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ON THE EMBEDDING AND COMPACTIFICATION OF q-COMPLETE MANIFOLDS

by Ionuţ CHIOSE

Abstract. — We characterize intrinsically two classes of manifolds that can be properly embedded into spaces of the form $\mathbb{P}^N \setminus \mathbb{P}^{N-q}$. The first theorem is a compactification theorem for pseudoconcave manifolds that can be realized as $\overline{X} \setminus (\overline{X} \cap \mathbb{P}^{N-q})$ where $\overline{X} \subset \mathbb{P}^N$ is a projective variety. The second theorem is an embedding theorem for holomorphically convex manifolds into $\mathbb{P}^1 \times \mathbb{C}^N$.

Résumé. — On caractérise intrinsèquement deux classes de variétés qui peuvent être incluses proprement dans des espaces de la forme $\mathbb{P}^N \setminus \mathbb{P}^{N-q}$. Le premier théorème est un théorème de compactification pour les variétés pseudoconcaves qui peuvent être réalisées comme $\mathbb{P}^N \setminus \mathbb{P}^{N-q}$, où $\overline{X} \subset \mathbb{P}^N$ est une variété projective. Le deuxième théorème est un théorème d'inclusion pour les variétés holomorphiquement convexes dans l'espace $\mathbb{P}^1 \times \mathbb{C}^N$.

Introduction

Two of the important problems in complex geometry are the *compactification problem* — to characterize complex manifolds which are isomorphic with a Zariski open subset of a compact variety, and the *embedding problem* — to characterize complex manifolds which can be realized as submanifolds of some standard spaces — usually projective spaces or affine spaces.

The compactification problem has had various solutions, both from the point of view of Riemannian geometry (Mok and Zhang, Yeung, Siu and Yau) and from the point of view of analytic geometry (Demailly, Nadel and Tsuji). Demailly [5] showed that if a complex manifold X of finite topological type carries a $\mathcal{C}^\infty$ strictly plurisubharmonic exhaustion function which

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satisfies two aditional conditions (finiteness of the volume and an estimate involving a Ricci curvature), then X is biholomorphic to an affine algebraic manifold. Therefore X can be compactified by adding a hyperplane at infinity. Nadel's result [10] settles the other extreme case, when X can be compactified by adding finitely many points. Nadel's theorem states that if X is a hyper 1-concave manifold which carries a line bundle whose ring of sections separates the points of X and gives local coordinates on X, and if X can be covered by Zariski open subsets which are uniformized by Stein manifolds, then X is biholomorphic to a quasi-projective manifold which can be compactified by adding finitely many points.

Our first result can be thought of as an "interpolation" between Demailly's result and Nadel's result:

Theorem 0.1. — *Let X be a connected complex manifold of dimension n and let $q \geqslant 2$. Suppose that:*

(i) *there exists a map $\pi : X \to \mathbb{P}^{q-1}$*

(ii) *there exists a $\mathcal{C}^\infty$ exhaustion function $\varphi : X \to \mathbb{R}$ such that*

$$(*) \qquad \omega := i\partial\bar{\partial}\varphi + \pi^* i\Theta(\mathcal{O}_{\mathbb{P}^{q-1}}(1)) > 0$$

(iii) *there exist $\mu \in \mathcal{C}^\infty(X,\mathbb{R})$ and $k_0 \in \mathbb{N}$ such that $k_0\omega + \mathrm{Ricci}(\omega) \geqslant -i\partial\bar{\partial}\mu$*

(iv) *X is $(n-q+1)$-concave*

(v) *$\dim H^{2p}(X,\mathbb{R}) < \infty$ for $q \leqslant p < \frac{n+q}{2}$.*

Then there exist a projective variety $\overline{X} \subset \mathbb{P}^N$ and $L \simeq \mathbb{P}^{N-q}$ a linear subspace of codimension q in $\mathbb{P}^N$ such that X is isomorphic to $\overline{X} \setminus (\overline{X} \cap L)$. Moreover, all the conditions except (iv) *are necessary conditions, while* (iv) *is a "generically" necessary condition.*

Note that the conditions appearing in Theorem 0.1 are similar to those in the above mentioned theorems. Note also that when $q = n$ condition (v) is empty and we obtain a particular case of Nadels' theorem. When $q = n+1$ we obtain the class of compact projective manifolds of dimension n.

The two most famous embedding theorems are Kodaira's embedding theorem which characterizes the projective manifolds in terms of the positivity of a line bundle, and the theorem on the proper embedding of Stein manifolds into affine spaces $\mathbb{C}^N$. An intermediate result between the two embedding theorems is Takayama's theorem [12]: a complex manifold can be properly embedded into a product $\mathbb{P}^N \times \mathbb{C}^M$ if and only if it is holomorphically convex and it carries a positive line bundle.

Our second result is a refined version of Takayama's theorem:

THEOREM 0.2. — *Let X be a connected complex manifold of dimension n. Then X is biholomorphic to a proper submanifold of $\mathbb{P}^1 \times \mathbb{C}^N$ if and only if:*

(i) *X is holomorphically convex; we let $f : X \to Y$ be the Remmert reduction of X*

(ii) *there exists a map $\pi : X \to \mathbb{P}^1$*

(iii) *there exists a $\mathcal{C}^\infty$ plurisubharmonic function $\psi : Y \to \mathbb{R}$ such that*

$$(*) \qquad \omega := i\partial\bar{\partial}\varphi + \pi^* i\Theta(\mathcal{O}_{\mathbb{P}^1}(1)) > 0$$

where $\varphi = \psi \circ f$.

Note that the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^N \hookrightarrow \mathbb{P}^M$, $M = 2(N+1) - 1$ restricts to $\mathbb{P}^1 \times \mathbb{C}^N$ to give a proper embedding into $\mathbb{P}^M \setminus \mathbb{P}^{M-2}$. Therefore in Theorem 0.2 we characterize a special class of holomorphically convex manifolds which can be embedded into $\mathbb{P}^N \setminus \mathbb{P}^{N-2}$.

The two conditions (i) and (ii) in Theorem 0.1 and (ii) and (iii) in Theorem 0.2 appear also in the theory of q-Stein spaces introduced by Barlet and Silva in [3]. And indeed, Theorem 0.1 implies in particular that (X, π^k) becomes a q-Stein manifold (1-Stein=Stein) for k sufficiently large, where π^k is the composition between π and $\mathbb{P}^{q-1} \ni [z_0 : \cdots : z_{q-1}] \to [z_0^k : \cdots : z_{q-1}^k] \in \mathbb{P}^{q-1}$, and Theorem 0.2 implies that (X, π) is a 2-Stein manifold.

The original motivation of this paper was the problem raised by Harvey and Lawson in [7]:

PROBLEM 0.3. — *Characterize (intrinsically) the proper submanifolds of $\mathbb{P}^N \setminus \mathbb{P}^{N-q}$.*

Such an intrinsic characterization should be an interpolation between Kodaira's embedding theorem (case $q = N + 1$) and the embedding of Stein manifolds into an affine space (case $q = 1$). Our theorems mentioned above provide such characterizations in two "extreme" special cases.

We now sketch the proofs of the two Theorems 0.1 and 0.2. There are several main ingrediendts in the proof of the pseudoconcave case. The first one is Demailly's Theorem 1.1; it allows us to construct sufficiently many sections in high powers of a positive line bundle. We will be able to "embed" any compact subset of X. The second one is Andreotti's theory of pseudoconcave spaces. It provides us with a Siegel-type theorem, with a compactification theorem for pseudoconcave spaces and some other results about the structure of our embedding. The third ingredient is a theorem of Dingoyan which says that if an open subset of a projective manifold is both "pseudoconcave" and "locally pseudoconvex", then its complement consists of a finite number of hypersurfaces. In our case the “pseudoconcavity” condition is given in the hypothesis, while the “local pseudoconvexity” condition is a consequence of $(*)$. The finite dimensionality of the singular cohomology groups will permit us to embed the “infinity” of X, via an elementary but important proposition due to Demailly. Finally we use Mok’s method to show that the embedding has the desired form. It consists essentially of showing that a certain Stein manifold is holomorphically convex with respect to the algebra of “algebraic” functions on that manifold.

For the pseudoconvex case we use a technical lemma to show that the only compact subvarieties of X are either points or rational curves isomorphic to $\mathbb{P}^1$ through the projection π. Then we consider the Remmert reduction of X. The problem is that in general a singular analytic Stein space cannot be embedded into an affine space. But a relatively compact subset of a Stein space can always be embedded, and we use this to show that X can be embedded into the desired space. Along the way we use an approximation theorem and some category arguments.

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1. The pseudoconcave case

In this section we prove Theorem 0.1 in the case $q = 2$. The proof of the general case for an arbitrary $q \geqslant 2$ follows similarly with only minor changes.

1.1. Preliminaries

In this section we recall some definitions and theorems needed for proving Theorem 0.1.

