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THE CHERN CHARACTER FOR LIE-RINEHART ALGEBRAS

by Helge MAAKESTAD

0. Introduction.

Classical Galois theory setting up a one to one correspondence between intermediate field-extensions of a Galois extension $E \subseteq F$ and subgroups of the Galois group $\mathrm{Gal}(F/E)$ was generalized by N. Jacobson in [12] to give a Galois-correspondence for purely inseparable field-extensions $k \subseteq K$ of exponent one of a field k of characteristic $p > 0$. This is a one to one correspondence between intermediate fields and $p - K/k$-sub-Lie algebras of $\mathrm{Der}_k(K)$. Jacobsons $p - K/k$-Lie algebra is the characteristic p version of a structure called a Lie-Rinehart algebra.

For an arbitrary k-algebra A, there exists the notion of a (k, A)-Lie-Rinehart algebra: it is a k-Lie algebra and A-module $\mathfrak{g}$ with a map of k-Lie algebras and A-modules $\alpha : \mathfrak{g} \to \mathrm{Der}_k(A)$, i.e a Lie-algebra acting on A in terms of vector fields. There exists the notion of a $\mathfrak{g}$-connection ∇ on an A-module W: this is an action

$$\nabla : \mathfrak{g} \to \mathrm{End}_k(W)$$

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generalizing the notion of a covariant derivation. There exists a complex $\mathrm{C}^{\bullet}(\mathfrak{g}, W, \nabla)$ – the Lie-Rinehart complex – generalizing simultaneously the algebraic de Rham complex of A and the Chevalley-Eilenberg complex of $\mathfrak{g}$. The main result of this paper is the following (see Theorem 2.12): There exists a ring homomorphism

$$ch^{\mathfrak{g}} : \mathrm{K}_0(\mathfrak{g}) \to \mathrm{H}^*(\mathfrak{g}, A)$$

from the Grothendieck ring $\mathrm{K}_0(\mathfrak{g})$ to the cohomology ring $\mathrm{H}^*(\mathfrak{g}, A)$. Here $\mathrm{K}_0(\mathfrak{g})$ is the Grothendieck ring of locally free A-modules with a $\mathfrak{g}$-connection and $\mathrm{H}^*(\mathfrak{g}, A)$ is the Lie-Rinehart cohomology of A with respect to $\mathfrak{g}$. We prove furthermore in Theorem 3.10 that the Chern character from Theorem 2.12 is independent with respect to choice of $\mathfrak{g}$-connection. This generalizes the construction of the classical Chern character (see Corollary 3.11.) Note that J. Huebschmann has in [10] considered a Chern-Weil construction in a similar situation, and it would be interesting to relate the construction in [10] to the construction in this note.

The notion of a (k, A)-Lie-Rinehart algebra is closely related to the notion of a groupoid in schemes. One constructs from a groupoid in schemes a Lie-Rinehart algebra in the same way as one constructs the Lie algebra from a group scheme. Much of the theory for group schemes and Lie algebras can be generalized to this new situation.

Lie-Rinehart algebras appear in topology and knot theory: T. Kohno has in [13] computed the Alexander polynomial of an irreducible plane curve C in $\mathbf{C}^2$ using the logarithmic deRham complex $\Omega^{\bullet}_{\mathbf{C}^2}(*C)$ which is just the standard complex where we let $\mathfrak{g}$ be the Lie algebra of derivations preserving the ideal of C in $\mathbf{C}^2$.

Groupoids and Lie-Rinehart algebras appear in the theory of motives: Let $\mathbf{T}$ be a Tannakian category over a field F of characteristic zero, and let ω be a fiber functor over the algebraic closure $\overline{F}$ of F. Then $Aut^{\otimes}(\omega)$ is represented by a groupoid S/S_0 and there exists an equivalence of categories

$$\mathbf{T} \cong \mathbf{Rep}(S/S_0),$$

(see [21]).

The paper is organized as follows: In the first section we define and sum up various general properties of Lie-Rinehart algebras, connections and the Lie-Rinehart complex. In the second section we prove existence of the Chern character. In the third section we prove that the Chern character is independent with respect to choice of connection.

1. Lie-Rinehart algebras, connections and the Lie-Rinehart complex.

In this section we introduce objects in the theory of modules on Lie-Rinehart algebras and state some general facts on the following: Let A be a commutative ring over a field k. Let furthermore $\mathfrak{g}$ be an (k, A)-Lie-Rinehart algebra and let (W, ∇) be a $\mathfrak{g}$-module. We introduce the Lie-Rinehart complex $\mathrm{C}^\bullet(\mathfrak{g}, W, \nabla)$. If ∇ is flat, $\mathrm{C}^\bullet(\mathfrak{g}, W, \nabla)$ is a DG-module, hence $\mathrm{H}^\bullet(\mathfrak{g}, W, \nabla)$ is a graded left $\mathrm{H}^\bullet(\mathfrak{g}, A)$-module.

Definition 1.1. — *Let A be a commutative k-algebra where k is a commutative ring. A (k, A)-*Lie-Rinehart algebra *on A is a k-Lie algebra and an A-module $\mathfrak{g}$ with a map $\alpha : \mathfrak{g} \to \mathrm{Der}_k(A)$ satisfying the following properties:*

$$\alpha(a\delta) = a\alpha(\delta) \tag{1.1.1}$$

$$\alpha([\delta, \eta]) = [\alpha(\delta), \alpha(\eta)] \tag{1.1.2}$$

$$[\delta, a\eta] = a[\delta, \eta] + \alpha(\delta)(a)\eta \tag{1.1.3}$$

for all $a \in A$ and $\delta, \eta \in \mathfrak{g}$. Let W be an A-module. A $\mathfrak{g}$-connection ∇ on W, is an A-linear map $\nabla : \mathfrak{g} \to \mathrm{End}_k(W)$ which satisfies the Leibniz-property, *i.e.*

$$\nabla(\delta)(aw) = a\nabla(\delta)(w) + \alpha(\delta)(a)w$$

for all $a \in A$ and $w \in W$. The $\mathfrak{g}$-connection ∇ is flat *if it is a map of Lie algebras. If ∇ is flat, we say that the pair (W, ∇) is a* $\mathfrak{g}$-module.

When it is clear from the context the notion Lie-Rinehart algebra will be use instead of (k, A)-Lie-Rinehart algebra. A Lie-Rinehart algebra is also referred to as a a *Lie-Cartan pair* or a *foliation*.

Definition 1.2. — *Let A be a k-algebra, $\mathfrak{g}$ a Lie-Rinehart algebra and (W, ∇) an A-module with a $\mathfrak{g}$-connection. Define a sequence of A-modules $\tilde{C}^\bullet(\mathfrak{g}, W, \nabla)$ and k-linear differentials $d^\bullet$ in the following way: Let $\tilde{C}^p(\mathfrak{g}, W, \nabla) = \mathrm{Hom}_k(\wedge^p \mathfrak{g}, W)$ where $\wedge^p \mathfrak{g}$ is wedge product over A. Define differentials*

$$d^p : \tilde{C}^p(\mathfrak{g}, W, \nabla) \to \tilde{C}^{p+1}(\mathfrak{g}, W, \nabla)$$

by

$$(d^p\psi)(\delta_1 \wedge \cdots \wedge \delta_{p+1}) = \sum_{i=1}^{p+1}(-1)^{i+1}\nabla_{\delta_i}\psi(\delta_1 \wedge \cdots \wedge \hat{\delta_i} \wedge \cdots \wedge \delta_{p+1})$$

$$+ \sum_{1\leqslant i<j\leqslant p+1}(-1)^{i+j}\psi([\delta_i,\delta_j] \wedge \cdots \wedge \hat{\delta_i} \wedge \cdots \wedge \hat{\delta_j} \wedge \cdots \wedge \delta_{p+1}). \tag{1.2.1}$$

Put $\tilde{C}^0 = W$ *and define* $d^0(w)(\delta) = \nabla(\delta)(w)$. *Let* $R_\nabla = d^1 \circ d^0$ *be the* curvature *of the connection* ∇.

Notice that $R_\nabla(\delta \wedge \eta) = [\nabla_\delta, \nabla_\eta] - \nabla_{[\delta,\eta]}$ hence W is a $\mathfrak{g}$-module if and only if the curvature is zero. Note also: if the connection ∇ is flat and $A = k$, the sequence of modules and differentials defined in 1.2 is just the ordinary *Chevalley-Eilenberg* complex of the representation W for the k-Lie algebra $\mathfrak{g}$.

