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ASYMPTOTICS OF THE PARTITION FUNCTION OF A RANDOM MATRIX MODEL

by Pavel M. BLEHER (*) & Alexander R. ITS (**)

1. Introduction.

The central object of our analysis is the partition function of a random matrix model,

$$Z_N = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{1\le j<k\le N} (z_j - z_k)^2 \, \mathrm{e}^{-N\sum_{j=1}^N V(z_j)} \mathrm{d}z_1 \cdots \mathrm{d}z_N$$
$$= N! \prod_{n=0}^{N-1} h_n,$$

where $V(z)$ is a polynomial,

$$V(z) = \sum_{j=1}^{2d} v_j z^j, \quad v_{2d} > 0, \tag{1.1}$$

and h_n are the normalization constants of the orthogonal polynomials on the line with respect to the weight $\mathrm{e}^{-NV(z)}$,

$$\int_{-\infty}^{\infty} P_n(z) P_m(z) \, \mathrm{e}^{-NV(z)} \, \mathrm{d}z = h_n \delta_{nm}, \quad P_n(z) = z^n + \cdots. \tag{1.2}$$

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In this work we are interested in the asymptotic expansion of the free energy,

$$F_N = -\frac{1}{N^2}\ln Z_N, \tag{1.3}$$

as $N\to\infty$. Our approach is based on the deformation τ_t of $V(z)$ to z^2,

$$\tau_t : V(z) \to (1-t^{-1})z^2 + V(t^{-\frac{1}{2}}z), \qquad 1\le t<\infty, \tag{1.4}$$

so that

$$\tau_1 V(z) = V(z), \qquad \tau_\infty V(z) = z^2, \tag{1.5}$$

and our main reasults are the following:

1) under the assumption that V is one-cut regular (for definitions see Section 4 below), we obtain a full asymptotic expansion of the recurrence coefficients γ_n, β_n of orthogonal polynomials in powers of N^{-2}, and we show the analyticity of the coefficients of these asymptotic expansions with respect to the coefficients v_k, $k=1,\dots,2d$;
2) under the assumption that $\tau_t V$ is one-cut regular for $t\ge 1$, we prove the full asymptotic expansion of F_N in powers of N^{-2}, and we show the analyticity of the coefficients of the asymptotic expansion with respect to v_k, $k=1,\dots,2d$;
3) under the assumptions that (i) V is singular, (ii) the equilibrium measure of V is nondegenerate at the end-points, and (iii) $\tau_t V$ is one-cut regular for $t>1$, we prove that the coefficients of the asymptotic expansion of the free energy of $\tau_t V$ for $t>1$ can be analytically continued to $t=1$;
4) for the singular quartic polynomial, $V(z) = (\frac{1}{4}z^4) - z^2$, we obtain the double scaling asymptotics of the free energy of $\tau_t V(z)$ where $t-1$ is of the order of $N^{-\frac{2}{3}}$; we prove that this asymptotics is a sum of a regular term, coming as a limit of the asymptotic expansion for $t>1$, and a singular term, which has the form of the logarithm of the Tracy-Widom distribution function.

In result 2), the existence of a full asymptotic expansion of the free energy in powers of N^{-2} was first proved by Ercolani and McLaughlin [EM], under the assumption that the coefficients of V are small. It was used in [EM] to make rigorous the Bessis-Itzykson-Zuber topological expansion [BIZ] related to counting Feynman graphs on Riemannian surfaces. Apparently, the results of [EM] can be used to obtain the asymptotic expansion of the recurrent coefficients, i.e. our result 1); however, the important specific structure of the series corresponding to the coefficients β_n, which is obtained in our Theorem 5.2, seems to be a challenge fot the methos of [EM]. The approach of Ercolani and McLaughlin is based on a direct use of the asymptotic solution of the Riemann-Hilbert problem, and it is very different from our approach, which is based on a combination of the Riemann-Hilbert analysis and the deformation equations. Also in result 2), our proof of the analyticity of the coefficients of the asymptotic expansion with respect to v_k uses an important result of Kuijlaars and McLaughlin [KM], that the Jacobian of the map of the end-points of the equlibrium measure to the basic set of integrals (see Section 4 below) is nonzero. In result 3), the existence of an analytic continuation of the free energy to the critical point (the one-sided analyticity) from the one-cut side was proved by Bleher and Eynard for a nonsymmetric singular quartic polynomial, see the paper [BE], where, in fact, the one-sided analyticity was proved from the both sides, one-cut and two-cut, and a phase transition of the third order was shown. Thus, result 3) gives an extension of the result of [BE] to a general singular V from the one-cut side. Observe that the analytic behavior of the free energy from the multi-cut side can be different for different singular V and it requires a special investigation. In result 4), to derive and to prove the double scaling asymptotics of the free energy we use and slightly extend the double scaling asymptotics of the recurrent coefficients, obtained in our paper [BI2]. In addition, we extend the Riemann-Hilbert approach of [DKMVZ] to the case when $t = 1 + cN^{-\frac{2}{3}+\varepsilon}$, where $c, \varepsilon > 0$. In this case the lenses thickness vanishes as $N^{-\frac{1}{3}}$ but this is enough to estimate the jump on the lenses by $\mathrm{e}^{-CN^{\varepsilon}}$ and to apply the methods of [DKMVZ].

The set up of the rest of the paper is the following. In Section 2 we derive formulas which describe the deformation of the recurrence coefficients and the free energy for a finite N, under deformations of V. Here, we make use of the integrability of the matrix model, and we refer the reader to excellent recent surveys of van Moerbeke [vM1], [vM2] on different modern aspects as well as the history of the matter. In Section 3 we use the deformation $\tau_t V(z)$ to obtain an integral representation of the free energy for a finite N. In Section 4 we obtain different results concerning the analyticity of the equilibrium measure for the q-cut regular case. In Section 5 we obtain one of our main results about the asymptotic expansion of recurrence coefficients in the one-cut regular case. This is applied then in

Section 6 to obtain the asymptotic expansion of the free energy, assuming that $\tau_t V$ is one-cut regular for $t \in [1, \infty)$. In Section 7 we derive an exact formula for the limiting free energy in the case when V is an even one-cut regular polynomial. In Section 8 we obtain a number of results concerning the one-sided analyticity for singular V. Finally, in Section 9 we obtain the double scaling asymptotics of the free energy for the singular quartic polynomial V.

2. Deformation equations for recurrence coefficients and partition function.

Define the psi-functions as

$$\psi_n(z) = \frac{1}{\sqrt{h_n}} P_n(z) \,\mathrm{e}^{-\frac{1}{2}V(z)}. \tag{2.1}$$

Then

$$\int_{-\infty}^{\infty} \psi_n(z)\psi_m(z)\,\mathrm{d}z = \delta_{nm}. \tag{2.2}$$

The psi-functions satisfy the three term recurrence relation,

$$z\psi_n(z) = \gamma_{n+1}\psi_{n+1}(z) + \beta_n\psi_n(z) + \gamma_n\psi_{n-1}(z), \tag{2.3}$$

where

$$\gamma_n = \sqrt{h_n/h_{n-1}}. \tag{2.4}$$

Set

$$\overrightarrow{\Psi}(z) = \begin{pmatrix} \psi_0(z) \\ \psi_1(z) \\ \psi_2(z) \\ \vdots \end{pmatrix}. \tag{2.5}$$

Then (2.3) can be written in the matrix form as

$$z\,\overrightarrow{\Psi}(z) = Q\,\overrightarrow{\Psi}(z), \quad Q = \begin{pmatrix} \beta_0 & \gamma_1 & 0 & 0 & 0 & \dots \\ \gamma_1 & \beta_1 & \gamma_2 & 0 & 0 & \dots \\ 0 & \gamma_2 & \beta_2 & \gamma_3 & 0 & \dots \\ 0 & 0 & \gamma_3 & \beta_3 & \gamma_4 & \dots \\ 0 & 0 & 0 & \gamma_4 & \beta_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.6}$$

Observe that $Z_N, h_n, \gamma_n, \beta_n$ are functions of the coefficients $v_1, \dots, v_{2d}$ of the polynomial $V(z)$. We will be interested in exact expressions for the derivatives of Z_N, h_n, γ_n, β_n with respect to v_k. Set

$$\widetilde{v}_k = N v_k, \quad k = 1, \dots, 2d. \tag{2.7}$$

Proposition 2.1. — *We have the following relations*:

$$\frac{\partial \ln h_n}{\partial \widetilde{v}_k} = -[Q^k]_{n,n}, \tag{2.8}$$

$$\frac{\partial \gamma_n}{\partial \widetilde{v}_k} = \frac{\gamma_n}{2}\big([Q^k]_{n-1,n-1} - [Q^k]_{n,n}\big), \tag{2.9}$$

$$\frac{\partial \beta_n}{\partial \widetilde{v}_k} = \gamma_n[Q^k]_{n,n-1} - \gamma_{n+1}[Q^k]_{n+1,n}, \tag{2.10}$$

where $[Q^k]_{n,m}$ *denotes the nm-th element of the matrix* Q^k; $n,m = 0,1,2,\dots$.

