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Proof of the Treves theorem on the KdV hierarchy

<http://aif.cedram.org/item?id=AIF_2005__55_6_2015_0>
PROOF OF THE TREVES THEOREM
ON THE KdV HIERARCHY

by Leonid A. Dickey

1. Necessity of the Treves condition for KdV.

Here we give a shorter proof of the Treves theorem [1] and some addition to the theorem (Theorem 2 below). A discussion of the significance of the theorem, and a part of the present proof (necessity) one can find in [2] along with an attempt to generalize the theorem.

**Theorem 1** (Treves). — A differential polynomial of $u$: $P[u] = P(u, u', u'', \ldots)$ is, up to an exact derivative, a linear combination of $\text{res}_\partial L^{m/2}$ where $L = \partial^2 + u$ if and only if

$$\text{res}_x P(\tilde{u}(x), \tilde{u}'(x), \tilde{u}''(x), \ldots) = 0,$$

where $\tilde{u}(x)$ is an arbitrary formal Laurent series of the form

$$\tilde{u}(x) = -2x^{-2} + \sum_0^\infty u_i(x^i/i!), \quad u_1 = 0.$$

(The following notations are used: $\text{res}_\partial$ symbolizes the coefficient in $\partial^{-1}$, and $\text{res}_x$ the coefficient in $x^{-1}$).

We also prove the following addition to the Treves theorem:

**Theorem 2.** — A differential polynomial $P[u] = P(u, u', u'', \ldots)$ is exactly a linear combination of $\text{res}_\partial L^{m/2}$ (without additional derivative

*Keywords: Kdv hierarchy, first integrals, Treves’ criterion.*

*Math. classification: 35Q53.*
The beginning of the proof of the theorem 1. — In this section we prove the necessity of the Treves condition (1).

Let us try to “undress” the operator \( L = \partial^2 + u \):
\[
\partial^2 + u = w\partial^2 w^{-1} = (1 + w_1\partial^{-1} + w_2\partial^{-2} + \cdots)\partial^2(1 + w_1\partial^{-1} + w_2\partial^{-2} + \cdots)^{-1}.
\]
Rewrite this as
\[
(1 + w_1\partial^{-1} + w_2\partial^{-2} + \cdots)\partial^2 = (\partial^2 + u)(1 + w_1\partial^{-1} + w_2\partial^{-2} + \cdots)
\]
which yields the recurrence relations
\[
2w_1' + u = 0
\]
\[
2w_{k+1}'' + w_k' + uw_k = 0, \quad k > 0.
\]

First, a lemma will be proven:

**Lemma 1.** — If a formal Laurent series \( \tilde{u} = -2/x^2 + \sum_{0}^{\infty} u_i(x^i/i!) \), \( u_1 = 0 \), is taken for \( u \), then all \( w_k \) can be found in the form of Laurent series.

The necessity of the Treves condition immediately follows from this lemma. Indeed,
\[
\text{res}_\partial L^{m/2} = \text{res}_\partial w\partial^m w^{-1} = \text{res}_\partial[w\partial^m, w^{-1}] + \text{res}_\partial w^{-1} w\partial^m
\]
\[
= \text{res}_\partial[w\partial^m, w^{-1}] = \partial( \quad )
\]
since the residue of the commutator of any two operators is an exact derivative. In this case this is an exact derivative of a Laurent series. Therefore, it cannot contain a term with \( x^{-1} \), i.e., its residue with respect to the variable \( x \) is zero.

**Proof of the lemma 1.** — We have \( 2w_1' + \tilde{u} = 0 \) whence
\[
w_1 = -\frac{1}{x} - \frac{1}{2} \sum_{0}^{\infty} u_i \frac{x^{i+1}}{(i+1)!} = -\frac{1}{x} - \sum_{1}^{\infty} b_i \frac{x^i}{i!}, \quad (b_2 = 0).
\]
Further, \(-w_2 = w_1'' - 2w_1' w_1 \) and \(-w_2 = w_1' - w_1^2 \). It is easy to calculate (taking into account that \( b_2 = 0 \)) that \( w_2 = (5b_3/6 + b_1^2)x^2 + O(x^3) = Ax^2 + O(x^3) = O(x^2) \). Here, \( O(x^n) \) means a power series starting with the
term involving \( x^n \). Using the recurrence formula, it is not difficult to show by induction that all the next terms have the same form:

\[
-w'_{k+1} = w''_k - 2w'_1w_k = 2A + O(x) - 2(x^{-2} + b_1 + O(x^2))(Ax^2 + O(x^3))
\]

whence \( w_{k+1} = O(x^2) \). \( \square \)