Lemma 1.3. — *Let* $\mathfrak{g}$ *be a Lie-Rinehart algebra and let* (W,∇) *be a* $\mathfrak{g}$*-connection. Consider the sequence of modules from definition 1.2,* $\tilde{C}^\bullet(\mathfrak{g},W,\nabla)$. *Then for all* $p \geqslant 0$ *the following holds:*

$$(d^{p+1}\circ d^p)(\delta_1 \wedge \cdots \wedge \delta_{p+2})$$

$$= \sum_{1\leqslant i<j\leqslant p+2}(-1)^{i+j+1}R_\nabla(\delta_i \wedge \delta_j)(\delta_1 \wedge \cdots \wedge \hat{\delta_i} \wedge \cdots \wedge \hat{\delta_j} \wedge \cdots \wedge \delta_{p+2}).$$

Furthermore the maps d^p *induce maps*

$$d^p : \mathrm{Hom}_A(\wedge^p\mathfrak{g}, W) \to \mathrm{Hom}_A(\wedge^{p+1}\mathfrak{g}, W)$$

i.e $d^p\phi(aw) = ad^p\phi(x)$.

Proof. — Standard fact. □

Definition 1.4. — *Define the* Lie-Rinehart complex $C^\bullet(\mathfrak{g},W,\nabla)$ *as follows:*

$$C^p(\mathfrak{g},W,\nabla) = \mathrm{Hom}_A(\wedge^p\mathfrak{g}, W),$$

with differentials

$$d^p : \mathrm{Hom}_A(\wedge^p\mathfrak{g}, W) \to \mathrm{Hom}_A(\wedge^{p+1}\mathfrak{g}, W)$$

defined by equation 1.2.1. Put $C^0 = W$ *and define* $d^0(w)(\delta) = \nabla(\delta)(w)$. *Let* $R_\nabla = d^1 \circ d^0$ *be the* curvature *of the connection* ∇. *Then from Lemma 1.3 it follows that we get a sequence of maps of* k*-vector spaces.*

The Lie-Rinehart complex is sometimes referred to as the *Chevalley-Hochschild complex*. We see from Lemma 1.3 that $C^\bullet(\mathfrak{g}, W, \nabla)$ is a complex if and only if the curvature R_∇ is zero, hence if the curvature R_∇ is zero, we get well defined cohomology spaces.

Definition 1.5. — *Assume* $\mathfrak{g}$ *is a Lie-Rinehart algebra and* (W, ∇) *is a a flat* $\mathfrak{g}$*-connection* ∇. *We define the cohmology of* (W, ∇) *as follows:*

$$\mathrm{H}^p(\mathfrak{g}, W, \nabla) = \mathrm{H}^p(C^\bullet(\mathfrak{g}, W, \nabla)),$$

where $C^\bullet(\mathfrak{g}, W, \nabla)$ *is the Lie-Rinehart complex.*

The maps d^p from 1.4 are k-linear, hence the abelian groups $\mathrm{H}^p(\mathfrak{g}, W, \nabla)$ are k-vector spaces. Note furthermore that the cohomology $\mathrm{H}^*(\mathfrak{g}, A, \nabla)$ depends on the choice of connection $\nabla : \mathfrak{g} \to \mathrm{End}_k(A)$.

If the ring A is a smooth k-algebra of finite type, i.e the module of differentials Ω^1_A is locally free of finite rank, it follows that the Lie-Rinehart complex is isomorphic to the *algebraic de Rham complex*, hence the Lie-Rinehart complex generalizes simultaneously the algebraic de Rham complex and the Chevalley-Eilenberg complex.

Proposition 1.6. — *Let* A *be a* k*-algebra and* $\mathfrak{g}$ *a Lie-Rinehart algebra. Let furthermore* (W, ∇_1) *and* (W, ∇_2) *be* A*-modules with* $\mathfrak{g}$*-connections. There exists an exterior-product*

$$C^*(\mathfrak{g}, W) \otimes_A C^*(\mathfrak{g}, W') \to C^*(\mathfrak{g}, W \otimes_A W')$$

with the following property:

$$d(xy) = d(x)y + (-1)^p x d(y), \tag{1.6.1}$$

for all elements x *in* $\mathrm{Hom}_A(\wedge^p \mathfrak{g}, W)$ *and* y *in* $\mathrm{Hom}_A(\wedge^q \mathfrak{g}, W)$.

Proof. — Standard fact. □

Recall some general definitions and standard facts on DG-algebras (for a reference see [25]). A DG-algebra $B^* = \oplus_{p \geqslant 0} B^p$ is a graded associative algebra equipped with a graded derivation d of degree 1. If we do not require $d^2 = 0$ we say that B^* is a *quasi-differential graded algebra.* We can define the *cohomology* $\mathrm{H}^*(B^*)$ of B^* and it follows that $\mathrm{H}^*(B^*)$ is a graded associative k-algebra. If B^* is graded commutative, so is $\mathrm{H}^*(B^*)$. A graded left B^*-module $M^* = \oplus_{p\geqslant 0} M^p$ is a *differential graded module* if it is equipped with a graded derivation of degree one with $d^2 = 0$. We say that M^* is a *quasi-differential graded module* if we do not require $d^2 = 0$. It follows that $\mathrm{H}^*(M^*)$ is a graded left $\mathrm{H}^*(B^*)$-module. If we are given two DG-algebras B^* and E^* over a field k, then a *map of DG-algebras*, is just a map $\phi^* : B^* \to E^*$ of graded associative algebras, commuting with the differentials. One easily verifies that such a map ϕ^* induces a map of graded associative algebras $\mathrm{H}(\phi^*) : \mathrm{H}^*(B^*) \to \mathrm{H}^*(E^*)$. Also, given two DG-modules M^* and N^* on a DG-algebra B^*, a *map of DG-modules*, is a map $\psi^* : M^* \to N^*$ commuting with the differentials. It is trivial to check that such a map ψ^* induces a map $\mathrm{H}(\psi^*) : \mathrm{H}^*(M^*) \to \mathrm{H}^*(N^*)$ of graded $\mathrm{H}^*(B^*)$-modules.

PROPOSITION 1.7. — *Let A be a k-algebra, $\mathfrak{g}$ a Lie-Rinehart algebra. Let furthermore (W, ∇) be an A-module with a $\mathfrak{g}$-connection. Then $C^*(\mathfrak{g}, A)$ is a DG-algebra and $C^*(\mathfrak{g}, W)$ is a quasi-DG-module on $C^*(\mathfrak{g}, A)$. If W is a $\mathfrak{g}$-module, then $C^*(\mathfrak{g}, W)$ is a DG-module, hence* $\mathrm{H}^*(\mathfrak{g}, A)$ *is a graded associative k-algebra and* $\mathrm{H}^*(\mathfrak{g}, W)$ *is in a natural way a graded left module on* $\mathrm{H}^*(\mathfrak{g}, A)$.

Proof. — This follows from the previous discussion and Proposition 1.6. □

PROPOSITION 1.8. — *Let A be a k-algebra, and $\mathfrak{g}$ an Lie-Rinehart algebra. Let furthermore (W, ∇) be a $\mathfrak{g}$-connection. The connection ∇ induces a connection* $\mathrm{ad}\nabla$ *on* $\mathrm{End}_A(W)$, *hence $C^*(\mathfrak{g}, \mathrm{End}_A(W))$ becomes in a natural way a quasi-DG-algebra. If W is a $\mathfrak{g}$-module, $C^*(\mathfrak{g}, \mathrm{End}_A(W))$ is DG-algebra.*

Proof. — This follows from the previous discussion. □

2. A construction of the Chern character.

This section contains proofs of the following results: Let k be a field of characteristic zero, and let A be a k-algebra. Let furthermore $\mathfrak{g}$ be an

(k, A)-Lie-Rinehart algebra, and (W, ∇) be a $\mathfrak{g}$-connection wich is of finite presentation as an A-module. There exists a *Chern character* $ch^{\mathfrak{g}}(W)$ in $\mathrm{H}^*(\mathfrak{g}|_U, \mathcal{O}_U)$ where U is the open subset of $\mathrm{Spec}(A)$ where W is locally free. We apply this to prove the existence of a ring homomorphism

$$ch^{\mathfrak{g}} : \mathrm{K}_0(\mathfrak{g}) \to \mathrm{H}^*(\mathfrak{g}, A)$$

where $\mathrm{K}_0(\mathfrak{g})$ is the *Grothendieck ring* of locally free A-modules with a $\mathfrak{g}$-connection.