1.1.1. We will repeatedly make use of the following theorem of Demailly [4]

THEOREM 1.1. — *Let* (E, h) *be a Hermitian holomorphic line bundle with semi-positive curvature (i.e.,* $i\Theta(E, h) \geqslant 0$*) on a complete Kähler manifold* (X, ω) *of dimension* n*. Suppose* $\varphi : X \to [-\infty, 0]$ *is a function which is of class* $\mathcal{C}^\infty$ *outside a discrete subset* S *of* X *and near each point* $p \in S$*,* $\varphi(z) = A_p \ln |z|^2$ *where* A_p *is a positive constant and* $z = (z_1, \ldots, z_n)$

are local coordinates centered at p. *Assume that* $i\Theta(E, e^{-\varphi}h) = i\Theta(E,h) + i\partial\bar\partial\varphi \geqslant 0$ *on* $X \setminus S$ *and let* $\lambda : X \to [0,1]$ *be a continuous function such that* $i\Theta(E,h) + i\partial\bar\partial\varphi \geqslant \lambda\omega$ *on* $X \setminus S$. *Then for every* $\mathcal{C}^\infty$ *form* v *of type* $(n,1)$ *with values in* E *on* X *such that* $\bar\partial v = 0$ *and*

$$\int_X \frac{1}{\lambda}|v|^2 e^{-\varphi} dV_\omega < \infty$$

there exists a $\mathcal{C}^\infty$ *form* u *of type* $(n,0)$ *with values in* E *on* X *such that* $\bar\partial u = v$ *and*

$$\int_X |u|^2 e^{-\varphi} dV_\omega \leqslant \int_X \frac{1}{\lambda}|v|^2 e^{-\varphi} dV_\omega$$

If E is a line bundle on X a complex manifold then we say that the ring

$$\mathcal{A}(X,E) = \bigoplus_{k=0}^{\infty} H^0(X, E^k)$$

separates the points of X if $\forall x \neq y \in X, \exists k \in \mathbb{N}, \exists s \in H^0(X,E^k)$ s.t. $s(x) = 0 \neq s(y)$ and that it *gives local coordinates* on X if $\forall x \in X, \exists k \in \mathbb{N}, \exists s_0, s_1, \ldots, s_n \in H^0(X,E^k)$ s.t. $s_0(x) \neq 0$ and

$$d\left(\frac{s_1}{s_0}\right) \wedge \cdots \wedge d\left(\frac{s_n}{s_0}\right)(x) \neq 0.$$

The following lemma is a simple application of the above Theorem 1.1

Lemma 1.2. — *Let* (X,ω) *be a complete Kähler manifold of dimension* n *and* (E,h) *a positive Hermitian line bundle on* X. *Assume that there exists* $k_0 \in \mathbb{N}$ *such that* $E^{k_0} \otimes K_X^*$ *is semipositive. Then* $\mathcal{A}(X,E)$ *separates the points of* X *and gives local coordinates on* X.

1.1.2. We will also make use of the theory of pseudoconcave manifolds as developed by Andreotti.

Definition 1.3. — *A manifold* X *of dimension* n *is said to be* q-complete, $1 \leqslant q \leqslant n$ *if* X *has a* $\mathcal{C}^\infty$ *exhaustion function* $\varphi : X \to [0,\infty)$ *such that* $i\partial\bar\partial\varphi(x)$ *has at least* $n-q+1$ *positive eigenvalues* $\forall x \in X$.

A manifold X *is said to be* p-concave, $1 \leqslant p \leqslant n$, *if* X *has a* $\mathcal{C}^\infty$ *exhaustion (i.e., proper) function* $\psi : X \to [a,b)$ *such that* $i\partial\bar\partial\psi(x)$ *has at least* $n-p+1$ *negative eigenvalues,* $\forall x \in X \setminus K$ *where* K *is some compact subset of* X.

The results that we need from the theory of pseudoconcave spaces can be summarized in the following

Theorem 1.4 (Andreotti [1], Andreotti and Tomassini [2]). — *Let X be a connected p-concave manifold of dimension n, $p \leqslant n-1$. Then the field of meromorphic functions $\mathcal{K}(X)$ on X has* $\mathrm{tr.deg}_{\mathbb{C}}\mathcal{K}(\mathrm{X}) \leqslant \mathrm{n}$. *If F is a line bundle on X, then $\dim H^j(X,F) < \infty$ for $j \leqslant n-p-1$. If X is embedded as a locally closed subset in some projective space $\mathbb{P}^N$, then X is included into an algebraic variety Z in $\mathbb{P}^N$, which is irreducible and of the same dimension n. There is a unique maximal analytic subset of Z of pure codimension 1 with support in $\overline{Z \setminus X}$.*

1.1.3. For the proof of the fact that the birational embedding in Theorem 0.1 is quasi-projective we will use the following result of Dingoyan [6].

Definition 1.5. — *Let V be a projective variety and U an open subset of V. Then U is said to be* locally pseudoconvex *in V if there exists a covering $\mathcal{W}$ of V by open Stein sets such that for every $W \in \mathcal{W}$, the connected components of $U \cap W$ are Stein.*

Theorem 1.6 (Dingoyan [6]). — *Let V be a projective manifold and X an open pseudoconcave, locally pseudoconvex subset of V. Then the topological boundary of X consists of a finite union of hypersurfaces.*

For the proof of Theorem 1.6 one uses the fact that X is locally pseudoconvex in V to construct a section s of an ample line bundle on V such that X is the domain of existence for s, and then the pseudoconcavity condition on X implies that s is algebraic on V, therefore the boundary of X consists of the polar set of s.

1.1.4. In order to prove that the birational embedding in Theorem 0.1 can be "resolved" in a finite number of steps, we will use the following proposition of Demailly [5]

Proposition 1.7. — *Let X be a complex manifold of dimension n and let Y be a subvariety of dimension p in X and $d = n-p = \mathrm{codim}_X Y$. Then*

$$H^q(X, X \setminus Y; \mathbb{C}) = 0 \text{ if } q < 2d$$

and

$$H^{2d}(X, X \setminus Y; \mathbb{C}) \simeq \mathbb{C}^J$$

where $(Y_j)_{j\in J}$ is the family of irreducible components of dimension p in Y.

1.2. The necessity of the conditions

We show that all the conditions in Theorem 0.1 except (iv) are necessary conditions and that (iv) is a "generically" necessary condition.

On $\mathbb{P}^N$ fix homogeneous conditions $[z_0 : z_1 : \cdots : z_N]$ and assume that $\mathbb{P}^{N-q} = \{z_0 = z_1 = \cdots = z_{q-1} = 0\}$. Let $\pi : \mathbb{P}^N \setminus \mathbb{P}^{N-q} \to \mathbb{P}^{q-1}$ be the projection away from $\mathbb{P}^{N-q}$ given by

$$\pi([z_0 : \cdots : z_N]) = [z_0 : \cdots : z_{q-1}].$$

On $\mathbb{P}^N \setminus \mathbb{P}^{N-q}$ consider the exhaustion function $\varphi : \mathbb{P}^N \setminus \mathbb{P}^{N-q} \to \mathbb{R}$,

$$\varphi([z_0 : \cdots : z_N]) = \ln\left(\frac{|z_0|^2 + \cdots + |z_N|^2}{|z_0|^2 + \cdots + |z_{q-1}|^2}\right).$$

Since

$$i\Theta(\mathcal{O}_{\mathbb{P}^{q-1}}(1)) = i\partial\bar{\partial}\ln(|z_0|^2 + \cdots + |z_{q-1}|^2)$$

is the curvature of $\mathcal{O}_{\mathbb{P}^{q-1}}(1)$ on $\mathbb{P}^{q-1}$, we have

$$i\partial\bar{\partial}\varphi + \pi^* i\Theta(\mathcal{O}_{\mathbb{P}^{q-1}}(1)) = i\Theta(\mathcal{O}_{\mathbb{P}^N}(1))|_{\mathbb{P}^N \setminus \mathbb{P}^{N-q}} > 0$$

Therefore any manifold X that can be properly embedded into $\mathbb{P}^N \setminus \mathbb{P}^{N-q}$ comes equipped with a projection $\pi : X \to \mathbb{P}^{q-1}$ and an exhaustion function $\varphi : X \to [0, \infty)$ such that

$$(*) \qquad i\partial\bar{\partial}\varphi + \pi^* i\Theta(\mathcal{O}_{\mathbb{P}^{q-1}}(1)) > 0$$

Condition $(*)$ implies in particular that $E = \pi^*\mathcal{O}_{\mathbb{P}^{q-1}}(1)$ is positive and that X is q-complete with respect to φ.

If moreover the manifold X can be compactified in $\mathbb{P}^N$ then X is a quasi-projective variety, therefore it is of finite topological type.

If $\overline{X}$ denotes the compactification of X, then $\mathcal{O}_{\mathbb{P}^N}(1)|_{\overline{X}}$ is ample on $\overline{X}$, and since the dualizing sheaf $\omega_{\overline{X}}$ is coherent, it follows that there exists $k_0 \in \mathbb{N}$ such that $\mathcal{O}_{\mathbb{P}^N}(k_0)|_{\overline{X}} \otimes \omega^*_{\overline{X}}$ is globally generated. Restricting to X, we obtain that $E^{k_0} \otimes K^*_X$ is globally generated, in particular it is semi-positive (*i.e.,* there exists a Hermitian metric such that its curvature is semi-positive definite).

For a given projective variety $\overline{X}$ of dimension n in $\mathbb{P}^N$, its intersection with a general linear subspace of of $\mathbb{P}^N$ of dimension $N - q$ has dimension $n - q$. Therefore if $\overline{X} \cap \mathbb{P}^{N-q}$ is of pure dimension $n - q$, then by Ohsawa's theorem [11] it follows that X is $(n - q + 1)$-concave.

1.3. Andreotti's theory on pseudoconcave spaces

Let X be a manifold as in Theorem 0.1 with $q = 2$. In this section we use Andreotti's results on pseudoconcave manifolds to construct a birational embedding of X. Then in Section 1.4 we show that the embedding is quasi-projective. Next in 1.5 we prove that the birational embedding can be resolved in a finite number of steps. Finally we use Mok's method [9] to show that the embedding that we get has the form $\overline{X} \setminus (\overline{X} \cap \mathbb{P}^{N-2})$.

In order to use Lemma 1.2, we have to show that X carries a complete Kähler metric:

Lemma 1.8. — *Let X be a manifold as above above. We can assume that $\varphi \geqslant 1$. Let $f(t) = t - \frac{1}{2}\ln t$ and $\eta = f \circ \varphi$. Set*

$$\widetilde{\omega} = i\partial\bar{\partial}\eta + \pi^* i\Theta(\mathcal{O}_{\mathbb{P}^1}(1)).$$

Then $\widetilde{\omega}$ is a complete Kähler metric on X.