2. Proof of the sufficiency of the Treves condition.

There is a grading in the differential algebra \( \mathcal{A} \) of polynomials in symbols \( u^{(k)} \), \( P[u] = P(u, u', u'', \ldots) \): \( w(u^{(n)}) = n + 2, w(\partial) = 1 \). If all terms of a polynomial \( P \) have the same weight \( k \) then

\[
P(\lambda^2 u, \lambda^3 u', \lambda^4 u'' \ldots) = \lambda^k P(u, u', u'', \ldots).
\]

**Lemma 2.** — If a differential polynomial \( P \) satisfies the Treves condition (1) then so does each homogeneous in weight component of this polynomial.

**Proof of the lemma 2.** — Let \( P = \sum P_\kappa \) where \( P_\kappa \) a homogeneous polynomial of weight \( \kappa \). Since \( \{u_n\} \) are arbitrary, we can replace them by \( u_n\lambda^{n+2} \). Now,

\[
\text{res}_x \sum \lambda^\kappa P_\kappa \left[ -2/x^2 + \lambda^2 u_0 + \sum_2^\infty \lambda^{n+2} u_n x^n/n! \right]
\]

\[
= \text{res}_x \sum \lambda^\kappa P_\kappa \left[ -2/(\lambda x)^2 + u_0 + \sum_2^\infty u_n (\lambda x)^n/n! \right]
\]

\[
= \sum \lambda^{\kappa-1} \text{res}_x P_\kappa \left[ -2/x^2 + u_0 + \sum_2^\infty u_n x^n/n! \right].
\]

If this is zero, then each term is zero since \( \lambda \) is arbitrary. \( \square \)

Therefore, we can consider each component of weight \( \kappa \) separately. The first integral \( \text{res}_\partial L^{m/2} \) where \( m = 2k - 1 \) has the weight 2\( k \). It is possible to prove that it contains a term \( Cu^k \) with a non-zero coefficient \( C \).

Indeed, dealing with the terms in \( \text{res}_\partial L^{m/2} \) where \( u \) is not differentiated, one can consider \( \partial \) and \( u \) as commuting. Then

\[
\text{res}_\partial (\partial^2 + u)^{m/2} = \text{res}_\partial \partial^m (1 + u\partial^{-2})^{m/2} = \text{res}_\partial \partial^m \sum_0^\infty \binom{m/2}{k} (u\partial^{-2})^k
\]

\[
= \binom{m/2}{(m+1)/2} u^{(m+1)/2}.
\]
If $P$ is a differential polynomial satisfying the Treves condition (1), then subtracting from it a linear combination of first integrals $\text{res}_L L^{m/2}$, one can achieve that it does not contain terms $Cu^k$ preserving the property to satisfy the condition (1). Then we reduce this polynomial. Namely, we reduce the order of the highest derivative involved in a differential monomial by “integration by parts” as much as possible: if a differential monomial $(u^{(i_1)})^{p_1} (u^{(i_2)})^{p_2} \cdots (u^{(i_k)})^{p_k}$ where $i_1 < i_2 < \cdots < i_k$ has $p_k = 1$ then the highest order of the derivative, $i_k$, can be reduced if $i_k \neq 0$ by addition of an exact derivative, for example $(u')^2 u''' = -2u'(u'')^2 + \partial((u')^2 u'')$. The second term is an exact derivative and the highest derivative involved in the first term is the second one. Another example: $uu' u'' = u(u'^2)' / 2 = \partial(uu'^2 / 2) - u'^3 / 2$. One can proceed doing this until all the monomials will contain their highest derivatives in power $> 1$ (with a possible exception: a term $Cu$). We call this the reduced form of a differential polynomial. It is unique.

