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## STABILITY OF HIGHER ORDER SINGULAR POINTS OF POISSON MANIFOLDS AND LIE ALGEBROIDS

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ABSTRACT. — We study the stability of singular points for smooth Poisson structures as well as general Lie algebroids. We give sufficient conditions for stability lying on the first-order approximation (not necessarily linear) of a given Poisson structure or Lie algebroid at a singular point. The main tools used here are the classical Lichnerowicz-Poisson cohomology and the deformation cohomology for Lie algebroids recently introduced by Crainic and Moerdijk. We also provide several examples of stable singular points of order  $k \geq 1$  for Poisson structures and Lie algebroids. Finally, we apply our results to pre-symplectic leaves of Dirac manifolds.

RÉSUMÉ. — Nous étudions la stabilité des singularités de structures de Poisson lisses et des algèbroïdes de Lie générales. Nous donnons des conditions suffisantes de stabilité reposant sur la première approximation (pas nécessairement linéaire) d'une structure de Poisson ou d'algèbroïde de Lie en un point singulier. Les principaux outils utilisés ici sont la cohomologie de Lichnerowicz-Poisson classique et la cohomologie de déformation introduite récemment par Crainic et Moerdijk. De plus, nous fournissons plusieurs exemples de points singuliers stables d'ordre  $k \geq 1$  pour des structures de Poisson et des algèbroïdes de Lie. Finalement, nous appliquons nos résultats aux feuilles pré-symplectiques des variétés de Dirac.

### 1. Introduction and main results

Every bivector field on a smooth finite-dimensional manifold  $M$  will be considered as a map  $\Lambda : M \rightarrow \bigwedge^2 TM$ . We endow the set  $\mathcal{V}_2(M)$  of all smooth bivector fields with the  $\mathcal{C}^s$ -topology, i.e., the topology of uniform convergence on compact sets for  $\mathcal{C}^s$ -differentiable maps and their derivatives up to order  $s$ . The integer  $s$  will be made more precise later on.

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A smooth Poisson structure on  $M$  is a bivector field  $\pi$  that satisfies the equation  $[\pi, \pi] = 0$ , where  $[\ , \ ]$  is the Schouten-Nijenhuis bracket on multi-vector fields (see [13]). A *singular point* of  $\pi$  is a point  $m \in M$  satisfying  $\pi(m) = 0$ . Such a point  $m$  is said to be *stable* if every Poisson structure  $\tilde{\pi}$ , which is close enough to  $\pi$ , has a singular point near  $m$ .

The following result, due to M. Crainic and R.-L. Fernandès, was announced at the 4th international conference on Poisson geometry (held in Luxembourg, June 2004):

**THEOREM 1.1** ([4]). — *Let  $\mathfrak{g}$  be the Lie algebra corresponding to the 1-jet of  $\pi$  at a singular point  $m_0$ . If the second scalar cohomology group of  $\mathfrak{g}$  vanishes then  $m_0$  is stable. Conversely, suppose  $\mathfrak{g}$  is a Lie algebra such that, every singular point of a degenerate Poisson structure whose 1-jet at that point corresponds to  $\mathfrak{g}$  is stable, then  $H^2(\mathfrak{g}, \mathbb{R}) = \{0\}$ .*

After a quick reading of this result, one may think that  $H^2(\mathfrak{g}, \mathbb{R}) = \{0\}$  is a necessary and sufficient condition for the stability of a singular point of a Poisson structure. However, a *curious phenomenon* occurs: there are plenty of stable singular points for Poisson structures having a zero 1-jet, hence  $H^2(\mathfrak{g}, \mathbb{R}) \neq 0$ . In Section 2, we will establish a stability criterion for general singularities which improves the one given by Crainic and Fernandès.

First, we recall some definitions. Two smooth maps from  $M$  to a manifold  $N$  have the same  $k$ -jet at a point  $m \in M$  if they coincide at  $m$  and the derivatives of their local expressions, relative to any local coordinate system, agree at that point up to order  $k$  ( $k \geq 0$ ). We denote by  $J_{\Phi}^k(m)$  the  $k$ -jet at  $m$  of a function  $\Phi \in C^\infty(M, N)$ . Precisely,  $J_{\Phi}^k(m)$  is the equivalence class of  $\Phi$  under the equivalence relation “to have the same  $k$ -jet at  $m$ ”.

A *singularity of order  $k$*  for a Poisson structure  $\pi$  on  $M$  is a point  $m \in M$ , where the  $(k - 1)$ -jet of  $\pi$  vanishes but not its  $k$ -jet ( $k > 0$ ). We say that such a singularity is  *$k$ -stable* if, for every neighborhood  $\Omega$  of  $m$  in  $M$ , there is a neighborhood  $\mathcal{W}$  of  $\pi$  in  $\mathcal{V}_2(M)$  such that every Poisson structure in  $\mathcal{W}$  has a singularity of order  $k$  in  $\Omega$ .