Proof. — Clearly $\widetilde{\omega}$ is closed. We have $i\partial\bar{\partial}\eta = f' \circ \varphi \, i\partial\bar{\partial}\varphi + f'' \circ \varphi \, i\partial\varphi \wedge \bar{\partial}\varphi$ and $f'(t) = 1 - \frac{1}{2t}$, $f''(t) = \frac{1}{2t^2}$. Hence

$$\widetilde{\omega} = \left(1 - \frac{1}{2\varphi}\right)\omega + \frac{1}{2\varphi}\pi^* i\Theta(\mathcal{O}_{\mathbb{P}^1}(1)) + \frac{1}{2\varphi^2} i\partial\varphi \wedge \bar{\partial}\varphi$$

so $\widetilde{\omega}$ is positive and $\widetilde{\omega} > \frac{1}{2\varphi^2} i\partial\varphi \wedge \bar{\partial}\varphi = \frac{1}{2} i\partial(\ln\varphi) \wedge \bar{\partial}(\ln\varphi)$. Therefore $|\partial(\ln\varphi)|^2_{\widetilde{\omega}} < 2$ and since $\ln\varphi$ is an exhaustion function, it follows that $\widetilde{\omega}$ is complete. ☐

Now X has a complete Kähler metric, $E = \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ is positive and $E^{k_0} \otimes K_X^*$ is semi-positive, therefore we can use Lemma 1.2 to show that $\mathcal{A}(X, E)$ separates the points of X and gives local coordinates on X.

Let s_0, s_1 be a basis of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ and denote by the same symbols s_0 and s_1 their pull-back to X. They are sections in E with $Z(s_0, s_1) = \{x \in X | s_0(x) = s_1(x) = 0\} = \emptyset$. They play a role analogue to the constant function 1 for Stein manifolds.

Since X is connected, the ring $\mathcal{A}(X, E)$ is an integral domain. We consider the field

$$Q(X, E) = \left\{ \frac{s}{t} \middle| \exists k \in \mathbb{N} \text{ s.t. } s, t \in H^0(X, E^k), t \neq 0 \right\} \subset \mathcal{K}(X).$$

The transcendence degree $\text{tr.deg}_{\mathbb{C}} Q(X, E) \geqslant n$ since $\mathcal{A}(X, E)$ gives local coordinates on X, and $\text{tr.deg}_{\mathbb{C}} \mathcal{K}(X) \leqslant n$ since X is $(n-1)$-concave. Therefore $Q(X, E) \subset \mathcal{K}(X)$ is a finite extension. Moreover, since X is smooth (in particular it is normal), $Q(X, E)$ is algebraically closed in the field $\mathcal{K}(X)$ of all meromorphic functions on X. This implies that $Q(X, E) = \mathcal{K}(X)$.

Let $s_0^k, s_1^k, s_2, \dots, s_n \in H^0(X, E^k)$ (where s_0 and s_1 are as above) such that $s_0^k(x) \neq 0$ and

$$d\left(\frac{s_1^k}{s_0^k}\right) \wedge \cdots \wedge d\left(\frac{s_n}{s_0^k}\right)(x) \neq 0$$

for some $x \in X$. Then

$$\operatorname{tr.deg} \mathbb{C}\left(\frac{s_1^k}{s_0^k}, \dots, \frac{s_n}{s_0^k}\right) = n$$

so

$$\mathbb{C}\left(\frac{s_1^k}{s_0^k}, \dots, \frac{s_n}{s_0^k}\right) \subset \mathcal{K}(X)$$

is a finite extension. Therefore there is a $g \in \mathcal{K}(X) = Q(X, E)$ such that

$$\mathcal{K}(X) = \mathbb{C}\left(\frac{s_1^k}{s_0^k}, \dots, \frac{s_n}{s_0^k}\right)(g)$$

so by taking k sufficiently large we can assume that

$$\mathcal{K}(X) = \mathbb{C}\left(\frac{s_1^k}{s_0^k}, \dots, \frac{s_{N_k}}{s_0^k}\right) \tag{1.1}$$

where $s_0^k, s_1^k, s_2, \dots, s_{N_k}$ is a basis of $H^0(X, E^k)$.

Let $\psi : X \to [a, b)$ be a $\mathcal{C}^\infty$ function that gives the $(n-1)$-concavity of X and let $K \subset X$ be a compact subset of X such that $i\partial\bar{\partial}\psi$ has 2 negative eigenvalues on $X \setminus K$. Let $c \in (\sup_K \psi, b)$ and $X_c = \{x \in X | \psi(x) < c\}$ which is relatively compact in X. Then there exists $k \in \mathbb{N}$ such that $\tau_k = [s_0^k : s_1^k : \cdots : s_{N_k}] : X \to \mathbb{P}^{N_k}$ is an embedding of $\overline{X}_c$. Note that τ_k is well-defined on X since $Z(s_0, s_1) = \emptyset$. We can assume that (1.1) is true for k.

Since $\tau_k(X_c)$ is pseudoconcave and locally closed in $\mathbb{P}^{N_k}$, there exists a projective compactification Z_k of $\tau_k(X_c)$ of the same dimension n. Obviously $\tau_k(X) \subset Z_k$.

Let $\nu_k : Z_k^\nu \to Z_k$ be the normalization of Z_k. Since ν_k is finite, it follows that $\nu_k^* \mathcal{O}_{\mathbb{P}^{N_k}}(1)$ is ample on Z_k^ν. Denote by $\tau_k^\nu : X \to Z_k^\nu$ the lifting of $\tau_k : X \to Z_k$.

Put

$$A_k = \{x \in X | \operatorname{rank} d\tau_k^\nu(x) < n\}$$

which is an analytic subset of X. Since $\tau_k^\nu|_{X_c}$ is an embedding, it follows that $A_k \subset X \setminus \overline{X}_c$. Since $i\partial\bar{\partial}\psi$ has 2 negative eigenvalues on $X \setminus \overline{X}_c$, it follows that $\dim A_k \leqslant n - 2$.

Lemma 1.9. — *τ_k^ν is injective on $X \setminus A_k$.*

Proof. — Let $x, y \in X \setminus A_k, x \neq y$. If $s_0(x) = 0$ and $s_0(y) \neq 0$ then clearly $\tau_k^\nu(x) \neq \tau_k^\nu(y)$. If $s_0(x) \neq 0, s_0(y) \neq 0$, then let $t \in H^0(X, E^l)$ such that $t(x) = 0, t(y) \neq 0$. Let $g = \frac{t}{s_0^l} \in \mathcal{K}(X)$; then g is defined at x and y and $g(x) \neq g(y)$. Condition (1) implies that there exist two homogeneous polynomials P and Q of the same degree such that

$$g = \frac{P(s_0^k, \dots, s_{N_k})}{Q(s_0^k, \dots, s_{N_k})}.$$

Then

$$\widehat{g} = \frac{P(z_0, \dots, z_{N_k})}{Q(z_0, \dots, z_{N_k})}$$

is a rational function on Z_k and set $\widetilde{g} = \nu_k^* \widehat{g}$ the pull-back of $\widehat{g}$ to Z_k^ν. Then $(\tau_k^\nu)^* \widetilde{g} = g$ and since g is defined at x and y and τ_k^ν is an isomorphism around x and y, it follows that $\widetilde{g}$ is defined at $\tau_k^\nu(x)$ and $\tau_k^\nu(y)$ and $\widetilde{g}(\tau_k^\nu(x)) = g(x) \neq g(y) = \widetilde{g}(\tau_k^\nu(y))$, hence $\tau_k^\nu(x) \neq \tau_k^\nu(y)$. □

Lemma 1.10. — $\tau_k^\nu(A_k) = \tau_k^\nu(X) \cap \operatorname{Sing}(Z_k^\nu)$.

Proof. — Let $x \in A_k$ and suppose that $\tau_k^\nu(x) \in \operatorname{Reg}(Z_k^\nu)$. Pick local coordinates $(w_1, \dots, w_n)$ on Z_k^ν centered at $\tau_k^\nu(x)$ and $(z_1, \dots, z_n)$ local coordinates on X centered at x. Then on a neighborhood of x, A_k is given by $\det \left(\frac{\partial (w_j \circ \tau_k^\nu)}{\partial z_l} \right)_{j,l=1,n} = 0$ which is an analytic subset of dimension $n-1$. This contradicts $\dim A_k \leqslant n-2$. Conversely, let $x \in X$ such that $\tau_k^\nu(x) \in \operatorname{Sing}(Z_k^\nu)$; if $x \in X \setminus A_k$, then $\tau_k^\nu(U)$ is a germ of a manifold at $\tau_k^\nu(x)$ for a sufficiently small neighborhood U of x, and since Z_k^ν is normal, it follows that τ_k^ν is a local isomorphism around $\tau_k^\nu(x)$, therefore $\tau_k^\nu(x) \in \operatorname{Reg}(Z_k^\nu)$. Contradiction. □

Since $\tau_k(X_c)$ is $(n-1)$-concave, it follows that there exists a unique maximal analytic subset H_k of pure dimension $n-1$ in Z_k with support in $Z_k \setminus \tau_k(X_c)$. Put $H_k^\nu = \nu_k^{-1}(H_k)$.

Lemma 1.11. — *Let $s \in H^0(X, E^{kl})$; then there exists a meromorphic section $\widetilde{s}$ of $\nu_k^* \mathcal{O}_{\mathbb{P}^N}(l)$ on Z_k^ν with polar set in H_k^ν such that $\widetilde{s} \circ \tau_k^\nu|_{X \setminus A_k} = s|_{X \setminus A_k}$.*

Proof. — $\frac{s}{s_0^{kl}} \in \mathcal{K}(X)$ so there exist two homogeneous polynomials P and Q of the same degree such that

$$\frac{s}{s_0^{kl}} = \frac{P(s_0^k, \dots, s_{N_k})}{Q(s_0^k, \dots, s_{N_k})}.$$

Set

$$\widetilde{s} = \nu_k^* \left(z_0^l \frac{P(z_0, \dots, z_{N_k})}{Q(z_0, \dots, z_{N_k})} \right)$$

where $[z_0 : \cdots : z_{N_k}]$ are homogeneous coordinates on $\mathbb{P}^{N_k}$. Then $\widetilde{s}\circ\tau_k^\nu|_{X_c} = s|_{X_c}$ is holomorphic so the polar set of $\widetilde{s}$ in Z_k^ν does not intersect $\tau_k^\nu(X_c)$. Since Z_k^ν is normal, the polar set of $\widetilde{s}$ is of pure dimension $n-1$ and therefore it has to be included in H_k^ν. □

Lemma 1.12. — $\tau_k^\nu(A_k) = \tau_k^\nu(X) \cap H_k^\nu$.