The reduced polynomial preserves the property (1) and does not contain the terms $Cu^k$. It remains to prove the following:

**Lemma 3.** — A reduced differential polynomial homogeneous with respect to the weight which satisfies the condition (1) and does not contain the term $Cu^k$ is zero.

Suppose that it is not zero. Let us write Eq. (1) in more detail:

\[
\text{res}_x Q \left( -\frac{2}{x^2} + u_0 + \sum_{i=2}^{\infty} u_i (x^i/i!), \frac{4}{x^3} + \sum_{i=2}^{\infty} u_i \partial(x^i/i!), \ldots \right) = 0.
\]

This equality can be differentiated with respect to $u_0$ which is the same as $\text{res}_x \partial Q[u]/\partial u = 0$. This operation can be repeated until there will be no factor $u$ at all, and, nevertheless, the polynomial is not zero since the term $Cu^k$ is absent. More than that, the polynomial preserves all the properties assumed in Lemma 3. We have

\[
\text{res}_x Q \left( \frac{4}{x^3} + \sum_{i=2}^{\infty} u_i \partial(x^i/i!), \ldots \right) = 0.
\]

Now let us take the derivative with respect to an arbitrary $u_k$, $k = 2, 3, \ldots$:

\[
\text{res}_x \sum_{i=1}^{\infty} \frac{\partial Q \left( 4/x^3 + \sum_{i=2}^{\infty} u_i \partial(x^i/i!), \ldots \right)}{\partial u^{(n)}(u^k/k!)} \times \partial^n(x^k/k!) = 0.
\]
Integrating by parts, we get:

\[
\text{res}_x \sum_1^\infty (-\partial)^{n-1} \partial Q \left( \frac{4}{x^3} + \sum_2^\infty u_i \partial \left( \frac{x^i}{i!} \right), -\frac{12}{x^4} + \sum_2^\infty u_i \partial^2 \left( \frac{x^i}{i!} \right), \ldots \right) \cdot \frac{x^{k-1}}{(k-1)!} = 0
\]
or

\[
\text{res}_x \frac{\delta Q \left( \frac{4}{x^3} + \sum_2^\infty u_i \partial \left( \frac{x^i}{i!} \right), -\frac{12}{x^4} + \sum_2^\infty u_i \partial^2 \left( \frac{x^i}{i!} \right), \ldots \right)}{\delta u'} \cdot \frac{x^{k-1}}{(k-1)!} = 0
\]

where \( k = 2, 3, \ldots \). Denoting the variational derivative \( \delta Q/\delta u' \) as \( R \), we have

\[
\text{res}_x xR(\frac{4}{x^3} + \sum_2^\infty u_i \partial \left( \frac{x^i}{i!} \right), -\frac{12}{x^4} + \sum_2^\infty u_i \partial^2 \left( \frac{x^i}{i!} \right), \ldots) x^{k-2} = 0
\]

whence

\[
(3) \quad \left( xR(\frac{4}{x^3} + \sum_2^\infty u_i \partial \left( \frac{x^i}{i!} \right), -\frac{12}{x^4} + \sum_2^\infty u_i \partial^2 \left( \frac{x^i}{i!} \right), \ldots) \right) = 0.
\]

Thus, the problem is now the following: to show that if \( R[v] \ (v = u') \) is a homogeneous polynomial satisfying Eq. (3), then \( R = 0 \). If this is proven, then \( \delta Q/\delta u' = 0 \) and \( Q \) is an exact derivative which is incompatible with the fact that it is a non-zero reduced polynomial.

We shall prove a slightly more general lemma, the generalization is needed in the proof of the theorem 2.

**LEMMA 4.** — Let \( R(u, u', u'', \ldots) \) be a homogeneous polynomial satisfying

\[
\left( xR(\frac{4}{x^3} + \sum_2^\infty u_i \partial \left( \frac{x^i}{i!} \right), 4/x^3 + \sum_2^\infty u_i \partial \left( \frac{x^i}{i!} \right), \ldots) \right) = 0,
\]

where \( u_1 = 0 \) and other \( u_i \) are arbitrary. Then \( R = 0 \).