Associated to any singular point  $m$  of order  $k$  for a Poisson structure  $\pi$ , there is a  $k$ -homogeneous Poisson structure on  $T_m M$  which is determined, up to isomorphism, by the  $k$ th-order terms of the Taylor expansion of  $\pi$  at  $m$ . This new Poisson tensor  $\pi^{(k)}$  will be called the  *$k$ -homogeneous part* of  $\pi$  at  $m$ . We associate to  $\pi^{(k)}$  the *homogeneous Lichnerowicz-Poisson cohomology complex* (see [9]):

$$\mathcal{V}_1^{(s-k+1)}(T_m M) \xrightarrow{\partial_1^{s-k+1}} \mathcal{V}_2^{(s)}(T_m M) \xrightarrow{\partial_2^s} \mathcal{V}_3^{(s+k-1)}(T_m M) \dots$$

where  $\mathcal{V}_r^{(s)}(T_m M)$  is the space of  $s$ -homogeneous  $r$ -vector fields on  $T_m M$  (with  $\mathcal{V}_r^{(s)}(T_m M) = \{0\}$  for  $s < 0$ ), and the operators  $\partial_r^\ell$  are defined by

$$\partial_r^\ell(A) = [\pi^{(k)}, A],$$

for all  $\ell$ -homogeneous multi-vector field  $A$ . The corresponding second cohomology group is

$$H_{\text{LP}}^{2,s}(\pi^{(k)}) = \frac{\text{Ker}(\partial_2^s)}{\text{Im}(\partial_1^{s-k+1})}.$$

Under these notations, we have the following theorem, where  $\mathcal{V}_2(M)$  is equipped with the  $\mathcal{C}^{2k}$ -topology.

**THEOREM 1.2.** — *Let  $m$  be a singularity of order  $k$  for a given Poisson structure whose  $k$ -homogeneous part at  $m$  is  $\pi^{(k)}$ . If  $H_{\text{LP}}^{2,s}(\pi^{(k)}) = \{0\}$ , for any  $s = 0, \dots, k - 1$ , then  $m$  is a  $k$ -stable singularity.*

Our second main result concerns the stability problem for general Lie algebroids. A *Lie algebroid* ([17]) is a triple  $(F, [\ , \ ], \varrho)$  formed by a vector bundle  $F$  over a smooth manifold  $M$  together with a Lie bracket on the space  $\Gamma(F)$  of smooth sections of  $F$ , and a bundle map  $\varrho : F \rightarrow TM$  called *the anchor map* such that

$$[X, fY] = f[X, Y] + (\varrho(X) \cdot f)Y, \quad \forall X, Y \in \Gamma(F), f \in C^\infty(M).$$

The distribution  $\text{Im} \varrho$  induces a singular foliation on  $M$  which is called *the foliation of the Lie algebroid*. A *singular point for a Lie algebroid* is, by definition, a point  $m$  of the base manifold  $M$  where the leaf of the associated foliation reduces to that point.

Our goal is to give a criterion for the stability of these singularities. But first, we have to put a suitable topology on the set of Lie algebroid structures on a given vector bundle. To do so and to present our results, we begin by recalling that there is a one-to-one correspondence between Lie algebroids and some special Poisson structures.

A *fiber-wise linear Poisson structure* on a vector bundle  $p : E \rightarrow M$  is a Poisson structure on  $E$  which has the following properties:

- The bracket of two fiber-wise linear functions is fiber-wise linear.
- The bracket of a fiber-wise linear function and a basic one is basic (every basic function on  $E$  has the form  $f \circ p$ , where  $f$  is a function on  $M$ ).
- The bracket of two basic functions vanishes.

A fiber-wise linear Poisson structure  $\Pi$  has the local form

$$\Pi = \sum_{1 \leq i < j \leq r} f_{i,j}^s(x) y_s \partial y_i \wedge \partial y_j + \sum_{1 \leq i \leq r} \sum_{1 \leq j \leq d} g_{i,j}(x) \partial y_i \wedge \partial x_j,$$

in any system of fibered coordinates  $(x, y)$ , where  $x = (x_1, \dots, x_d)$  is a coordinate system on the base manifold  $M$  and  $y = (y_1, \dots, y_r)$  consists of functions that are linear on the fibers ( $\partial z$  stands for  $\frac{\partial}{\partial z}$ ).

It is well-known that there is a one-to-one correspondence between Lie algebroid structures on a vector bundle  $F$  and fiber-wise linear Poisson structures on the total space of its dual  $E = F^*$  (see for instance [17]). Hereafter, we identify these two concepts: the concept of a Lie algebroid will be sometimes employed instead of that of a fiber-wise linear Poisson structure. In particular, the topology we put on the set of Lie algebroid structures on a given vector bundle  $F$  is the one which corresponds to the  $C^s$ -topology for bivector fields on the total space of the dual vector bundle  $E = F^*$  (the integer  $s$  will be specified later on). A point  $m$  of the base manifold  $M$  is a singular point of a Lie algebroid if and only if  $0_m$ , the origin of the fiber over  $m$ , is a singular point of the associated fiber-wise linear Poisson structure  $\Pi$ . In the above local expression of  $\pi$ , such a singular point corresponds to a point  $x$  with  $g_{i,j}(x) = 0$ .

It makes sense to speak of *singularities of order  $k$  for a Lie algebroid*. By definition, these are singular points  $m$  such that  $0_m$  is a singularity of order  $k$  for  $\Pi$ . The first non-zero terms in the Taylor expansion of  $\Pi$  at  $0_m$  have the local form

$$\Pi^{(k)} = \sum f_{i,j}^{(k-1),\ell}(x) y_\ell \partial y_i \wedge \partial y_j + \sum g_{i,j}^{(k)}(x) \partial y_i \wedge \partial x_j,$$

where  $g_{i,j}^{(k)}$  (resp.  $f_{i,j}^{(k-1),\ell}$ ) are  $k$ -homogeneous (resp.  $(k-1)$ -homogeneous) polynomials. When  $k = 1$  we have

$$\Pi^{(1)} = \sum a_{i,j}^\ell y_\ell \partial y_i \wedge \partial y_j + \sum b_{i,j}^\ell x_\ell \partial y_i \wedge \partial x_j,$$

where  $a_{i,j}^\ell$  are structure constants for a Lie algebra  $\mathfrak{g}$  and  $b_{i,j}^\ell$  are constants. It corresponds to the algebroid associated to a linear action of the Lie algebra  $\mathfrak{g}$  on the tangent space of the base manifold at a given point (see for example [17]).