Proof. — Formula (2.8) is proven in [Ey]. By (2.4), it implies (2.9). Let us prove (2.10). Introduce the vector function

$$\overrightarrow{\Psi}_n(z) = \begin{pmatrix} \psi_{n-1}(z) \\ \psi_n(z) \end{pmatrix}. \tag{2.11}$$

As shown in [Ey] (see also [BEH]), it satisfies the deformation equation

$$\frac{\partial \overrightarrow{\Psi}_n}{\partial \widetilde{v}_k} = U_k(z;n)\,\overrightarrow{\Psi}_n(z), \tag{2.12}$$

where

$$U_k(z;n) = \frac{1}{2}\begin{pmatrix} z^k - [Q^k]_{n-1,n-1} & 0 \\ 0 & [Q^k]_{n,n} - z^k \end{pmatrix} + \gamma_n \begin{pmatrix} [Q(z;k-1)]_{n,n-1} & -[Q(z;k-1)]_{n-1,n-1} \\ [Q(z;k-1)]_{n,n} & -[Q(z;k-1)]_{n,n-1} \end{pmatrix}, \tag{2.13}$$

and

$$Q(z;k-1) = \sum_{j=0}^{k-1} z^j Q^{k-1-j}. \tag{2.14}$$

From (2.3),

$$\overrightarrow{\Psi}_{n+1} = \frac{1}{\gamma_{n+1}} U(z;n)\overrightarrow{\Psi}_n(z), \quad U(z;n) = \begin{pmatrix} 0 & \gamma_{n+1} \\ -\gamma_n & z-\beta_n \end{pmatrix}. \tag{2.15}$$

The compatibility condition of (2.13) and (2.15) is

$$\frac{\partial U(z;n)}{\partial \widetilde{v}_k} = U_k(z;n+1)U(z;n) - U(z;n)U_k(z;n) + \frac{1}{\gamma_{n+1}}\frac{\partial \gamma_{n+1}}{\partial \widetilde{v}_k} U(z;n). \tag{2.16}$$

By restricting this equation to the element 22, we obtain (2.10). Proposition 2.1 is proved. □

We will be especially interested in the derivatives with respect to $\widetilde{v}_2$. For $k=2$, Proposition 2.1 gives that

$$\frac{\partial \ln h_n}{\partial \widetilde{v}_2} = -\gamma_n^2 - \beta_n^2 - \gamma_{n+1}^2, \tag{2.17}$$

$$\frac{\partial \gamma_n}{\partial \widetilde{v}_2} = \frac{\gamma_n}{2}(\gamma_{n-1}^2 + \beta_{n-1}^2 - \gamma_{n+1}^2 - \beta_n^2), \tag{2.18}$$

$$\frac{\partial \beta_n}{\partial \widetilde{v}_2} = \gamma_n^2\beta_{n-1} + \gamma_n^2\beta_n - \gamma_{n+1}^2\beta_n - \gamma_{n+1}^2\beta_{n+1}. \tag{2.19}$$

Observe that all these expressions are local in n, so that they depend only on the recurrent coefficients with indices which differ from n by a fixed number. Our next step will be to get a local expression for the second derivative of Z_n.

Proposition 2.2. — *We have the following relation*:

$$\frac{\partial^2 \ln Z_N}{\partial \widetilde{v}_2^2} = \gamma_N^2(\gamma_{N-1}^2 + \gamma_{N+1}^2 + \beta_N^2 + 2\beta_N\beta_{N-1} + \beta_{N-1}^2). \tag{2.20}$$

Proof. — For the sake of brevity we denote $(') = \partial/\partial\widetilde{v}_2$. From (2.17)–(2.19) we obtain that

$$\begin{aligned}(\ln h_n)'' &= -2\gamma_n\gamma_n' - 2\beta_n\beta_n' - 2\gamma_{n+1}\gamma_{n+1}' \\ &= -\gamma_n^2(\gamma_{n-1}^2 + \beta_{n-1}^2 - \gamma_{n+1}^2 - \beta_n^2) \\ &\quad - 2\beta_n(\gamma_n^2\beta_{n-1} + \gamma_n^2\beta_n - \gamma_{n+1}^2\beta_n - \gamma_{n+1}^2\beta_{n+1}) \\ &\quad - \gamma_{n+1}^2(\gamma_n^2 + \beta_n^2 - \gamma_{n+2}^2 - \beta_{n+1}^2) \\ &= I_{n+1} - I_n,\end{aligned} \tag{2.21}$$

where

$$I_n = \gamma_n^2(\gamma_{n-1}^2 + \gamma_{n+1}^2 + \beta_n^2 + 2\beta_n\beta_{n-1} + \beta_{n-1}^2). \tag{2.22}$$

From (1.1) and (2.21) we obtain now the telescopic sum,

$$(\ln Z_N)'' = \sum_{n=0}^{N-1}(\ln h_n)'' = \sum_{n=0}^{N-1}(I_{n+1} - I_n) = I_N - I_0. \tag{2.23}$$

Observe that $I_0 = 0$, because $\gamma_0 = 0$, hence (2.20) follows. □

Remark . — When $k = 1$, Proposition 2.1 gives that

$$\frac{\partial \ln h_n}{\partial \widetilde{v}_1} = -\beta_n, \quad \frac{\partial \gamma_n}{\partial \widetilde{v}_1} = \frac{\gamma_n}{2}(\beta_{n-1} - \beta_n), \quad \frac{\partial \beta_n}{\partial \widetilde{v}_1} = \gamma_n^2 - \gamma_{n+1}^2, \tag{2.24}$$

hence

$$\frac{\partial^2 \ln Z_N}{\partial \widetilde{v}_1^2} = \gamma_N^2. \tag{2.25}$$

Similar formulae can be derived also for $k \geq 3$, but they become complicated.

Remark . — For the case of even potentials, equations (2.11)–(2.16), as well as the statement of Proposition 2.1 were obtained in [FIK]. It is also worth noticing that in the even case, differential-difference equation (2.9) is the well-known Volterra hierarchy whose integrability was first established in 1974–1975 in the pioneering works of Flaschka [Fl], Kac and van Moerbeke [KvM], and Manakov [Ma], and whose particular case (2.18) is the classical Kac-van Moerbeke discrete version of the KdV equation [KvM].

Remark . — Proposition 2.2 was proven in [IKF] for the case of the even quartic potential $V(z) = v_2 z^2 + v_4 z^4$.

3. Free energy for a finite N.

In terms of v_2 formula (2.20) reduces to the following:

$$\frac{\partial^2 F_N}{\partial v_2^2} = -\gamma_N^2(\gamma_{N-1}^2 + \gamma_{N+1}^2 + \beta_N^2 + 2\beta_N\beta_{N-1} + \beta_{N-1}^2), \tag{3.1}$$

where F_N is the free energy, see (1.4).

The main problem we will be interested in is an asymptotics of the free energy as $N \to \infty$. Our approach will be based on a deformation of the polynomial $V(z)$ to the quadratic polynomial z^2. To that end we set

$$W(z) = V(z) - z^2, \tag{3.2}$$

and we define a one-parameter family of polynomials,

$$V(z;t) = z^2 + W\left(\frac{z}{\sqrt{t}}\right), \qquad t \geq 1. \tag{3.3}$$

Then obviously,

$$V(z;1) = V(z), \qquad V(z;\infty) = z^2.$$

It is convenient to introduce the operator τ_t, see (1.5). Then $V(z;t) = \tau_t V(z)$. The operators τ_t satisfy the group property.

Proposition 3.1. — *One has*

$$\tau_t\tau_s = \tau_{ts}. \tag{3.4}$$

Proof. — We have that

$$\begin{aligned}\tau_t\big(\tau_s\big(V(z)\big)\big) &= \tau_t\big((1-s^{-1})z^2 + V(s^{-\frac12}z)\big)\\ &= (1-t^{-1}z^2) + (1-s^{-1})t^{-1}z^2 + V(t^{-\frac12}s^{-\frac12}z)\\ &= (1-t^{-1}s^{-1})z^2 + V(t^{-\frac12}s^{-\frac12}z) = \tau_{ts}\big(V(z)\big).\end{aligned} \tag{3.5}$$

Proposition 3.1 is proved. □

Let $Z_N = Z_N(t)$ be the partition function (1.1) for the polynomial $V(z;t)$ and $F_N = F_N(t)$ the corresponding free energy.

Proposition 3.2. — *One has*

$$\begin{aligned}\frac{\partial^2 F_N(t)}{\partial t^2} = -\frac{1}{t^2}\Big\{\gamma_N^2(t)\big[\gamma_{N-1}^2(t) + \gamma_{N+1}^2(t) + \beta_N^2(t)\\ + 2\beta_N(t)\beta_{N-1}(t) + \beta_{N-1}^2(t)\big] - \frac12\Big\},\end{aligned} \tag{3.6}$$

where $\gamma_n(t)$,$\beta_n(t)$ are the recurrence coefficients of orthogonal polynomials with respect to the weight $\mathrm{e}^{-NV(z;t)}$.