Proof. — Let $z = \tau_k^\nu(x) \in H_k^\nu$. If $x \in X \setminus A_k$, then $(\tau_k^\nu)^{-1}(H_k^\nu)$ has a component of dimension $n-1$ included in $X \setminus X_c$. This is a contradiction, so $x \in A_k$, *i.e.*, $\tau_k^\nu(X) \cap H_k^\nu \subset \tau_k^\nu(A_k)$. Conversely, suppose $x \in A_k$ and $\tau_k^\nu(x) \notin H_k^\nu$. Let U be a neighborhood of x such that $\tau_k^\nu(U) \cap H_k^\nu = \emptyset$. Let $x_1, x_2 \in U, x_1 \neq x_2$ and $s \in H^0(X, E^{kl})$ such that $s(x_1) \neq s(x_2)$. Then $\widetilde{s}$ the corresponding section on Z_k^ν is well-defined at $\tau_k^\nu(x_1)$ and $\tau_k^\nu(x_2)$ and $\widetilde{s}(\tau_k^\nu(x_1)) \neq \widetilde{s}(\tau_k^\nu(x_2))$ so $\tau_k^\nu(x_1) \neq \tau_k^\nu(x_2)$. Therefore $\tau_k^\nu|_U$ is injective. Since Z_k^ν is normal, $\tau_k^\nu|_U$ is open. Therefore $\tau_k^\nu|_U : U \to \tau_k^\nu(U)$ is a homeomorphism and $\tau_k^\nu(U)$ is an open neighborhood of $\tau_k^\nu(x)$. Then, since Z_k^ν is normal, $\tau_k^\nu(U)$ is also normal, and $\tau_k^\nu|_U : U \to \tau_k^\nu(U)$ is the normalization of $\tau_k^\nu(U)$ so $\tau_k^\nu|_U$ is an analytic isomorphism. Therefore $\tau_k^\nu(x) \in \operatorname{Reg}(Z_k^\nu)$, contradiction with Lemma 1.10. □

1.4. Quasi-projectivity of the embedding

So far we have a morphism $\tau_k^\nu : X \to Z_k^\nu$ which is an embedding outside an analytic subset A_k of codimension $\geqslant 2$. In this section we will show that $\tau_k^\nu(X \setminus A_k)$ is a Zariski open set in Z_k^ν.

Let $x_0 \in \mathbb{P}^1$ such that $s_0(x_0) = 0$ and $X_0 = X \setminus \pi^{-1}(x_0)$. Let

$$\varphi_0 = \varphi + \ln\left(\frac{|s_0|^2 + |s_1|^2}{|s_0|^2}\right)$$

which is an exhaustion function on X_0. Moreover, $i\partial\bar{\partial}\varphi_0 = \omega|_{X_0} > 0$, therefore X_0 is a Stein manifold.

Let $\pi_k : X \to \mathbb{P}^1, \pi_k = [s_0^k : s_1^k]$ and $\varphi_k \in \mathcal{C}^\infty(X, \mathbb{R})$,

$$\varphi_k = \varphi + \ln\left(\frac{(|s_0|^2 + |s_1|^2)^k}{|s_0|^{2k} + |s_1|^{2k}}\right) \tag{1.2}$$

Then φ_k is an exhaustion function and $i\partial\bar{\partial}\varphi_k + \pi_k^* i\Theta(\mathcal{O}_{\mathbb{P}^1}(1)) = i\partial\bar{\partial}\varphi + k\pi^* i\Theta(\mathcal{O}_{\mathbb{P}^1}(1)) > 0$.

By Hironaka's theorem on the resolution of singularities, there exists a projective manifold $\overline{Z}_k$ and a proper morphism $\lambda_k : \overline{Z}_k \to Z_k^\nu$ such that

$\lambda_k^{-1}(\mathrm{Sing}(Z_k^\nu) \cup H_k^\nu \cup \nu_k^{-1}(Z(z_0, z_1)))$ is a hypersurface $\overline{H}_k$ having normal crossings and

$$\lambda_k|_{\overline{Z}_k \setminus \overline{H}_k} : \overline{Z}_k \setminus \overline{H}_k \to Z_k^\nu \setminus (\mathrm{Sing}(Z_k^\nu) \cup H_k^\nu \cup \nu_k^{-1}(Z(z_0, z_1)))$$

is an isomorphism, where $[z_0 : \cdots : z_{N_k}]$ are homogeneous coordinates on $\mathbb{P}^{N_k}$. Set $\overline{\tau}_k : X \setminus A_k \to \overline{Z}_k, \overline{\tau}_k = (\lambda_k|_{\overline{Z}_k \setminus \overline{H}_k})^{-1} \circ \tau_k^\nu$. Then we have the following diagram:

(1.3)

$$\begin{tikzcd}
X \setminus A_k \arrow[r, "\overline{\tau}_k"] \arrow[d, hook] & \overline{Z}_k \arrow[d, "\lambda_k"] & \\
X \arrow[r, "\tau_k^\nu"] \arrow[dr, "\tau_k"] & Z_k^\nu \arrow[d, "\nu_k"] & \\
 & Z_k \arrow[r, hook] & \mathbb{P}^{N_k}
\end{tikzcd}$$

The following lemma is well-known, but we give its proof since a similar method will be used in Lemma 1.14:

Lemma 1.13. — *Let X be a Stein manifold and $f : X \to Y$ a holomorphic map to a complex manifold Y. Let $U \subset Y$ be a connected open Stein subset of Y. Then $f^{-1}(U) \subset X$ is Stein.*

Proof. — Let φ be an exhaustion strictly plurisubharmonic function on X and ψ an exhaustion strictly plurisubharmonic function on U. Set $\mu = \varphi|_{f^{-1}(U)} + \psi \circ f|_{f^{-1}(U)}$ on $f^{-1}(U)$. Then μ is clearly strictly plurisubharmonic and an exhaustion function on U, therefore $f^{-1}(U)$ is a Stein manifold. ☐

Lemma 1.14. — $\overline{\tau}_k(X \setminus A_k) = \overline{Z}_k \setminus \overline{H}_k$

Proof. — First we are going to show that $\overline{Z}_k \setminus \overline{\tau}_k(X \setminus A_k)$ is a hypersurface. In order to use Theorem 1.6, we have to show that $\overline{\tau}_k(X \setminus A_k)$ is locally pseudoconvex in $\overline{Z}_k$, *i.e.*, that any $z \in \overline{Z}_k$ has a Stein neighborhood U_z such that $U_z \cap \overline{\tau}_k(X \setminus A_k)$ is Stein. Let $z \in \overline{Z}_k$. If $\nu_k(\lambda_k(z)) \notin Z(z_0, z_1)$, assume $\nu_k(\lambda_k(z)) \notin Z(z_0)$ and let U_z be a small ball centered at z such that $\nu_k(\lambda_k(U_z)) \cap Z(z_0) = \emptyset$. Then $U_z \setminus \overline{H}_k$ is Stein, therefore $\lambda_k(U_z \setminus \overline{H}_k)$ is Stein (because λ_k is an isomorphism on $\overline{Z}_k \setminus \overline{H}_k$), therefore from Lemma 1.13 $(\tau_k^\nu)^{-1}(\lambda_k(U_z \setminus \overline{H}_k))$ is Stein in X_0 and is included in $X_0 \setminus A_k$. Hence $\overline{\tau}_k(X \setminus A_k) \cap U_z$ is Stein.

If $\nu_k(\lambda_k(z)) \in Z(z_0, z_1)$, then let U_z be a small ball centered at z such that $(\nu_k \circ \lambda_k)^* \mathcal{O}_{\mathbb{P}^N}(1)|_{U_z}$ is trivial. Let $\overline{s}_0^k$ and $\overline{s}_1^k$ be the pull-backs of z_0 and z_1 to $\overline{Z}_k$ and $\overline{H}_{1k} = Z(\overline{s}_0^k, \overline{s}_1^k) \subset \overline{H}_k$ and $\overline{H}_{2k}$ the rest of the components of $\overline{H}_k$. On U_z the two sections $\overline{s}_0^k$ and $\overline{s}_1^k$ give two holomorphic functions h_0 and h_1 such that $Z(h_0, h_1) = U_z \cap \overline{H}_{1k}$. Since $\overline{H}_k$ has normal crossings, we can assume that $\overline{H}_{1k} \cap U_z = \{w_1 w_2 \cdots w_l = 0\}$ and $\overline{H}_{2k} \cap U_z = \{w_{l+1} \cdots w_{l+p} = 0\}$ where $(w_1, \ldots, w_n)$ are local coordinates on U_z centered at z. Since $Z(h_0, h_1) = Z(h)$ where $h = w_1 \cdots w_l$, from Hilbert's Nullstellensatz it follows that there exist $m \in \mathbb{N}$ and g_0, g_1 holomorphic functions on U_z such that $g_0 h_0 + g_1 h_1 = h^m$. In particular there exists a constant C such that $|h|^{2m} \leqslant C(|h_0|^2 + |h_1|^2)$. Let

$$\overline{\mu} = \ln\left(\frac{|h_0|^2 + |h_1|^2}{|h|^{2m}}\right)$$

on $U_z \setminus \overline{H}_{1k}$ which is a function bounded from below. Let

$$\overline{\eta} = \ln\left(\frac{1}{|w_{l+1} \cdots w_{l+p}|^2}\right)$$

on $U_z \setminus \overline{H}_{2k}$ and $\overline{\theta} = \frac{1}{1-|w|^2}$. Denote by μ, η and θ the pull-back of $\overline{\mu}, \overline{\eta}$ and $\overline{\theta}$ to $\overline{\tau}_k^{-1}(U_z) \subset X \setminus A_k$. Let φ_k be the function given in (1.2) and on $\overline{\tau}_k^{-1}(U_z)$ consider the function $\gamma = \varphi_k + \mu + \eta + \theta$. Then it follows that $i\partial\overline{\partial}(\varphi_k + \mu) = \omega|_{\overline{\tau}_k^{-1}(U_z)} > 0$ and therefore γ is strictly plurisubharmonic on $\overline{\tau}_k^{-1}(U_z)$. It is easy to check that γ is an exhaustion function on $\overline{\tau}_k^{-1}(U_z)$, therefore $\overline{\tau}_k^{-1}(U_z)$ is Stein so $U_z \cap \overline{\tau}_k(X \setminus A_k)$ is Stein. >From Theorem 1.6 it follows that $\overline{Z}_k \setminus \overline{\tau}_k(X \setminus A_k) = \overline{H}_k'$ is a hypersurface which is included in $\overline{H}_k$.

