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GEOMETRIC AND p -ADIC MODULAR FORMS OF HALF-INTEGRAL WEIGHT

by Nick RAMSEY

ABSTRACT. — In this paper we introduce a geometric formalism for studying modular forms of half-integral weight. We then use this formalism to define p -adic modular forms of half-integral weight and to construct p -adic Hecke operators.

RÉSUMÉ. — Nous nous proposons ici de présenter un formalisme géométrique ayant pour but l'étude des formes modulaires des poids demi-entiers. Ce formalisme est mis à contribution pour définir les formes modulaires p -adiques des poids demi-entiers, et dans la construction des opérateurs de Hecke p -adiques.

1. Introduction

The aim of this article is to generalize the notion of a classical (holomorphic) modular form of half-integral weight to rings other than \mathbb{C} . In particular we define, for any positive integers N and k , the space $M_{k/2}(4N, R)$ of modular forms of weight $k/2$ and level $4N$ over the ring R , for any ring R in which the level $4N$ is invertible. The definition is geometric in nature in the sense that it involves the modular curves $X_1(4N)$, and admits a modification to a p -adic theory by “deleting the supersingular locus”.

The author's primary motivation for studying such objects lies in their relation to (central) special values of modular L -functions. This remarkable relationship, discovered by Waldspurger, expresses these values (appropriately normalized) in terms of *squares* of the Fourier coefficients of modular forms of half-integral weight. Investigating the p -adic nature of modular forms of half-integral, and in particular the nature of p -adic families of modular forms of half-integral weight, represents a first step in one approach to understanding the p -adic variation of (square roots of) these

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L -values. Eying the integral weight case, one expects the notion of over-convergence to play a central role in questions about families of modular forms of half-integral weight. This, in turn, is currently best understood in geometric terms.

In the integral weight case, a geometric theory of modular forms is achieved by considering sections of tensor powers of the line bundle ω which is the push-forward of the sheaf of relative differentials from the universal elliptic curve over $X_1(N)$ (supposing $N \geq 5$). Lacking such a “natural” bundle to work with, we resort to an alternative technique which exploits well-understood modular forms of weight $k/2$ and reduce (by division) to consideration of certain rational functions on $X_1(4N)$. This method is reminiscent of the use of the Eisenstein family to define families of p -adic modular forms of integral weight (as in [3] and [2]).

Let

$$\theta(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2}, \quad q = e^{2\pi i \tau}$$

denote the usual Jacobi theta function. It is well-known that θ obeys a transformation law with respect to the group

$$\Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \text{ such that } 4|c \right\},$$

namely

$$\theta\left(\frac{a\tau + b}{c\tau + d}\right) = \varepsilon_d^{-1} \left(\frac{c}{d}\right) (c\tau + d)^{1/2} \theta(\tau), \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4),$$

where

$$(1.1) \quad \varepsilon_d = \begin{cases} 1 & d \equiv 1 \pmod{4} \\ i & d \equiv -1 \pmod{4} \end{cases}$$

and the square root is chosen to have argument in the interval $[-\pi/2, \pi/2)$. The Jacobi symbol here is taken with the conventions of [8].

We will take the θ^k as the “well-understood” forms by which we divide as indicated above. Indeed, in section 3 we single out a family of divisors $\Sigma_{4N,k}$ on the curves $X_1(4N)$ with the property that the rational functions F on $X_1(4N)$ such that $F\theta^k$ is a holomorphic modular forms of weight $k/2$ are exactly those F for which $(F) \geq -\Sigma_{4N,k}$. The caveat is that $\Sigma_{4N,k}$ has *non-integral* (rational) coefficients. The divisor $\Sigma_{4N,k}$ is supported on the cusps and makes perfect sense as a relative \mathbb{Q} -divisor on $X_1(4N)_R$ over any base R in which $4N$ is invertible. We will accordingly identify $M_{k/2}(4N, R)$ with the R -module of rational functions F on $X_1(4N)_R$ with $(F) \geq -\Sigma_{4N,k}$. In

other words, if $\lfloor \Sigma_{4N,k} \rfloor$ is the relative effective Cartier divisor obtained by taking the floor of the coefficients of $\Sigma_{4N,k}$, then

$$M_{k/2}(4N, R) = H^0(X_1(4N)_R, \mathcal{O}(\lfloor \Sigma_{4N,k} \rfloor)).$$

In what follows, we will adopt the suggestive notation

$$H^0(X_1(4N)_R, \Sigma_{4N,k})$$

for the right-hand side of this equation.

If this definition is to be a useful one, it must enjoy some the host of properties possessed by the integral weight analog. To this end, we will define q -expansions, prove a q -expansion principle, prove some basic “change-of-ring” compatibilities, and furnish a *geometric* construction of the Hecke operators. Because of its central importance, we briefly expound on this last point.

Let $X_1(N, m)$ denote the modular curve over $\mathbb{Z}[1/(Nm)]$ (roughly) classifying elliptic curves with a point of order N and a cyclic subgroup of order m having trivial intersection with that generated by the point. Let

$$\pi_1, \pi_2 : X_1(N, m) \longrightarrow X_1(N)$$

denote the degeneracy maps which forget and quotient by, respectively, the cyclic subgroup of order m . These pairs of maps form the usual Hecke correspondences on $X_1(N)$, and the geometric manifestation of the integral weight Hecke operator T_m (for, say, m prime) is

$$\begin{aligned} H^0(X_1(N), \omega^k) &\longrightarrow H^0(X_1(N), \omega^k) \\ f &\longmapsto \pi_{1*}(\pi_2^* f), \end{aligned}$$

where we have exploited the canonical identification

$$\pi_1^* \omega \cong \pi_2^* \omega.$$

A little reflection on the definition of a modular form of half-integral weight given above leads one to look for a map of the form

$$\begin{aligned} H^0(X_1(4N), \Sigma_{4N,k}) &\longrightarrow H^0(X_1(4N), \Sigma_{4N,k}) \\ F &\longmapsto \pi_{1*}(\pi_2^* F \cdot \Theta) \end{aligned}$$

for some rational function Θ on $X_1(4N, m)$ satisfying

$$(1.2) \quad \text{div}(\Theta) \geq \pi_2^* \Sigma_{4N,k} - \pi_1^* \Sigma_{4N,k}.$$

Note that this last divisor is of degree 0, so if such a Θ exists, then one has equality above. It then follows that Θ is necessarily unique up to a constant factor, and the divisor $\pi_2^* \Sigma_{4N,k} - \pi_1^* \Sigma_{4N,k}$ actually has integral coefficients.

The following is proven in section 6, and involves a slightly refined space of forms $M_{k/2}(4N, \chi, R)$ with Nebentypus character $\chi : (\mathbb{Z}/4N\mathbb{Z})^\times \rightarrow R^\times$.

THEOREM 1.1. — *Suppose $\gcd(4N, m) = 1$. Then there exists a Θ satisfying (1.2) if and only if m is a square. If $m = l^2$ for a prime l not dividing $4N$, the corresponding endomorphism T_{l^2} of $M_{k/2}(4N, \chi, R)$ (appropriately normalized) has the following effect: if $\sum a_n q^n$ is the q -expansion of F at ∞ , then the q -expansion of $T_{l^2}F$ at ∞ is $\sum b_n q^n$ where*

$$b_n = a_{l^2 n} + \left(\frac{-1}{l}\right)^{(1-k)/2} l^{(k-1)/2-1} \binom{n}{l} \chi(l)a_n + l^{k-2} \chi(l^2)a_{n/l^2}.$$

This theorem furnishes a geometric explanation of the classical fact that there are no good (i.e. prime to the level) Hecke operators of non-square index, and shows that the operators obtained by the procedure above at square indices are the same as the classical ones (as in [8]).

Let K be a discretely valued subfield of \mathbb{C}_p . In section 7 we define K -Banach space $M_{k/2}(4N, K, r)$ of p -adic modular forms of weight $k/2$, level $4N$, and growth condition $r \in [0, 1]$, and prove the following theorem.

THEOREM 1.2. — *Let p be a prime number not dividing $4N$ and let r further satisfy $r > p^{-1/p(1+p)}$. There is a completely continuous endomorphism U_{p^2} of $M_{k/2}(4N, K, r)$ having the effect*

$$\sum a_n q^n \mapsto \sum a_{p^2 n} q^n$$

on q -expansions at ∞ . Moreover, the norm of U_{p^2} is at most p^2 .

The definition of the space $M_{k/2}(4N, K, r)$ is a straightforward modification of that of $M_{k/2}(4N, R)$ achieved by deleting part of the supersingular locus. In particular, it still relies on properties of the forms θ^k (equivalently, the divisors $\Sigma_{4N,k}$). We note that it is perhaps more natural in this p -adic context to use Eisenstein series in place of θ^k , since much of their divisors lie *outside of the range of consideration*. This point of view is adopted in a forthcoming work investigating p -adic families of modular forms of half-integral weight ([6]).