Let \( R \) be of weight \( \kappa \). From the homogeneity, it follows that the given equality can be written as

\[
(4) \quad \left( x^{1-\kappa} R(-2 + \sum_0^\infty u_i x^{i+2}/i! + 4 + \sum_2^\infty u_i x^{i+2}/(i-1)! \ldots) \right) = 0.
\]

Now, one must expand \( R(-2 + \sum_0^\infty u_i x^{i+2}/i! + 4 + \sum_2^\infty u_i x^{i+2}/(i-1)! \ldots) \) in powers of \( x \) and write that all terms of power less than \( \kappa - 1 \)
vanish. For that, it is more convenient to expand this expressions in powers of \( u_i \), it automatically will be an expansion in powers of \( x \).

The arguments of \( R \): \( u, u', \ldots \) we denote as \( \xi_0, \xi_1, \ldots, \) thus, \( R = R(\xi_0, \xi_1, \ldots, \xi_\mu) \). We have (denoting \( \partial_{u_j} = \partial/\partial u_j \) etc)

\[
\partial_{u_j} R \left(-2 + \sum_0^{\infty} \frac{x^{i+2}}{i!}, 4 + \sum_2^{\infty} \frac{x^{i+2}}{(i-1)!}, \ldots \right) = \partial_{u_j} R(\xi_1, \xi_2, \ldots)
\]  

\[
= D_j R(\xi_0, \xi_1, \ldots) x^{j+2} \text{ where } D_j = \frac{1}{j!} \partial_{\xi_0} + \frac{1}{(j-1)!} \partial_{\xi_1} + \cdots + \partial_{\xi_j}
\]

for \( j = 0, 2, 3, \ldots \). Now:

\[
x^{1-\kappa} R \left(-2 + \sum_0^{\infty} \frac{x^{i+2}}{i!}, 4 + \sum_2^{\infty} \frac{x^{i+2}}{(i-1)!}, -12 + \sum_2^{\infty} \frac{x^{i+2}}{(i-2)!}, \ldots \right)
\]  

\[
= \sum_{q_0, q_2, \ldots} \frac{D_{q_0}^0 D_{q_2}^2 \cdots R(\theta)}{q_0! q_2! \cdots x^{\kappa-1-\lambda(q)}} u_0^{q_0} u_2^{q_2} \cdots
\]

where

\[
(\theta) = (-2 \cdot 1!, 2 \cdot 2!, -2 \cdot 3!, \ldots, (-1)^{\mu+1} 2 \cdot (\mu + 1)!, \; \lambda(q) = \sum (j+2) q_j.
\]

The condition that the expression \( (5) \) does not contain negative powers of \( x \) becomes

\[
D_0^{q_0} D_2^{q_2} \cdots R(\theta) = 0 \text{ when } \lambda(q) < \kappa - 1.
\]

**Lemma 5 on homogeneous polynomials.** — Let

\[
R(\xi_0, \xi_1, \xi_2, \ldots, \xi_\mu) = \sum_{(p)} a_{(p_0 p_1 p_2 \cdots p_\mu)} \xi_0^{p_0} \xi_1^{p_1} \xi_2^{p_2} \cdots \xi_\mu^{p_\mu}
\]

where \( 2p_0 + 3p_1 + 4p_2 + \cdots + (\mu + 2)p_\mu = \kappa \geq 2 \). Suppose the equation \( (6) \) where \( (\theta) = (\xi_0^*, \xi_1^*, \xi_2^*, \ldots) \) is a fixed set of nonzero values of the corresponding variables is satisfied for all sets of integers \( \{q_0, q_2, q_3, \ldots\} \) such that \( \lambda(q) \equiv \sum (2 + j) q_j \leq \kappa - 2 \). If \( \kappa \geq 4 \), we consider only sets \( \{q_i\} \) such that not all of \( q_i \) are zero. Then all coefficients \( a_{(p)} \) are zero.

Notice that if \( \kappa < 4 \), the polynomial \( R \) contains only one term, and the proof is obvious.
3. Proof of the Lemma 5 and the end of the proof of the sufficiency of the Treves condition.