If $\overline{H}_k \neq \overline{H}_k'$ then one component of $\overline{H}_k$ intersects $\overline{\tau}_k(X \setminus A_k)$, so we obtain a subvariety in X of dimension $n-1$ which is properly included in $\{\psi > c\}$, which is a contradiction. Therefore $\overline{Z}_k \setminus \overline{\tau}_k(X \setminus A_k) = \overline{H}_k$. ☐

1.5. Holomorphically convex spaces and the algebra of algebraic functions

In this section we show first that the birational embedding can be resolved in a finite number of steps, and then that the embedding that we get can be adjusted to have the desired form.

We have that $\overline{\tau}_k : X \setminus A_k \to \overline{Z}_k \setminus H_k$ is an isomorphism, in particular $X \setminus A_k$ is of finite topological type.

Condition $(*)$ implies that X is a 2-complete manifold; this implies that

$$H^{n+2}(X;\mathbb{C}) = H^{n+3}(X;\mathbb{C}) = \ldots = H^{2n}(X;\mathbb{C}) = 0.$$

Together with condition (v) we get that $\dim H^{2p}(X;\mathbb{C}) < \infty$, for $2 \leqslant p \leqslant n$.

Let $(Y_j)_{j\in J}$ be the irreducible components of A_k of codimension 2 in X. We have the exact sequence of the pair $(X, X\setminus A_k)$:

$$H^3(X\setminus A_k;\mathbb{C})\to H^4(X,X\setminus A_k;\mathbb{C})\to H^4(X;\mathbb{C})$$

>From Proposition 1.7 we have that $H^4(X,X\setminus A_k;\mathbb{C})\simeq\mathbb{C}^J$. Since $\dim H^4(X,\mathbb{C})<\infty$ and $\dim H^3(X\setminus A_k;\mathbb{C})<\infty$, it follows that $|J|<\infty$, *i.e.*, A_k has finitely many irreducible components of dimension $n-2$. Pick $x_j\in Y_j$ and then we can find k' sufficiently large such that $E^{k'}$ "resolves" the points x_j, *i.e.*, $x_j\notin A_{k'}$. Therefore all the irreducible components of $A_{k'}$ have dimension $\leqslant n-3$. It is clear now that we can repeat the above procedure to get that for k sufficiently large the "bad" set $A_k=\emptyset$.

Our whole discussion so far can be summarized in the following

Proposition 1.15. — *Let X be a manifold as in Theorem 0.1. Then there exists a $k\in\mathbb{N}$ such that $\tau_k^\nu : X\to Z_k^\nu$ is an embedding and $\tau_k^\nu(X)=Z_k^\nu\setminus(H_k^\nu\cup\operatorname{Sing}(Z_k^\nu)\cup\nu_k^{-1}(Z(z_0,z_1)))$.*

In order to complete the proof of Theorem 0.1, we have to show that the complement of $\tau_k^\nu(X)$ can be realized as the intersection between Z_k^ν and a linear subspace of codimension 2. We will use Mok's method [9] (see also [5]); first we will show that a certain Stein manifold is holomorphically convex with respect to the algebraic functions, and then we show that the Stein manifold is actually affine.

On $X_0=X\setminus\pi^{-1}(x_0)=\{x\in X|s_0(x)\neq 0\}$ consider the algebra

$$\mathcal{H}_0=\left\{f\in H^0(X_0,\mathcal{O}_{X_0})\middle|\exists l\in\mathbb{N},\exists s\in H^0(X,E^l)\text{ s.t. } f=\frac{s}{s_0^l}\right\}\subset H^0(X_0,\mathcal{O}_{X_0})$$

It obviously separates the points of X_0 and gives local coordinates on X_0 and we are going to prove that X_0 is *holomorphically convex* with respect to $\mathcal{H}_0$, *i.e.*, for any compact $K\subset X_0$, $\widehat{K}_{\mathcal{H}_0}=\{x\in X_0||f(x)|\leqslant\sup_K|f|,\forall f\in\mathcal{H}_0\}$ is also compact.

On X_0 we have the strictly plurisubharmonic exhaustion function

$$\varphi_0=\varphi|_{X_0}+\ln\left(\frac{|s_0|^2+|s_1|^2}{|s_0|^2}\right).$$

Set

$$\omega_0=i\partial\bar\partial\left(\varphi_0-\frac{1}{2}\ln\varphi_0\right)$$

which is a complete Kähler metric on X_0 (proof as in Lemma 1.8) and

$$\omega_0=\left(1-\frac{1}{2\varphi_0}\right)\omega|_{X_0}+\frac{1}{2\varphi_0^2}i\partial\varphi_0\wedge\bar\partial\varphi_0$$

so

$$\omega_0^n \geqslant \left(1 - \frac{1}{2\varphi_0}\right)^n \omega^n|_{X_0} \geqslant \frac{1}{2^n}\omega^n|_{X_0}.$$

Let μ be the function that appears in Theorem 0.1 in condition (iii). Denote by $dV_{\omega_0} = \omega_0^n$ the volume form of ω_0.

LEMMA 1.16. — *Let $f \in H^0(X_0, \mathcal{O}_{X_0})$ such that*

$$\int_{X_0} |f|^2 e^{-\mu - l\varphi_0} dV_{\omega_0} < \infty$$

for some $l \in \mathbb{N}$. Then $f \in \mathcal{H}_0$.

Proof. — We are going to show that $s_0^l f$ (which is a section in E^l on X_0) can be extended to a holomorphic section in E^l over X. Let $x \in \pi^{-1}(x_0)$ and $(z_1, \ldots, z_n)$ local coordinates centered at x on U a small neighborhood of x. Let $g_0 = \frac{s_0}{s_1}$ on $U \cap X_0$. Then

$$\varphi_0|_{U \setminus Z(g_0)} = \varphi|_{U \setminus Z(g_0)} + \ln\left(\frac{1 + |g_0|^2}{|g_0|^2}\right).$$

The function μ is bounded on U, so we can assume that

$$\int_{U \setminus Z(g_0)} |f|^2 e^{-l\varphi_0} dV_{\omega_0} < \infty.$$

Then the integrability condition for f implies

$$\int_{U \setminus Z(g_0)} |f|^2 |g_0|^{2l} |dz_1 \wedge \cdots \wedge dz_n|^2 < \infty$$

This implies that fg_0^l can be extended to U and therefore $s_0^l f$ can be extended to X so $f = \frac{s_0^l f}{s_0^l} \in \mathcal{H}_0$. □

LEMMA 1.17. — *X_0 is holomorphically convex with respect to $\mathcal{H}_0$.*

Proof. — Let K be a compact subset of X_0 and $c_0 = \sup_K \varphi_0$. We are going to show that $\widehat{K}_{\mathcal{H}_0} \subset \{\varphi_0 \leqslant c_0\}$. Let $x \in X$, $\varphi_0(x) > c_0$ and $\varepsilon > 0$ such that $\varphi_0(x) > c_0 + 3\varepsilon$. We want to construct $f \in \mathcal{H}_0$ such that $|f(x)| > \sup_K |f|$. Let $(z_1, \ldots, z_n)$ be local coordinates centered at x on $U = \{|z| < 2\} \subset \{\varphi_0 > c_0 + 2\varepsilon\}$ and let $V = \{|z| < 1\}$ and $\eta \in \mathcal{C}_0^\infty(X, \mathbb{R}), 0 \leqslant \eta \leqslant 1$, supp $\eta \subset U$, $\eta|_V = 1$ and $\gamma = n\eta \ln|z|^2$ defined to be 0 on $X \setminus U$. On X_0 consider the trivial line bundle $\underline{\mathbb{C}}$ with the metric $e^{-\mu - l(\varphi_0 - c_0 - 2\varepsilon)}$ and the dual of the canonical line bundle $K_{X_0}^*$ with the metric induced by $\omega|_{X_0}$. Denote by h_l the Hermitian metric induced on $\underline{\mathbb{C}} \otimes K_{X_0}^* \simeq K_{X_0}^*$; then

$$i\Theta(K_{X_0}^*, h_l) = i\partial\bar{\partial}\mu|_{X_0} + l\omega|_{X_0} + \mathrm{Ricci}(\omega)|_{X_0} > 0$$

for l sufficiently large. For l large enough we have $i\Theta(K^*_{X_0}, e^{-\gamma}h_l) = i\partial\bar{\partial}\gamma + i\Theta(K^*_{X_0}, h_l) \geqslant \omega|_{X_0}$ so we can find a continuous function $\lambda : X_0 \to (0,1]$ which does not depend on l such that $i\Theta(K^*_{X_0}, e^{-\gamma}h_l) \geqslant \lambda\omega_0$. Let $v = \bar{\partial}\eta$. Then $\bar{\partial}v = 0$ and $v|_V = 0$ so

$$\int_{X_0} \frac{1}{\lambda}|v|^2 e^{-\gamma-\mu-l(\varphi_0-c_0-2\varepsilon)} dV_{\omega_0} < \infty$$

and moreover the above integral is bounded from above by $\int_{X_0} \frac{1}{\lambda}|v|^2 e^{-\gamma-\mu} dV_{\omega_0}$ since $\varphi_0 - c_0 - 2\varepsilon > 0$ on U. Note that the above integral does not depend on l. From Theorem 1.1 it follows that there exists u_l a $\mathcal{C}^\infty$ function such that $\bar{\partial}u_l = v = \bar{\partial}\eta$ and

$$\int_{X_0} |u_l|^2 e^{-\gamma-\mu-l(\varphi_0-c_0-2\varepsilon)} dV_{\omega_0} \leqslant \int_{X_0} \frac{1}{\lambda}|v|^2 e^{-\gamma-\mu} dV_{\omega_0}.$$