Proof. — By the change of variables $z_j = \sqrt{t}u_j$, we obtain from (1.1) that

$$Z_N(t) = t^{N^2/2}\widehat{Z}_N(t), \tag{3.7}$$

where

$$\widehat{Z}_N(t) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{1\le j<k\le N}(u_j-u_k)^2\,\mathrm{e}^{-\Sigma_{j=1}^N N\widehat{V}(u_j;t)}\,\mathrm{d}u_1\cdots\mathrm{d}u_N \tag{3.8}$$

is the partition function for

$$\widehat{V}(u;t) = V\big(\sqrt{t}u;t\big) = tu^2 + W(u). \tag{3.9}$$

Hence

$$\widehat{F}_N(t) \equiv -\frac{1}{N^2}\ln\widehat{Z}_N(t) = -\frac{1}{N^2}\ln\big[-t^{N^2/2}Z_N(t)\big] = F_N(t) + \frac{\ln t}{2}\cdot \tag{3.10}$$

By (3.1),

$$\frac{\partial^2 \widehat{F}_N(t)}{\partial t^2} = -\widehat{\gamma}_N^2(t)\big[\widehat{\gamma}_{N-1}^2(t)+\widehat{\gamma}_{N+1}^2(t)+\widehat{\beta}_N^2(t) + 2\widehat{\beta}_N(t)\widehat{\beta}_{N-1}(t)+\widehat{\beta}_{N-1}^2(t)\big], \tag{3.11}$$

where $\widehat{\gamma}_n(t), \widehat{\beta}_n(t)$ are the recurrence coefficients of the orthogonal polynomials

$$\widehat{P}_n(u;t) = t^{-\frac{1}{2}n} P_n\big(\sqrt{t}\,u;t\big) \tag{3.12}$$

with respect to the weight $\mathrm{e}^{-N\widehat{V}(u;t)}$. Since

$$\big(\sqrt{t}\,u\big)P_n\big(\sqrt{t}\,u;t\big) = P_{n+1}\big(\sqrt{t}\,u;t\big) + \beta_n(t)P_n\big(\sqrt{t}\,u;t\big) + \gamma_n^2(t)P_{n-1}(\sqrt{t}u;t), \tag{3.13}$$

we obtain that

$$\hat{\gamma}_n(t) = t^{-\frac{1}{2}}\gamma_n(t), \quad \widehat{\beta}_n(t) = t^{-\frac{1}{2}}\beta_n(t), \tag{3.14}$$

hence from (3.11), (3.10) we obtain that

$$\frac{\partial^2}{\partial t^2}\Big(F_N(t)+\frac{\ln t}{2}\Big) = -\frac{\gamma_N^2(t)}{t^2}\big[\gamma_{N-1}^2(t)+\gamma_{N+1}^2(t)+\beta_N^2(t) + 2\beta_N(t)\beta_{N-1}(t)+\beta_{N-1}^2(t)\big], \tag{3.15}$$

which implies (3.6). Proposition 3.2 is proven. □

We would like to integrate formula (3.6). To that end we need an asymptotic behavior of the recurrence coefficients γ_n, β_n, $n = N-1, N, N+1$ as $t \to \infty$. For a finite N it is easy.

Proposition 3.3. — *Assume that n and N are fixed. Then, as $t \to \infty$,*

$$\gamma_n(t) = \sqrt{\frac{n}{2N}} + O(t^{-\frac{1}{2}}), \quad \beta_n = O(t^{-\frac{1}{2}}). \tag{3.16}$$

Proof. — When $t = \infty$, $V(z;t) = z^2$, hence $\gamma_n(\infty)$, $\beta_n(\infty)$ are recurrence coefficients for the Hermite polynomials,

$$\gamma_n(\infty) = \sqrt{\frac{n}{2N}}, \quad \beta_n(\infty) = 0. \tag{3.17}$$

When t is finite, the orthogonal polynomials and recurrence coefficients can be obtained through the Gramm-Schmidt orthogonalization algorithm. Since for any moment we have the relation,

$$\int_{-\infty}^{\infty} z^k\,\mathrm{e}^{-N(z^2+W(z/\sqrt{t}))}\,\mathrm{d}z = \int_{-\infty}^{\infty} z^k\,\mathrm{e}^{-Nz^2}\,\mathrm{d}z + O(t^{-\frac{1}{2}}), \tag{3.18}$$

estimation (3.16) follows. □

From Propositions 3.2 and 3.3 we obtain the following formula for F_N.

Theorem 3.4. — *One has*

$$F_N(t) = F_N^{\mathrm{Gauss}} + \int_t^\infty \frac{t-\tau}{\tau^2}\Big\{\gamma_N^2(\tau)\big[\gamma_{N-1}^2(\tau)+\gamma_{N+1}^2(\tau) + \beta_N^2(\tau)+2\beta_N(\tau)\beta_{N-1}(\tau)+\beta_{N-1}^2(\tau)\big]-\frac{1}{2}\Big\}\,\mathrm{d}\tau, \tag{3.19}$$

where $\gamma_n(\tau)$, $\beta_n(\tau)$, $n=N-1,N,N+1$, *are the recurrence coefficients for orthogonal polynomials with respect to the weight* $\mathrm{e}^{-NV(z;\tau)}$, *and*

$$F_N^{\mathrm{Gauss}} = -\frac{1}{N^2}\ln\left(\frac{(2\pi)^{\frac12 N}}{(2N)^{\frac12 N^2}}\prod_{n=1}^N n!\right) \tag{3.20}$$

is the free energy of the Gaussian ensemble.

Proof. — From Proposition 3.3 we obtain that

$$\gamma_N^2(\tau)\big[\gamma_{N-1}^2(\tau)+\gamma_{N+1}^2(\tau)+\beta_N^2(\tau) + 2\beta_N(\tau)\beta_{N-1}(\tau)+\beta_{N-1}^2(\tau)\big]-\frac12 = O(\tau^{-\frac12}), \tag{3.21}$$

hence the integral in (3.19) converges. In addition, $F_N(\infty)$ is the free energy for the Gaussian ensemble. Since for the Gaussian ensemble,

$$h_n = h_0\gamma_1^2\cdots\gamma_n^2 = \frac{\sqrt{\pi}}{\sqrt{N}}\,\frac{1}{(2N)}\cdots\frac{n}{(2N)} = \frac{\sqrt{\pi}}{\sqrt{N}}\,\frac{n!}{(2N)^n}, \tag{3.22}$$

we obtain from (1.1) that

$$Z_N(\infty) = N!\prod_{n=0}^{N-1}\left[\frac{\sqrt{\pi}}{\sqrt{N}}\,\frac{n!}{(2N)^n}\right] = \frac{(2\pi)^{\frac12 N}}{(2N)^{\frac12 N^2}}\prod_{n=1}^N n!, \tag{3.23}$$

so that $F_N(\infty)=F_N^{\mathrm{Gauss}}$. Denote the function on the right in (3.19) by $\widetilde{F}_N(t)$. Then by Proposition 3.2,

$$\frac{\partial^2 F_N(t)}{\partial t^2} = -\frac{1}{t^2}\Big\{\gamma_N^2(t)\big[\gamma_{N-1}^2(t)+\gamma_{N+1}^2(t)+\beta_N^2(t) + 2\beta_N(t)\beta_{N-1}(t)+\beta_{N-1}^2(t)\big]-\frac12\Big\} = \frac{\partial^2\widetilde{F}_N(t)}{\partial t^2}, \tag{3.24}$$

hence $F_N(t)-\widetilde{F}_N(t)=at+b$. Since $F_N(\infty)=\widetilde{F}_N(\infty)=F_N^{\mathrm{Gauss}}$, we obtain that $a=b=0$, hence $F_N(t)=\widetilde{F}_N(t)$. Theorem 3.4 is proven. □

4. Analyticity of the equilibrium measure for a regular V.

This section is auxiliary. We prove a general theorem on the analyticity of the equilibrium measure with respect to perturbations of a regular V. The proof will follow directly from a result of Kuijlaars and McLaughlin [KM], that the Jacobian of the map of the end-points of the equilibrium measure to the integrals $\{T_j, N_k\}$ is nonzero. Let us introduce the main definitions. We will assume that $V(z)$ is a real analytic function satisfying the growth condition,

$$\lim_{|x|\to\infty} \frac{V(x)}{\log|x|} = \infty. \tag{4.1}$$

The weighted energy of a Borel probability measure μ on the line is

$$I_V(\nu) = -\iint_{\mathbb{R}^2} \log|x-y|\,\mathrm{d}\nu(x)\,\mathrm{d}\nu(y) + \int_{\mathbb{R}^1} V(x)\,\mathrm{d}\nu(x). \tag{4.2}$$