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2. Notation and Conventions

For an integer $N \geq 4$ and a ring R in which N is invertible, $Y_1(N)$ will denote the fine moduli scheme classifying pairs $(E/S, P)$ where E is an elliptic curve over an R -algebra S and P is a point of order N (as in [5]). For such N , $X_1(N)_R$ will denote the compactified moduli scheme, which is a fine moduli scheme for the *generalized* $\Gamma_1(N)$ problem if $N \geq 5$, and coarse moduli scheme if $N = 4$. We will have no occasion to make use of generalized elliptic curves in what follows.

Choosing another positive integer m which is invertible in R , we may also form the moduli scheme $Y_1(N, m)_R$ classifying triples $(E/S, P, C)$ consisting of an elliptic curve E over an R -algebra S , a point P of order N , and a cyclic subgroup C of order m not meeting that generated by P , as well as its compactification $X_1(N, m)_R$. All of these moduli schemes are smooth over R .

If K is the complex field \mathbb{C} or a p -adic field, we will denote by $X_1(N)_K^{\text{an}}$ (and similarly $X_1(N, m)_K^{\text{an}}$, etc.) the associated complex-analytic or rigid-analytic space, respectively. Over \mathbb{C} , we choose to uniformize $Y_1(N)$ via the map

$$\begin{aligned} \Gamma_1(N) \backslash \mathbb{H} &\xrightarrow{\sim} Y_1(N)_{\mathbb{C}}^{\text{an}} \\ \tau &\longmapsto \left(E_\tau, \frac{1}{N} \right) \end{aligned}$$

where \mathbb{H} denotes the complex upper half-plane and E_τ denote the elliptic curve $\mathbb{C}/\langle 1, \tau \rangle$ over \mathbb{C} . Similarly, we choose to uniformize $Y_1(N, m)$ via the map

$$\begin{aligned} (\Gamma_1(N) \cap \Gamma^0(m)) \backslash \mathbb{H} &\xrightarrow{\sim} Y_1(N, m)_{\mathbb{C}}^{\text{an}} \\ \tau &\longmapsto \left(E_\tau, \frac{1}{N}, \left\langle \frac{\tau}{m} \right\rangle \right). \end{aligned}$$

If $\Gamma \subseteq \text{SL}_2(\mathbb{Z})$ is any subgroup, we will also use the notation $X(\Gamma)_{\mathbb{C}}^{\text{an}}$ for the compactification of the quotient $\Gamma \backslash \mathbb{H}$, ignoring any algebraic model of this Riemann surface (i.e. this is not intended to denote the analytification of anything in this case).

We will also need to consider the moduli spaces $X_1(N, p^n)_R$ in the case that p is *not* invertible in R (but N is). By such a space we will always mean the Katz-Mazur model (see [5]). In particular, $X_1(N, p^n)_R$ is not smooth over R and will even have non-reduced fibers when $n > 1$. We note for later use that $X_1(N, p^n)_R$ is Cohen-Macaulay over R as its singularities are all local complete intersections.

We will denote the Tate elliptic curve over $\mathbb{Z}((q))$ by $\underline{\text{Tate}}(q)$ (see [4]). Our conventions concerning the Tate curve differ from the standard ones as follows. In the presence of, for example, level N structure, previous authors (e.g., [4]) have preferred to consider the curve $\underline{\text{Tate}}(q^N)$ over the base $\mathbb{Z}((q))$. Points of order N on this curve are used to characterize the behavior of a modular form at the cusps, and are all defined over the fixed ring $\mathbb{Z}((q))[\zeta_N]$ (where ζ_N is some primitive N^{th} root of 1). We prefer to fix the curve $\underline{\text{Tate}}(q)$ and instead consider *extensions* of the base. Thus, in the presence of level N structure, we introduce the formal variable q_N , and *define* $q = q_N^N$. Then the curve $\underline{\text{Tate}}(q)$ is defined over the sub-ring $\mathbb{Z}((q))$ of $\mathbb{Z}((q_N))$ and all of its N -torsion is defined over the ring $\mathbb{Z}((q_N))[\zeta_N]$. To be precise, the N -torsion is given by

$$\zeta_N^i q_N^j, \quad 0 \leq i, j \leq N - 1.$$

The reason for this formal relabeling is that, with these conventions, the q -expansions of modular forms look like they do complex-analytically if we identify q_N with $e^{2\pi i\tau/N}$. In particular, the “ q -order” of vanishing (i.e. the smallest, generally fractional, power of q appearing in the expansion) at a cusp does not change when one pulls back through a degeneracy map, even in the presence of ramification.

For a prime l and an l^{th} root of unity ζ we will denote the associated quadratic Gauss sum by

$$\mathfrak{g}_l(\zeta) := \sum_{a=1}^{l-1} \left(\frac{a}{l}\right) \zeta^a.$$

3. Definitions

Given that θ is non-vanishing in \mathbb{H} , we must examine the behavior of θ at the cusps of $X_1(4)_{\mathbb{C}}^{\text{an}}$ in order to determine the divisor $\Sigma_{4N,k}$ mentioned in the introduction. There are three such cusps, and using standard formulas for transforming “theta functions with characteristics” one finds that the q -expansions of θ at these cusps are as follows (defined up to a constant

involving powers of 2 and roots of unity, except in the first case).

∞	q -expansion
$1/2$	$\theta_1(q) := \sum_{n \in \mathbb{Z}} q^{n^2}$
0	$\theta_3(q) := q_4 \sum_{n \in \mathbb{Z}} q^{n^2+n}$
0	$\theta_2(q) := \sum_{n \in \mathbb{Z}} q_4^{n^2}$

Recall that in the complex-analytic setting, q_h is shorthand for $e^{2\pi i \tau/h}$.

If F is a rational function on $X_1(4N)_{\mathbb{C}}^{\text{an}}$, it follows that $F \cdot \theta^k$ is holomorphic if and only if $\text{div}(F) \geq -\Sigma_{4N,k}$, where

$$\Sigma_{4N,k} = \sum_{c \sim 1/2} \frac{k w_c}{4} \cdot c,$$

the sum is taken over all cusps c on $X_1(4N)_{\mathbb{C}}^{\text{an}}$ mapping to $1/2$ on $X_1(4)_{\mathbb{C}}^{\text{an}}$ under the degeneracy map which multiplies the point of order $4N$ by N , and w_c denotes the width of the cusp c (which coincides with the ramification index of this map at c since $1/2 \in X_1(4)_{\mathbb{C}}^{\text{an}}$ has width 1).

To determine the appropriate replacement for $\Sigma_{4N,k}$ on the algebraic curve $X_1(4N)_R$ (R a $\mathbb{Z}[1/(4N)]$ -algebra), we consider the form θ^4 . By the transformation formula (1.1), this is a modular form of weight 2 for $\Gamma_1(4)$. By GAGA and the q -expansion principle, θ^4 furnishes a section of ω^2 on the curve $X_1(4)_{\mathbb{Z}[1/2]}$. Thus we may evaluate the corresponding rule on pairs $(\text{Tate}(q), P)$ where P runs through the points of order 4 on $\text{Tate}(q)$ (recall the conventions Section 2). There are six such pairs up to isomorphism, and the values are as follows,

	point on $\underline{\text{Tate}}(q)$	cusps on $X_1(4)_{\mathbb{C}}^{\text{an}}$	q -expansion
	ζ_4	∞	$\left(\sum_{n \in \mathbb{Z}} q^{n^2}\right)^4$
	$\zeta_4 q_2$	$1/2$	$q \left(\sum_{n \in \mathbb{Z}} q^{n^2+n}\right)^4$
(3.3)	q_4	0	$-\frac{1}{4} \left(\sum_{n \in \mathbb{Z}} q_4^{n^2}\right)^4$
	$\zeta_4 q_4$	0	$-\frac{1}{4} \left(\sum_{n \in \mathbb{Z}} (\zeta_4 q_4)^{n^2}\right)^4$
	$\zeta_4^2 q_4 = -q_4$	0	$-\frac{1}{4} \left(\sum_{n \in \mathbb{Z}} (-q)^{n^2}\right)^4$
	$\zeta_4^3 q_4 = -\zeta_4 q_4$	0	$-\frac{1}{4} \left(\sum_{n \in \mathbb{Z}} (\zeta_4^3 q_4)^{n^2}\right)^4$

where we have filled in the middle column by comparison with table (3.1).

It is now clear from the table that the cusp associated to the pair $(\underline{\text{Tate}}(q), \zeta_4 q_2)$ is the correct replacement for the cusp $1/2$ in general. To simplify notation, we will simply denote the cusp on $X_1(4)_R$ corresponding to this pair by the symbol $1/2$, and accordingly *define* a divisor $\Sigma_{4N,k}$ on the curve $X_1(4N)_R$ by the formula (3.2) for any ring R in which $4N$ is invertible.

Now that we have a good version of $\Sigma_{4N,k}$ on $X_1(4N)_R$, we enshrine the identification of the Introduction in a *definition*.