Set $f_l = \eta - u_l$. Then $\int_U |u_l|^2 e^{-\gamma} dV_{\omega_0} < \infty$ implies $u_l(x) = 0$ so $f_l(x) = 1$. On $\{\varphi_0 < c_0 + \varepsilon\}$ we have $\varphi_0 - c_0 - 2\varepsilon < -\varepsilon$ so

$$\int_{\{\varphi_0 < c_0+\varepsilon\}} |u_l|^2 e^{-\mu+l\varepsilon} dV_{\omega_0} \leqslant \int_{X_0} \frac{1}{\lambda}|v|^2 e^{-\gamma-\mu} dV_{\omega_0}.$$

Now u_l is holomorphic on $\{\varphi_0 < c_0 + 2\varepsilon\}$ because $\bar{\partial}u_l = \bar{\partial}\eta = 0$ on $\{\varphi_0 < c_0 + 2\varepsilon\}$. An application of the Cauchy's inequalities shows that $||u_l||_{\{\varphi_0 \leqslant c_0\}} \to 0$ when $l \to \infty$. Now it is clear that for l large enough the function $f_l = \eta - u_l$ has the property $|f_l(x)| > \sup_K |f_l|$. Moreover the functions f_l satisfy the L^2 condition $\int_{X_0} |f_l|^2 e^{-\mu-l\varphi_0} dV_{\omega_0} < \infty$ and from Lemma 1.16 it follows that $f_l \in \mathcal{H}_0$. ☐

We can replace E by E^k and then besides the properties (i)−(v) we also have: Let $s_0, s_1, \ldots, s_N$ be a basis of $H^0(X, E)$. Set $\tau = [s_0 : \cdots : s_N] : X \to Z \subset \mathbb{P}^N$; then $\tau^\nu : X \to Z^\nu$ is an embedding such that $Z^\nu \setminus \tau^\nu(X) = \nu^{-1}(Z(z_0, z_1)) \cup H^\nu \cup \operatorname{Sing}(Z^\nu)$.

Set $Z^\nu_0 = Z^\nu \setminus \nu^{-1}(Z(z_0))$. Any function $f \in \mathcal{H}_0$ can be written $f = \frac{s}{s_0^l}$ where $s \in H^0(X, E^l)$. From Lemma 1.11 it follows that s can be extended to a meromorphic section $\widetilde{s}$ in $\nu^*\mathcal{O}_{\mathbb{P}^N}(l)$ with polar set in H^ν. Then $\widetilde{f} = \frac{\widetilde{s}}{s_0^l}$ is a meromorphic function on Z^ν_0 which extends f and the polar set of $\widetilde{f}$ is included in $H^\nu \cap Z^\nu_0$.

As an easy application of Lemma 1.17 we get that

$$\operatorname{Sing}(Z^\nu) \subset \nu^{-1}(Z(z_0, z_1)) \cup H^\nu.$$

The proof now proceeds along the lines of Mok [9]. Denote by H' the union of the irreducible components of H^ν which are not included in $\nu^{-1}(Z(z_0, z_1))$. If H' is a $\mathbb{Q}$-Cartier divisor (*i.e.*, set-theoretically locally complete intersection) then let t be a section in some line bundle L such that the support of the zero divisor of t is H'. Then $\nu^*\mathcal{O}_{\mathbb{P}^N}(l) \otimes L$ is very ample for some large l and then $Z^\nu \setminus \tau^\nu(X) = Z(z_0^l \otimes t, z_1^l \otimes t)$. But in general H' does not have to be a $\mathbb{Q}$-Cartier divisor.

Actually one can prove the following

Lemma 1.18. — *If $H' \cap (Z^\nu \setminus \nu^{-1}(Z(z_0, z_1)))$ is locally complete intersection in $Z^\nu \setminus \nu^{-1}(Z(z_0, z_1))$ then the conclusion of Theorem 0.1 is true.*

Proof. — Indeed, let $x \in H' \cap (Z^\nu \setminus \nu^{-1}Z(z_0, z_1))$ and let $s_x \in H^0(Z^\nu, \nu^*\mathcal{O}_{\mathbb{P}^N}(l))$ and U_x a Zariski open neighborhood of x such that $H' \cap (Z^\nu \setminus \nu^{-1}Z(z_0, z_1)) \cap U_x = Z(s_x) \cap U_x$. Let W be the union of the irreducible components of $Z(s_x)$ which are not contained in H'. Let $t_x \in H^0(Z^\nu, \nu^*\mathcal{O}_{\mathbb{P}^N}(m))$ such that $t_x|_W = 0$, $t_x(x) \neq 0$. Then for s sufficiently large $\frac{t_x^s}{s_x}$ is a holomorphic section in $\nu^*\mathcal{O}_{\mathbb{P}^N}(sm - l)$ on $Z^\nu \setminus H'$. Since $H' \cap (Z^\nu \setminus \nu^{-1}Z(z_0, z_1))$ is quasi-compact, it follows that we can find $k \in \mathbb{N}$ such that τ_k^ν is a proper embedding into $Z_k^\nu \setminus \nu_k^* Z(z_0, z_1)$. ☐

We will construct subvarieties Y_j in Z^ν, $j = \overline{1, n}$ such that Y_j is of pure dimension j and $Y_j \cap H'$ is a hypersurface in Y_j for all $j = 1, n$. Put $Y_n = Z^\nu$. Suppose Y_j has been constructed. Then pick a section s_j in $\nu^*\mathcal{O}_{\mathbb{P}^N}(l)$ for some large l which vanishes on H' but does not vanish identically on any of the irreducible components of Y_j. Then Y_{j-1} is the union of the irreducible components of $Y_j \cap Z(s_j)$ which are not contained in H'.

We can complete now the proof of Theorem 0.1. We prove by induction on j that there exist $k_j \in \mathbb{N}$ such that the restriction of $\tau_{k_j}^\nu : X \to Z_{k_j}^\nu \setminus \nu_{k_j}^{-1}(Z(z_0, z_1))$ to $X \cap Y_j$ is a proper embedding in $Z_{k_j}^\nu \setminus \nu_{k_j}^{-1}(Z(z_0, z_1))$. For $j = n$ we get the proof of Theorem 0.1. If $j = 1$ then $\dim Y_1 = 1$ and let $x_1, \ldots, x_m$ be the intersection points of Y_1 and H' which are not contained in $\nu^{-1}(Z(z_0, z_1))$. Suppose $x_1 \in Z_0^\nu = Z^\nu \setminus \nu^{-1}(Z(z_0))$; then from Lemma 1.17 and the maximum principle it follows that there exists a holomorphic function f_1 in $\mathcal{H}_0$ whose restriction to Y_1 has a pole at x_1. Similarly for the other points we get some functions $f_2, \ldots, f_m$ whose restrictions to Y_1 have poles at $x_2, \ldots, x_m$ respectively. These functions induce some sections in some power k_1 of E and then clearly the restriction of $\tau_{k_1}^\nu : X \to Z_{k_1}^\nu \setminus \nu_{k_1}^{-1}(Z(z_0, z_1))$ to $Y_1 \cap X$ is a proper embedding.

Suppose k_j has been constructed such that

$$\tau_{k_j}^\nu : X \to Z_{k_j}^\nu \setminus \nu_{k_j}^{-1}(Z(z_0, z_1))$$

when restricted to $Y_j \cap X$ is a proper embedding. We have a map $\phi_j : Z_{k_j}^\nu \setminus \nu_{k_j}^{-1}(Z(z_0, z_1)) \to Z^\nu \setminus \nu^{-1}(Z(z_0, z_1))$ such that $\phi_j^{-1}(H') = H'_{k_j} \cap$

$(Z^{\nu}_{k_j} \setminus \nu^{-1}_{k_j}(Z(z_0,z_1)))$. Set $\overline{Y}_j = \tau^{\nu}_{k_j}(Y_j \cap X)$ and $\overline{Y}_{j+1} = \overline{\tau^{\nu}_{k_j}(Y_{j+1} \cap X)} \setminus \nu^{-1}_{k_j}(Z(z_0,z_1))$. By the induction hypothesis we have that $\overline{Y}_j$ is a proper subvariety of $\overline{Y}_{j+1}$. Since $\overline{Y}_{j+1} \cap \phi_j^{-1}(Z(s_j))$ is the disjoint union $(\overline{Y}_{j+1} \cap H'_{k_j}) \cup \overline{Y}_j$, where s_j is the section that appears in the construction of Y_{j-1}, it follows that $\overline{Y}_{j+1} \cap H'_{k_j}$ is locally complete intersection in $\overline{Y}_{j+1}$. Let $x \in \overline{Y}_{j+1} \cap H'_{k_j}$. Then there exists a section t in $\nu^{*}_{k_j}\mathcal{O}_{\mathbb{P}^N}(l)$ such that $t(x) \neq 0$ and $t = 0$ on the irreducible components of $\phi_j^{-1}(Z(s_j))$ which do not intersect $\overline{Y}_{j+1} \cap H'_{k_j}$. Like in Lemma 1.18 we can find k_{j+1} such that $\tau^{\nu}_{k_{j+1}}|_{Y_{j+1}\cap X}$ is a proper embedding in $Z^{\nu}_{k_{j+1}} \setminus \nu^{-1}_{k_{j+1}}(Z(z_0,z_1))$.