There exists a unique equilibrium probability measure ν_{eq}, which minimizes the functional $I_V(\nu)$,

$$I_V(\nu_{\mathrm{eq}}) = \min\Big\{I_V(\nu):\ \nu \geq 0,\ \int_{\mathbb{R}^1} \mathrm{d}\nu = 1\Big\}. \tag{4.3}$$

As shown in [DKM], the equilibrium measure is absolutely continuous and it is supported by a finite number of intervals,

$$\operatorname{supp}\nu_{\mathrm{eq}} = \bigcup_{i=1}^{q} [a_i, b_i]. \tag{4.4}$$

The density of ν_{eq} on the support is given by the formula

$$\rho(x) = \frac{1}{2\pi i} h(x)\sqrt{R_+(x)}, \quad x \in \bigcup_{i=1}^{q} [a_i, b_i], \tag{4.5}$$

where the function h is real analytic and

$$R(z) = \prod_{i=1}^{q} \big[(z-a_i)(z-b_i)\big]. \tag{4.6}$$

For $\sqrt{R(z)}$ the principal sheet is taken, with cuts on $\bigcup_{i=1}^{q}[a_i, b_i]$, and $\sqrt{R_+(x)}$ means the value on the upper cut. The equilibrium measure,

$\nu = \nu_{\mathrm{eq}}$, satisfies the following variational conditions: there exists a real constant $\ell = \ell_V$ such that

$$L\nu(x) - \frac{1}{2}V(x) = \ell, \quad x \in \bigcup_{i=1}^{q}[a_i, b_i], \tag{4.7}$$

$$L\nu(x) - \frac{1}{2}V(x) \le \ell, \quad x \in \mathbb{R}^1 \setminus \bigcup_{i=1}^{q}[a_i, b_i], \tag{4.8}$$

where

$$L\nu(z) = \int_{\bigcup_{i=1}^{q}[a_i,b_i]} \log|z-x|\,\mathrm{d}\nu(x), \quad z \in \mathbb{C}. \tag{4.9}$$

Set

$$\omega(z) = \int_{\mathbb{R}^1} \frac{\mathrm{d}\nu(x)}{z-x}, \quad z \in \mathbb{C}. \tag{4.10}$$

Then

$$\omega(z) = \frac{1}{2}V'(z) - \frac{1}{2}h(z)\sqrt{R(z)}, \tag{4.11}$$

see, e.g., [DKMVZ]. This implies that

$$h(z) = \frac{1}{2\pi i}\oint_{\Gamma} \frac{V'(s)}{\sqrt{R(s)}(s-z)}\,\mathrm{d}s, \tag{4.12}$$

where Γ is a positively oriented contour in Ω around $\{z\}\bigcup_{i=1}^{q}[a_i, b_i]$. Also, for $j = 0, \dots, q$,

$$T_j \equiv \frac{1}{2\pi i}\oint_{\Gamma} \frac{V'(z)z^j}{\sqrt{R(z)}}\,\mathrm{d}z = 2\delta_{jq}. \tag{4.13}$$

where Γ is a positively oriented contour in Ω around $\bigcup_{i=1}^{q}[a_i, b_i]$.

A real analytic V is called *regular* if

1) inequality (4.8) is strict for all $x \in \mathbb{R}^1 \setminus \bigcup_{i=1}^{q}[a_i, b_i]$,

2) $h(x) > 0$ for all $x \in \bigcup_{i=1}^{q}[a_i, b_i]$.

Otherwise V is called *singular*.

We formulate now the main result of this section.

THEOREM 4.1. — *Suppose $V(z;t)$, $t \in [-t_0,t_0]$, $t_0 > 0$, is a one-parameter family of real analytic functions such that*

(a) *there exists a domain $\Omega \subset \mathbb{C}$ such that $\mathbb{R} \subset \Omega$ and such that $V(z;t)$ is analytic on $\Omega \times [-t_0,t_0]$,*

(b) *$V(x,t)$ satisfies the uniform growth condition,*

$$\lim_{|x|\to\infty} \frac{\min\{V(x;t)\colon\ |t| \le t_0\}}{\log|x|} = \infty, \tag{4.14}$$

(c) *$V(z;0)$ is regular.*

Then there exists $t_1 > 0$ such that if $t \in [-t_1,t_1]$, then

1) *$V(z;t)$ is regular,*

2) *the number q of the intervals of the support of the equilibrium measure of $V(z;t)$ is independent of t, and*

3) *the end-points of the support intervals, $a_i(t),b_i(t)$, $i = 1,\dots,q$, are real analytic functions on $[-t_1,t_1]$.*

Proof. — The regularity of $V(z;t)$ and t-independence of q are proved in [KM]. To prove the analyticity consider the system of equations on $\{a_i, b_i,\ i = 1,\dots,q\}$,

$$T_j = 2\delta_{kq}\ \ (j = 0,1,\dots,q), \quad N_k = 0\ \ (k = 1,\dots,q-1), \tag{4.15}$$

where T_j is defined in (4.13) and

$$N_k = \frac{1}{2\pi i}\oint_{\Gamma_k} h(z)\sqrt{R(z)}\,\mathrm{d}z, \tag{4.16}$$

where Γ_k is a positively oriented contour around $[b_k, a_{k+1}]$, which lies in a small neighborhood of $[b_k, a_{k+1}]$, so that $\Gamma_k \subset \Omega$ and it does not contain the other end-points. In (4.16) it is assumed that the function $\sqrt{R(z)}$ is defined in such a way that it has a cut on $[b_k, a_{k+1}]$. As shown in [KM], the Jacobian of the map $\{[a_i, b_i]\} \mapsto \{T_j, N_k\}$ is nonzero. The functions T_j, N_k are analytic with respect to a_i, b_i and t. By the implicit function theorem, this implies the analyticity of $a_i(t), b_i(t)$. Theorem 4.1 is proved. □

When applied to a polynomial V, Theorem 4.1 gives the following result.

Corollary 4.2. — *Suppose* $V(z) = v_1 z + \cdots + v_{2d} z^{2d}$, $v_{2d} > 0$, *is q-cut regular. Then for any* $p \le 2d$ *there exists* $t_1 > 0$ *such that for any* $t \in [-t_1, t_1]$,

1) $V(z;t) = V(z) + tz^p$ *is q-cut regular,*

2) *the end-points of the support intervals,* $a_i(t), b_i(t)$, $i = 1, \dots, q$, *are real analytic functions on* $[-t_1, t_1]$.

Theorem 4.1 can be applied to prove the analyticity of the $(N = \infty)$-free energy,

$$F = \lim_{N\to\infty} -\frac{1}{N^2} \ln Z_N. \tag{4.17}$$

If V is real analytic satisfying growth condition (4.1), then the limit on the right exists, see [Joh], and

$$F = I_V(\nu_{\rm eq}). \tag{4.18}$$

Theorem 4.3. — *Under the conditions of Theorem* 4.1, *the free energy* $F = F(t)$ *is analytic on* $[-t_1, t_1]$.

Proof. — The density of the equilibrium measure has form (4.5), where $h(x)$ is a real analytic function, which is found by formula (4.12). By Theorem 4.1 the end-points of the support of $\nu_{\rm eq}$ depend analytically on t, hence (4.12) implies that h depends analytically on t, and, therefore, $\nu_{\rm eq}$ depends analytically on t. Formula (4.17) implies the analyticity of F. Theorem 4.3 is proved. □

Theorem 4.3 implies that the critical points of the random matrix model, the points of nonanalyticity of the free energy, are at singular V only.

5. Asymptotic expansion of the recurrence coefficients for a one-cut regular polynomial V.

In this section we will assume that $V(z)$ is a polynomial, which possesses a one-cut regular equilibrium measure. The equilibrium measure is one-cut means that its support consists of one interval $[a, b]$, and if it is one-cut regular then

$$d\nu_{\rm eq}(x) = \frac{1}{2\pi} h(x)\sqrt{(b-x)(x-a)}, \quad x \in [a, b], \tag{5.1}$$

where $h(x)$ is a polynomial such that $h(x) > 0$ for all real x (see the work of Deift, Kriecherbauer and McLaughlin [DKM]). For the sake of brevity, we will say that $V(x)$ is one-cut regular if its equilibrium measure is one-cut regular.

As shown by Kuijlaars and McLaughlin [KM], if $V(x)$ is one-cut regular then there exists $\varepsilon > 0$ such that for any s in the interval $1-\varepsilon \le s \le 1+\varepsilon$, the polynomial $s^{-1}V(x)$ is one-cut regular, and the end-points, $a(s), b(s)$, are analytic functions of s such that $a(s)$ is decreasing and $b(s)$ is increasing. (In fact, the result of Kuijlaars and McLaughlin is much more general and it includes multi-cut V as well.)