Recall briefly classical Chern-Weil theory: Let A be a k-algebra, where k is a field of characteristic 0, and let E be a locally free A-module. Any connection

$$\nabla : E \to \Omega^1_A \otimes E$$

gives rise to a connection

$$\mathrm{ad}\nabla : \mathrm{End}_A(E) \to \Omega^1_A \otimes \mathrm{End}_A(E),$$

and we get

$$R^k_\nabla \in \Omega^{2k}_A \otimes \mathrm{End}_A(E).$$

Since E is locally free there exists a trace map $\mathrm{tr} : \mathrm{End}_A(E) \to A$ and we get Chern-classes

$$ch_k(E, \nabla) \in \mathrm{H}^{2k}_{\mathrm{DR}}(A).$$

This construction defines a group-homomorphism

$$ch^A : \mathrm{K}_0(A) \to \mathrm{H}^*_{\mathrm{DR}}(A).$$

Theorem 2.1.— *The map* $ch^A : \mathrm{K}_0(A) \to \mathrm{H}^*_{\mathrm{DR}}(A)$ *is a ring-homomorphism.*

Proof.— See Theorem 8.1.7 in [16]. □

Notice the following: If ∇ and ∇' are two $\mathfrak{g}$-connections on an A-module A, where $\mathfrak{g}$ is an Lie-Rinehart algebra, then the difference $\nabla - \nabla'$ is an element of the module $\mathrm{Hom}_A(\mathfrak{g}, \mathrm{End}_A(W))$. We express this by saying that the set of $\mathfrak{g}$-connections on W form a *principal homogeneous space* (or a *torsor*) on $\mathrm{Hom}_A(\mathfrak{g}, \mathrm{End}_A(W))$. This means that given a $\mathfrak{g}$-connection ∇ on W, any other connection ∇' can be obtained from ∇ by adding an element ϕ from $\mathrm{Hom}_A(\mathfrak{g}, \mathrm{End}_A(W))$, that is $\nabla' = \nabla + \phi$ for a unique ϕ.

Lemma 2.2. — *Let A be a k-algebra, $\mathfrak{g}$ a Lie-Rinehart algebra and W a $\mathfrak{g}$-connection which is free as an A-module. The trace map*

$$\mathrm{tr}^* : C^*(\mathfrak{g}, \mathrm{End}_A(W)) \to C^*(\mathfrak{g}, A)$$

is a morphism of complexes.

Proof. — The only thing we have to prove is that for all $p \geqslant 0$ we have commutative diagrams

$$\begin{array}{ccc} C^p(\mathfrak{g}, \mathrm{End}_A(W)) & \xrightarrow{d^p} & C^{p+1}(\mathfrak{g}, \mathrm{End}_A(W)) \\ \downarrow{\scriptstyle \mathrm{tr}} & & \downarrow{\scriptstyle \mathrm{tr}} \\ C^p(\mathfrak{g}, A) & \xrightarrow{d^p} & C^{p+1}(\mathfrak{g}, A). \end{array}$$

We may assume that we have chosen a basis $\{e_i\}$ for W as an A-module and we can write $W = \oplus_{i=1}^n Ae_i$. Then in this basis we have a connection $\nabla'_{\delta_i}(\sum a_i e_i) = \sum \alpha(\delta_i)(a_i)e_i$, and one verifies that $R_{\nabla'} = 0$, hence the connection ∇' is integrable. The connection ∇ which defines the $\mathfrak{g}$-structure structure on W can now be written in a unique way as $\nabla = \nabla' + \phi$, where ϕ is an element of $\mathrm{Hom}_A(\mathfrak{g}, \mathrm{End}_A(W))$, since $\mathfrak{g}$-connections are a torsor on $\mathrm{Hom}_A(\mathfrak{g}, \mathrm{End}_A(W))$. The induced connection $ad\nabla$ on $\mathrm{End}_A(W)$ then becomes

$$ad\nabla = [\nabla, -] = [\nabla' + \phi, -] = [\nabla', -] + [\phi, -].$$

The rest is straightforward calculation: Let ψ be an element of $C^p(\mathfrak{g}, \mathrm{End}_A(W)) = \mathrm{Hom}_A(\wedge^p \mathfrak{g}, \mathrm{End}_A(W))$. Put also $\omega = \delta_1 \wedge \cdots \wedge \delta_{p+1}$, $\omega(i) = \delta_1 \wedge \cdots \wedge \hat{\delta_i} \wedge \cdots \wedge \delta_{p+1}$ for $1 \leqslant i \leqslant p+1$, and $\omega(i,j) = [\delta_i, \delta_j] \wedge \delta_1 \wedge \cdots \wedge \hat{\delta_i} \wedge \cdots \wedge \hat{\delta_j} \wedge \cdots \wedge \delta_{p+1}$.

Then we have that

$$\begin{aligned} &\mathrm{tr}((d^p\psi)(\omega)) \\ &= \mathrm{tr}\left(\sum_{i=1}^{p+1}(-1)^{i+1} ad\nabla_{\delta_i}\psi(\omega(i))\right) + \mathrm{tr}\left(\sum_{1\leqslant i<j\leqslant p+1}(-1)^{i+j}\psi(\omega(i,j))\right) \\ &= \mathrm{tr}\left(\sum_{i=1}^{p+1}(-1)^{i+1}[\nabla'_{\delta_i}, \psi(\omega(i))]\right) + \mathrm{tr}\left(\sum_{i=1}^{p+1}(-1)^{i+1}[\phi(\delta_i), \psi(\omega(i))]\right) \\ &\qquad + \mathrm{tr}\left(\sum_{1\leqslant i<j\leqslant p+1}(-1)^{i+j}\psi(\omega(i,j))\right) \end{aligned}$$

$$= \mathrm{tr}\left(\sum_{i=1}^{p+1}(-1)^{i+1}(\alpha(\delta_i)(\psi(\omega(i)))_{kl})\right) + \mathrm{tr}\left(\sum_{i<j}(-1)^{i+j}\psi(\omega(i,j))\right)$$

$$= \sum_{i=1}^{p+1}(-1)^{i+1}\mathrm{tr}(\alpha(\delta_i)(\psi(\omega(i))_{kl})) + \sum_{i<j}(-1)^{i+j}\mathrm{tr}(\psi(\omega(i,j)))$$

$$= \sum_{i=1}^{p+1}(-1)^{i+1}\alpha(\delta_i)(\mathrm{tr}(\psi(\omega(i)))) + \sum_{i<j}(-1)^{i+j}(\mathrm{tr}\circ\psi)(\omega(i,j))$$

$$= d^p(\mathrm{tr}\circ\psi)(\omega)$$

and we see that $\mathrm{tr}\circ d^p = d^p\circ\mathrm{tr}$ and we have proved the assertion. □

Corollary 2.3. — *Assume x^* is an element of $C^*(\mathfrak{g}, \mathrm{End}_A(W))$ with the property that $d^*(x^*) = 0$, then $\mathrm{tr}^*(x^*)$ gives rise to a cohomology-class $\overline{\mathrm{tr}^*(x^*)}$ in $\mathrm{H}^*(\mathfrak{g}, A)$.*

Proof. — We show that $d^*(\mathrm{tr}^*(x^*)) = 0$: For all $p \geqslant 0$ we have commutative diagrams

$$\begin{array}{ccc} C^p(\mathfrak{g}, \mathrm{End}_A(W)) & \xrightarrow{d^p} & C^{p+1}(\mathfrak{g}, \mathrm{End}_A(W)) \\ \downarrow{\scriptstyle \mathrm{tr}} & & \downarrow{\scriptstyle \mathrm{tr}} \\ C^p(\mathfrak{g}, A) & \xrightarrow{d^p} & C^{p+1}(\mathfrak{g}, A) \end{array}$$

by lemma 2.2. We see that $d^p(\mathrm{tr}^p(x^p) = \mathrm{tr}(d^p(x^p)) = \mathrm{tr}(0) = 0$, hence we have that $d^*(\mathrm{tr}^*(x^*)) = 0$ and we get a well-defined cohomology-class $\overline{\mathrm{tr}^*(x^*)}$ in $\mathrm{H}^*(\mathfrak{g}, A)$. □

Given a $\mathfrak{g}$-connection W, where $\mathfrak{g}$ is an Lie-Rinehart algebra, one verifies that the curvature R_∇ is an element of $\mathrm{Hom}_A(\mathfrak{g}\wedge\mathfrak{g}, \mathrm{End}_A(W)) = C^2(\mathfrak{g}, \mathrm{End}_A(W))$.