DEFINITION 3.1. — *Let k be an odd positive integer, let N be any positive integer, and let R be a ring in which $4N$ is invertible. A modular form of weight $k/2$ and level $4N$ is an element of the R -module*

$$M_{k/2}(4N, R) := H^0(X_1(4N)_R, \Sigma_{4N,k}).$$

We recall from the Introduction that this notation is shorthand for the collection of rational functions F on $X_1(4N)_R$ with $\text{div}(F) \geq -\Sigma_{4N,k}$. In other words, this is the space of sections of the sheaf $\mathcal{O}([\Sigma_{4N,k}])$ associated to the relative effective Cartier divisor $[\Sigma_{4N,k}]$ on $X_1(4N)_R$. The reason for carrying around the \mathbb{Q} -divisor $\Sigma_{4N,k}$ instead of uniformly resorting to its floor $[\Sigma_{4N,k}]$ (or equivalently the associated sheaf) is that the latter do not behave well with respect to degeneracy maps and the former does. For

example, if $N|M$ and

$$\pi : X_1(4M) \rightarrow X_1(4N)$$

denotes the map which multiplies the point by M/N , then

$$\pi^* \Sigma_{4N,k} = \Sigma_{4M,k}.$$

The precise behavior of $\Sigma_{4N,k}$ under various degeneracy maps will be of critical importance in the sequel.

For $d \in (\mathbb{Z}/(4N)\mathbb{Z})^\times$, let $\langle d \rangle$ denote the diamond automorphism of $X_1(4N)$ corresponding to d . Note that

$$\langle d \rangle^* \Sigma_{4N,k} = \Sigma_{4N,k}, \text{ for all } d \in (\mathbb{Z}/(4N)\mathbb{Z})^\times.$$

It follows that the group $(\mathbb{Z}/(4N)\mathbb{Z})^\times$ acts on the space $M_{k/2}(4N, R)$. For a character

$$\chi : (\mathbb{Z}/(4N)\mathbb{Z})^\times \rightarrow R^\times,$$

we will denote the χ -isotypic component of $M_{k/2}(4N, R)$ by $M_{k/2}(4N, \chi, R)$, and refer to such forms as having “nebenotypus χ ”.

4. Properties

As in the integral weight case, it will be useful to reinterpret the elements of $M_{k/2}(4N, R)$ as “rules” which take test objects as input and give some output subject to a few restraints. In particular, we will take as test objects pairs $(E/S, P)$ where E is an elliptic curve over an R -algebra S and P is a point of order $4N$ on E . An element F of $M_{k/2}(4N, R)$ can then be identified with a rule (also called F) which takes as input a test object $(E/S, P)$ and outputs an element of S such that

- if the test objects $(E/S, P)$ and $(E'/S, P')$ are isomorphic over S , then $F(E, P) = F(E', P')$,
- if $\phi : S \rightarrow S'$ is any map of R -algebras and $(E/S, P)$ is a test object over S , then

$$F((E, P) \times_S S') = \phi(F(E, P)),$$

and

- if P is a point of order $4N$ on $\text{Tate}(q)$, then

$$F(\text{Tate}(q), P) \in q_{4N}^{-e} \mathbb{Z}[[q_{4N}]]$$

where

$$e = \frac{4N}{w_c} \text{ord}_c \Sigma_{4N,k} = \begin{cases} Nk & c \sim 1/2 \\ 0 & c \not\sim 1/2 \end{cases}$$

and c is the cusp associated to $(\underline{\text{Tate}}(q), P)$.

With this in mind, we can define the q -expansions of a modular form of half-integral weight by evaluating the corresponding rule on pairs $(\underline{\text{Tate}}(q), P)$ for various points P of order $4N$. By the definition of the space $M_{k/2}(4N, R)$, these q -expansions will not agree with the classical ones, and will in fact be off exactly by a factor of the corresponding q -expansion of θ^k , and we must adjust accordingly.

To be precise, let P be a point of order 4 on $\underline{\text{Tate}}(q)$. Then, up to isomorphism, P is one of the points listed in the first column of Table 3.3. Up to roots of unity, the corresponding q -expansions of θ are as follows (see Table 3.1).

P	q -expansion of θ
ζ_4	$\sum_{n \in \mathbb{Z}} q^{n^2}$
$\zeta_4 q_2$	$q_4 \sum_{n \in \mathbb{Z}} q^{n^2+n}$
$\zeta_4^k q_4, 0 \leq k \leq 3$	$\frac{1}{1 + \zeta_4^k} \sum_{n \in \mathbb{Z}} \zeta_4^{kn^2} q_4^{n^2}$

DEFINITION 4.1. — Let $F \in M_{k/2}(4N, R)$ and let P be a point of order $4N$ on $\underline{\text{Tate}}(q)$. Let θ_P be the q -expansion of θ in the above table corresponding to the point NP of order 4. The q -expansion of F at $(\underline{\text{Tate}}(q), P)$ is

$$F(\underline{\text{Tate}}(q), P)\theta_P^k \in \mathbb{Z}((q_{4N})) \otimes R[\zeta_{4N}].$$

We remark in particular that the definition of $\Sigma_{4N,k}$ exactly suffices to ensure that all of the q -expansions of an $F \in M_{k/2}(4N, R)$ are in fact in the subring

$$\mathbb{Z}[[q_{4N}]] \otimes R[\zeta_{4N}].$$

That is, such an F is “holomorphic at the cusps.” We will often fix a root of unity ζ_{4N} , and refer to the q -expansion at $(\underline{\text{Tate}}(q), \zeta_{4N})$ as the “ q -expansion at ∞ .” The reason is that if $R = \mathbb{C}$ and we have fixed $\zeta_{4N} = e^{2\pi i/N}$, then under the uniformizations detailed in Section 2 we recover the classical q -expansion at “ $i\infty$.”

Half-integral weight modular forms enjoy a q -expansion principle much like the integral-weight version. As in that case, one must first define modular forms over *modules* as opposed to *rings*. For a $\mathbb{Z}[1/(4N)]$ -module K , one defines the space of modular forms of weight $k/2$ and level $4N$ with coefficients in K as the collection of rules which associate to each test datum

(E, P) over a $\mathbb{Z}[1/(4N)]$ -algebra R , an element of $R \otimes_{\mathbb{Z}[1/(4N)]} K$, subject to the usual compatibilities and conditions on the Tate curve. Note that, in the case that K is in fact a ring, the space so defined is canonically isomorphic to the space we have already defined. However, in contrast to the case of rings, the q -expansions of elements of $M_{k/2}(4N, K)$ for a $\mathbb{Z}[1/(4N)]$ -module K , lie in the ring

$$\mathbb{Z}[[q_{4N}]] \otimes_{\mathbb{Z}} \mathbb{Z}[1/(4N), \zeta_{4N}] \otimes_{\mathbb{Z}[1/(4N)]} K,$$

In particular, these q -expansion *do not* agree with the ones defined for rings (even when K is a ring), though one can still deduce useful information about the “ring” q -expansions from the “module” q -expansions. At any rate, it is in the context of the “module” q -expansion that the q -expansion principle is most naturally stated.

THEOREM 4.2. — *Let $K \subset L$ be an inclusion of $\mathbb{Z}[1/(4N)]$ -modules and let $F \in M_{k/2}(4N, L)$. Let P be a points of order $4N$ on $\underline{\text{Tate}}(q)$ and suppose that the q -expansion of F at $(\underline{\text{Tate}}(q), P)$ has coefficients in $\mathbb{Z}[1/(4N), \zeta_{4N}] \otimes_{\mathbb{Z}[1/(4N)]} K$. Then $F \in M_{k/2}(4N, K)$.*

Proof. — Referring to Table 4.1, we see that each of θ_P and θ_P^{-1} has coefficients in $\mathbb{Z}[1/(4N), \zeta_{4N}]$. It follows that the q -expansion of F at $(\underline{\text{Tate}}(q), P)$ has coefficients in $\mathbb{Z}[1/(4N), \zeta_{4N}] \otimes_{\mathbb{Z}[1/(4N)]} K$ if and only if $F(\underline{\text{Tate}}(q), P)$ does. If the latter is true, then $F \in M_{k/2}(4N, K)$ by the ordinary q -expansion principle in weight 0 (the poles are of no consequence). □

We wrap up this section with a basic result about changes-of-ring. Any map $R \rightarrow S$ of $\mathbb{Z}[1/(4N)]$ -algebras induces a map

$$M_{k/2}(4N, R) \otimes_R S \longrightarrow M_{k/2}(4N, S).$$

We would like conditions under which this map is an isomorphism. As these are sections of the sheaf $\mathcal{O}([\Sigma_{4N,k}])$, it suffices by standard base-changing results to see that

$$H^1(X_1(4N)_R, \mathcal{O}([\Sigma_{4N,k}])) = 0.$$

THEOREM 4.3. — *This vanishing of H^1 holds whenever $k \geq 5$.*

Proof. — By Serre duality it suffices to show that

$$\text{deg}([\Sigma_{4N,k}]) > \text{deg}(\Omega^1_{X_1(4N)_R}).$$

Note that the form θ^4 furnishes a section of ω^2 on $X_1(4N)_R$ with divisor $\Sigma_{4N,4}$. Also, by the Kodaira-Spencer isomorphism,

$$\Omega^1_{X_1(4N)} \cong \omega^2(-C)$$

where C is the divisor of cusps. Thus

$$\deg(\Omega_{X_1(4N)}^1) = \deg(\omega^2) - |C| < \deg(\Sigma_{4N,4}) \leq \deg(\lfloor \Sigma_{4N,k} \rfloor)$$

whenever $k \geq 4$. □

Remark 4.4. — In weights $1/2$ and $3/2$ this inequality of degrees does not hold in general, the first counterexamples occurring at levels 20 and 68 , respectively. The author does not know if base change holds in these cases.