Proposition 5.1. — *Suppose $V(x)$ is one-cut regular. Then there exists $\varepsilon > 0$ such that for all n in the interval*

$$1-\varepsilon \le \frac{n}{N} \le 1+\varepsilon, \tag{5.2}$$

the recurrence coefficients admit the uniform asymptotic representation,

$$\gamma_n = \gamma\Big(\frac{n}{N}\Big) + O(N^{-1}), \quad \beta_n = \beta\Big(\frac{n}{N}\Big) + O(N^{-1}). \tag{5.3}$$

The functions $\gamma(s)$,$\beta(s)$ are expressed as

$$\gamma(s) = \frac{b(s)-a(s)}{4}, \quad \beta(s) = \frac{a(s)+b(s)}{2}, \tag{5.4}$$

where $[a(s),b(s)]$ is the support of the equilibrium measure for the polynomial $s^{-1}V(x)$.

Proof. — For $n = N$ the result follows from [DKMVZ]. For a general n, we can write $NV = ns^{-1}V$, $s = nN^{-1}$, and the result follows from the mentioned above result from [KM], that $s^{-1}V$ is one-cut regular, and from [DKMVZ]. The uniformity of the estimate of the error term follows from the result from [KM] on the analytic dependence of the equilibrium measure of $s^{-1}V$ on s and from the proof in [DKMVZ]. □

We can now formulate the main result of this section.

Theorem 5.2. — *Suppose that $V(x)$ is a one-cut regular polynomial. Then there exists $\varepsilon > 0$ such that for all n in the interval* (5.2), *the recurrence coefficients admit the following uniform asymptotic expansion* *as $N \to \infty$ in powers of N^{-2}:*

$$(5.5)\qquad \begin{cases} \gamma_n \sim \gamma\Big(\dfrac{n}{N}\Big) + \displaystyle\sum_{k=1}^{+\infty} N^{-2k} f_{2k}\Big(\dfrac{n}{N}\Big), \\ \beta_n \sim \beta\Big(\dfrac{n+\frac{1}{2}}{N}\Big) + \displaystyle\sum_{k=1}^{+\infty} N^{-2k} g_{2k}\Big(\dfrac{n+\frac{1}{2}}{N}\Big), \end{cases}$$

where $f_{2k}(s)$, $g_{2k}(s)$, $k \geq 1$, are analytic functions on $[1-\varepsilon, 1+\varepsilon]$.

Proof. — Let us remind that the proof in [DKMVZ] of the asymptotic formula for the recurrence coefficients is based on a reduction of the Riemann-Hilbert (RH) problem for orthogonal polynomials to a RH problem in which all the jumps are of the order of N^{-1}. By iterating the reduced RH problem, one obtains an asymptotic expansion of the recurrence coefficients,

$$(5.6)\qquad \gamma_N \sim \gamma + \sum_{k=1}^{\infty} N^{-k} f_k, \quad \beta_N \sim \beta + \sum_{k=1}^{\infty} N^{-k} \widehat{g}_k.$$

For a general n, let us write $NV = ns^{-1}V$, $s = nN^{-1}$. Then, as shown in [KM], the equilibrium meassure of $s^{-1}V$ is one-cut regular and it depends analytically on s in the interval $[1-\varepsilon, 1+\varepsilon]$. As follows from the iterations of the reduced RH problem, the coefficients f_k, $\hat{g}_k$ are expressed analytically in terms of the equilibrium measure and hence they analytically depend on nN^{-1}, so that

$$(5.7)\quad \gamma_n \sim \gamma\Big(\frac{n}{N}\Big) + \sum_{k=1}^{\infty} N^{-k} f_k\Big(\frac{n}{N}\Big), \;\; \beta_n \sim \beta\Big(\frac{n}{N}\Big) + \sum_{k=1}^{\infty} N^{-k} \widehat{g}_k\Big(\frac{n}{N}\Big),$$

where $f_k(s)$, $\widehat{g}_k(s)$ are analytic functions on $[1-\varepsilon, 1+\varepsilon]$. We can rewrite the expansion of β_n in the form

$$(5.8)\qquad \begin{cases} \gamma_n \sim \gamma\Big(\dfrac{n}{N}\Big) + \displaystyle\sum_{k=1}^{\infty} N^{-k} f_k\Big(\dfrac{n}{N}\Big), \\ \beta_n \sim \beta\Big(\dfrac{n+\frac{1}{2}}{N}\Big) + \displaystyle\sum_{k=1}^{\infty} N^{-k} g_k\Big(\dfrac{n+\frac{1}{2}}{N}\Big), \end{cases}$$

where $g_k(s)$ are analytic on $[1-\varepsilon, 1+\varepsilon]$. What we have to prove is that $f_k = g_k = 0$ for odd k. This will be done by using the string equations.

Recall the string equations for the recurrence coefficients,

$$(5.9)\qquad \gamma_n \big[V'(Q)\big]_{n,n-1} = \frac{n}{N}, \quad \big[V'(Q)\big]_{n,n} = 0.$$

where $[V'(Q)]_{n,m}$ is the element (n,m) of the matrix $V'(Q)$. We have, in particular, that

$$(5.10)\qquad \begin{cases} [Q]_{n,n-1} = \gamma_n, & [Q]_{n,n} = \beta_n, \\ [Q^2]_{n,n-1} = \beta_{n-1}\gamma_n + \beta_n\gamma_n, & [Q^2]_{n,n} = \gamma_n^2 + \beta_n^2 + \gamma_{n+1}^2, \\ [Q^3]_{n,n-1} = \gamma_{n-1}^2\gamma_n + \gamma_n^3 + \gamma_n\gamma_{n+1}^2 & \\ \qquad\qquad + \beta_{n-1}^2\gamma_n + \beta_{n-1}\beta_n\gamma_n + \beta_n^2\gamma_n, & \\ [Q^3]_{n,n} = \beta_{n-1}\gamma_n^2 + 2\beta_n\gamma_n^2 + 2\beta_n\gamma_{n+1}^2 + \beta_{n+1}\gamma_{n+1}^2 + \beta_n^3, & \end{cases}$$

and so on.

LEMMA 5.3. — *For any $k \geq 1$, the expression of $[Q^k]_{n,n-1}$ in terms of γ_j, β_j is invariant with respect to the change of variables*

$$(5.11)\qquad \sigma_0 = \{\gamma_j \mapsto \gamma_{2n-j},\, \beta_j \mapsto \beta_{2n-j-1},\, j = 0,1,2,\ldots\},$$

provided $n > j + k$. Similarly, the expression of $[Q^k]_{n,n}$ in terms of γ_j, β_j is invariant with respect to the change of variables

$$(5.12)\qquad \sigma_1 = \{\gamma_{n+j} \mapsto \gamma_{n-j+1},\, \beta_{n+j} \mapsto \beta_{n-j},\, j = 0,1,2,\ldots\},$$

provided $n > j + k$.

Proof. — Observe that the matrix Q^k is symmetric. By the rule of multiplication of matrices,

$$(5.13)\quad [Q^k]_{n,n-1} = \sum Q_{n,j_1} \cdots Q_{j_{k-1},n-1} = \sum Q_{n,\sigma(j_{k-1})} \cdots Q_{\sigma(j_1),n-1},$$

where $\sigma(j) \equiv 2n - j - 1$. Observe that $\sigma(n) = n - 1$, $\sigma(\sigma(j)) = j$ and

$$Q_{j,j} = \beta_j,\ \ Q_{\sigma(j),\sigma(j)} = \beta_{2n-j-1};\quad Q_{j,j-1} = \gamma_j,\ \ Q_{\sigma(j),\sigma(j-1)} = \gamma_{2n-j}.$$

This proves the invariance of $[Q^k]_{n,n-1}$ with respect to σ_0. The invariance of $[Q^k]_{n,n}$ with respect to σ_1 is established similarly. Lemma 5.3 is proved. □

Since $V'(Q)$ is a linear combination of powers of Q, we obtain the following corrolary of Lemma 5.3.

COROLLARY 5.4. — *The expression of $\gamma_n[V'(Q)]_{n,n-1}$ (respectively, $[V'(Q)]_{n,n}$) in terms of γ_j, β_j is invariant with respect to the change of variables σ_0 (respectively, σ_1).*

Let us

1) substitute asymptotic expansions (5.8) into equations (5.9) and expand into powers series in N^{-1};

2) expand $\gamma(\frac{n+j}{N})$, $f_k(\frac{n+j}{N})$, $\beta(\frac{n+\frac{1}{2}+j}{N})$, $g_k(\frac{n+\frac{1}{2}+j}{N})$ in the Taylor series at $s=n/N$;

3) equate coefficients at powers of N^{-1}.

