geq E(h)$ .  $\square$

Once again, using the computation in [15], we get:

- If  $\text{Im}(\varphi) \subset \mathcal{E}_m$ , then it is contained in the elements of the filtration corresponding to (strictly) negative eigenvalues of  $s_{\text{op}}$ , so that the path  $t \mapsto e^{t s_{\text{op}}}(\mathcal{E}, \varphi)$  converges as in the classical case to the object

$$(\mathcal{E}_1/\mathcal{E}_0, \dots, \mathcal{E}_k/\mathcal{E}_{k-1}).$$

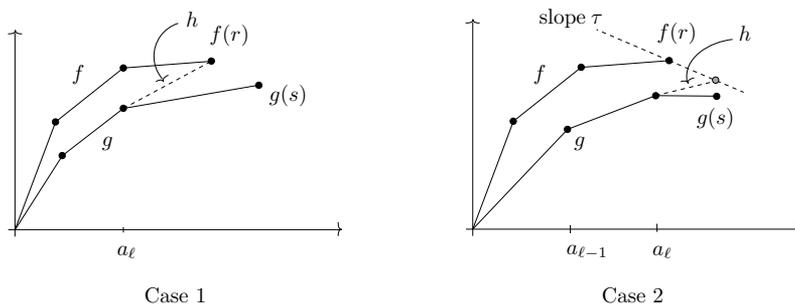


Figure 3.2. Proof of technical Lemma 3.7

- Else, if  $\text{Im}(\varphi) \not\subset \mathcal{E}_m$ , then it converges to the object

$$(\mathcal{E}_1/\mathcal{E}_0, \dots, \mathcal{E}_m/\mathcal{E}_{m-1}, (\mathcal{E}_{m+1}/\mathcal{E}_m, \bar{\varphi}), \mathcal{E}_{m+2}/\mathcal{E}_{m+1}, \dots, \mathcal{E}_k/\mathcal{E}_{k-1})$$

where  $\bar{\varphi}$  is the map induced from  $\varphi$  (remark that in this case, the eigenvalue of  $s_{\text{op}}$  corresponding to  $F_m$  is vanishing).

This illustrates once again our general principle. By Theorem 3.1 the limit object is semistable with respect to the gauge group  $\prod_{i=1}^k \text{Aut}(E_i/E_{i-1})$ .

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