As for general Poisson structures, the first homogeneous terms of the Taylor expansion at  $0_m$ , denoted  $\Pi^{(k)}$ , determine a  $k$ -homogeneous Lie algebroid. Moreover, we can see that  $\Pi^{(k)}$  is a fiber-wise linear Poisson structure on  $V = T_{0_m}E$ , viewed as a trivial vector bundle with base  $T_mM$  and fiber  $T_{0_m}(p^{-1}(m)) \simeq p^{-1}(m)$ . We say that the algebroid singularity  $m$  is  *$k$ -stable* if, for every neighborhood  $\Omega$  of  $m$  in  $M$ , there is a neighborhood  $\mathcal{W}$  of  $\Pi$  in  $\mathcal{V}_2(E)$  such that every fiber-wise linear Poisson structure in  $\mathcal{W}$  has a singularity of order  $k$  in  $\Omega$ . Even though Lie algebroids can be viewed as specific Poisson structures, the Lichnerowicz-Poisson cohomology used in Theorem 1.2 is not suitable for the study of the stability problem for

Lie algebroids. One needs to use the *deformation cohomology* introduced recently by Crainic and Moerdijk (see [5]). This is also called *the linear fiber-wise Poisson cohomology* for Lie algebroids (see [9]). Hereafter, we will use a local and homogeneous version. We attach to  $\Pi^{(k)}$  the deformation complex

$$\mathcal{V}_{1,lin}^{(s-k+1)}(T_{0_m}E) \xrightarrow{\partial_1^{s-k+1}} \mathcal{V}_{2,lin}^{(s)}(T_{0_m}E) \xrightarrow{\partial_2^s} \mathcal{V}_{3,lin}^{(s+k-1)}(T_{0_m}E) \dots$$

which is by definition the sub-complex of the Lichnerowicz-Poisson complex. Precisely, we replace  $\mathcal{V}_r^{(s)}(T_{0_m}E)$  by its sub-space  $\mathcal{V}_{r,lin}^{(s)}(T_{0_m}E)$  formed by  $s$ -homogeneous  $r$ -vector fields  $K$  which satisfies the following properties:

- $K(df_1 \dots, df_r)$  is fiber-wise linear if  $f_1, \dots, f_r$ , are fiber-wise linear,
- $K(df_1 \dots, df_r)$  is basic if  $f_1, \dots, f_{r-1}$ , are fiber-wise linear and  $f_r$  is basic,
- $K(df_1 \dots, df_r) = 0$  if at least two of the  $f_i$  are basic.

In any local coordinates  $(x, y)$  chosen as above, elements of  $\mathcal{V}_{r,lin}^{(s)}(V)$  have the form

$$\sum f_{i_1 \dots i_r}^{(s-1),u}(x) y_u \partial y_{i_1} \wedge \dots \wedge \partial y_{i_r} + \sum g_{j_1 \dots j_{r-1},t}^{(s)}(x) \partial x_t \wedge \partial y_{j_1} \wedge \dots \wedge \partial y_{j_{r-1}}$$

where  $g_{j_1 \dots j_{r-1},t}^{(s)}$  are  $s$ -homogeneous polynomials and  $f_{i_1 \dots i_r}^{(s-1),u}$  are  $(s - 1)$ -homogeneous polynomials on  $T_m M$ . Denote by  $H_{lin}^{2,s}(\Pi^{(k)})$  the second group of cohomology for the above cohomology complex.

Under these notations we have the following theorem, where  $\mathcal{V}_2(E)$  is equipped with the  $\mathcal{C}^{2k}$ -topology.

**THEOREM 1.3.** — *Let  $m$  be a singularity of order  $k$  for the algebroid  $\Pi$  and let  $\Pi^{(k)}$  be its  $k$ -homogeneous part at  $m$ . If  $H_{lin}^{2,s}(\Pi^{(k)}) = \{0\}$ , for any  $s = 0, \dots, k - 1$ , then  $m$  is a  $k$ -stable algebroid singularity.*

Section 2 contains the proofs of our main results (Theorems 1.2 and 1.3). In Section 3, we will give examples and shows why most of the singularities of higher order are stable (at least in dimension 3). Section 4 gives a method for generalizing the above results to singular leaves (not necessarily reduced to a point).