Lemma 2.4 *(The Bianchi identity). — Let A be a k-algebra, $\mathfrak{g}$ a Lie-Rinehart algebra, and W a $\mathfrak{g}$-connection. Then $d^2(R_\nabla) = 0$.*

Proof. — This is straightforward calculation: Let

$$\omega = \alpha\wedge\beta\wedge\gamma$$

be an element of $\wedge^3\mathfrak{g}$. Then we see that

$$d^2 R_\nabla(\alpha\wedge\beta\wedge\gamma)$$

$$= ad\nabla_\alpha R_\nabla(\beta\wedge\gamma) - ad\nabla_\beta R_\nabla(\alpha\wedge\gamma) + ad\nabla_\gamma R_\nabla(\alpha\wedge\beta)$$
$$- R_\nabla([\alpha,\beta]\wedge\gamma) + R_\nabla([\alpha,\gamma]\wedge\beta) - R_\nabla([\beta,\alpha]\wedge\alpha)$$
$$= \nabla_\alpha R_\nabla(\beta\wedge\gamma) - R_\nabla(\beta\wedge\gamma)\nabla_\alpha$$
$$- (\nabla_\beta R_\nabla(\alpha\wedge\gamma) - R_\nabla(\alpha\wedge\gamma)\nabla_\beta) + \nabla_\gamma R_\nabla(\alpha\wedge\beta) - R_\nabla(\alpha\wedge\beta)\nabla_\gamma$$
$$- ([\nabla_{[\alpha,\beta]},\nabla_\gamma] - \nabla_{[[\alpha,\beta],\gamma]}) + [\nabla_{[\alpha,\gamma]},\nabla_\beta] - \nabla_{[[\alpha,\gamma],\beta]}$$
$$- ([\nabla_{[\beta,\gamma]},\nabla_\alpha] - \nabla_{[[\beta,\gamma],\alpha]})$$
$$= [\nabla_\alpha,[\nabla_\beta,\nabla_\gamma]] + [\nabla_\beta,[\nabla_\gamma,\nabla_\alpha]] + [\nabla_\gamma,[\nabla_\alpha,\nabla_\beta]]$$
$$+ \nabla_{[[\alpha,\beta],\gamma]} + \nabla_{[[\beta,\gamma],\alpha]} + \nabla_{[[\gamma,\alpha],\beta]}$$
$$+ [\nabla_\beta,\nabla_{[\alpha,\gamma]}] - [\nabla_\alpha,\nabla_{[\beta,\gamma]}]$$
$$- [\nabla_\gamma,\nabla_{[\alpha,\beta]}]$$
$$- [\nabla_{[\alpha,\beta]},\nabla_\gamma] + [\nabla_{[\alpha,\gamma]},\nabla_\beta] - [\nabla_{[\beta,\gamma]},\nabla_\alpha] = 0$$

and we have proved the assertion. □

PROPOSITION 2.5. — *Let A be a k-algebra, $\mathfrak{g}$ a Lie-Rinehart algebra and (W,∇) be a $\mathfrak{g}$-connection. Let furthermore R_∇ be the curvature of ∇. Then $d^{2n}(R_\nabla^n) = 0$ for all $n \geqslant 1$.*

Proof. — We prove this by induction on n: By lemma 2.4 we see that the lemma is true for $n = 1$. Assume it is true for $n = k$. We see that

$$d(R_\nabla^{k+1}) = d(R_\nabla^k \wedge R_\nabla) = d(R_\nabla^k)\wedge R_\nabla + (-1)^{2k} R_\nabla^k \wedge d(R_\nabla)$$

and $d(R_\nabla^k)\wedge R_\nabla + (-1)^{2k} R_\nabla^k \wedge d(R_\nabla)$ is zero by the induction hypothesis and lemma 2.4, and we have proved the assertion. □

Let in the following A be a k-algebra, where k is a field of characteristic 0. Let $\mathfrak{g}$ be a Lie-Rinehart algebra and (W,∇) a $\mathfrak{g}$-connection, where W is an A-module of finite presentation. Let $\exp(R_\nabla)$ be defined as $\sum_{n\geqslant 0}\frac{1}{n!}R_\nabla^n$. Consider the open set $U \subseteq \mathrm{Spec}A$ where W is locally free, which exists since W is of finite presentation. By lemma 2.2 we have trace maps

$$\mathrm{tr}^* : C^p(\mathfrak{g}_\mathfrak{p}, \mathrm{End}_{A_\mathfrak{p}}(W_\mathfrak{p})) \to C^*(\mathfrak{g}_\mathfrak{p}, A_\mathfrak{p})$$

defined for all $\mathfrak{p}$ in U, since $W|_U$ is locally free, and these maps glue to give a map of sheaves of complexes

$$\mathrm{tr}^* : C^*(\mathfrak{g}|_U, \mathrm{End}_{\mathcal{O}_U}(W|_U)) \to C^*(\mathfrak{g}|_U, \mathcal{O}_U).$$

We have that $R_\nabla|_U$ is an element of $C^2(\mathfrak{g}|_U, \mathrm{End}_{\mathcal{O}_U}(W|_U))$ and we obtain an element $\exp(R_\nabla|_U)$ in $C^*(\mathfrak{g}|_U, \mathrm{End}_{\mathcal{O}_U}(W|_U))$. By lemma 2.2 we see that the element $d^*(\exp(R_\nabla|_U))$ equals zero, since it vanishes when we localize at all prime-ideals $\mathfrak{p}$ in U. Consider the element $x^* = \mathrm{tr}^*(\exp(R_\nabla|_U))$, which lives in $C^*(\mathfrak{g}|_U, \mathcal{O}_U)$.

Theorem 2.6. — *The following holde: $d^*(x^*) = 0$. Hence $\overline{x^*}$ defines a cohomology-class in* $\mathrm{H}^*(\mathfrak{g}|_U, \mathcal{O}_U)$.

Proof. — It follows from corollary 2.3 that $d^*(x^*) = 0$, since we have already seen that $d^*(\exp(R_\nabla)) = 0$, hence we get a cohomology class as claimed. □

Definition 2.7. — *Let A be a k-algebra where k is a field of characteristic 0 and let $\mathfrak{g}$ be an Lie-Rinehart algebra. Let furthermore W be a $\mathfrak{g}$-connection, where W is an A-module of finite presentation. We let the element $ch^{\mathfrak{g}}(W, \nabla) = \overline{x^*}$ in $\mathrm{H}^*(\mathfrak{g}|_U, \mathcal{O}_U)$ from theorem 2.6 be the* Chern character *of the $\mathfrak{g}$-connection (W, ∇).*

By theorem 2.6 the class $ch^{\mathfrak{g}}(W, \nabla)$ in $\mathrm{H}^*(\mathfrak{g}|_U, \mathcal{O}_U)$ is an invariant of the pair (W, ∇). Given any k-algebra A, where k is a field of characteristic 0, and $\mathfrak{g}$ an Lie-Rinehart algebra, we consider $\mathrm{K}_0(\mathfrak{g})$, the *Grothendieck ring* of $\mathfrak{g}$. This is defined as follows: $\mathrm{K}_0(\mathfrak{g})$ is the free abelian group on the symbols $[W, \nabla]$ module a subgroup D wich we will define below. Here (W, ∇) is a $\mathfrak{g}$-connection which is a locally free A-module of finite rank. The symbol $[W, \nabla]$ denotes the isomorphism-class of the pair (W, ∇). The subgroup D is the group generated by the relations

$$[W \oplus W', \nabla \oplus \nabla'] - [W, \nabla] - [W', \nabla'].$$

That is: $\mathrm{K}_0(\mathfrak{g}) = \oplus\mathbb{Z}[W, \nabla]/D$. (We obviously have that the direct sum of two $\mathfrak{g}$-connections is again a $\mathfrak{g}$-connection.) Given two $\mathfrak{g}$-connections (W, ∇) and (W', ∇'), there exists a natural connection $\nabla \otimes \nabla' = \nabla \otimes 1 + 1 \otimes \nabla'$ on $W \otimes_A W'$, hence $W \otimes_A W'$ is in a natural way a $\mathfrak{g}$-connection. Define a map

$$\otimes : \oplus\mathbb{Z}[W, \nabla] \times \oplus\mathbb{Z}[W, \nabla] \to \mathrm{K}_0(\mathfrak{g})$$

by the following

$$\otimes(\sum_i n_i[W_i, \nabla_i], \sum_j m_j[V_j, \nabla'_j]) = \sum_{i,j} n_i m_j [W_i \otimes_A V_j, \nabla_i \otimes \nabla'_j].$$

Lemma 2.8. — *The map $\otimes$ defines a $\mathbb{Z}$-bilinear product*

$$\mathrm{K}_0(\mathfrak{g}) \otimes_{\mathbb{Z}} \mathrm{K}_0(\mathfrak{g}) \to \mathrm{K}_0(\mathfrak{g})$$

making $\mathrm{K}_0(\mathfrak{g})$ into a commutative $\mathbb{Z}$-algebra.