We use the induction with respect to the number of variables \( \xi_0, \xi_1, \ldots, \xi_{\mu} \). For \( \mu = 0 \) the statement is trivial. Let it be proven for \( \mu - 1 \).

The Euler formula for the weight-homogeneous polynomial \( R \) reads

\[
2\xi_0 \partial_{\xi_0} R + 3\xi_1 \partial_{\xi_1} R + 4\xi_2 \partial_{\xi_2} R + \cdots + (\mu + 2)\xi_{\mu} \partial_{\xi_{\mu}} R = \kappa R.
\]

Solving \( \mu \) equations with \( \mu \) unknowns, one can express \( \partial_{\xi_0}, \ldots, \partial_{\xi_{\mu - 1}} \) in terms of \( D_0, D_2, \ldots, D_\mu \) and \( \partial_{\xi_\mu} \):

\[
\partial_{\xi_j} = \tau_{j0} D_0 + \sum_{k=2}^{\mu} \tau_{jk} D_k + \sigma_j \partial_{\xi_\mu}, \quad j = 0, \ldots, \mu - 1
\]

with constant coefficients. Substituting this for \( \partial_{\xi_i} \) in the Euler equation, we get

\[
\left( a_0(\xi) D_0 + \sum_{i=2}^{\mu} a_i(\xi) D_i + \beta(\xi) \partial_{\xi_\mu} \right) R = \kappa R
\]

where coefficients linearly depend on \( \{ \xi_i \} \). Hence,

\[
\partial_{\xi_\mu} R = \left( b(\xi) + a_0(\xi) D_0 + \sum_{i=2}^{\mu} a_i(\xi) D_i \right) R
\]

with coefficients which are rational functions of \( \{ \xi_i \} \). Iterating this formula, we have

\[
(7) \quad \partial_{\xi_\mu}^m R = \sum_{j_0+j_2+\cdots+j_\mu \leq m} \alpha_{j_0,j_2,\ldots,j_\mu}(\xi) D_{j_0}^{q_0} D_{j_2}^{q_2} \cdots D_{j_\mu}^{q_\mu} R.
\]

One can write \( R = \sum_{0}^{d} a_j(\xi_0, \ldots, \xi_{\mu-1}) \xi_j^i / j! \) where \( a_d \) is not the identical zero. We consider two cases: when \( a_d \) is not a constant and when it is a constant. If \( a_d \) is not a constant, its weight is at least 2 (since this is the smallest weight of variables \( \xi_i \)).

Now we can prove that the weight-homogeneous differential polynomial \( a_d(\xi_0, \ldots, \xi_{\mu-1}) \) of weight \( \kappa_1 = \kappa - (\mu + 2)d \), if \( \kappa_1 \geq 2 \), has the property

\[
D_{j_0}^{q_0} D_{j_2}^{q_2} \cdots D_{j_{\mu-1}}^{q_{\mu-1}} a_d(\theta) = 0 \text{ if } \lambda(q) = 2q_0 + 4q_2 + \cdots + (\mu + 1)q_{\mu-1} \leq \kappa_1 - 2.
\]

Indeed, using (7), we obtain

\[
D_{j_0}^{q_0} D_{j_2}^{q_2} \cdots D_{j_{\mu-1}}^{q_{\mu-1}} a_d(\theta) = D_{j_0}^{q_0} D_{j_2}^{q_2} \cdots D_{j_{\mu-1}}^{q_{\mu-1}} \partial_{\xi_\mu}^d R(\theta)
\]

\[
= \sum_{j_0+j_2+\cdots+j_\mu \leq d} \alpha_{j_0,j_2,\ldots,j_\mu}(\xi) D_{j_0}^{q_0} D_{j_2}^{q_2} \cdots D_{j_\mu}^{q_\mu} D_{j_0}^{q_0} D_{j_2}^{q_2} \cdots D_{j_{\mu-1}}^{q_{\mu-1}} R(\theta).
\]
Since
\[(2j_0 + 4j_2 + \cdots + (\mu + 2)j_\mu) + (2q_0 + 4q_2 + \cdots + (\mu + 1)q_{\mu-1}) \leq (\mu + 2)(j_0 + j_2 + \cdots + j_\mu) + \kappa_1 - 2 \leq (\mu + 2)d + \kappa - (\mu + 2)d - 2 = \kappa - 2,
\]
all terms of the sum vanish. Since \(a_d\) depends on \(\mu - 1\) variables, the lemma 5 is supposed to be true for \(a_d\), and \(a_d\) is identically zero. Therefore, \(R = 0\).