## 2. Proofs of the main theorems

We will start by establishing Theorem 1.2. First, one can notice the following:

*Remark 2.1.* — The hypothesis of Theorem 1.2 says that  $\partial_2^s$  are one-to-one for  $s = 0, \dots, k - 2$ , and

$$\text{Ker} \partial_2^{k-1} = \text{Im} \partial_1^0.$$

Given a neighborhood  $\Omega$  of  $m$  in  $M$ , we choose a local coordinate system  $(x_1, \dots, x_n)$  defined on an open neighborhood  $U$  of  $m$  that is contained in  $\Omega$ . For any bivector field  $\Lambda$  on  $M$  and for any point  $p$  in  $U$ , we denote by  $\Lambda_p^{(s)}$  the  $s$ -homogeneous terms of the Taylor expansion of  $\Lambda$  at  $p$  in the above coordinates. We can view  $\Lambda_p^{(s)}$  as an element of  $\mathcal{V}_2^{(s)}(\mathbb{R}^n)$ , then the  $(k - 1)$ -jet extension of  $\Lambda$  is given by:

$$\begin{aligned} \mathcal{J}_\Lambda^{k-1} : U &\longrightarrow \mathcal{V}_2^{(0)}(\mathbb{R}^n) \times \dots \times \mathcal{V}_2^{(k-1)}(\mathbb{R}^n) \\ p &\longmapsto (\Lambda_p^{(0)}, \dots, \Lambda_p^{(k-1)}). \end{aligned}$$

Accordingly  $\pi^{(k)} = \pi_m^{(k)}$  may be thought of as an element of  $\mathcal{V}_2^{(k)}(\mathbb{R}^n)$  and the associated linear map  $\partial_1^0$  sends  $\mathcal{V}_1^{(0)}(\mathbb{R}^n)$  into  $\mathcal{V}_2^{(k-1)}(\mathbb{R}^n)$ . From now on, we will use the notation

$$A = \{0\} \times \dots \times \{0\} \times \text{Im}(\partial_1^0)$$

which is a subspace of  $\mathcal{V}_2^{(0)}(\mathbb{R}^n) \times \dots \times \mathcal{V}_2^{(k-1)}(\mathbb{R}^n)$ . Then, one gets

LEMMA 2.2. — *The image of the differential of  $\mathcal{J}_\pi^{k-1}$  at  $m$  coincides with  $A$ .*

The proof of this lemma is straightforward. It is left to the reader.

Given a point  $p$  in  $U$  and a bivector field  $\Lambda$  on  $M$ , we define

$$\mathcal{F}_{p,\Lambda} : \mathcal{V}_2^{(0)}(\mathbb{R}^n) \times \dots \times \mathcal{V}_2^{(k-1)}(\mathbb{R}^n) \longrightarrow \mathcal{V}_3^{(k-1)}(\mathbb{R}^n) \times \dots \times \mathcal{V}_3^{(2k-2)}(\mathbb{R}^n)$$

such that the components of  $\mathcal{F}_{p,\Lambda}(v_0, \dots, v_{k-1})$  are given by

$$\mathcal{F}_{p,\Lambda}^{(\ell-1)}(v_0, \dots, v_{k-1}) = \sum_{i \leq \ell-k} [v_i, \Lambda_p^{(\ell-i)}] + \frac{1}{2} \sum_{\ell-k < i, j \leq k-1, i+j=\ell} [v_i, v_j],$$

for all  $k \leq \ell \leq 2k - 1$ , and where  $[ , ]$  is the Schouten-Nijenhuis bracket. One has:

LEMMA 2.3. — *Under the hypothesis of Theorem 1.2,  $A$  is the kernel of the differential of  $\mathcal{F}_{m,\pi}$  at the origin  $\mathbf{O}$ .*

*Proof.* — When  $k = 1$ , one gets  $\mathcal{F}_{m,\pi}(v_0) = [v_0, \pi^{(1)}]$ . Differentiating, one can easily check that  $\text{Ker}((\mathcal{F}_{m,\pi})_*(\mathbf{O})) = \text{Ker}(\partial_2^0) = \text{Im}(\partial_1^0)$ . Now, assume that  $k > 1$ , then the equation

$$((\mathcal{F}_{m,\pi})_*(\mathbf{O}))(w_0, \dots, w_{k-1}) = 0$$

gives the system of equations

$$\begin{aligned}
 [w_0, \pi^{(k)}] &= 0, & [w_1, \pi^{(k)}] + [w_0, \pi^{(k+1)}] &= 0, \dots, \\
 [w_{k-1}, \pi^{(k)}] + \dots + [w_0, \pi^{(2k-1)}] &= 0
 \end{aligned}$$

The first equation of this system says that  $w_0 \in \ker(\partial_2^0)$ . By Remark 2.1, one gets  $w_0 \in \text{Im}(\partial_1^{1-k})$ . Using the fact that  $\text{Im}(\partial_1^{1-k}) = \{0\}$  for  $k > 1$ , one obtains  $w_0 = 0$ . By an iteration procedure, one gets

$$w_0 = \dots = w_{k-2} = 0 \quad \text{and} \quad \partial_2^{k-1}(w_{k-1}) = 0, \quad \forall k > 1.$$

This shows that the kernel of the differential of  $\mathcal{F}_{m,\pi}$  at  $\mathbf{O}$  is exactly  $A$ .  $\square$

*Proof of Theorem 1.2.* — Once for all, we fix a complement subspace  $B$  of  $A$ , i.e.,

$$\mathcal{V}_2^{(0)}(\mathbb{R}^n) \times \dots \times \mathcal{V}_2^{(k-1)}(\mathbb{R}^n) = A \oplus B.$$

Lemma 2.3 implies that the differential of  $\mathcal{F}_{m,\pi}$  at the origin is one-to-one, when restricted to  $B$ . Applying the implicit function theorem, one gets that  $\mathcal{F}_{m,\pi}$  is an injective immersion when it is restricted to a neighborhood  $\mathcal{B}_0$  of the origin in  $B$ . Note that  $\mathcal{F}_{p,\Lambda}$  depends continuously on  $\Lambda$  in the  $\mathcal{C}^{2k}$ -topology. A classical singularity technique (see Lemma A, p. 61, [11]) implies that there is a neighborhood  $\mathcal{B}_1 \subset \mathcal{B}_0$  of the origin in  $B$ , a neighborhood  $\Omega_1 \subset U$  of  $m$ , and a neighborhood  $\mathcal{W}_1$  of  $\pi$  in  $\mathcal{V}_2(M)$  such that, for every  $p$  in  $\Omega_1$  and for every  $\Lambda$  in  $\mathcal{W}_1$ ,  $\mathcal{F}_{p,\Lambda}$  is one-to-one when restricted to  $\mathcal{B}_1$ .