Proof. — This is straightforward. □

Lemma 2.9. — *Let (W,∇) and (W',∇') be two $\mathfrak{g}$-connections. The the following holds:*

$$R_{\nabla\oplus\nabla'} = R_\nabla \oplus R_{\nabla'} \tag{2.9.1}$$

$$R_{\nabla\otimes\nabla'} = R_\nabla \otimes 1 + 1 \otimes R_{\nabla'} \tag{2.9.2}$$

$$R_\nabla \otimes 1 \wedge 1 \otimes R_{\nabla'} = 1 \otimes R_{\nabla'} \wedge R_\nabla \otimes 1 \tag{2.9.3}$$

$$(R_{\nabla\oplus\nabla'})^n = R^n_\nabla \oplus R^n_{\nabla'}. \tag{2.9.4}$$

Proof. — We first prove equation 2.9.1:

$$R_{\nabla\oplus\nabla'}(\delta\wedge\eta) = [\nabla\oplus\nabla'_\delta, \nabla\oplus\nabla'_\eta] - \nabla\oplus\nabla'_{[\delta,\eta]}.$$

It follows that if we pick (w,w') in $W\oplus W'$, we get

$$R_{\nabla\oplus\nabla'}(\delta\wedge\eta)(w,w')$$

$$\begin{aligned}
&= [\nabla\oplus\nabla'_\delta, \nabla\oplus\nabla'_\eta](w,w') - \nabla\oplus\nabla'_{[\delta,\eta]}(w,w') \\
&= \nabla\oplus\nabla'_\eta \circ \nabla\oplus\nabla'_\delta(w,w') - \nabla\oplus\nabla'_\delta \circ \nabla\oplus\nabla'_\eta(w,w') - \nabla\oplus\nabla'_{[\delta,\eta]}(w,w') \\
&= \nabla\oplus\nabla'_\eta(\nabla_\delta(w),\nabla'_\delta(w')) - \nabla\oplus\nabla'_\delta(\nabla_\eta(w),\nabla'_\eta(w')) - (\nabla_{[\delta,\eta]}(w),\nabla'_{[\delta,\eta]}(w')) \\
&= (\nabla_\eta\nabla_\delta(w),\nabla'_\eta\nabla'_\delta(w')) - (\nabla_\delta\nabla_\eta(w),\nabla'_\delta\nabla'_\eta(w')) - (\nabla_{[\delta,\eta]}(w),\nabla'_{[\delta,\eta]}(w')) \\
&= (R_\nabla(\delta\wedge\eta)(w), R_{\nabla'}(\delta\wedge\eta)(w')) \\
&= R_\nabla \oplus R_{\nabla'}(w,w')
\end{aligned}$$

and equation 2.9.1 follows. We prove equation 2.9.2: Let $w\otimes w'$ be an element of $W\otimes_A W'$, and let $\nabla\otimes\nabla' = \nabla\otimes 1 + 1\otimes\nabla'$ be the $\mathfrak{g}$-connection on $W\otimes_A W'$. We get

$$R_{\nabla\otimes\nabla'}(\delta\wedge\eta)(w\otimes w')$$

$$= [\nabla\otimes\nabla'_\delta, \nabla\otimes\nabla'_\eta](w\otimes w') - \nabla\otimes\nabla'_{[\delta,\eta]}(w\otimes w')$$

$$= \nabla\otimes\nabla'_\eta\circ\nabla\otimes\nabla'_\delta(w\otimes w') - \nabla\otimes\nabla'_\delta\circ\nabla\otimes\nabla'_\eta(w\otimes w') - \nabla\otimes\nabla'_{[\delta,\eta]}(w\otimes w')$$
$$= \nabla\otimes\nabla'_\eta(\nabla_\delta(w)\otimes w' + w\otimes\nabla'_\delta(w')) - \nabla\otimes\nabla'_\delta(\nabla_\eta(w)\otimes w' + w\otimes\nabla'_\eta(w')$$
$$-(\nabla_{[\delta,\eta]}(w)\otimes w' + w\otimes\nabla_{[\delta,\eta]}(w'))$$
$$= \nabla_\eta\nabla_\delta(w)\otimes w' + \nabla_\delta(w)\otimes\nabla'_\eta(w') + \nabla_\eta(w)\otimes\nabla'_\delta(w') + w\otimes\nabla'_\eta\nabla'_\delta(w')$$
$$-(\nabla_\delta\nabla_\eta(w)\otimes w' + \nabla_\eta(w)\otimes\nabla'_\delta(w') + \nabla_\delta(w)\otimes\nabla'_\eta(w') + w\otimes\nabla'_\delta\nabla'_\eta(w'))$$
$$-\nabla_{[\delta,\eta]}(w)\otimes w' - w\otimes\nabla_{[\delta,\eta]}(w')$$
$$= [\nabla_\delta,\nabla_\eta](w)\otimes w' + w\otimes[\nabla'_\delta,\nabla'_\eta](w') - \nabla_{[\delta,\eta]}(w)\otimes w' - w\otimes\nabla'_{[\delta,\eta]}(w')$$
$$= R_\nabla(\delta\wedge\eta)(w)\otimes w' + w\otimes R_{\nabla'}(\delta\wedge\eta)(w')$$

and equation 2.9.2 follows. We prove equation 2.9.3: Let ω be an element of $\wedge^4\mathfrak{g}$. We get

$$R_\nabla\otimes 1\wedge 1\otimes R_{\nabla'}(w)$$
$$= \sum_{(2,2)} sgn(\sigma)(R_\nabla\otimes 1, 1\otimes R_{\nabla'})\sigma(\omega)$$
$$= \sum_{(2,2)} sgn(\sigma)R_\nabla(\sigma(\omega))\otimes 1\circ 1\otimes R_{\nabla'}(\sigma(\omega))$$
$$= \sum_{(2,2)} sgn(\sigma)1\otimes R_{\nabla'}(\sigma(\omega))\circ R_\nabla(\sigma(\omega))\otimes 1$$
$$= \sum_{(2,2)} sgn(\sigma)(1\otimes R_{\nabla'}, R_\nabla\otimes 1)\sigma(\omega)$$
$$= 1\otimes R_{\nabla'}\wedge R_\nabla\otimes 1(\omega)$$

and equation 2.9.3 follows. Finally we prove equation 2.9.4 by induction on n. For n=2 we get the following: Let $\omega = \delta_1\wedge\cdots\wedge\delta_4$, and for any $(2,2)$-shuffle σ put $\sigma(\omega)^1 = \delta_{\sigma(1)}\wedge\delta_{\sigma(2)}$ and $\sigma(\omega)^2 = \delta_{\sigma(3)}\wedge\delta_{\sigma(4)}$. We get

$$(R_\nabla\oplus R_{\nabla'})^2(\omega)$$

$$= \sum_{(2,2)} sgn(\sigma)(R_\nabla\oplus R_{\nabla'}, R_\nabla\oplus R_{\nabla'})\sigma(\omega)$$
$$= \sum_{(2,2)} sgn(\sigma)R_\nabla\oplus R_{\nabla'}(\sigma(\omega)^1)\circ R_\nabla\oplus R_{\nabla'}(\sigma(\omega)^2)$$
$$= \sum_{(2,2)} sgn(\sigma)R_\nabla(\sigma(\omega)^1)\oplus R_{\nabla'}(\sigma(\omega)^1)\circ R_\nabla(\sigma(\omega)^2)\oplus R_{\nabla'}(\sigma(\omega)^2)$$
$$= \sum_{(2,2)} sgn(\sigma)R_\nabla(\sigma(\omega)^1)R_\nabla(\sigma(\omega)^2)\oplus R_{\nabla'}(\sigma(\omega)^1)R_{\nabla'}(\sigma(\omega)^2)$$

$$= (\sum_{(2,2)} sgn(\sigma)(R_\nabla, R_\nabla)\sigma(\omega)) \oplus (\sum_{(2,2)} sgn(\sigma)(R_{\nabla'}, R_{\nabla'})\sigma(\omega))$$
$$= R^2_\nabla(\omega) \oplus R^2_{\nabla'}(\omega)$$

and we have proved equation 2.9.4 for $n = 2$. Assume the equation is true for $n = k$. Put $n = k + 1$, and let $\omega = \delta_1 \wedge \cdots \wedge \delta_{2k+2}$. Put also for any $(2k, 2)$-shuffle σ, $\sigma(\omega)^1 = \delta_{\sigma(1)} \wedge \cdots \wedge \delta_{\sigma(2k)}$ and $\sigma(\omega)^2 = \delta_{\sigma(2k+1)} \wedge \delta_{\sigma(2k+2)}$. We get

$$R^k_{\nabla\oplus\nabla'} R_{\nabla\oplus\nabla'}(\omega)$$
$$= \sum_{(2k,2)} sgn(\sigma)(R^k_{\nabla\oplus\nabla'}, R_{\nabla\oplus\nabla'})\sigma(\omega).$$

By the induction hypothesis we get

$$= \sum_{(2k,2)} sgn(\sigma)(R^k_\nabla \oplus R^k_{\nabla'}, R_\nabla \oplus R_{\nabla'})\sigma(\omega)$$
$$= \sum_{(2k,2)} sgn(\sigma) R^k_\nabla \oplus R^k_{\nabla'}(\sigma(\omega)^1) \circ R_\nabla \oplus R_{\nabla'}(\sigma(\omega)^2)$$
$$= \sum_{(2k,2)} sgn(\sigma) R^k_\nabla(\sigma(\omega)^1) \oplus R^k_{\nabla'}(\sigma(\omega)^1) \circ R_\nabla(\sigma(\omega)^2) \oplus R_{\nabla'}(\sigma(\omega)^2)$$
$$= \sum_{(2k,2)} sgn(\sigma) R^k_\nabla(\sigma(\omega)^1) R_\nabla(\sigma(\omega)^2) \oplus R^k_{\nabla'}(\sigma(\omega)1) \circ R_{\nabla'}(\sigma(\omega)^2)$$
$$= (\sum_{(2k,2)} sgn(\sigma)(R^k_\nabla, R_\nabla)\sigma(\omega)) \oplus (\sum_{(2k,2)} sgn(\sigma)(R^k_{\nabla'}, R_{\nabla'})\sigma(\omega))$$
$$= (R^{k+1}_\nabla \oplus R^{k+1}_{\nabla'})(\omega)$$

and equation 2.9.4 follows, and we have proved the lemma. □

Lemma 2.10. — *Let W and W' be two free A-modules, and let ϕ in $\mathrm{End}_A(W)$ and ψ in $\mathrm{End}_A(W')$ be two endomorphisms. Then the following holds*

$$\mathrm{tr}(\phi \otimes \psi) = \mathrm{tr}(\phi)\mathrm{tr}(\psi).$$

Proof. — Let $W = \oplus_{i=1}^n Ae_i$ and $W' = \oplus_{j=1}^m Af_j$ be two direct-sum decompositions of W and W'. Put also $\phi = (a_{ij})$ and $\psi = (b_{ij})$ where a_{ij} and b_{ij} are elements of A. One verifies that for instance $\mathrm{tr}(\phi) = \sum_i e_i \phi e_i$. We get

$$\mathrm{tr}(\phi \otimes \psi) = \sum_{i,j} e_i \otimes f_j(\phi \otimes \psi) e_i \otimes f_j.$$