5. The Modular Units Θ_m

Let m be an odd positive integer. In this section we introduce a family of rational functions on $X_1(4m)$ whose divisors are supposed on the cusps. These functions will later be used to construct the Hecke operators.

Let Θ_m be the holomorphic function on \mathbb{H} defined by

$$\Theta_m(\tau) = \frac{\theta(\tau/m)}{\theta(\tau)}.$$

Note that, for

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(4) \cap \Gamma^0(m),$$

we have

$$\Theta_m|_\gamma = \left(\frac{m}{d}\right) \Theta_m.$$

Thus Θ_m furnishes a holomorphic function on $(\Gamma_1(4) \cap \Gamma^1(m)) \backslash \mathbb{H}$ which descends through the natural map

$$(5.1) \quad (\Gamma_1(4) \cap \Gamma^1(m)) \backslash \mathbb{H} \longrightarrow (\Gamma_1(4) \cap \Gamma^0(m)) \backslash \mathbb{H}$$

exactly when m is a square. Let d denote the extension of this map to

$$X(\Gamma_1(4) \cap \Gamma^1(m))_{\mathbb{C}}^{\text{an}} \longrightarrow X(\Gamma_1(4) \cap \Gamma^0(m))_{\mathbb{C}}^{\text{an}} = X_1(4, m)_{\mathbb{C}}^{\text{an}}.$$

For $N \geq 1$ we define two maps

$$\pi_i : Y_1(4N, m) \longrightarrow Y_1(4N)$$

by the following two transformations of moduli functors

$$\begin{aligned} \pi_1 : (E, P, C) &\longmapsto (E, P) \\ \pi_2 : (E, P, C) &\longmapsto (E/C, P/C). \end{aligned}$$

We remark that in the complex-analytic uniformizations of these curves given in Section 2, these maps are given by $\pi_1(\tau) = \tau$ and $\pi_2(\tau) = \tau/m$.

PROPOSITION 5.1. — *The divisor of Θ_m as a meromorphic function on $X(\Gamma_1(4) \cap \Gamma^1(m))_{\mathbb{C}}^{\text{an}}$ is*

$$\text{div}(\Theta_m) = (\pi_2 \circ d)^* \Sigma_{4,1} - (\pi_1 \circ d)^* \Sigma_{4,1}$$

Proof. — It is tempting to deduce this directly from Table (3.1) and the “formula” $\Theta_m = (\pi_2 \circ d)^* \theta / (\pi_1 \circ d)^* \theta$. The problem is that θ has not been realized as the section of any bundle. One way around this is to raise both sides to the fourth power and use the fact that

$$\theta^4 \in H^0(X_1(4)_{\mathbb{C}}^{\text{an}}, \omega^2) = M_2(\Gamma_1(4), \mathbb{C}).$$

We have

$$\Theta_m^4(\tau) = \frac{\theta(\tau/m)^4}{\theta(\tau)^4},$$

so that

$$\Theta_m^4 = (\pi_2 \circ d)^* \theta^4 \otimes ((\pi_1 \circ d)^* \theta^4)^{-1}$$

where we have used the canonical identification

$$\mathcal{O} = \omega^2 \otimes (\omega^2)^{-1}.$$

It follows that

$$\begin{aligned} 4\text{div}_{\mathcal{O}} \Theta_m &= \text{div}_{\mathcal{O}} \Theta_m^4 \\ &= \text{div}_{\omega^2} (\pi_2 \circ d)^* \theta^4 - \text{div}_{\omega^2} (\pi_1 \circ d)^* \theta^4 \\ &= (\pi_2 \circ d)^* \text{div}_{\omega^2} \theta^4 - (\pi_1 \circ d)^* \text{div}_{\omega^2} \theta^4. \end{aligned}$$

The desired result will then follow from the fact that $\text{div}_{\omega^2} \theta^4 = 4\Sigma_{4,1}$ upon division by 4. This fact is easily read off from Table (3.3). \square

Since $\text{gcd}(4, m) = 1$, the groups $\Gamma_1(4) \cap \Gamma^1(m)$ and $\Gamma_1(4m)$ are conjugate, and the curves $X(\Gamma_1(4) \cap \Gamma^1(m))_{\mathbb{C}}^{\text{an}}$ and $X_1(4m)_{\mathbb{C}}^{\text{an}}$ are isomorphic. One way to realize this isomorphism explicitly is via the moduli interpretation of the latter curve and the map

$$(5.2) \quad X(\Gamma_1(4) \cap \Gamma^1(m))_{\mathbb{C}}^{\text{an}} \longrightarrow X_1(4m)_{\mathbb{C}}^{\text{an}}$$

given (on noncuspidal points) by

$$\tau \longmapsto (E_{\tau}, P)$$

where P is the unique point on E_{τ} with $mP = 1/4$ and $4P = \tau/m$. Translating the natural map d into these terms one arrives at the map

$$d' : X_1(4m)_{\mathbb{C}}^{\text{an}} \rightarrow X_1(4, m)_{\mathbb{C}}^{\text{an}}$$

given on noncuspidal points by

$$(E, P) \longmapsto (E, mP, \langle 4P \rangle).$$

We denote the pullback of Θ_m through the (inverse of) the isomorphism (5.2) by the same name. Its divisor is then

$$\operatorname{div}(\Theta_m) = (\pi_2 \circ d')^* \Sigma_{4,1} - (\pi_1 \circ d')^* \Sigma_{4,1}.$$

Recall that the divisor $\Sigma_{4N,k}$ is defined more generally on the algebraic curve $X_1(4N)_R$ for any R in which $4N$ is invertible. Applying GAGA to Θ_m thought of as a section of

$$\mathcal{O}_{X_1(4m)_{\mathbb{C}}^{\text{an}}}((\pi_2 \circ d')^* \Sigma_{4,1} - (\pi_1 \circ d')^* \Sigma_{4,1}),$$

we see that Θ_m comes from a section (also denoted Θ_m) of the corresponding sheaf on the algebraic curve $X_1(4m)_{\mathbb{C}}$. Repeating this argument with Θ_m^{-1} as a section of

$$\mathcal{O}_{X_1(4m)_{\mathbb{C}}^{\text{an}}}((\pi_1 \circ d')^* \Sigma_{4,1} - (\pi_2 \circ d')^* \Sigma_{4,1})$$

furnishes the opposite inequality of the divisor of Θ_m and shows that Θ_m is a rational function on the algebraic curve $X_1(4m)_{\mathbb{C}}$ with

$$(5.3) \quad \operatorname{div}(\Theta_m) = (\pi_2 \circ d')^* \Sigma_{4,1} - (\pi_1 \circ d')^* \Sigma_{4,1}.$$

Finally, we note that, as in the analytic case, Θ_m descends through the map

$$d' : X_1(4m) \longrightarrow X_1(4, m)$$

(given by the same transformation of moduli functors as above) exactly when m is a square. To see this, note that the group

$$(\mathbb{Z}/4m\mathbb{Z})^{\times} \cong (\mathbb{Z}/4\mathbb{Z})^{\times} \times (\mathbb{Z}/m\mathbb{Z})^{\times}$$

acts on the curve $X_1(4m)_{\mathbb{C}}$ in the usual way by scaling the point of order $4m$. One sees by passage to the analytic world that the action of $(\mathbb{Z}/m\mathbb{Z})^{\times}$ on Θ_m by pullback is again given by the character (m/\cdot) .