It follows from Lemma 2.2 that the map  $\mathcal{J}_\pi^{k-1}$  is transversal to  $B$  at the origin. By transversality, there is a neighborhood  $\mathcal{W}_2$  of  $\pi$  in  $\mathcal{V}_2(M)$  such that, for every  $\Lambda$  in  $\mathcal{W}_2$ ,  $\mathcal{J}_\Lambda^{k-1}$  intersects (transversally)  $\mathcal{B}_1$  at a point  $p \in \Omega_1$ . Set  $\mathcal{W} = \mathcal{W}_1 \cap \mathcal{W}_2$ . Now pick a *Poisson structure*  $\Lambda$  in  $\mathcal{W}$ . We know that there is a point  $p$  in  $\Omega_1$  where  $\mathcal{J}_\Lambda^{k-1}(p)$  intersects  $\mathcal{B}_1$ . Considering the equation  $[\Lambda, \Lambda] = 0$ , we can notice that the terms of order  $k - 1, \dots, 2k - 2$  in the Taylor expansion of  $[\Lambda, \Lambda]$  at that point  $p$  give

$$\mathcal{F}_{p,\Lambda}(\mathcal{J}_\Lambda^{k-1}(p)) = 0.$$

The injectivity of  $(\mathcal{F}_{p,\Lambda})|_{\mathcal{B}_1}$  implies

$$\mathcal{J}_\Lambda^{k-1}(p) = 0.$$

If necessary, we may replace  $\mathcal{W}$  by a smaller neighborhood of  $\pi$  to ensure that the  $k$ -jet of  $\Lambda$  at  $p$  is non-trivial. So,  $p$  is the singularity point for  $\Lambda$  we sought. This completes the proof of Theorem 1.2.  $\square$



*Remark 2.4.* — In our study there is no need for the singularity to be isolated. In fact, from the proof of Theorem 1.2, one sees that the set of singularities of order  $k$  for Poisson structures  $\Lambda$  sufficiently close to  $\pi$  is (locally) formed by points  $p$  such that  $\mathcal{J}_\Lambda^{k-1}(p) \in \mathcal{B}_1$ , for each  $k$  fixed. So, this set is, by transversality, a sub-manifold of codimension  $r = \dim(A)$ . One can also notice that this manifold depends continuously on  $\Lambda$ , in a natural sense.

The proof of Theorem 1.3 is a direct adaptation of that of Theorem 1.2. Essentially, one replaces the symbol  $\mathcal{V}_u^{(s)}$  by  $\mathcal{V}_{u,lin}^{(s)}$  and

$$\begin{aligned} \mathcal{J}_\Lambda^{k-1} : U &\longrightarrow \mathcal{V}_2^{(0)}(\mathbb{R}^n) \times \cdots \times \mathcal{V}_2^{(k-1)}(\mathbb{R}^n) \\ p &\longmapsto (\Lambda_p^{(0)}, \dots, \Lambda_p^{(k-1)}) \end{aligned}$$

by

$$\begin{aligned} \mathcal{J}_\Lambda^{k-1} : U &\longrightarrow \mathcal{V}_{2,lin}^{(0)}(\mathbb{R}^d \times \mathbb{R}^r) \times \cdots \times \mathcal{V}_{2,lin}^{(k-1)}(\mathbb{R}^d \times \mathbb{R}^r). \\ p &\longmapsto (\Lambda_{0_p}^{(0)}, \dots, \Lambda_{0_p}^{(k-1)}), \end{aligned}$$

where  $d$  and  $r$  are the dimensions of the base manifold and the fiber, respectively.

### 3. Examples

#### 3.1. Singularities of Poisson structures in dimension 2

In dimension 2, singularities of Poisson structures are singularities of functions for which there are specific classical techniques (see [14]). For instance, singularities of topologically stable Poisson structures in Radko’s sense (see [16]) are 1-stable.

#### 3.2. Singularities of order 1 for Poisson structures

Let  $m$  be a singularity of order 1 for a Poisson structure  $\pi$ . Denote by  $\mathfrak{g}$  the Lie algebra corresponding to the linear part of  $\pi$  at  $m$ . In this case, the hypothesis of Theorem 1.2 reduces to  $H_{LP}^{2,0}(\pi^{(1)}) = \{0\}$ . A straightforward calculation shows that this condition is equivalent to  $H^2(\mathfrak{g}, \mathbb{R}) = 0$ . This is exactly the condition given by Crainic and Fernandès (see Section 1).

One can notice that if the Lie algebra associated to the linear part of  $\pi$  at  $m$  is strongly rigid in the sense of Bordemann, Makhlof, and Petit (see [1]) then  $m$  is 1-stable.