This gives a system of equations on γ,β,f_k,g_k. The zeroth order equations read

$$\gamma\big[V'(Q_0)\big]_{n,n-1}=s,\quad [V'(Q_0)]_{n,n}=0, \tag{5.14}$$

where Q_0 is a constant infinite Jacobi (tridiagonal) matrix, such that

$$[Q_0]_{n,n}=\beta,\quad [Q_0]_{n,n-1}=[Q_0]_{n-1,n}=\gamma,\quad n\in\mathbb{Z}. \tag{5.15}$$

equations (5.14) are written as

$$A(\gamma,\beta)=s,\quad B(\gamma,\beta)=0, \tag{5.16}$$

where

$$\begin{cases} A(\gamma,\beta)=\gamma\sum_{j=2}^{2d} jv_j \sum_{k=0}^{[\frac{1}{2}(j-2)]} \beta^{j-2k-2}\gamma^{2k+1}\binom{j-1}{2k+1}\binom{2k+1}{k}, \\ B(\gamma,\beta)=\sum_{j=1}^{2d} jv_j \sum_{k=0}^{[\frac{1}{2}(j-1)]} \beta^{j-2k-1}\gamma^{2k}\binom{j-1}{2k}\binom{2k}{k}. \end{cases} \tag{5.17}$$

Observe that γ,β given in (5.4) solve equations (5.16). The k-th order equations for $k\geq 1$ have the form

$$\frac{\partial A(\gamma,\beta)}{\partial\gamma}f_k+\frac{\partial A(\gamma,\beta)}{\partial\beta}g_k=p,\quad \frac{\partial B(\gamma,\beta)}{\partial\gamma}f_k+\frac{\partial B(\gamma,\beta)}{\partial\beta}g_k=q, \tag{5.18}$$

where p,q are expressed in terms of the previous coefficients, $\gamma,\beta,f_1,g_1,\dots,$ f_{k-1},g_{k-1}, and their derivatives. Here the partial derivatives on the left are evaluated at γ,β given in (5.4).

Lemma 5.5. — *The first order equations are*

$$\frac{\partial A(\gamma,\beta)}{\partial\gamma}f_1+\frac{\partial A(\gamma,\beta)}{\partial\beta}g_1=0,\quad \frac{\partial B(\gamma,\beta)}{\partial\gamma}f_1+\frac{\partial B(\gamma,\beta)}{\partial\beta}g_1=0. \tag{5.19}$$

Proof. — Observe that the terms with f_1, g_1 are the only first order terms which appear at step 1) above. All the other terms appear at step 2), in the expansion of $\gamma(\frac{n+j}{N})$, $f_k(\frac{n+j}{N})$, $\beta(\frac{n+\frac{1}{2}+j}{N})$, $g_k(\frac{n+\frac{1}{2}+j}{N})$ in the Taylor series at $s = n/N$. Consider any monomial on the left in the first equation in (5.9),

$$C\gamma_{n+j_1}\cdots\gamma_{n+j_p}\beta_{n+\ell_1}\cdots\beta_{n+\ell_q}.$$

By Lemma 5.3, there is a partner to this term of the form

$$C\gamma_{n-j_1}\cdots\gamma_{n-j_p}\beta_{n-\ell_1-1}\cdots\beta_{n-\ell_q-1}.$$

When we substitute expansions (5.8), we obtain

$$C\Big(\gamma\Big(s+\frac{j_1}{N}\Big)+\cdots\Big)\cdots\Big(\gamma\Big(s+\frac{j_p}{N}\Big)+\cdots\Big)\Big(\beta\Big(s+\frac{\ell_1+\frac{1}{2}}{N}\Big)+\cdots\Big) \cdots\Big(\beta\Big(s+\frac{\ell_q+\frac{1}{2}}{N}\Big)+\cdots\Big)$$

and

$$C\Big(\gamma\Big(s-\frac{j_1}{N}\Big)+\cdots\Big)\cdots\Big(\gamma\Big(s-\frac{j_p}{N}\Big)+\cdots\Big)\Big(\beta\Big(s-\frac{\ell_1+\frac{1}{2}}{N}\Big)+\cdots\Big) \cdots\Big(\beta\Big(s-\frac{\ell_q+\frac{1}{2}}{N}\Big)+\cdots\Big)$$

for the partner. When we expand these expressions in powers of N^{-1}, the first order terms cancel each other in the sum of the partners (in fact, all the odd terms cancel). This proves the first equation in (5.19). The second one is proved similarly. Lemma 5.5 is proved. □

Lemma 5.5 implies that $f_1(s) = g_1(s) = 0$ for all s such that

$$\det\begin{pmatrix}\partial A(\gamma,\beta)/\partial\gamma & \partial A(\gamma,\beta)/\partial\beta\\ \partial B(\gamma,\beta)/\partial\gamma & \partial B(\gamma,\beta)/\partial\beta\end{pmatrix}\neq 0, \tag{5.20}$$

where all the partial derivatives are evaluated at $\gamma(s), \beta(s)$ given in (5.4).

LEMMA 5.6. — *If for a given* $s \in [1-\varepsilon, 1+\varepsilon]$, *condition* (5.20) *holds, then all odd coefficients* $f_{2k+1}(s)$, $g_{2k+1}(s)$ *are zero.*

Proof. — By Lemma 5.5 $f_1(s) = g_1(s) = 0$. If we consider terms of the order of N^{-3} then we obtain the equations

$$\frac{\partial A(\gamma,\beta)}{\partial\gamma}f_3+\frac{\partial A(\gamma,\beta)}{\partial\beta}g_3=0,\quad \frac{\partial B(\gamma,\beta)}{\partial\gamma}f_3+\frac{\partial B(\gamma,\beta)}{\partial\beta}g_3=0. \tag{5.21}$$

Indeed, the same argument as in Lemma 5.5 proves that all other terms of the third order cancel out. Since condition (5.20) holds, it implies that $f_3(s) = g_3(s) = 0$. By continuing this argument we prove that all odd $f_{2k+1}(s)$, $g_{2k+1}(s)$ vanish. Lemma 5.6 is proved. □

Lemma 5.7. — *Condition* (5.20) *holds for all* $s \in [1-\varepsilon, 1+\varepsilon]$.

Proof. — By differentiating equations (5.16) in s we obtain that

$$\left\{ \begin{aligned} &\frac{\partial A(\gamma,\beta)}{\partial\gamma}\frac{\partial\gamma}{\partial s} + \frac{\partial A(\gamma,\beta)}{\partial\beta}\frac{\partial\beta}{\partial s} = 1, \\ &\frac{\partial B(\gamma,\beta)}{\partial\gamma}\frac{\partial\gamma}{\partial s} + \frac{\partial B(\gamma,\beta)}{\partial\beta}\frac{\partial\beta}{\partial s} = 0. \end{aligned} \right. \tag{5.22}$$

By differentiating equations (5.16) in t_1 we obtain that

$$\left\{ \begin{aligned} &\frac{\partial A(\gamma,\beta)}{\partial\gamma}\frac{\partial\gamma}{\partial t_1} + \frac{\partial A(\gamma,\beta)}{\partial\beta}\frac{\partial\beta}{\partial t_1} = 0, \\ &\frac{\partial B(\gamma,\beta)}{\partial\gamma}\frac{\partial\gamma}{\partial t_1} + \frac{\partial B(\gamma,\beta)}{\partial\beta}\frac{\partial\beta}{\partial t_1} = -1. \end{aligned} \right. \tag{5.23}$$

By rewriting equations (5.22), (5.23) in the matrix form, we obtain that

$$\begin{pmatrix} \partial A(\gamma,\beta)/\partial\gamma & \partial A(\gamma,\beta)/\partial\beta \\ \partial B(\gamma,\beta)/\partial\gamma & \partial B(\gamma,\beta)/\partial\beta \end{pmatrix} \begin{pmatrix} \partial\gamma/\partial s & \partial\gamma/\partial t_1 \\ \partial\gamma/\partial s & \partial\gamma/\partial t_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.24}$$

Since

$$\det \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -1 \neq 0,$$

this implies (5.20). Lemma 5.7 is proved. □

From Lemmas 5.6 and 5.7 we obtain that the odd coefficients f_{2k+1}, g_{2k+1} vanish. Theorem 5.2 is proved. □

6. Asymptotic expansion of the free energy for a one-cut regular *V*.

We have the following extension of Theorem 5.2.

Theorem 6.1. — *Suppose that* $V(z;t)$, $t \in [-t_0, t_0]$, $t_0 > 0$, *is a one-parameter analytic family of polynomials of degree* $2d$, *such that* $V(z;0)$ *is one-cut regular. Then there exist* $t_1 > 0$ *and* $\varepsilon > 0$ *such that for all* $t \in [-t_1, t_1]$ *and all* $n \in [(1-\varepsilon)N, (1+\varepsilon)N]$, *the recurrence coefficients corresponding to* $V(z;t)$, *admit the following uniform asymptotic expansion as* $N \to \infty$:

$$\left\{ \begin{aligned} &\gamma_n \sim \gamma\Big(\frac{n}{N};t\Big) + \sum_{k=1}^{\infty} N^{-2k} f_{2k}\Big(\frac{n}{N};t\Big), \\ &\beta_n \sim \beta\Big(\frac{n+\frac{1}{2}}{N};t\Big) + \sum_{k=1}^{\infty} N^{-2k} g_{2k}\Big(\frac{n+\frac{1}{2}}{N};t\Big), \end{aligned} \right. \tag{6.1}$$

where $\gamma(s;t)$, $\beta(s;t)$, $f_{2k}(s;t)$, $g_{2k}(s;t)$, $k \geq 1$, *are analytic functions of* s,t *on* $[1-\varepsilon, 1+\varepsilon] \times [-t_1, t_1]$.