In the next section, we will need a few q -expansions of Θ_m in terms of Tate curves. To compute these expansions, one must translate this data into complex-analytic data and compute with standard transformation formulas for θ . To do this, we proceed as follows. Let \mathcal{R} denote the subring $\mathbb{C}((q_{4m}))$ consisting of q -expansions of holomorphic functions in the upper half-plane \mathbb{H} which are invariant under translation by $4m$, and for $\tau \in \mathbb{H}$ let ϕ_{τ} denote the map

$$\begin{aligned} \mathcal{R} &\longrightarrow \mathbb{C} \\ \sum a_n q_{4m}^n &\longmapsto \sum a_n e^{\frac{2\pi i n \tau}{4m}}. \end{aligned}$$

For any point P of order $4m$ on $\underline{\text{Tate}}(q)$, the pair $(\underline{\text{Tate}}(q), P)$ is defined over $\mathbb{Z}[1/(4m), \zeta_4, \zeta_m]((q_{4m}))$, and if we pick an embedding of the coefficient ring

$$\iota : \mathbb{Z}[1/(4m), \zeta_4, \zeta_m] \hookrightarrow \mathbb{C}$$

and base change $(\underline{\text{Tate}}(q), P)$ to $\mathbb{C}((q_{4m}))$ accordingly, the result is actually defined over \mathcal{R} (simply look at the defining equations of $\underline{\text{Tate}}(q)$). Moreover, if we then base change to \mathbb{C} via the map ϕ_τ , we arrive at a pair (E_τ, P') for some point P' of order $4m$ on E_τ . Thus

$$\phi_\tau(\iota(\Theta_m(\underline{\text{Tate}}(q), P))) = \Theta_m(E_\tau, P') = \Theta_m(\gamma\tau)$$

where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$$

is chosen so that the isomorphism

$$\begin{array}{ccc} E_\tau & \xrightarrow{\sim} & E_{\gamma\tau} \\ z & \mapsto & \frac{z}{c\tau + d} \end{array}$$

carries the point P' to the unique point P'' on $E_{\gamma\tau}$ such that $4P'' = \gamma\tau/m$ and $mP'' = 1/4$. Now $\Theta_m(\gamma\tau)$ can be worked out explicitly from the definition of Θ_m , and the desired expansion is easily read off.

We illustrate this technique by computing one of these q -expansions. Fix a primitive 4^{th} root of 1, ζ_4 . If ζ_m is any m^{th} root of 1 (not necessarily primitive), and P is the unique point of order $4m$ with $4P = \zeta_m q_m$ and $mP = \zeta_4$, then we claim that

$$\Theta_m(\underline{\text{Tate}}(q), P) = \frac{\sum_{n \in \mathbb{Z}} \zeta_m^{n^2} q_m^{n^2}}{\sum_{n \in \mathbb{Z}} q^{n^2}}.$$

We specify the embedding ι by sending ζ_4 to i and ζ_m to $e^{2\pi i k/m}$ for some integer k . Thus

$$\phi_\tau(\iota(\Theta_m(\underline{\text{Tate}}(q), P))) = \Theta_m(E_\tau, P')$$

where P' is the unique point on E_τ such that $4P' = (k + \tau)/m$ and $mP' = 1/4$. One then notes that matrix

$$\gamma = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

has the desired property. That is, that the isomorphism

$$\begin{array}{ccc} E_\tau & \xrightarrow{\sim} & E_{\tau+k} \\ z & \mapsto & z \end{array}$$

sends P' to the unique point on $E_{\tau+k}$ with $4P' = (k + \tau)/m = \gamma\tau/m$ and $mP' = 1/4$. Thus we are lead to

$$\Theta_m(\tau + k) = \frac{\theta(\frac{\tau+k}{m})}{\theta(\tau + k)} = \frac{\sum_{n \in \mathbb{Z}} e^{2\pi i n^2 \frac{\tau+k}{m}}}{\sum_{n \in \mathbb{Z}} e^{2\pi i n^2 (\tau+k)}} = \frac{\sum_{n \in \mathbb{Z}} (e^{2\pi i k/m})^{n^2} (e^{2\pi i \tau/m})^{n^2}}{\sum_{n \in \mathbb{Z}} (e^{2\pi i \tau})^{n^2}}.$$

This implies that the q -expansion is as claimed, by the definitions of ι and ϕ_τ .

Note in particular that this q -expansion with $\zeta_m = 1$ corresponds to a $\mathbb{Z}[1/(4m)]$ -rational cusp and has rational coefficients. By the q -expansion principle (in weight zero), we see that Θ_m is in fact defined over $\mathbb{Z}[1/(4m)]$. That is, Θ_m extends to a rational function on $X_1(4m)_{\mathbb{Z}[1/(4m)]}$ with divisor given by (5.3).

In the special case $m = l^2$ with l and odd prime, we have two more q -expansions. The procedure used to derive these is exactly as illustrated above. We omit the details for brevity. Recall that Θ_{l^2} descends to $X_1(4, l^2)$ in this case. Let ζ_{l^2} be a primitive $(l^2)^{\text{th}}$ root of unity. Then we have

$$\Theta_{l^2}(\text{Tate}(q), \zeta_4, \langle \zeta_{l^2} \rangle) = l \frac{\sum_{n \in \mathbb{Z}} q^{l^2 n^2}}{\sum_{n \in \mathbb{Z}} q^{n^2}}$$

and

$$\Theta_{l^2}(\text{Tate}(q), \zeta_4, \langle \zeta_{l^2} q_l \rangle) = \left(\frac{-1}{l} \right) \mathfrak{g}_l(\zeta_{l^2}^l) \frac{\sum_{n \in \mathbb{Z}} \zeta_{l^2}^{ln^2} q^{n^2}}{\sum_{n \in \mathbb{Z}} q^{n^2}}$$

Finally, we will need one more expansion in the special case $m = l$, an odd prime. Let ζ_l be a primitive l^{th} root of unity, and P be the unique point of order $4l$ with $4P = \zeta_l$ and $lP = \zeta_4$, then

$$\Theta_l(\text{Tate}(q), P) = \mathfrak{g}_l(\zeta_l) \frac{\sum_{n \in \mathbb{Z}} q^{ln^2}}{\sum_{n \in \mathbb{Z}} q^{n^2}}.$$

6. Hecke Operators

As explained in the Introduction, in order to construct Hecke operators on the space $M_{k/2}(4N, R)$, we look for rational functions Θ on $X_1(4N, m)$ with the property that

$$\text{div}(\Theta) = \pi_2^* \Sigma_{4N,k} - \pi_1^* \Sigma_{4N,k}.$$

The following theorem is the manifestation in this formalism of the classical fact that in half-integral weight, good Hecke operators occur only for square indices.

THEOREM 6.1. — *Let m be a positive integer with $\gcd(4N, m) = 1$, and let R be a ring in which $4Nm$ is invertible. Then there exists a rational function Θ on $X_1(4N, m)_R$ with*

$$\operatorname{div}(\Theta) = \pi_2^* \Sigma_{4N, k} - \pi_1^* \Sigma_{4N, k}$$

if and only if m is a square. In case m is a square, such a Θ is unique up to multiplication by a constant in R^\times .

Proof. — The uniqueness statement is clear. Suppose that Θ has the specified divisor. Then $d'^* \Theta$ is a rational function on $X_1(4Nm)$ with divisor

$$d'^*(\pi_2^* \Sigma_{4N, k} - \pi_1^* \Sigma_{4N, k}) = (\pi_2 \circ d')^* \Sigma_{4N, k} - (\pi_1 \circ d')^* \Sigma_{4N, k}.$$

By Proposition 5.1 and the compatibility of $\Sigma_{4N, k}$ with pullback through the standard map

$$X_1(4Nm) \longrightarrow X_1(4m),$$

this is the divisor of Θ_m^k on $X_1(4Nm)$. Thus Θ and Θ_m^k agree up to a constant, and the “only if” statement follows from the fact that Θ_m^k descends through d' if and only if m is a square (note that we are using that k is odd here, as we must). The “if” statement follows simply because Θ_m^k furnishes a function on $X_1(4N, m)$ with the desired divisor when m is a square. \square

Let l be a prime not dividing $4N$ and let R be a ring in which $4Nl$ is invertible. We define T_{l^2} to be the endomorphism of $M_{k/2}(4N, R)$ defined by

$$F \longmapsto \frac{1}{l^2} \pi_{1*}(\pi_2^* F \cdot \Theta_{l^2}^k).$$

Note that this endomorphism preserves the subspaces $M_{k/2}(4N, R, \chi)$ for each character χ . The following result says that this definition agrees with the classical one, as in ([8]).

THEOREM 6.2. — *Fix a primitive $(4N)^{\text{th}}$ root of unity ζ_{4N} and let ∞ denote the corresponding cusp. Let $F \in M_{k/2}(4N, R, \chi)$ have q -expansion $\sum a_n q^n$ at ∞ . Then the modular form $T_{l^2} F$ has q -expansion $\sum b_n q^n$ at ∞ where*

$$b_n = a_{l^2 n} + \chi(l) \left(\frac{-1}{l}\right)^{\frac{k-1}{2}} l^{\frac{k-1}{2}-1} \binom{n}{l} a_n + \chi(l^2) l^{k-2} a_{n/l^2}.$$

Proof. — We are given that

$$F(\underline{\text{Tate}}(q), \zeta_{4N}) \cdot \left(\sum_{n \in \mathbb{Z}} q^{n^2}\right)^k = \sum a_n q^n.$$

The quantity

$$\pi_{1*}(\pi_2^* F \cdot \Theta_{l^2}^k)(\underline{\text{Tate}}(q), \zeta_{4N})$$

is a sum over the cyclic subgroups of order l^2 on $\underline{\text{Tate}}(q)$. These subgroups are of three types, and we shall compute the corresponding contributions to the above sum one by one. Choose a primitive root ζ_{l^2} as above.