**3.3. Singularities of order 2 for Poisson structures**

It is shown in [6] that Poisson structures on a vector space  $V$  having the form

$$\pi^{(2)} = \sum a_{i,j} x_i x_j \partial x_i \wedge \partial x_j$$

are generic among quadratic Poisson structures on  $V$ . We consider such a Poisson structure  $\pi^{(2)}$ . It is known that the  $a_{i,j}$  are invariants for this Poisson structure (see [8]). Set  $\lambda_i = \sum_j a_{i,j}$ .

LEMMA 3.1. — *If  $\lambda_i \neq 0$  for every  $i$ , and  $\lambda_i + \lambda_j \neq 0$  for  $i < j$ . then  $H_{LP}^{2,s}(\pi^{(2)}) = \{0\}$ , for  $s = 0$  and  $s = 1$ .*

Note that the hypothesis in the above lemma may occur only when the dimension is at least 3. The proof of this lemma can be found in [14]. This lemma and Theorem 1.2 imply that that singularities of order 2 of quadratic Poisson structures are, in general, stable (in dimension  $\geq 3$ ).

**3.4. Singularities of Poisson structures in dimension 3**

Let  $\omega$  be a volume form on a 3-dimensional manifold  $M$ . The map  $\pi \mapsto \alpha = i_\pi \omega$  establishes a one-to-one correspondence between Poisson structures and integrable 1-forms (i.e., 1-forms  $\alpha$  such that  $\alpha \wedge d\alpha = 0$ ). On the open set  $U$  where  $\alpha$  is nonzero, the distribution formed by vector fields  $X$  satisfying  $i_X \alpha = 0$  defines a regular foliation of codimension 1. If  $U$  is strictly contained in  $M$ , we may impose that the leaves are single points outside  $U$ . Then, we get a singular foliation on  $M$  which coincides with the symplectic foliation of  $\pi$ . In [2], Camacho and Lins-Neto proved that singularities of order  $k$  for these integrable forms are, in general, stable. Hence, the same conclusion holds for Poisson structures in dimension 3.

In fact, the proof of Theorem 1.2 is inspired by methods used for integrable 1-forms. We will interpret, in terms of Poisson structures, two results for integrable 1-forms obtained by Camacho and Lins-Neto. First, recall that a vector field  $X = X_1 \partial x + X_2 \partial y + X_3 \partial z$  on  $\mathbb{R}^3$ , which vanishes at the origin, has an *algebraically isolated zero* at the origin if the ideal generated by the germs of its components  $X_1, X_2$  and  $X_3$  has a finite codimension in the space of germs of functions of  $x, y$  and  $z$  at the origin.

LEMMA 3.2. — *Let  $\pi^{(k)}$  be a  $k$ -homogeneous Poisson structure on  $\mathbb{R}^3$ . Suppose its modular vector field  $X^{(k-1)}$  with respect to  $\omega = dx \wedge dy \wedge dz$  has an algebraically isolated zero at the origin. Then the hypothesis of Theorem 1.2 (i.e.,  $H_{LP}^{2,s}(\pi^{(k)}) = \{0\}$ , for any  $s = 0, \dots, k - 1$ ) holds true.*

Hence any order  $k$  singularity, with  $\pi^{(k)}$  as in this lemma, is  $k$ -stable.

LEMMA 3.3. — *Under the hypothesis of Lemma 3.2 and for all  $k > 2$ , we have*

$$\pi^{(k)} = \frac{1}{k + 1} I \wedge X^{(k-1)},$$

where  $I = x\partial x + y\partial y + z\partial z$  is the Liouville vector field.

This last lemma allows to construct many examples of  $k$ -stable singularities for  $k > 2$ . For instance, the origin is a  $k$ -stable singularity for

$$\pi = (xy^{k-1} - yx^{k-1})\partial x \wedge \partial y + (yz^{k-1} - zy^{k-1})\partial y \wedge \partial z + (zx^{k-1} - xz^{k-1})\partial z \wedge \partial x.$$

### 3.5. Singularities of order 1 for Lie algebroids

We recall that if  $m$  is a singularity order 1 for a Lie algebroid then the linear part of this algebroid at that singularity is a Lie algebroid corresponding to a linear action of a Lie algebra  $\mathfrak{g}$  on a vector space  $V$ . We will see that, in this case, the condition of stability of Theorem 1.3 is equivalent to  $H^1(\mathfrak{g}, V^*) = 0$ .

Consider a fiber-wise linear Poisson structure that corresponds to an action Lie algebroid  $g \times V$ . Suppose that it is given by the following expression in a fibered coordinate system  $(x_1, \dots, x_d, y_1, \dots, y_r)$ :

$$\Pi^{(1)} = \sum_k a_{i,j}^k y_k \partial y_i \wedge \partial y_j + \sum_\ell b_{p,q}^\ell x_\ell \partial y_p \wedge \partial x_q,$$

where  $a_{i,j}^k$  are structure constants of the Lie algebra  $\mathfrak{g}$  and  $b_{p,q}^\ell$  are constants which determine the action  $\mathfrak{g}$  on  $V$ . Set

$$\mu = \sum \mu_{u,v} \partial y_u \wedge \partial x_v,$$

where the  $\mu_{u,v}$  are constants. Consider the equation  $[\Pi^{(1)}, \mu] = 0$ . In this equation, the coefficient of  $\partial x_q \wedge \partial y_i \wedge \partial y_j$  gives

$$(1) \quad \sum_v \left( \mu_{i,v} b_{j,q}^v - \mu_{j,v} b_{i,q}^v \right) - \sum_u a_{i,j}^u \mu_{u,q} = 0.$$