Proof. — Theorem 4.1 implies that the equilibrium measure of $V(z;t)$ is analytic in $t \in [-t_1, t_1]$ (see the proof of Theorem 4.3). This implies the analyticity of γ, β in s and t. From equations (5.18) we obtain the analyticity of f_k, g_k, $k \geq 1$. By Theorem 5.2 all odd f_{2k+1}, g_{2k+1} vanish. This proves Theorem 6.1. □

Let us return to the polynomial $V(z;t) = \tau_t V(z)$. We will assume the following hypothesis.

Hypothesis (R). — For all $t \geq 1$ the polynomial $\tau_t V(z)$ is one-cut regular.

Theorem 6.2. — *If a polynomial* $V(z)$ *satisfies Hypothesis* (R), *then its free energy admits the asymptotic expansion,*

$$F_N - F_N^{\mathrm{Gauss}} \sim F + N^{-2}F^{(2)} + N^{-4}F^{(4)} + \cdots, \tag{6.2}$$

where F_N^{Gauss} *is defined in* (3.20). *The leading term of the asymptotic expansion is*

$$F = \int_1^\infty \frac{1-\tau}{\tau^2}\left[2\gamma^4(\tau) + 4\gamma^2(\tau)\beta^2(\tau) - \frac{1}{2}\right] \mathrm{d}\tau, \tag{6.3}$$

where

$$\gamma(\tau) = \frac{b(\tau) - a(\tau)}{4}, \quad \beta(\tau) = \frac{a(\tau) + b(\tau)}{2}, \tag{6.4}$$

and $[a(\tau), b(\tau)]$ *is the support of the equilibrium measure for the polynomial* $V(z;\tau)$. *The quantities* $\gamma = \gamma(\tau)$, $\beta = \beta(\tau)$ *solve the equations,*

$$A(\gamma, \beta; \tau) = 1, \quad B(\gamma, \beta; \tau) = 0, \tag{6.5}$$

where

$$\begin{cases} A(\gamma,\beta;\tau) = 2\Big(1 - \dfrac{1}{\tau}\Big)\gamma^2 \\ \qquad\qquad + \gamma \displaystyle\sum_{j=1}^{2d} \frac{j v_j}{\tau^{\frac{1}{2}j}} \sum_{k=0}^{[\frac{1}{2}(j-2)]} \beta^{j-2k-2}\gamma^{2k+1}\binom{j-1}{2k+1}\binom{2k+1}{k}, \\ B(\gamma,\beta;\tau) = 2\Big(1 - \dfrac{1}{\tau}\Big)\beta \\ \qquad\qquad + \displaystyle\sum_{j=1}^{2d} \frac{j v_j}{\tau^{\frac{1}{2}j}} \sum_{k=0}^{[\frac{1}{2}(j-1)]} \beta^{j-2k-1}\gamma^{2k}\binom{j-1}{2k}\binom{2k}{k}. \end{cases} \tag{6.6}$$

Proof. — By applying Theorem 6.1 to $\tau_t V(z)$, we obtain the uniform asymptotic expansions,

$$\left\{\begin{aligned}&\gamma_n(t)\sim\gamma\Big(\frac{n}{N};t\Big)+\sum_{k=1}^{\infty}N^{-2k}f_k\Big(\frac{n}{N};t\Big),\\&\beta_n(t)\sim\beta\Big(\frac{n+\frac12}{N};t\Big)+\sum_{k=1}^{\infty}N^{-2k}g_k\Big(\frac{n+\frac12}{N};t\Big).\end{aligned}\right.\tag{6.7}$$

From (6.6) with $\tau=t$, as $t\to\infty$,

$$A(\gamma,\beta;t)=2\gamma^2+O(t^{-\frac12}),\quad B(\gamma,\beta;t)=2\beta+O(t^{-\frac12}),\tag{6.8}$$

hence the solutions to system (6.5) are

$$\gamma(t)=\frac{\sqrt2}{2}+O(t^{-\frac12}),\quad \beta(t)=O(t^{-\frac12}).\tag{6.9}$$

By differentiating equations (6.5) in $\tau=t$ we obtain the equations,

$$\left\{\begin{aligned}&\frac{\partial A(\gamma,\beta;t)}{\partial\gamma}\gamma'(t)+\frac{\partial A(\gamma,\beta;t)}{\partial\beta}\beta'(t)=p,\\&\frac{\partial B(\gamma,\beta;t)}{\partial\gamma}\gamma'(t)+\frac{\partial B(\gamma,\beta;t)}{\partial\beta}\beta'(t)=q,\end{aligned}\right.\tag{6.10}$$

where p,q are expressed in terms of $\gamma(t),\beta(t)$ and $p,q=O(t^{-\frac12})$. From this system of equations we obtain that $\gamma'(t),\beta'(t)=O(t^{-\frac12})$. By differentiating equations (6.5) many times we obtain the estimates for $j\ge1$,

$$\gamma^{(j)}(t)=O(t^{-\frac12}),\quad \beta^{(j)}(t)=O(t^{-\frac12}).\tag{6.11}$$

From equations (5.18) we obtain the estimates on f_k,g_k,

$$f_k^{(j)}(t)=O(t^{-\frac12}),\quad g_k^{(j)}(t)=O(t^{-\frac12}),\quad j\ge0,\tag{6.12}$$

and from the reduced RH problem, that for any $K\ge0$,

$$\left\{\begin{aligned}&\Big|\gamma_n(t)-\gamma\Big(\frac{n}{N};t\Big)-\sum_{k=1}^{K}N^{-2k}f_k\Big(\frac{n}{N};t\Big)\Big|\\&\qquad\qquad\le C(K)N^{-2K-2}t^{-\frac12},\\&\Big|\beta_n(t)-\beta\Big(\frac{n+\frac12}{N};t\Big)-\sum_{k=1}^{K}N^{-2k}g_k\Big(\frac{n+\frac12}{N};t\Big)\Big|\\&\qquad\qquad\le C(K)N^{-2K-2}t^{-\frac12}.\end{aligned}\right.\tag{6.13}$$

(cf. the derivation of the estimates (A.77) and (A.78) in Appendix A.) Let us substitute expansions (6.7) into (3.19) and expand the terms on the right in the Taylor series at $n/N=1$. In this way we obtain the asymptotic expansion,

$$\begin{aligned}&\Theta_N(\tau)\\ &\quad\equiv\gamma_N^2(\tau)\big[\gamma_{N-1}^2(\tau)+\gamma_{N+1}^2(\tau)+\beta_N^2(\tau)\\ &\qquad\qquad+2\beta_N(\tau)\beta_{N-1}(\tau)+\beta_{N-1}^2(\tau)\big]-\frac{1}{2} \qquad (6.14)\\ &\quad\sim\Theta(\tau)+\sum_{k=1}^{\infty}N^{-k}\Theta^{(k)}(\tau) \qquad (6.15)\end{aligned}$$

where

$$\Theta(\tau)=2\gamma^4(\tau)+4\gamma^2(\tau)\beta^2(\tau)-\frac{1}{2}, \tag{6.16}$$

$$\Theta^{(k)}(\tau)\le C(k)\tau^{-\frac{1}{2}},\quad k\ge 1. \tag{6.17}$$

Observe that the expression (6.14) is invariant with respect to the transformation

$$\sigma_0=\big\{\gamma_j\mapsto\gamma_{2N-j},\ \beta_j\mapsto\beta_{2N-j-1}\big\}. \tag{6.18}$$

Therefore, as in the proof of Lemmas 5.5, 5.6, we obtain that all odd $\Theta^{(2k+1)}=0$. Theorem 6.2 is proved. □

The following parametric extension of Theorem 6.2 is useful for applications.

Theorem 6.3. — *Suppose* $\{V(z;u),\ u\in[-u_0,u_0]\}$ *is a one-parameter analytic family of one-cut regular polynomials of degree* $2d$ *such that the polynomial* $V(z;0)$ *satisfies Hypothesis* (R). *Then there exists* $u_1>0$ *such that the coefficients* $F(u),F^{(2)}(u),F^{(4)}(u),\dots$ *of the asymptotic expansion of the free energy for* $V(z;u)$ *are analytic on* $[-u_1,u_1]$.