It is trivial to check that $e_k \otimes f_l(\phi \otimes \psi) e_m \otimes f_k$ equals $a_{km} b_{ln}$, hence we get

$$\sum_{i,j} a_{ii} b_{jj} = (\sum_i a_{ii})(\sum_j b_{jj}) = (\mathrm{tr}\phi)(\mathrm{tr}\psi)$$

and the lemma follows. □

Lemma 2.11. — *Let (W, ∇) and (W', ∇') be two locally free $\mathfrak{g}$-connections, then*

$$\mathrm{tr}(R^n_\nabla \otimes 1 \wedge 1 \otimes R^m_{\nabla'}) = (\mathrm{tr}(R^n_\nabla)) \wedge (\mathrm{tr}(R^m_{\nabla'})).$$

Proof. — Let $\omega = \delta_1 \wedge \cdots \wedge \delta_{2(n+m)}$, and put for any $(2n, 2m)$ shuffle σ, $\sigma(\omega)^1 = \delta_{\sigma(1)} \wedge \cdots \wedge \delta_{\sigma(2n)}$ and $\sigma(\omega)^2 = \delta_{\sigma(2n+1)} \wedge \cdots \wedge \delta_{\sigma(2(n+m))}$. We see that

$$\begin{aligned} &R^n_\nabla \otimes 1 \wedge 1 \otimes R^m_{\nabla'}(\omega) \\ &= \sum_{(2n,2m)} sgn(\sigma)(R^n_\nabla \otimes 1, 1 \otimes R^m_{\nabla'})\sigma(\omega) \\ &= \sum_{(2n,2m)} sgn(\sigma) R^n_\nabla(\sigma(\omega)^1) \otimes 1 \circ 1 \otimes R^m_{\nabla'}(\sigma(\omega)^2) \\ &= \sum_{(2n,2m)} sgn(\sigma) R^n_\nabla(\sigma(\omega)^1) \otimes R^m_{\nabla'}(\sigma(\omega)^2). \end{aligned}$$

By lemma 2.10 we get

$$\begin{aligned} &\mathrm{tr}(R^n_\nabla \otimes 1 \wedge 1 \otimes R^m_{\nabla'}(\omega)) \\ &= \mathrm{tr}(\sum_{(2n,2m)} sgn(\sigma) R^n_\nabla(\sigma(\omega)^1) \otimes R^m_{\nabla'}(\sigma(\omega)^2)) \\ &= \sum_{(2n,2m)} sgn(\sigma)(\mathrm{tr} \circ R^n_\nabla)(\sigma(\omega)^1)(\mathrm{tr} \circ \mathrm{R}^m_{\nabla'})(\sigma(\omega)^2) \\ &= \sum_{(2n,2m)} sgn(\sigma)(\mathrm{tr} \circ R^n_\nabla, \mathrm{tr} \circ R^m_{\nabla'})\sigma(\omega) = (\mathrm{tr} \circ R^n_\nabla) \wedge (\mathrm{tr} \circ R^m_{\nabla'})(\omega) \end{aligned}$$

and we have proved the assertion. □

We can now prove the existence of the Chern character.

Theorem 2.12. — *There exists a ring homomorphism*

$$ch^{\mathfrak{g}} : \mathrm{K}_0(\mathfrak{g}) \to \mathrm{H}^*(\mathfrak{g}, A)$$

from the Grothendieck ring $\mathrm{K}_0(\mathfrak{g})$ *to the cohomology ring* $\mathrm{H}^*(\mathfrak{g}, A)$.

Proof. — For every locally free $\mathfrak{g}$-connection W of finite rank

we obtain by Theorem 2.6 a cohomology class

$ch^{\mathfrak{g}}(W)$ in $\mathrm{H}^*(\mathfrak{g}, A)$. Define a map $\phi : \oplus\mathbb{Z}[W, \nabla] \to \mathrm{H}^*(\mathfrak{g}, A)$ by the formula

$$\phi(\sum_i n_i[W_i, \nabla_i] = \sum_i n_i ch(W_i, \nabla_i).$$

We want to show that the map ϕ gives rise to a well-defined map

$$ch^{\mathfrak{g}} : \mathrm{K}_0(\mathfrak{g}) \to \mathrm{H}^*(\mathfrak{g}, A).$$

Let $[W \oplus W', \nabla \oplus \nabla'] - [W, \nabla] - [W', \nabla']$ be a generator of the group D, where $\mathrm{K}_0(\mathfrak{g}) = \oplus\mathbb{Z}[W, \nabla]/D$. We get

$$ch^{\mathfrak{g}}([W \oplus W', \nabla \oplus \nabla'] - [W, \nabla] - [W', \nabla']) =$$

$$ch^{\mathfrak{g}}(W \oplus W', \nabla \oplus \nabla') - ch^{\mathfrak{g}}(W, \nabla) - ch^{\mathfrak{g}}(W', \nabla')$$
$$= \sum_{n \geqslant 0} \frac{1}{n!} \mathrm{tr}(R_{\nabla \oplus \nabla'})^n - \sum_{k \geqslant 0} \frac{1}{k!} \mathrm{tr} R^k_{\nabla} - \sum_{l \geqslant 0} \frac{1}{l!} \mathrm{tr} R^l_{\nabla'}.$$

By lemma 2.9, equation 2.9.1 and 2.9.4 we get

$$\sum_{n \geqslant 0} \frac{1}{n!} \mathrm{tr}(R^n_{\nabla} \oplus R^n_{\nabla'}) - \sum_{k \geqslant 0} \frac{1}{k!} \mathrm{tr}(R^k_{\nabla}) - \sum_{l \geqslant 0} \frac{1}{l!} \mathrm{tr}(R^l_{\nabla'})$$

$$= \sum_{n \geqslant 0} \frac{1}{n!} (tr R^n_{\nabla} + \mathrm{tr} R^n_{\nabla'}) - \sum_{k \geqslant 0} \frac{1}{k!} \mathrm{tr} R^k_{\nabla} - \sum_{l \geqslant 0} \mathrm{tr} R^l_{\nabla'} = 0$$

hence ϕ gives rise to a map $ch^{\mathfrak{g}} : \mathrm{K}_0(\mathfrak{g}) \to \mathrm{H}^*(\mathfrak{g}, A)$, and obviously $ch^{\mathfrak{g}}$ is a group-homomorphism. We show that $ch^{\mathfrak{g}}$ is a ring homomorphism: Put for any $\mathfrak{g}$-connection (W, ∇), $ch_n(W, \nabla) = \frac{1}{n!} \mathrm{tr} R^n_{\nabla}$. We have that $ch^{\mathfrak{g}}(W, \nabla) = \sum_{n \geqslant 0} ch_n(W, \nabla)$. Since $C^*(\mathfrak{g}, \mathrm{End}_A(W \otimes_A W'))$ is an associative A-algebra and by lemma 2.9, equation 2.9.3 we have that $R_{\nabla} \otimes 1 \wedge 1 \otimes R_{\nabla'} = 1 \otimes R_{\nabla'} \wedge R_{\nabla} \otimes 1$, we can apply the binomial-theorem. We get

$$ch_n(W \otimes W', \nabla \otimes \nabla') = \frac{1}{n!} (R_{\nabla \otimes \nabla'})^n$$

and by lemma 2.9, equation 2.9.2 we get

$$\frac{1}{n!}\operatorname{tr}(R_\nabla \otimes 1 + 1 \otimes R_{\nabla'})^n = \sum_{i+j=n} \frac{1}{i!j!}\operatorname{tr}(R_\nabla \otimes 1)^i (1 \otimes R_{\nabla'})^j.$$