Now, we discuss the second alternative: \(a_d\) is a constant. Then, instead of \(a_d = \partial_d^\mu R\), we consider
\[\tilde{a}_d = \partial_{\xi_\mu}^{d-1} R = a_{d-1}(\xi_0, \ldots, \xi_{\mu-1}) + c_{\xi_\mu}, c = \text{const}.\]
We have \(\kappa_2 = w(\tilde{a}_d) = \mu + 2 \geq 2\). Moreover, if \(a_{d-1} \neq 0\), \(\kappa_2 \geq 4\) since \(a_{d-1}\) cannot be linear. Just as in the first case, we can prove that \(D_0^n D_2^n \cdots \tilde{a}_d(\theta) = 0\) if \(\lambda(q) \leq \kappa_2 - 2\). If not all of \(q_i\) are zero, then \(D_0^n D_2^n \cdots \tilde{a}_d(\theta) = D_0^n D_2^n \cdots a_{d-1}(\theta) = 0\), and we conclude that \(a_{d-1} = 0\). Then \(c = 0\) since \(\partial_{\xi_\mu}^{d-1} R(\theta) = 0\), and \(a_d = 0\). Thus, \(R = 0\) and the lemmas 5, 4, and also the theorem 1 are proven.

\[\square\]

4. Proof of the Theorem 2.

In the same way as before, one can restrict himself to homogeneous polynomials. Let \(\text{res}_x R(\tilde{u}(x), \tilde{u}'(x), \tilde{u}''(x), \ldots) = 0\) for all series (2), \(R\) being a differential polynomial. One can differentiate this equality with respect to \(u_0, u_2, u_3, \ldots:\)
\[\text{res}_x \frac{\partial R}{\partial u} = 0, \text{res}_x \left(\frac{\partial R}{\partial u} \cdot \frac{x^2}{2} + \frac{\partial R}{\partial u'} \cdot \frac{x^2}{2} + \frac{\partial R}{\partial u''} \cdot \frac{\partial^2 x^2}{2}\right) = 0, \ldots,
\]
or
\[\text{res}_x \frac{\delta R}{\delta u} = 0, \text{res}_x \frac{\delta R}{\delta u} \cdot \frac{x^2}{2} = 0, \text{res}_x \frac{\delta R}{\delta u} \cdot \frac{x^3}{3!} = 0, \ldots,
\]
which means that \(P = \delta R/\delta u\), after the substitution (2), can have, as a Laurent polynomial, only one singular term \(ax^{-2}\). It is well-known that any polynomial \(\text{res}_\theta L^{m/2}\) is the variational derivative of the next one, \(\text{res}_\theta L^{(m+2)/2}\) (see [3], Proposition 3.5.2.). The theorem is proven one way.

Conversely, let \(P\) be a homogeneous differential polynomial such that after the substitution (2) there is only one singular term, \(ax^{-2}\). In particular, \(\text{res}_x P(\tilde{u}, \tilde{u}', \ldots) = 0\). Then \(P\) is a sum of \(c \cdot \text{res}_\theta L^{m/2}\) and a derivative, \(\partial S(u, u', \ldots)\). Unless \(S\) is identically zero, it must contain,
after the substitution (2), singular terms of order $x^{-2}$ or higher since if it were not so, i.e., $(xS(\tilde{u}, \tilde{u}', \ldots))_\infty = 0$, then, according to the Lemma 4, $S$ would be zero. Then $\partial S$ has nonzero singular terms of order at least $x^{-3}$ in contradiction to the assumption. Thus, $S = 0$, and the given differential polynomial is just $c \cdot \text{res}_\partial L^m/2$.

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