Firstly, there is the lone subgroup $\langle \zeta_{l^2} \rangle$. Passing to the quotient we find

$$(\underline{\text{Tate}}(q)/\langle \zeta_{l^2} \rangle, \zeta_{4N}/\langle l^2 \rangle) \cong (\underline{\text{Tate}}(q^{l^2}), \zeta_{4N}^{l^2})$$

so that the contribution of this subgroup to the sum is

$$\begin{aligned} F(\underline{\text{Tate}}(q^{l^2}), \zeta_{4N}^{l^2}) \Theta_{l^2}(\underline{\text{Tate}}(q), \zeta_{4N}, \langle \zeta_{l^2} \rangle)^k &= F(\underline{\text{Tate}}(q^{l^2}), \zeta_{4N}^{l^2}) \left(l \frac{\sum_{n \in \mathbb{Z}} q^{l^2 n^2}}{\sum_{n \in \mathbb{Z}} q^{n^2}} \right)^k \\ &= \chi(l^2) l^k \frac{\sum_{n \in \mathbb{Z}} a_n q^{l^2 n^2}}{(\sum_{n \in \mathbb{Z}} q^{n^2})^k} \end{aligned}$$

where we have used the q -expansion formulas for Θ_{l^2} from the previous section.

Secondly, there are the subgroups $\langle \zeta_{l^2}^i q_{l^2} \rangle$ for $0 \leq i \leq l^2 - 1$. Note that

$$(\underline{\text{Tate}}(q)/\langle \zeta_{l^2}^i q_{l^2} \rangle, \zeta_{4N}/\langle \zeta_{4N}^i q_{l^2} \rangle) \cong (\underline{\text{Tate}}(\zeta_{4N}^i q_{l^2}), \zeta_{4N}),$$

so that these subgroups collectively contribute

$$\begin{aligned} \sum_{i=0}^{l^2-1} F(\underline{\text{Tate}}(\zeta_{l^2}^i q_{l^2}), \zeta_{4N}) \Theta_{l^2}(\underline{\text{Tate}}(q), \zeta_{4N}, \langle \zeta_{l^2}^i q_{l^2} \rangle)^k &= \sum_{i=0}^{l^2-1} F(\underline{\text{Tate}}(\zeta_{l^2}^i q_{l^2}), \zeta_{4N}) \left(\frac{\sum_{n \in \mathbb{Z}} \zeta_{l^2}^{in^2} q_{l^2}^{n^2}}{\sum_{n \in \mathbb{Z}} q^{n^2}} \right)^k \\ &= \sum_{i=0}^{l^2-1} \frac{\sum a_n (\zeta_{l^2}^i q_{l^2})^n}{(\sum_{n \in \mathbb{Z}} q^{n^2})^k} = l^2 \frac{\sum a_{l^2 n} q^n}{(\sum_{n \in \mathbb{Z}} q^{n^2})^k}. \end{aligned}$$

Lastly, there are the subgroups $\langle \zeta_{l^2}^j q_l \rangle$ for $1 \leq j \leq l - 1$. Note that

$$(\underline{\text{Tate}}(q)/\langle \zeta_{l^2}^j q_l \rangle, \zeta_{4N}/\langle \zeta_{l^2}^j q_l \rangle) \cong (\underline{\text{Tate}}(\zeta_l^j q), \zeta_{4N}^l).$$

Thus the collective contribution of these terms is

$$\begin{aligned} \sum_{j=1}^{l-1} F(\underline{\text{Tate}}(\zeta_l^j q), \zeta_{4N}^l) \Theta_{l^2}(\underline{\text{Tate}}(q), \zeta_{4N}, \langle \zeta_{l^2}^j q_l \rangle)^k &= \sum_{j=1}^{l-1} F(\underline{\text{Tate}}(\zeta_l^j q), \zeta_{4N}^l) \left(\left(\frac{-1}{l} \right) \mathfrak{g}_l(\zeta_l^j) \frac{\sum_{n \in \mathbb{Z}} \zeta_l^{jn^2} q^{n^2}}{\sum_{n \in \mathbb{Z}} q^{n^2}} \right)^k. \end{aligned}$$

Note that

$$\mathfrak{g}_l(\zeta_l^j) = \left(\frac{j}{l} \right) \mathfrak{g}_l(\zeta_l),$$

so that the above continues as

$$\begin{aligned}
 & \chi(l) \left(\frac{-1}{l}\right) \left(\frac{\mathfrak{g}_l(\zeta_l)}{\sum_{n \in \mathbb{Z}} q^{n^2}}\right)^k \sum_{j=1}^{l-1} \left(\frac{j}{l}\right) F(\text{Tate}(\zeta_l q) \cdot \zeta_{4N}) \sum_{n \in \mathbb{Z}} (\zeta_l^j q)^{n^2} \\
 &= \chi(l) \left(\frac{-1}{l}\right) \left(\frac{\mathfrak{g}_l(\zeta_l)}{\sum_{n \in \mathbb{Z}} q^{n^2}}\right)^k \sum_{j=1}^{l-1} \left(\frac{j}{l}\right) \sum a_n (\zeta_l^j q)^n \\
 &= \chi(l) \left(\frac{-1}{l}\right) \left(\frac{\mathfrak{g}_l(\zeta_l)}{\sum_{n \in \mathbb{Z}} q^{n^2}}\right)^k \sum_n a_n \left(\sum_{j=1}^{l-1} \left(\frac{j}{l}\right) \zeta_l^{jn}\right) q^n \\
 &= \chi(l) \left(\frac{-1}{l}\right) \frac{\mathfrak{g}_l(\zeta_l)^{k+1}}{\left(\sum_{n \in \mathbb{Z}} q^{n^2}\right)^k} \sum_n \binom{n}{l} a_n q^n.
 \end{aligned}$$

It is well known that the Gauss sum above squares to

$$\mathfrak{g}_l(\zeta_l)^2 = \left(\frac{-1}{l}\right) l,$$

so we may further continue the above as

$$\chi(l) \left(\frac{-1}{l}\right)^{\frac{k-1}{2}} \frac{l^{\frac{k+1}{2}}}{\left(\sum_{n \in \mathbb{Z}} q^{n^2}\right)^k} \sum_n \binom{n}{l} a_n q^n.$$

Adding these three expressions together and dividing by l^2 we arrive at the desired q -expansion for $T_{l^2}F$. □

Suppose now that l is an odd prime dividing N . Consider the map e defined as follows.

$$\begin{aligned}
 e : X_1(4N, l) &\longrightarrow X_1(4l) \\
 (E, P, C) &\longmapsto (E/C, (N/l)P/C).
 \end{aligned}$$

By chasing degeneracy maps around, one verifies that

$$e^*(\text{div}(\Theta_l^{-1})) = \pi_2^* \Sigma_{4N,1} - \pi_1^* \Sigma_{4N,1},$$

so that we get a well-defined endomorphism of $M_{k/2}(4N, R)$, namely,

$$U_l(F) = \frac{1}{l} \pi_{1*}(\pi_2^* F \cdot e^* \Theta_l^{-k}).$$

THEOREM 6.3. — *Let l be an odd prime dividing N . Fix a primitive $(4N)^{\text{th}}$ root of unity ζ_{4N} and let ∞ denote the corresponding cusp. The endomorphism U_l of $M_{k/2}(4N, R)$ has the effect*

$$\sum a_n q^n \longmapsto \mathfrak{g}_l(\zeta_{4N}^{4N/l}) \sum a_{ln} q^n$$

on q -expansions at ∞ .