By lemma 2.11 we get

$$\begin{aligned}\sum_{i+j=n} \frac{1}{i!j!}(\operatorname{tr} R_\nabla)^i \wedge (\operatorname{tr} R_{\nabla'})^j &= \sum_{i+j=n} (\frac{1}{i!}\operatorname{tr} R^i_\nabla) \wedge (\frac{1}{j!}\operatorname{tr} R^j_{\nabla'}) \\ &= \sum_{i+j=n} ch_i(W,\nabla) ch_j(W',\nabla').\end{aligned}$$

The following holds

$$\begin{aligned} ch^{\mathfrak{g}}(W\otimes W', \nabla\otimes\nabla') &= \sum_{n\geqslant 0} ch_n(W\otimes W', \nabla\otimes\nabla') \\ &= \sum_{n\geqslant 0}\left(\sum_{i+j=n} ch_i(W,\nabla)ch_j(W',\nabla')\right) \\ &= \left(\sum_{k\geqslant 0} ch_k(W,\nabla)\right)\left(\sum_{l\geqslant 0} ch_l(W',\nabla')\right) \\ &= ch^{\mathfrak{g}}(W,\nabla)ch^{\mathfrak{g}}(W',\nabla'),\end{aligned}$$

and the theorem follows. □

3. On independence of choice of connection.

In this section we prove the fact that the Chern character $ch^{\mathfrak{g}}(W,\nabla)$ of an A-module with a $\mathfrak{g}$-connection from Theorem 2.6 is independent with respect to choice of connection ∇. Let in the following A be a k-algebra where k is a field of characteristic zero. Let furthermore $\mathfrak{g}$ be a Lie-Rinehart algebra with anchor map $\alpha : \mathfrak{g} \to \operatorname{Der}_k(A)$. We first prove a series of technical lemmas:

Lemma 3.1. — *We get in a natural way a map $\alpha\otimes 1 : \mathfrak{g}[t] \to \operatorname{Der}_k(A[t])$, making $\mathfrak{g}[t]$ into an $(k, A[t])$-Lie-Rinehart algebra.*

Proof. — Define a k-Lie algebra structure on $\mathfrak{g}[t]$ as follows: $[\sum_i \delta_i \otimes f_j, \sum_j \eta_j \otimes g_j] = \sum_{i,j}[\delta_i,\eta_j]\otimes f_i g_j$. Define furthermore a map $\alpha\otimes 1 : \mathfrak{g}[t] \to \operatorname{Der}_k(A[t])$ by $\alpha\otimes 1(\delta\otimes f)(a\otimes g) = \alpha(\delta)(a)\otimes fg$, then it is straightforward to check that $\mathfrak{g}[t]$ is a $(k, A[t])$-Lie-Rinehart algebra. □

Lemma 3.2. — *Let W be an A-module with a $\mathfrak{g}$-connection ∇. There exists a $\mathfrak{g}[t]$-connection $\nabla \otimes 1$ on the $A[t]$-module $W[t]$.*

Proof. — Define the following map: $\nabla \otimes 1 : \mathfrak{g}[t] \to \mathrm{End}_k(W[t])$, by letting $\nabla \otimes 1(\delta \otimes f)(w \otimes g) = \nabla(\delta)(w) \otimes fg$. Then it is straightforward to check that $\nabla \otimes 1$ is a $\mathfrak{g}[t]$-connection. □

Lemma 3.3. — *Let ∇_0 and ∇_1 be $\mathfrak{g}$-connections on W, then $\nabla = \nabla_1 \otimes t + \nabla_0 \otimes (1-t)$ is a $\mathfrak{g}[t]$-connection on $W[t]$.*

Proof. — This is straightforward. □

Lemma 3.4. — *Let ∇ be a $\mathfrak{g}$-connection on an A-module W. Let $\nabla \otimes 1$ be the induced $\mathfrak{g}[t]$-connection on $W[t]$. Then the curvature $R_{\nabla\otimes 1}$ defines a natural map*

$$R_{\nabla\otimes 1} : \wedge^2 \mathfrak{g}[t] \to \mathrm{End}_A(W)[t].$$

Proof. — Define $R_{\nabla\otimes 1}(\delta \otimes f \wedge \eta \otimes g) = R_\nabla(\delta \wedge \eta) \otimes fg$, then the lemma follows. □

Lemma 3.5. — *Let ∇ be a $\mathfrak{g}$-connection on the A-module W, and consider the induced connection $\nabla \otimes 1$ on $W[t]$. There exists a map $p^i_* : C^p(\mathfrak{g}[t], W[t]) \to C^p(\mathfrak{g}, W)$ making commutative diagrams*

$$\begin{array}{ccc} C^p(\mathfrak{g}[t], W[t]) & \xrightarrow{d} & C^{p+1}(\mathfrak{g}[t], W[t]) \\ \downarrow p^i_* & & \downarrow p^i_* \\ C^p(\mathfrak{g}, W) & \xrightarrow{d} & C^{p+1}(\mathfrak{g}, W) \end{array}$$

for all p.

Proof. — Define the maps

$$p^i_* : C^p(\mathfrak{g}[t], W[t]) \to C^p(\mathfrak{g}[t], W[t])$$

as follows: There exists an obvious map $q : \wedge^p \mathfrak{g} \to \wedge^p \mathfrak{g}[t]$ defined by mapping $\delta_1 \wedge \cdots \wedge \delta_p$ to $\delta_1 \otimes 1 \wedge \cdots \wedge \delta_p \otimes 1$. There exists a map $p^i : W[t] \to W$ defined by letting $p^i(t) = i$ for $i = 0, 1$. Put now for any A-linear map $\phi : \wedge^p \mathfrak{g}[t] \to W[t]$, $p^i_*(\phi) = p^i \circ \phi \circ q$. We show that we get commutative diagrams as claimed: Consider first $p^i_*(d\phi)(\delta_1 \wedge \cdots \wedge \delta_{p+1}) =$

$$p^i d(\phi)(\delta_1 \otimes 1 \wedge \cdots \wedge \delta_{p+1} \otimes 1) =$$

$$
\begin{aligned}
& p^i + (\sum_{k=1}^{p+1}(-1)^{k+1}\nabla(\delta_k)\otimes 1\phi(\cdots\wedge\delta_k\hat{\otimes}1\wedge\cdots) \\
& + \sum_{k<l}(-1)^{k+l}\phi([\delta_k\otimes 1,\delta_l\otimes 1]\wedge\cdots\delta_k\hat{\otimes}1\cdots\delta_l\hat{\otimes}1\cdots)) \\
& = \sum_{k=1}^{p+1}p^i\nabla(\delta_k)\otimes 1\phi(\cdots\delta_k\hat{\otimes}1\cdots) \\
& + \sum_{k<l}(-1)^{k+l}p^i\phi([\delta_k,\delta_l]\otimes 1\wedge\cdots\delta_k\hat{\otimes}1\cdots\delta_l\hat{\otimes}1\cdots).
\end{aligned}
\tag{3.5.1}
$$

Consider

$$
\begin{aligned}
& d(p^i_*\phi)(\delta_1\wedge\cdots\wedge\delta_{p+!}) \\
& = \sum_{k=1}^{p+1}(-1)^{k+1}\nabla(\delta_k)p^i_*\phi(\cdots\hat{\delta}_k\cdots) \\
& + \sum_{k<l}(-1)^{k+l}p^i_*\phi([\delta_k,\delta_l]\cdots\hat{\delta}_k\cdots\hat{\delta}_l\cdots) \\
& = \sum_{k=1}^{p+1}(-1)^{k+1}\nabla(\delta_k)p^i\phi(\cdots\delta_1\hat{\otimes}1\cdots) \\
& + \sum_{k<l}(-1)^{k+l}p^i\phi([\delta_k,\delta_l]\otimes 1\cdots\delta_k\hat{\otimes}1\cdots\delta_l\hat{\otimes}1\cdots).
\end{aligned}
\tag{3.5.2}
$$

One checks that $\nabla(\delta_k)p^i = p^i\nabla(\delta_k)\otimes 1$ hence equation 3.5.1 equals equation 3.5.2, and the claim follows. □

Lemma 3.6. — *Given two $\mathfrak{g}$-connections ∇_0, ∇_1 on W, and let $\nabla = \nabla_1\otimes t+\nabla_0\otimes(1-t)$ be the induced connection on $W[t]$. Then the curvature R_∇ is an element of $C^2(\mathfrak{g}[t], \mathrm{End}_A(W)[t])$, and it follows that $p^i_*(R_\nabla) = R_{\nabla_i}$ for $i = 0$ and 1.*

Proof. — This is straighforward. □

Lemma 3.7. — *Consider the map*

$$p^i_* : C^p(\mathfrak{g}[t], \mathrm{End}_A(W)[t]) \to C^p(\mathfrak{g}, \mathrm{End}_A(W)).$$

Let ϕ and ψ be elements of $C^p(\mathfrak{g}[t], \mathrm{End}_A(W)[t])$ and $C^q(\mathfrak{g}[t], \mathrm{End}_A(W)[t])$ respectively. The following holds:

$$p^i_*(\phi \wedge \psi) = p^i_*(\phi) \wedge p^i_*(\psi).$$

In particular it follows that $p^i_*(R^k_\nabla) = (p^i_* R_\nabla)^k$.