This completes the proof of Theorem 0.1 in the case $q = 2$.

For the proof of the general case $q \geqslant 2$, there is only one significant change one has to make: instead of two sections s_0 and s_1, one considers q sections $s_0, s_1, \ldots, s_{q-1}$ which form a basis of $H^0(\mathbb{P}^{q-1}, \mathcal{O}_{\mathbb{P}^{q-1}}(1))$.

2. The pseudoconvex case

In this section we prove Theorem 0.2.

2.1. The necessity of the conditions

In this section we show that conditions (i), (ii) and (iii) in Theorem 0.2 are necessary conditions.

Let X be a proper submanifold of $\mathbb{P}^1 \times \mathbb{C}^N$. It is obviously holomorphically convex. Denote by p_1 and p_2 the projections on $\mathbb{P}^1$ and $\mathbb{C}^N$. Denote by π the restriction of p_1 to X. Let $Z = p_2(X)$ which is an analytic subspace of $\mathbb{C}^N$ by the proper mapping theorem. Let $f : X \to Y$ be the Remmert reduction of X. There exists a holomorphic map $h : Y \to Z$ such that $h \circ f = p_2$. Define $\psi = \lambda \circ h$ where λ is the $\mathcal{C}^{\infty}$ function $\lambda : \mathbb{C}^N \to \mathbb{R}$, $\lambda(z) = |z|^2$. Then clearly λ is plurisubharmonic and if $\varphi = \psi \circ f$ then $i\partial\bar{\partial}\varphi = i\partial\bar{\partial}(\lambda \circ h \circ f) = i\partial\bar{\partial}(\lambda \circ p_2)$ so $i\partial\bar{\partial}\varphi + \pi^{*}i\Theta(\mathcal{O}_{\mathbb{P}^1}(1)) > 0$. We only have to prove that ψ is $\mathcal{C}^{\infty}$ on Y, *i.e.*, locally on Y, ψ is the restriction of a $\mathcal{C}^{\infty}$ function. Obviously λ is a $\mathcal{C}^{\infty}$ function. Our assertion will follow from the following simple

Lemma 2.1. — *Let $h : Y \to Z$ be a holomorphic map between analytic spaces and let λ be a $\mathcal{C}^{\infty}$ function on Z. Then $\varphi = \lambda \circ h$ is $\mathcal{C}^{\infty}$ on Y.*

Proof. — It is a local problem, so we can assume that both Y and Z are biholomorphic to analytic subsets of the unit balls $B_N(0,1)$ and $B_M(0,1)$ in some affine spaces $\mathbb{C}^N$ and $\mathbb{C}^M$. We can assume that λ is the restriction of a $\mathcal{C}^\infty$ function λ'. Consider the embedding $Y \hookrightarrow Y \times Z$ given by $y \to (y, h(y))$. Then $Y \times Z$ is biholomorphic to an analytic subset of $B_N(0,1) \times B_M(0,1)$. On $B_N(0,1) \times B_M(0,1)$ consider the $\mathcal{C}^\infty$ function $\widetilde{\lambda}$ given by $\widetilde{\lambda}(y,z) = \lambda'(z)$. Then obviously ψ is the restriction of $\widetilde{\lambda}$ through the above embedding $Y \hookrightarrow Y \times Z$. □

2.2. The proof of the pseudoconvex case

Let X be a manifold as in Theorem 0.2. First we will show that any compact subvariety of X is isomorphic to $\mathbb{P}^1$ through π, and then we will use the Remmert reduction theorem to construct a proper embedding into $\mathbb{P}^1 \times \mathbb{C}^N$.

Let $f : X \to Y$ be the Remmert reduction of X.

In general a Stein analytic space can not be properly embedded into an affine space $\mathbb{C}^N$, the main obstruction being the dimension of the tangent space at singular points. However, there is always a holomorphic homeomorphism of a Stein space onto a subvariety of some $\mathbb{C}^N$. Let $g : Y \to \mathbb{C}^N$ be this map.

We can choose the function ψ in Theorem 0.2 , (iii) to be an exhaustion function (replace ψ with $\psi + \lambda \circ g$ where λ is a suitable exhaustion function on $\mathbb{C}^N$), and then condition $(*)$ implies that φ is a 2-convex exhaustion function, *i.e.*, $i\partial\bar{\partial}\varphi(x)$ has at least $n-1$ strictly positive eigenvalues for any $x \in X$, so X is a 2-complete manifold.

Let $Y \subset X$ be a compact irreducible analytic subset of X. Then $\varphi|_Y$ is constant (since φ is plurisubharmonic) and because $i\partial\bar{\partial}\varphi(x)$ has at least $n-1$ strictly positive eigenvalues, it follows that $\dim Y \leqslant 1$.

The key result in proving Theorem 0.2 is the following Lemma, whose proof can be found in Section 2.3:

LEMMA 2.2. — *Let C be a curve, $C \subset \Delta^n = \{z \in \mathbb{C}^n \| z| < 1\}$ such that* $\mathrm{Sing}(C) = \{0\}$ *and let $\varphi \in \mathcal{C}^\infty(\Delta^n, \mathbb{R})$ be a plurisubharmonic function such that $\varphi|_C = 0$. Then $(i\partial\bar{\partial}\varphi(0))^{n-1} = 0$.*

Let $C \subset X$ be a compact irreducible curve. Then $\varphi|_C$ is constant and from Lemma 2.2 above it follows that $\mathrm{Sing}(C) = \emptyset$. Indeed, $0 < \omega^n = (i\partial\bar{\partial}\varphi + \pi^* i\Theta(\mathcal{O}_{\mathbb{P}^1}(1)))^n = (i\partial\bar{\partial}\varphi)^{n-1}(i\partial\bar{\partial}\varphi + n\pi^* i\Theta(\mathcal{O}_{\mathbb{P}^1}(1)))$ so $(i\partial\bar{\partial}\varphi)^{n-1} \neq 0$.

Let $C_1, C_2 \subset X$ be compact irreducible curves. If $C_1 \cap C_2 \neq \emptyset$ then again Lemma 2.2 applies to show that $C_1 = C_2$. In particular any connected analytic subset of X is irreducible.

Let $C \subset X$ be a compact irreducible curve and consider $\pi|_C : C \to \mathbb{P}^1$. Since $\varphi|_C$ is constant, from $(*)$ it follows that $d(\pi|_C)(x) \neq 0$ for any $x \in C$, and therefore $\pi|_C : C \to \mathbb{P}^1$ is a covering map. Since $\mathbb{P}^1$ is simply connected, $\pi|_C : C \to \mathbb{P}^1$ is an isomorphism.

Consider the map $\pi \times f : X \to \mathbb{P}^1 \times Y$. Then f is injective. Indeed, the fibres of f are connected and compact, therefore if $f(x) = f(y)$ and $x \neq y$ then $x, y \in f^{-1}(f(x))$ which is a compact irreducible curve in X; but then $\pi(x) \neq \pi(y)$.

Moreover, condition $(*)$ implies that $\pi \times f$ has maximal rank n everywhere on X. Indeed, the problem is local on X, so let $x \in X$ such that $s_0(x) \neq 0$ where s_0, s_1 is a basis for $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. On $(\mathbb{P}^1 \setminus \{s_0 = 0\}) \times Y$ we have the $\mathcal{C}^\infty$ function

$$\gamma = \ln\left(1 + \frac{|s_1|^2}{|s_0|^2}\right) + \psi$$

and condition $(*)$ implies that $i\partial\bar{\partial}\gamma' > 0$ where γ' is the pull back of the above function γ through $\pi \times f$. This easily implies that $\pi \times f$ has rank n on X.

Now let $Y_c = \{y \in Y | \lambda(g(y)) < c\}$ where $\lambda : \mathbb{C}^N \to \mathbb{R}$, $\lambda(z) = |z|^2$. Then since Y_c is relatively compact in Y it can be embedded into some affine space through $g_1, \dots, g_M \in H^0(Y_c, \mathcal{O}_{Y_c})$.

Put $h_1 = g_1 \circ f, \dots, h_M = g_M \circ f$. Then $\pi \times (h_1, \dots, h_M) : X_c \to \mathbb{P}^1 \times \mathbb{C}^M$ is an embedding, where $X_c = \{x \in X | f(x) \in Y_c\}$. The functions $h_1, \dots, h_M$ on X_c can be uniformly approximated on compacts by global functions $h'_1, \dots, h'_M \in H^0(X, \mathcal{O}_X)$. Therefore for any $c \in \mathbb{R}$ we can find $h'_1, \dots, h'_M \in H^0(X, \mathcal{O}_X)$ such that $\pi \times (h'_1, \dots, h'_M) : X \to \mathbb{P}^1 \times \mathbb{C}^M$ has rank n on X_c.

By means of category arguments (as in for instance [8]) we will show that the number of functions giving the embedding can be kept bounded by $2n+1$ and that there exists a map $\pi \times (h_1, \dots, h_{2n+1}) : X \to \mathbb{P}^1 \times \mathbb{C}^{2n+1}$ of rank n on X. First, we have the following lemma, whose proof is very similar to Lemma 5.3.5 in [8], so we omit it:

LEMMA 2.3. — *If $h \in H^0(X, \mathcal{O}_X)^{M+1}, M > 2n$ is such that $\pi \times h$ has rank n on a compact subset of X, then one can find $(a_1, \dots, a_M) \in \mathbb{C}^M$ arbitrarily close to* 0 *such that $\pi \times (h_1 - a_1 h_{M+1}, \dots, h_M - a_M h_{M+1})$ has rank n on K. In fact this is true for all $a \in \mathbb{C}^M$ outside a set of measure* 0.