But, if  $V^*$  is identified with the vector space of constant vector fields generated by the vector fields  $\partial x_i$ , then the dual action  $\varrho : \mathfrak{g} \rightarrow \text{End}(V^*)$  is given by

$$\varrho(y_i)(\partial x_j) = - \sum_v b_{i,v}^j \partial x_v.$$

Furthermore, if we set

$$\mu_i = \sum_u \mu_{i,\ell} \partial x_\ell,$$

we obtain that Equation (1) can be expressed as

$$\langle \varrho(y_i)(\mu_j) - \varrho(y_j)(\mu_i), dx_q \rangle = \sum_u a_{i,j}^u \mu_{u,q}.$$

More precisely, Equation (1) says that  $\mu$  corresponds to a 1-cocycle for the representation  $\varrho$ . This shows that constant 2-cocycles for the deformation cohomology complex can be identified with 1-cocycles relative to the representation  $\varrho$ . Similarly, 2-coboundaries for the deformation cohomology complex can be identified with 1-coboundaries relative to the representation  $\varrho$ . There follows that

$$H_{lin}^{2,0}(\Pi^{(1)}) \cong H^1(\mathfrak{g}, V^*).$$

In particular, when  $\Pi^{(1)}$  is the fiber-wise Poisson structure induced by the natural action of  $\mathfrak{g}$  on itself, then  $H_{lin}^{2,0}(\Pi^{(1)}) \cong H^1(\mathfrak{g}, \mathfrak{g}^*)$ . As R.-L. Fernandès points out (private communication), one can notice that, in this case, the hypothesis of Theorem 1.3 is stronger than that of Theorem 1.2. More precisely, the natural inclusion  $H^2(\mathfrak{g}, \mathbb{R}) \hookrightarrow H^1(\mathfrak{g}, \mathfrak{g}^*)$  is not onto, in general (since  $H^2(\mathfrak{g}, \mathbb{R})$  can be viewed as the skew-symmetric part of  $H^1(\mathfrak{g}, \mathfrak{g}^*)$ ). Consequently, the fact that  $m$  is a 1-stable singular point for a Poisson structure whose 1-jet at  $m$  corresponds to  $\mathfrak{g}$  does not guaranty that it is 1-stable for the associated Lie algebroid. For instance, if  $\mathfrak{g} = \mathfrak{aff}(1)$  (i.e., the Lie algebra of affine transformations on  $\mathbb{R}^2$ ) then the origin  $O$  is 1-stable for the associated Lie-Poisson structure on  $\mathfrak{g}^*$  since  $H^2(\mathfrak{aff}(1), \mathbb{R}) = \{0\}$ . But, the origin is not 1-stable for the corresponding Lie algebroid.

Furthermore, one can notice that, in the particular case where  $\mathfrak{g}$  is reductive and the  $X_i = \sum b_{ik}^\ell x_\ell \partial x_k$  are diagonal, the Hochschild-Serre factorization theorem (see [12]) gives  $H_{lin}^{2,0}(\Pi^{(1)}) = \{0\} \iff H^1(\mathfrak{g}, \mathbb{R}) = \{0\}$ .

### 3.6. Singularity of order 2 for Lie algebroids

Consider  $\mathbb{R}^{n+1}$  ( $n > 1$ ) with the standard coordinates  $(x_0, x_1, \dots, x_n)$  and the function  $\rho = x_0^2 + \dots + x_n^2$ . Let  $\Pi$  be the fiber-wise linear Poisson structure on  $\mathbb{R}^{2n+2}$  given by

$$\Pi^{(2)} = (\rho - 1) \sum_{i=0}^n \partial x_i \wedge \partial y_i + \sum_{i < j} (x_i y_j - x_j y_i) \partial y_i \wedge \partial y_j.$$

This is the Lie algebroid attached to  $(\rho - 1)$ , in the sense of Monnier (see [15]). Every point lying on the unit sphere is singular. It follows from results proven in [14] that  $H_{lin}^{2,s}(\Pi^{(2)})$  vanishes for all  $s$  (see Lemma 5.4.3 and Proposition 6.1.2 in [14]). Therefore, the unit sphere is 2-stable.

#### 4. Application: stability of non-trivial leaves

In this section, we will apply our previous results to Dirac manifolds and non-trivial leaves of Lie algebroids. Let  $M$  be a smooth  $n$ -dimensional manifold. We consider the vector bundle  $TM \oplus T^*M$  together with the Courant bracket defined by

$$[(X_1, \alpha_1), (X_2, \alpha_2)]_C = ([X_1, X_2], \mathcal{L}_{X_1}\alpha_2 - i_{X_2}d\alpha_1),$$

for any  $(X_1, \alpha_1), (X_2, \alpha_2) \in \Gamma(TM \oplus T^*M)$ . Recall that a *Dirac structure* on  $M$  (see [3]) is a sub-bundle  $L \subset TM \oplus T^*M$  of rank  $n$  which is closed under the Courant bracket and which is isotropic with respect to the bilinear symmetric operation given by

$$\langle (X_1, \alpha_1), (X_2, \alpha_2) \rangle = \frac{1}{2}(i_{X_2}\alpha_1 + i_{X_1}\alpha_2),$$

for any  $(X_1, \alpha_1), (X_2, \alpha_2) \in \Gamma(TM \oplus T^*M)$ . In this case,  $(M, L)$  is called a *Dirac manifold*. It is known that every Dirac manifold admits a foliation by pre-symplectic leaves (see [3]). We also know that the dimensions of the leaves have the same parity (see [7]).