Proof. — This is straighforward. □

Lemma 3.8. — *There exists for all p commutative diagrams*

$$\begin{array}{ccc} C^p(\mathfrak{g}[t], \mathrm{End}_A(W)[t]) & \xrightarrow{\mathrm{tr}\otimes 1} & C^p(\mathfrak{g}[t], A[t]) \\ \downarrow p^i_* & & \downarrow p^i_* \\ C^p(\mathfrak{g}, \mathrm{End}_A(W)) & \xrightarrow{\mathrm{tr}} & C^p(\mathfrak{g}, A) \end{array}$$

in particular we get $p^i_*(\mathrm{tr}(R^k_\nabla)) = \mathrm{tr}(p^i_* R^k_\nabla)$.

Proof. — Let $\phi : \wedge^p \mathfrak{g} \to \mathrm{End}_A(W)[t]$ be an A-linear map. Since W is locally free, we have a trace map $\mathrm{tr} : \mathrm{End}_A(W) \to A$, and we get a trace-map $\mathrm{tr} \otimes 1 : \mathrm{End}_A(W)[t] \to A[t]$, and we get $\mathrm{tr} \otimes 1 \circ \phi$ in $C^p(\mathfrak{g}[t], A[t])$. We see that $p^i_*(\mathrm{tr} \otimes 1 \circ \phi)(\delta_1 \wedge \cdots \wedge \delta_p) =$

$$p^i_* \circ \mathrm{tr} \otimes 1 \circ \phi(\delta_1 \otimes 1 \wedge \cdots \wedge \delta_p \otimes 1) \tag{3.8.1}$$

We also see that $\mathrm{tr}(p^i_*(\phi))(\delta_1 \wedge \cdots \wedge \delta_p) =$

$$\mathrm{tr} \circ p^i \circ \phi(\delta_1 \otimes 1 \wedge \cdots \wedge \delta_p \otimes 1) \tag{3.8.2}$$

and since $p^i_* \circ \mathrm{tr} \otimes 1 = \mathrm{tr} \circ p^i_*$ we see that equation 3.8.1 equals equation 3.8.2, and we have proved the assertion. □

Lemma 3.9. — *The maps $p^i_* : C^p(\mathfrak{g}[t], \mathrm{End}_A(W)[t]) \to C^p(\mathfrak{g}, \mathrm{End}_A(W))$ satisfy $p^i_*(\phi \wedge \psi) = p^i_*(\phi) \wedge p^i_*(\psi)$. In particular we get $p^i_*(R^k_\nabla) = (p^i_* R_\nabla)^k$.*

Proof. — $p^i_*(\phi \wedge \psi)(\delta_1 \wedge \cdots \wedge \delta_{p+q})$

$$= p^i(\phi \wedge \psi(\delta_1 \otimes 1 \wedge \cdots \wedge \delta_{p+q} \otimes 1)$$
$$= p_i \sum_{(p,q)} sgn(\sigma)\phi(\delta_{\sigma(1)} \otimes 1 \cdots \delta_{\sigma(p)} \otimes 1)\psi(\delta_{\sigma(p+1)} \otimes 1 \cdots \delta_{\sigma(p+q)} \otimes 1)$$

$$= \sum_{(p,q)} sgn(\sigma) p_*^i \phi(\delta_{\sigma(1)} \cdots \delta_{\sigma(p)}) p_*^i(\psi)(\delta_{\sigma(p+1)} \cdots \delta_{\sigma(p+q)})$$

$$= p_*^i(\phi) \wedge p_*^i(\psi)(\delta_1 \wedge \cdots \wedge \delta_{p+q})$$

and the lemma follows. □

We are now in position to prove the main theorem of this section.

THEOREM 3.10. — *Let A be any k-algebra where k is any field, and let $\mathfrak{g}$ be a Lie-Rinehart algebra. Let W be a locally free A-module with a $\mathfrak{g}$-connection ∇. The class $ch_n(W, \nabla)$ in $\mathrm{H}^{2n}(\mathfrak{g}, A)$ is independent with respect to choice of connection.*

Proof. — Consider the complex $C^*(\mathfrak{g}[t], A[t])$:

$$\cdots \to C^{p-1}(\mathfrak{g}[t], A[t]) \to C^p(\mathfrak{g}[t], A[t]) \to C^{p+1}(\mathfrak{g}[t], A[t]) \to \cdots$$

By functoriality we get:

$$C^p(\mathfrak{g}[t], A[t]) = \mathrm{Hom}_A(\wedge^p(\mathfrak{g} \otimes_A A[t]), A[t]) = \mathrm{Hom}_A((\wedge^p \mathfrak{g}) \otimes_A A[t], A[t])$$

$$= \mathrm{Hom}_A(\wedge^p \mathfrak{g}, A) \otimes_A A[t] = \mathrm{Hom}_A(\wedge^p \mathfrak{g}, A) \otimes_k k[t].$$

It follows that we get an isomorphism at the level of cohomology-groups

$$\mathrm{H}^i(\mathfrak{g}[t], A[t]) \cong \mathrm{H}^i(\mathfrak{g}, A).$$

We get induced maps on cohomology groups

$$p_*^i : \mathrm{H}^{2k}(\mathfrak{g}[t], A[t]) \to \mathrm{H}^{2k}(\mathfrak{g}, A)$$

with the property that

$$p_*^i(\overline{\mathrm{tr}(R_\nabla)}) = \overline{\mathrm{tr}(R_{\nabla_i})}.$$

It follows that

$$\overline{\mathrm{tr}(R_{\nabla_0})} = \overline{\mathrm{tr}(R_{\nabla_1})},$$

and the theorem follows. □

It follows from Theorem 3.10 that the Chern character from Theorem 2.12 is independent of choice of connection. We get a corollary:

Corollary 3.11. — *Let A be a smooth k-algebra of finite type where k is a field of characteristic zero. There exists a ring homomorphism*

$$ch^A : \mathrm{K}_0(A) \to \mathrm{H}^*_{\mathrm{DR}}(A).$$

Proof. — There exists a natural map

$$\Omega^p_A \to (\Omega^p_A)^{**} = \mathrm{Hom}_A(\wedge^p \mathrm{Der}_k(A), A)$$

hence we get when Ω^1_A is locally free an isomorphism $i_p : \mathrm{H}^p_{\mathrm{DR}}(A) \cong \mathrm{H}^p(\mathrm{Der}_k(A), A)$. Any connection

$$\nabla : E \to E \otimes \Omega^1_A$$

gives rise to a covariant derivation

$$\overline{\nabla} : \mathrm{Der}_k(A) \to \mathrm{End}_k(E).$$

One checks that the Chern class defined by $\overline{\nabla}$ agrees with the one defined by ∇ via i_p, and the claim follows. □

The ring homomorphism from Corollary 3.11 is the classical Chern character from Theorem 2.1.

Note that by functoriality there always exist a diagram

$$\begin{array}{ccc} \mathrm{K}_0(A) & \longrightarrow & \mathrm{K}_0(\mathfrak{g}) \\ \downarrow{\scriptstyle ch^A} & & \downarrow{\scriptstyle ch^A} \\ \mathrm{H}^*_{\mathrm{DR}}(A) & \longrightarrow & \mathrm{H}^*(\mathfrak{g}, A), \end{array}$$

but the map $\mathrm{K}_0(A) \to \mathrm{K}_0(\mathfrak{g})$ is not surjective in general: by the example in [18], section 2 the following holds. Let k be a field of characteristic zero and consider $\mathcal{O}(d)$ on $\mathbf{P}^1_k$. There exist a left $\mathcal{O}_{\mathbf{P}^1}$-linear splitting

$$\mathcal{P}^1(\mathcal{O}(d)) \cong \mathcal{O}(d-1) \oplus \mathcal{O}(d-1),$$

hence the Atiyah-sequence

$$0 \to \Omega^1 \otimes \mathcal{O}(d) \to \mathcal{P}^1(\mathcal{O}(d)) \to \mathcal{O}(d) \to 0$$

is not left split. It follows that $\mathcal{O}(d)$ does not have a connection. If we consider the linear Lie-Rinehart algebra $\mathbf{V}_{\mathcal{O}(d)}$ of $\mathcal{O}(d)$ introduced in section 1 in [18], we see that $\mathcal{O}(d)$ has a $\mathbf{V}_{\mathcal{O}(d)}$-connection. It follows that the natural map

$$\mathrm{K}_0(\mathbf{P}^1_k) \to \mathrm{K}_0(\mathbf{V}_{\mathcal{O}(d)})$$

is not surjective hence $ch^{\mathfrak{g}}$ is not determined by ch^A in general. Note also that the construction of the Chern-class $ch_n(W, \nabla)$ is valid for any S-algebra A, where S and A are commutative rings. The Chern character exists when S is a ring containing the rationals.

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