Proof. — Note that the collection of subgroups of $\underline{\text{Tate}}(q)$ of order l not meeting the subgroup generated by ζ_{4N} is $\langle \zeta_l^i q_l \rangle$, for $0 \leq i < l - 1$. We compute

$$\begin{aligned} & \pi_{1*}(\pi_2^* F \cdot e^* \Theta_l^{-k})(\underline{\text{Tate}}(q), \zeta_{4N}) \\ &= \sum_{i=0}^{l-1} (\pi_2^* F \cdot e^* \Theta_l^{-k})(\underline{\text{Tate}}(q), \zeta_{4N}, \langle \zeta_l^i q_l \rangle) \\ &= \sum_{i=0}^{l-1} F(\underline{\text{Tate}}(q) / \langle \zeta_l^i q_l \rangle, \zeta_{4N} / \langle \zeta_l^i q_l \rangle) \Theta(\underline{\text{Tate}}(q) / \langle \zeta_l^i q_l \rangle, \zeta_{4N}^{N/l} / \langle \zeta_l^i q_l \rangle)^{-k} \\ &= \sum_{i=0}^{l-1} F(\underline{\text{Tate}}(\zeta_l^i q_l), \zeta_{4N}) \Theta(\underline{\text{Tate}}(\zeta_l^i q_l), \zeta_{4N}^{N/l})^{-k} \\ &= \sum_{i=0}^{l-1} F(\underline{\text{Tate}}(\zeta_l^i q_l), \zeta_{4N}) \left(\mathfrak{g}_l(\zeta_{4N}^{4N/l}) \frac{\sum_{n \in \mathbb{Z}} q^{n^2}}{\sum_{n \in \mathbb{Z}} (\zeta_l^i q_l)^{n^2}} \right)^{-k} \\ &= \mathfrak{g}_l(\zeta_{4N}^{4N/l})^{-k} \sum_{i=0}^{l-1} \frac{\sum a_n (\zeta_l^i q_l)^n}{(\sum_{n \in \mathbb{Z}} q^{n^2})^k} = l \frac{\mathfrak{g}_l(\zeta_{4N}^{4N/l})}{(\sum_{n \in \mathbb{Z}} q^{n^2})^k} \sum a_{ln} q^n \end{aligned}$$

dividing by l we get the desired result. □

Note in particular that the effect of the map U_l on q -expansions depends on which ∞ is chosen, whereas that of its square $(U_l)^2$ does not (since the square of the Gauss sum is independent of the root of unity used). On a related note, U_l (unlike $(U_l)^2$ or T_{l^2}) does not preserve the subspaces $M_{k/2}(4N, R, \chi)$ but rather induces maps

$$U_l : M_{k/2}(4N, R, \chi) \longrightarrow M_{k/2}(4N, R, \chi \cdot (l/\cdot))$$

where (l/\cdot) is the usual Legendre character.

Remark 6.4. — One can also construct operators U_4 and (if N is even), U_2 . Predictably, the operator U_2 multiplies the Nebentypus by $(2/\cdot)$. One can also show that Kohnen’s $+$ -space makes good sense in this more general setting. That is, one can construct a natural operator (closely related to U_4) on the space of forms, of which Kohnen’s $+$ -space is an eigenspace. The details of these constructions are in the author’s thesis ([7]).

7. p -adic modular forms of half-integral weight

In this section we fix a rational prime $p \geq 5$. Let K be a complete subfield of \mathbb{C}_p and let $r \in \mathbb{Q} \cap [0, 1]$. In this section we will define the space

of $M_{k/2}(4N, K, r)$ of r -overconvergent p -adic modular forms of weight $k/2$ and level $4N$, as well as a number of natural operators on this space. For the sake of simplicity of exposition, we restrict ourselves to the case $p \nmid N$.

By our choice of p , the weight $p - 1$ and level 1 Eisenstein series E_{p-1} furnishes a lifting the Hasse invariant to characteristic zero. It is well-known that E_{p-1} has Fourier coefficients in $\mathbb{Q} \cap \mathbb{Z}_p$ and therefore furnishes a section of ω^{p-1} on $X_1(N)_{\mathbb{Z}_p}$ for any $N \geq 4$. Note that this makes sense even in case $N = 4$ because the bundle ω^2 does descend from the moduli stack for the $\Gamma_1(4)$ problem to the coarse moduli scheme $X_1(4)$ (even though ω does not).

For $N \geq 4$, let $X_1(4N)_K^{\text{an}}$ denote the rigid analytic space over \mathbb{Q}_p associated to the curve $X_1(4N)_K$. For any r as above, we may also consider the region in this curve defined by the inequality $|E_{p-1}| \geq r$. This is known to be a connected affinoid subdomain of $X_1(4N)_K^{\text{an}}$, and will be denoted by $X_1(4N)_{\geq r}^{\text{an}}$. Thus, in particular, $X_1(4N)_{\geq 1}^{\text{an}}$ is the “ordinary locus” in $X_1(4N)_K^{\text{an}}$. Similarly, one may consider $X_1(4N, m)_{\geq r}^{\text{an}}$. This space is connected exactly when $\text{gcd}(m, p) = 1$ (recall that we are assuming that $p \nmid 4N$). For these facts and others which we will freely use in the sequel, see [3] and [1].

DEFINITION 7.1. — *The space of p -adic modular forms of weight $k/2$, level $4N$ ($p \nmid N$), and growth condition r over K is*

$$M_{k/2}(4N, K, r) = H^0(X_1(4N)_{\geq r}^{\text{an}}, \Sigma_{4N, k}).$$

Each of the spaces $M_{k/2}(4N, K, r)$ is a (generally infinite-dimensional) Banach space over K . As usual, forms in the spaces above with $r < 1$ are said to be *overconvergent*. The space of all overconvergent forms is accordingly the inductive limit

$$M_{k/2}^\dagger(4N, K) = \lim_{r \rightarrow 1} M_{k/2}(4N, K, r).$$

The curve $\text{Tate}(q)$ can also be thought of as a p -adic analytic family of elliptic curves over the punctured unit disk $0 < |q| < 1$. As usual, in the presence of level $4N$ structure, we instead consider a parameter q_{4N} and define $q = q_{4N}^{4N}$. Then we may consider the family $\text{Tate}(q)$ over the punctured disk $0 < |q_{4N}| < 1$, so that all of the $4N$ -torsion is rational after adjoining the $4N^{\text{th}}$ roots of unity.

If $F \in M_{k/2}(4N, K, r)$ and P is a point of order $4N$ on $\text{Tate}(q)$ then we may evaluate F on the pair $(\text{Tate}(q), P)$ to obtain a meromorphic function on the punctured disk $0 < |q_{4N}| < 1$ which has a Laurent expansion in $\overline{K}((q_{4N}))$. Just as in the algebraic case, we can define the q -expansion of F

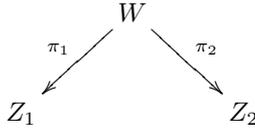
at this pair $(\text{Tate}(q), P)$ be adjusting this expansion by the corresponding expansion of θ from Table 4.1. Thus we find, just as in the algebraic case, that a meromorphic function on $X_1(4N)_{\geq r}^{\text{an}}$ is in $M_{k/2}(4N, K, r)$ if and only if it is analytic away from the cusps and all of these q -expansions are in $\overline{K}[[q_{4N}]]$.

8. p -adic Hecke operators

Our aim is to define a Hecke action on the spaces $M_{k/2}(4N, K, r)$. The situation is rather different depending on whether or not we are dealing with the Hecke operator “at p ”, but in all cases we use a variant of the “pull-back, multiply by a unit, and push forward” construction above relative to the usual correspondence

$$\pi_1, \pi_2 : X_1(4N, m)_K^{\text{an}} \rightrightarrows X_1(4N)_K^{\text{an}}.$$

Suppose Z_1 and Z_2 are admissible opens in $X_1(4N)_K^{\text{an}}$ and W is an admissible open in $X_1(4N, m)_K^{\text{an}}$ such that the π_i restrict to a pair of maps



Suppose further that $\pi_1^{-1}(Z_1) \subseteq \pi_2^{-1}(Z_2)$ and Θ is a meromorphic function on W with

$$\text{div}(\Theta) \geq \pi_2^* \Sigma_{4N, k} - \pi_1^* \Sigma_{4N, k}.$$

These data furnish us with a map

$$H^0(Z_2, \Sigma_{4N, k}) \longrightarrow H^0(Z_1, \Sigma_{4N, k})$$

defined as the composition

$$\begin{array}{c} H^0(Z_2, \Sigma_{4N, k}) \xrightarrow{\pi_2^*} H^0(\pi_2^{-1}(Z_2), \pi_2^* \Sigma_{4N, k}) \xrightarrow{\cdot \Theta} H^0(\pi_2^{-1}(Z_2), \pi_1^* \Sigma_{4N, k}) \\ \xrightarrow{\text{restriction}} H^0(\pi_1^{-1}(Z_1), \pi_1^* \Sigma_{4N, k}) \xrightarrow{\pi_{1*}} H^0(Z_1, \Sigma_{4N, k}) \end{array}$$

where the long arrow is restriction.

This setup suffices for the case $m = l^2$ with $\text{gcd}(l, 4Np) = 1$. Then we may take $Z_1 = Z_2 = X_1(4N)_{\geq r}^{\text{an}}$, $W = X_1(4N, l^2)_{\geq r}^{\text{an}}$, and $\Theta = \Theta_{l^2}^k$. The two conditions that must be satisfied, namely that the π_i restrict as above

and that $\pi_1(Z_1) \subseteq \pi_2^{-1}(Z_2)$, together are equivalent to the assertion that if C is a cyclic subgroup of E of order l^2 , then $|E_{p-1}(E)| \geq r$ if and only if $|E_{p-1}(E/C)| \geq r$. Of course, it suffices to verify the “only if” part since we may then apply the result to the pair $(E/C, E[l^2]/C)$ to verify the other implication. At any rate, the result is well known and holds for cyclic subgroups of all orders prime to p .