A pre-symplectic leaf  $S$  of  $(M, L)$  is *singular* if every neighborhood of  $S$  intersects leaves of higher dimension. We say that the singular leaf  $S$  is *stable* if, for any small neighborhood  $U$  of  $S$ , there is a neighborhood  $\mathcal{W}$  of  $L|_U$  such that all Dirac structures in  $\mathcal{W}$  admit a pre-symplectic leaf that has the same dimension as  $S$ . Here, we endow the set of Dirac structures on  $M$  with the topology induced by the  $C^2$ -topology on sections of the associated bundle of Grassmanians of  $n$ -planes in each fiber of  $TM \oplus T^*M$ , where  $n = \dim(M)$ .

Let  $S$  be a pre-symplectic leaf of  $(M, L)$ , it is proven in [7] that if  $N \subset M$  is an embedded submanifold that intersects transversally  $S$  at  $m_0$  (in the sense that  $T_{m_0}M = T_{m_0}S \oplus T_{m_0}N$ ) then  $L$  induces a Poisson structure on  $N$  which vanishes at  $m_0$ . Moreover, this transverse Poisson structure does not depend on  $m_0$ : it is unique up to Poisson isomorphisms. This is called *the transverse Poisson structure along  $S$* .

**PROPOSITION 4.1.** — *Assume that  $S$  is an embedded singular pre-symplectic leaf of a Dirac manifold  $(M, L)$  and  $m_0 \in S$  is a stable singular point for the transverse Poisson structure. Then  $S$  is stable.*

*Proof.* — Consider a tubular neighborhood of  $S$  that corresponds to the vector bundle  $p : E \rightarrow S$ . Denote by  $\pi_V$  the transverse Poisson structure defined on an open neighborhood  $U \subset E_{m_0}$  of  $m_0$  with  $\pi_V(m_0) = 0$ . We set  $N = E_{m_0}$ . Any Dirac structure  $L'$  sufficiently close to  $L|_E$  has

pre-symplectic leaves  $S'$  near  $S$  that intersect  $N$  with the transversality relation:

$$T_m E = T_m S' + T_m N$$

at points  $m \in U \subset N$ . A priori, the above sum is not direct. Using the transversality relation and the local normal form of  $L'$  given in [7], one deduces that  $L' \cap (TN \oplus T^*M)$  has a constant rank near  $m_0$ . It follows that  $L'$  induces a Dirac structure  $L'_\nu$  in a neighborhood of  $m_0$  in  $N$ . Moreover, the fact that  $L'$  is chosen to be very close to  $L|_E$  implies that  $L'_\nu$  is the graph of a Poisson structure  $\pi'_\nu$  which is close to  $\pi_\nu$  on  $U$  (if necessary, we replace  $U$  by a smaller open set). Since  $m_0$  is stable for  $\pi_\nu$ , the Poisson structure  $\pi'_\nu$  has a singular point in  $U$ . Obviously, singular points for  $\pi'_\nu$  are located on leaves of the Dirac manifold  $(E, L')$  that have the same dimension as  $S$ . □

*Remark 4.2.* — In the above proof, the singular leaf  $S'$  associated with  $L'$  is not, in general, diffeomorphic to the original leaf  $S$  (specially, when  $S$  is non-compact). For instance, let  $M$  be the product of  $\mathbb{R}^2$  by the torus  $\mathbb{T}^2$  and suppose that  $L_\pi$  is the graph of the Poisson structure  $\pi$  given by

$$\pi = y \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \frac{\partial}{\partial \theta_1} \wedge \frac{\partial}{\partial \theta_2},$$

where  $(x, y)$  are coordinates of  $\mathbb{R}^2$  and  $(\theta_1, \theta_2)$  coordinates of  $\mathbb{T}^2$ . The submanifolds given by equations  $x = x_0, y = 0$  are singular leaves of  $(M, L_\pi)$ . By Proposition 4.1, each of these leaves is stable. Note that they are diffeomorphic to  $\mathbb{T}^2$ . Now, we consider

$$\pi_\varepsilon = y \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \left( \frac{\partial}{\partial \theta_1} + \varepsilon \frac{\partial}{\partial x} \right) \wedge \frac{\partial}{\partial \theta_2},$$

where  $\varepsilon \in \mathbb{R}$  is a deformation parameter. For  $\varepsilon \neq 0$ , every 2-dimensional symplectic leaf of  $\pi_\varepsilon$  is (globally) diffeomorphic to  $S^1 \times \mathbb{R}$ . Hence, it is not (globally) diffeomorphic to the torus.

We should mention that Crainic and Fernandès have obtained (private communication) results on stability of compact leaves of Poisson structures in a stronger sense, i.e.,  $S$  diffeomorphic to  $S'$ .

Using the notion of transverse Lie algebroid structure along a leaf (see, for instance, [10]) and the above techniques, one can similarly show the following proposition:

**PROPOSITION 4.3.** — *Let  $S$  be an embedded singular leaf of the base manifold of a Lie algebroid. If  $m \in S$  is a stable singularity of the transversal algebroid structure along  $S$ , then  $S$  is stable.*

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