>From this lemma it follows easily that the set of all $h \in H^0(X, \mathcal{O}_X)^{2n+1}$ for which $\pi \times h$ does not have rank n on X is of the first category (*i.e.*, it is contained in the union of countably many closed sets with no interior point). Therefore there exists $h = (h_1, \dots, h_{2n+1}) \in H^0(X, \mathcal{O}_X)^{2n+1}$ such that $\pi \times h$ has rank n on X.

Now it is clear that $\pi \times (h_1, \dots, h_{2n+1}, g \circ f) : X \to \mathbb{P}^1 \times \mathbb{C}^{2n+1+N}$ is a proper embedding.

2.3. A technical lemma

In this section we prove the following

LEMMA 2.4. — *Let C be a curve, $C \subset \Delta^n = \{z \in \mathbb{C}^n \| z| < 1\}$ such that* $\mathrm{Sing}(C) = \{0\}$ *and let $\varphi \in \mathcal{C}^\infty(\Delta^n, \mathbb{R})$ be a plurisubharmonic function such that $\varphi|_C = 0$. Then $(i\partial\bar\partial\varphi(0))^{n-1} = 0$.*

Proof. — The fact that $(i\partial\bar\partial\varphi(0))^{n-1} = 0$ means that $i\partial\bar\partial\varphi(0)$ has two zero eigenvalues. Since C is singular at 0, we have three cases:

a) Two of the irreducible components of C at 0 are non-singular and they intersect transversally. Then we can assume that the two irreducible components are given by $\{z_2 = \cdots = z_n = 0\}$ and $\{z_1 = z_3 = \cdots = z_n = 0\}$. Then obviously $\frac{\partial^2\varphi}{\partial z_1 \partial \bar z_1}(0) = \frac{\partial^2\varphi}{\partial z_2 \partial \bar z_2}(0) = 0$ and since φ is plurisubharmonic, $\frac{\partial^2\varphi}{\partial z_1 \partial \bar z_2}(0) = \frac{\partial^2\varphi}{\partial z_2 \partial \bar z_1}(0) = 0$ which implies $(i\partial\bar\partial\varphi(0))^{n-1} = 0$

b) Two of the irreducible components of C at 0 are non-singular and they are tangent. Then we can assume that the two irreducible components are given by $\{z_2 = \cdots = z_n = 0\}$ and $\{z_2 = z_1^{p_2}\zeta_2, \dots, z_n = z_1^{p_n}\zeta_n\}$ where $2 \leqslant p_2 = \cdots = p_m < p_{m+1} \leqslant \cdots \leqslant p_n$ and $\zeta_2, \dots, \zeta_n$ are holomorphic functions of z_1 such that $\zeta_2(0) \cdots \zeta_n(0) \neq 0$. Set

$$\psi(z_1, \dots, z_n) = \varphi(z_1, \dots, z_n) + \varphi(z_1, z_1^{p_2}\zeta_2 - z_2, \dots, z_1^{p_n}\zeta_n - z_n).$$

Then ψ is a plurisubharmonic function and $\psi(z_1, 0, \dots, 0) = 0$,

$$\psi(z_1, z_1^{p_2}\zeta_2, \dots, z_1^{p_n}\zeta_n) = 0.$$

An easy computation shows that $i\partial\bar\partial\psi(0) = 2i\partial\bar\partial\varphi(0)$ and it is enough to prove that $(i\partial\bar\partial\psi(0))^{n-1} = 0$. Set

$$\mu(t, s) = \varphi(z_1, z_2 - tz_2 - s(z_2 - z_1^{p_2}\zeta_2), \dots, z_n - tz_n - s(z_n - z_1^{p_n}\zeta_n)).$$

Then one can show that

$$\begin{aligned}\psi(z_1,\dots,z_n) &= \int_0^1\int_0^1 \frac{\partial^2\mu}{\partial s\partial t}dsdt\\ &= \sum_{j,k=2}^{n} z_j(z_k - z_1^{p_k}\zeta_k)\alpha_{jk} + z_j(\bar z_k - \bar z_1^{p_k}\bar\zeta_k)\beta_{jk}\\ &\qquad\qquad + \bar z_j(z_k - z_1^{p_k}\zeta_k)\bar\beta_{jk} + \bar z_j(\bar z_k - \bar z_1^{p_k}\bar\zeta_k)\bar\alpha_{jk}\end{aligned}$$

where α_{jk},β_{jk} are $\mathcal{C}^\infty$ functions. Then for $l\geqslant 2$ and $z_2=\cdots=z_n=0$ we have

$$0=\frac{\partial^2\psi}{\partial z_l\partial\bar z_1} = \sum_{k=2}^{n} -z_1^{p_k}\zeta_k\frac{\partial\alpha_{lk}}{\partial\bar z_1} - p_k\bar z_1^{p_k-1}\bar\zeta_k\beta_{lk} - z_1^{p_k}\frac{\partial\bar\zeta_k}{\partial\bar z_1}\beta_{lk} - \bar z_1^{p_k}\bar\zeta_k\frac{\partial\beta_{lk}}{\partial\bar z_1}.$$

Set $z_1=\bar z_1$ in the above equation and then simplify it by $z_1^{p_2-1}$. Then let z_1 approach 0. We get $\sum_{k=2}^m p_k\bar\zeta_k(0)\beta_{lk}(0)=0,\forall l\geqslant 2$. On the other hand $\frac{\partial^2\psi}{\partial z_j\partial\bar z_k}(0)=\beta_{jk}(0)+\bar\beta_{kj}(0)$ for $j,k\geqslant 2$ which implies

$$\sum_{j,k=2}^{m}\frac{\partial^2\psi}{\partial z_j\partial\bar z_k}(0)p_j\zeta_j(0)p_k\bar\zeta_k(0)=0.$$

Since $\zeta_j(0)\neq 0$, the above equality implies that $i\partial\bar\partial\psi(0)$ has at least two zero eigenvalues: one corresponding to $(1,0,\dots,0)$, the other one to $(0,p_2\zeta_2(0),\dots,p_m\zeta_m(0),0,\dots,0)$.

c) One of the irreducible components of C at 0 is singular at 0. Then we can assume that C is locally irreducible at 0. Let $C^\nu\xrightarrow{\nu}C$ be the normalization of C and assume that ν is given locally by $\nu(t)=(t^{p_1},t^{p_2}\zeta_2,\dots,t^{p_n}\zeta_n)$ where $\zeta_2,\dots,\zeta_n$ are holomorphic functions such that $\zeta_2(0)\cdots\zeta_n(0)\neq 0$. Since C is singular at 0, we can assume that $2\leqslant p_1<p_2<p_3\leqslant\cdots\leqslant p_n\leqslant\infty$ and $p_2=qp_1+r$ where $0<r<p_1$

Set $\psi_0(t)=\varphi\circ\nu(t)=\varphi(t^{p_1},t^{p_2}\zeta_2,\dots,t^{p_n}\zeta_n)=0$. Then

$$\psi_1=\frac{1}{t^{p_1-1}\bar t^{p_1-1}}\frac{\partial^2\psi_0}{\partial t\partial\bar t}=\sum_{j,k=1}^{n}\frac{\partial^2\varphi}{\partial z_j\partial\bar z_k}t^{p_j-p_1}\bar t^{p_k-p_1}\zeta_j^1\bar\zeta_k^1=0$$

where $\zeta_j^1=p_j\zeta_j+t\frac{d\zeta_j}{dt}$. Notice that $\zeta_j^1(0)=p_j\zeta_j(0)\neq 0$ for $j=1,2$. If we let t approach 0 in $\psi_1=0$ we get $\frac{\partial^2\varphi}{\partial z_1\partial\bar z_1}(0)=0$. We want to show that $\frac{\partial^2\varphi}{\partial z_2\partial\bar z_2}(0)=0$.

Let $\Gamma(p)$ be the class of all $\mathcal{C}^\infty$ functions which can be written as a sum of functions of the form: $\lambda_{\alpha\beta}\circ\nu(t)t^\alpha\bar t^\beta\zeta_\alpha\bar\zeta_\beta$ where $\alpha,\beta\in\{0,p,p+1,\dots\}$, if $\alpha = \beta$ then $\alpha = \beta \neq p$ and $\zeta_0 = 1$. Then clearly

$$\psi_1 = \frac{\partial^2\varphi}{\partial z_2 \partial \bar{z}_2}\zeta_2^1\bar{\zeta}_2^1 + h_{p_2-p_1}$$

where $h_{p_2-p_1} \in \Gamma(p_2 - p_1)$.

If $h_{p_2-sp_1} \in \Gamma(p_2 - sp_1)$, $s < q$, then one can show that

$$\frac{1}{t^{p_1-1}\bar{t}^{p_1-1}} \frac{\partial^2 h_{p_2-sp_1}}{\partial t \partial \bar{t}} \in \Gamma(p_2 - (s+1)p_1)$$

By induction we get

$$\psi_q = \frac{\partial^2\varphi}{\partial z_2 \partial \bar{z}_2} t^{p_1-qp_1}\bar{t}^{p_2-qp_1}\zeta_2^q\bar{\zeta}_2^q + h_{p_2-qp_1} = 0$$

where $h_{p_2-qp_1} \in \Gamma(p_2 - qp_1)$, $\zeta_2^q(0) \neq 0$ and $p_2 = qp_1 + r, 0 < r < p_1$. In $\frac{\partial^2\psi_q}{\partial t \partial \bar{t}} = 0$ take $t = \bar{t}$ then divide the equation by $t^{2(r-1)}$ then let $t \to 0$. It follows that $\frac{\partial^2\varphi}{\partial z_2 \partial \bar{z}_2}(0) = 0$. □

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