THEOREM 8.1. — *There exists a continuous endomorphism T_{l^2} of $M_{k/2}(4N, K, \chi, r)$ having the effect*

$$\sum a_n q^n \longmapsto \sum b_n q^n$$

where

$$b_n = a_{l^2 n} + \chi(l) \left(\frac{-1}{l}\right)^{\frac{k-1}{2}} l^{\frac{k-1}{2}-1} \binom{n}{l} a_n + \chi(l^2) l^{k-2} a_{n/l^2}$$

on q -expansions at (any choice of) ∞ .

Proof. — Dividing the endomorphism of $M_{k/2}(4N, K, r)$ obtained from the above choice of Θ by l^2 we arrive at a new endomorphism denoted T_{l^2} . The effect of T_{l^2} on q -expansions at ∞ is exactly as in the algebraic case, as the same computation verifies.

We claim that $\|T_{l^2}\| \leq 1$, so that, in particular, T_{l^2} is continuous. That the norms of π_2^* and π_{1*} are at most one are generalities. It therefore suffices to show that multiplication by $\Theta_{l^2}^k$ as a map

$$H^0(\pi_2^{-1}(Z_2), \pi_2^* \Sigma_{4N,k}) \longrightarrow H^0(\pi_2^{-1}(Z_2), \pi_1^* \Sigma_{4N,k})$$

is bounded by 1.

This follows immediately from the fact that $\Theta_{l^2}^k$ is an *integral* section of the sheaf

$$\mathcal{O}_{X_1(4N, l^2)}(\pi_2^* \Sigma_{4N,k} - \pi_1^* \Sigma_{4N,k})$$

on the algebraic curve $X_1(4N, l^2)_K$ (i.e., that it extends to a section over all of $X_1(4N, l^2)_{\mathcal{O}_K}$) as the q -expansion principle readily demonstrates. \square

The more interesting case to consider is that of $m = p^2$. If $r > p^{-1/p(1+p)}$, then the existence of the canonical subgroup of order p^2 (see [4],[1]) furnishes a map

$$X_1(4N)_{\geq r}^{\text{an}} \longrightarrow X_1(4N, p^2)_{\geq r}^{\text{an}}$$

which is an isomorphism onto the connected component of right-hand space containing the cusp associated to $(\text{Tate}(q), \zeta_{4N}, \mu_{p^2})$ (for any primitive choice of ζ_{4N}), the inverse being π_1 . We will denote this component by

$W_{\geq r}$. Standard results on quotienting by canonical subgroups show that, for r as above, π_2 restricts to a map

$$\pi_2 : W_{\geq r} \longrightarrow X_1(4N)_{\geq rp^2}^{\text{an}}.$$

We will denote the restrictions of the various modular units and divisors on $X_1(4N, p^2)_K^{\text{an}}$ to $W_{\geq r}$ by their usual names. Then we have, as usual, that

$$\text{div}(\Theta_{p^2}^{-k}) = \pi_1^* \Sigma_{4N, k} - \pi_2^* \Sigma_{4N, k}.$$

Thus we have a well-defined map

$$\begin{aligned} M_{k/2}(4N, K, r) &\longrightarrow M_{k/2}(4N, K, rp^2) \\ F &\longmapsto \langle p^2 \rangle^* \pi_{2*}(\pi_1^* F \cdot (p\Theta_{p^2})^{-k}) \end{aligned}$$

Let us determine the effect of this map on q -expansions. To compute the trace π_{2*} we note that the fiber of π_2 above the pair (E, P) consists of all (isomorphism classes of) triple (E', P', C) such that C is the canonical subgroup of E' of order p^2 , and $(E'/C, P'/C) \cong (E, P)$. This is the same as the set of triples $(E/C', mP/C', E[p^2]/C')$ where C' runs through all order p^2 cyclic subgroups of E which have trivial intersection with its canonical subgroup of order p^2 (i.e. that do not contain the order p canonical subgroup of E), and m is an inverse of $p^2 \pmod{4N}$. So we may compute

$$\begin{aligned} &(\langle p^2 \rangle^* \pi_{2*}(\pi_1^* \langle p^2 \rangle^* F \cdot (p\Theta_{p^2})^{-k}))(\underline{\text{Tate}}(q), \zeta_{4N}) \\ &= \sum_{i=0}^{p^2-1} (\pi_1^* F \cdot (p\Theta_{p^2})^{-k})(\underline{\text{Tate}}(q) / \langle \zeta_{p^2}^i q_{p^2} \rangle, \zeta_{4N} / \langle \zeta_{p^2}^i q_{p^2} \rangle, \\ &\hspace{15em} \underline{\text{Tate}}(q)[p^2] / \langle \zeta_{p^2}^i q_{p^2} \rangle) \\ &= \sum_{i=0}^{p^2-1} F(\underline{\text{Tate}}(\zeta_{p^2}^i q_{p^2}), \zeta_{4N})(p\Theta_{p^2}(\underline{\text{Tate}}(\zeta_{p^2}^i q_{p^2}), \zeta_{4N}, \mu_{p^2}))^{-k} \\ &= \sum_{i=0}^{p^2-1} F(\underline{\text{Tate}}(\zeta_{p^2}^i q_{p^2}), \zeta_{4N}) \left(\frac{\sum_{n \in \mathbb{Z}} q^{n^2}}{\sum_{n \in \mathbb{Z}} (\zeta_{p^2}^i q_{p^2})^{n^2}} \right)^{-k} \\ &= p^2 \frac{\sum a_{p^2 n} q^n}{(\sum_{n \in \mathbb{Z}} q^{n^2})^k}, \end{aligned}$$

where $\sum a_n q^n$ is the q -expansion of F at $(\underline{\text{Tate}}(q), \zeta_{4N})$.

THEOREM 8.2. — *Suppose that $p^{-1/p(1+p)} < r$. Then there exists a continuous endomorphism U_{p^2} of $M_{k/2}(4N, K, r)$ of norm at most p^2 having the effect*

$$\sum a_n q^n \longmapsto \sum a_{p^2 n} q^n$$

on q -expansions at (any choice of) ∞ . Moreover, in case $r < 1$, U_{p^2} is in fact completely continuous.

Proof. — The endomorphism we seek is the map

$$M_{k/2}(4N, K, r) \longrightarrow M_{k/2}(4N, K, r^{p^2})$$

defined above divided by p^2 and post composed with the natural inclusion

$$M_{k/2}(4N, K, r^{p^2}) \subseteq M_{k/2}(4N, K, r).$$

As this latter map is completely continuous if $r < 1$, it suffices to show that the map

$$M_{k/2}(4N, K, r) \longrightarrow M_{k/2}(4N, K, r^{p^2})$$

is continuous. As in the case for T_{l^2} above, it suffices to see that multiplication by the unit $(p\Theta_{p^2})^{-k}$ is bounded as a map

$$H^0(W_{\geq r}, \pi_1^* \Sigma_{4N,k}) \longrightarrow H^0(W_{\geq r}, \pi_2^* \Sigma_{4N,k}).$$

This is more subtle than the case of T_{l^2} because of the bad reduction of $X_1(4N, p^2)$.

Consider the modular unit $(p\Theta_{p^2})^{-1}$ on the algebraic curve $X_1(4N, p^2)_K$, and let $X_1(4N, p^2)_{\mathcal{O}_K}$ denote the Katz-Mazur model of this curve over the ring of integers \mathcal{O}_K in K . We claim that $(p\Theta_{p^2})^{-1}$ extends to all of $X_1(4N, p^2)_{\mathcal{O}_K}$ as a section of the sheaf

$$\mathcal{O}_{X_1(4N, p^2)_{\mathcal{O}_K}}(\pi_1^* \Sigma_{4N,1} - \pi_2^* \Sigma_{4N,1}).$$

Since $X_1(4N, p^2)_{\mathcal{O}_K}$ is Cohen-Macaulay over \mathcal{O}_K it suffices by the q -expansion principle to show that $(p\Theta_{p^2})^{-1}$ has integral q -expansions at a collection of cusps which, together, meet all three components of the special fiber of $X_1(4N, p^2)$ in characteristic p . In particular, we fix primitive roots ζ_{4N} and ζ_{p^2} and take the three cusps corresponding to $(\underline{\text{Tate}}(q), \zeta_{4N}, \langle \zeta_{p^2} \rangle)$, $(\underline{\text{Tate}}(q), \zeta_{4N}, \langle \zeta_{p^2} q_p \rangle)$, and $(\underline{\text{Tate}}(q), \zeta_{4N}, \langle q_{p^2} \rangle)$. That the indicated q -expansions are integral follows readily from the results of Section 5.

It follows that multiplication by $(p\Theta_{p^2})^{-k}$ is of norm at most 1, and therefore continuous. Taking into account the division by p^2 , we see that $\|U_{p^2}\| \leq p^2$. □

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