Proof. — The functions are expressed in terms of integrals of finite combinations of the functions $\gamma^{(j)}(1;u)$, $\beta^{(j)}(1;u)$, $f_{2k}^{(j)}(1;u)$, $g_{2k}^{(j)}(1;u)$. By Theorem 6.1 these functions are analytic in u. By the same argument as in the proof of Theorem 6.2, we obtain that they behave like $O(t^{-\frac{1}{2}})$ as $t\to\infty$. Therefore, the integrals expressing $F^{(2n)}(u)$ converge and define an analytic function in u. Theorem 6.3 is proved. □

The following proposition is auxiliary: it gives first several terms of the asymptotic expansion of the free energy for the Gaussian ensemble.

PROPOSITION 6.4. — *The constant $F_N^{\rm Gauss}$ has the following expansion*:

$$F_N^{\rm Gauss} \sim \frac{\ln 2}{2} - \frac{3}{4} - \frac{\ln N}{N} + \big(1-\ln(2\pi)\big)\frac{1}{N} - \frac{5\ln N}{12N^2} - \Big(\zeta'(-1)+\frac{\ln(2\pi)}{2}\Big)\frac{1}{N^2} - \frac{1}{12N^3} + \frac{1}{240N^4} + \frac{1}{360N^5} + O\Big(\frac{1}{N^6}\Big). \tag{6.19}$$

Proof. — This is obtained from (3.20) with the help of Maple. □

7. Exact formula for the free energy for an even V.

For an even V, $\beta(\tau)=0$, and formula (6.3) simplifies,

$$F = \int_1^\infty \frac{1-\tau}{\tau^2}\Big[2R^2(\tau)-\frac{1}{2}\Big]\,\mathrm{d}\tau, \tag{7.1}$$

where

$$R(\tau)=\gamma^2(\tau). \tag{7.2}$$

From (6.5), (6.6) we obtain that $R=R(\tau)$ solves the equation

$$2\Big(1-\frac{1}{\tau}\Big)R + \sum_{j=1}^{d} j v_{2j}\binom{2j}{j}\Big(\frac{R}{\tau}\Big)^j = 1. \tag{7.3}$$

Set

$$\xi(\tau)=\frac{R(\tau)}{\tau}\cdot \tag{7.4}$$

Then equation (7.3) is rewritten as

$$\tau = \frac{1}{2\xi} + 1 - \frac{1}{2}\sum_{j=1}^{d} v_{2j} j \binom{2j}{j}\xi^{j-1} \equiv \tau(\xi). \tag{7.5}$$

From (7.1),

$$(7.6)\quad F=\int_1^\infty (1-\tau)\Big[2\xi^2(\tau)-\frac{1}{2\tau^2}\Big]\,\mathrm{d}\tau = \lim_{T\to\infty}\Big[\int_1^T 2(1-\tau)\xi(\tau)^2\,\mathrm{d}\tau+\frac{1}{2}\ln T-\frac{1}{2}\Big].$$

The change of variable $\tau=\tau(\xi)$ reduces the latter formula to

$$(7.7)\qquad F=\lim_{T\to\infty}\Big[\int_{\xi(1)}^{\xi(T)} 2(1-\tau(\xi))\xi^2\tau'(\xi)\,\mathrm{d}\xi+\frac{1}{2}\ln T-\frac{1}{2}\Big],$$

or, by (7.5), to

$$(7.8)\quad F=\lim_{T\to\infty}\frac{1}{2}\Big[\int_{\xi(1)}^{\xi(T)}\Big(\frac{1}{\xi}-\sum_{j=1}^d v_{2j}j\binom{2j}{j}\xi^{j-1}\Big) \times\Big(1+\sum_{j=1}^d v_{2j}j(j-1)\binom{2j}{j}\xi^j\Big)\,\mathrm{d}\xi+\ln T\Big].$$

By (7.4), $\xi(1)=R(1)$. By (3.16), $R(T)=\frac{1}{2}+O(T^{-\frac{1}{2}})$, (since $n=N$), hence

$$(7.9)\qquad \xi(T)=\frac{R(T)}{T}=\frac{1}{2T}+O(T^{-\frac{3}{2}}).$$

When we distribute on the right in (7.8), we have the following terms:

$$(1)\quad \lim_{T\to\infty}\Big[\int_{\xi(1)}^{\xi(T)}\frac{1}{\xi}\,\mathrm{d}\xi+\ln T\Big]=-\ln 2-\ln R(1),$$

$$(2)\quad -\int_{\xi(1)}^{0}\sum_{j=1}^d v_{2j}j\binom{2j}{j}\xi^{j-1}\,\mathrm{d}\xi=\sum_{j=1}^d v_{2j}\binom{2j}{j}R(1)^j,$$

$$(3)\quad \int_{\xi(1)}^{0}\sum_{j=1}^d v_{2j}j(j-1)\binom{2j}{j}\xi^{j-1}=-\sum_{j=1}^d v_{2j}(j-1)\binom{2j}{j}R(1)^j,$$

$$(4)\quad -\int_{\xi(1)}^{0}\Big(\sum_{j=1}^d v_{2j}j\binom{2j}{j}\xi^{j-1}\Big)\Big(\sum_{k=1}^d v_{2k}k(k-1)\binom{2k}{k}\xi^k\Big)\,\mathrm{d}\xi$$

$$(7.10)\qquad =\sum_{n=2}^{2d}\Big[\sum_{\substack{1\le j,\,k\le m\\ j+k=n}} v_{2j}v_{2k}jk(k-1)\binom{2j}{j}\binom{2k}{k}\Big]\frac{1}{n}R(1)^n,$$

hence (7.8) can be transformed into the following expression:

$$F = -\frac{\ln 2}{2} - \frac{\ln R}{2} + \sum_{j=1}^{d} u_j R^j + \sum_{n=2}^{2d} w_n R^n, \tag{7.11}$$

where $R = R(1)$,

$$u_j = -\frac{j-2}{2}\binom{2j}{j} v_{2j}, \tag{7.12}$$

$$w_n = \frac{1}{2n} \sum_{\substack{1\le j,\, k\le m \\ j+k=n}} v_{2j} v_{2k}\, jk(k-1)\binom{2j}{j}\binom{2k}{k}. \tag{7.13}$$

Example. — The quartic polynomial,

$$V(z) = \frac{z^4}{4} + tz^2, \quad t > -1. \tag{7.14}$$

The condition $t > -1$ is necessary and sufficient for one cut. By (7.3), $R = R(1) > 0$ solves the equation,

$$3R^2 + 2tR = 1, \tag{7.15}$$

hence

$$R = \frac{-t + \sqrt{t^2+3}}{3}\cdot \tag{7.16}$$

By (7.12), (7.13),

$$u_1 = t, \quad u_2 = 0, \quad w_2 = 0, \quad w_3 = t, \quad w_4 = \frac{9}{8}\cdot \tag{7.17}$$

Thus, by (7.11),

$$F = -\frac{\ln 2}{2} - \frac{\ln R}{2} + tR + tR^3 + \frac{9}{8} R^4. \tag{7.18}$$

8. One-sided analyticity for a singular V.

In this section we will prove general results on the one-sided analyticity for a singular V. We will assume the following hypothesis.

Hypothesis (S). — $V(z;t)$, $t \in [0, t_0]$, is a one-parameter family of real analytic functions such that

(a) there exists a domain $\Omega \subset \mathbb{C}$ such that $\mathbb{R} \subset \Omega$ and such that $V(z;t)$ is analytic on $\Omega \times [0, t_0]$,

(b) $V(x, t)$ satisfies the uniform growth condition,

$$\lim_{|x| \to \infty} \frac{\min\{V(x;t) : 0 \le t \le t_0\}}{\log |x|} = \infty, \tag{8.1}$$

(c) $V(z;t)$ is one-cut regular for $0 < t \le t_0$,

(d) $V(z;0)$ is one-cut singular and $h(a) \neq 0$, $h(b) \neq 0$, where $[a, b]$ is the support of the equilibrium measure for $V(z;0)$.

THEOREM 8.1. — *Suppose $V(z;t)$ satisfies Hypothesis (S). Then the end-points $a(t)$,$b(t)$ of the equilibrium measure for $V(z;t)$ are analytic on $[0,t_0]$.*

Proof. — Set

$$T_j(a, b;t) = \frac{1}{2\pi i} \oint_\Gamma \frac{V'(z;t) z^j}{\sqrt{(z-a)(z-b)}} \,\mathrm{d}z, \tag{8.2}$$

where Γ is a positively oriented closed contour around $[a, b]$ inside Ω. For $t \in [0, t_0]$, we have the following equations on $a = a(t)$, $b = b(t)$:

$$T_0(a, b;t) = 0, \quad T_1(a, b;t) = 2. \tag{8.3}$$

By differentiating (8.2) we obtain that at $a = a(t)$, $b = b(t)$,

$$\begin{aligned} \frac{\partial T_0(a, b;t)}{\partial a} &= \frac{1}{4\pi i} \oint_\Gamma \frac{V'(z)}{(z-a)\sqrt{R(z)}} \,\mathrm{d}z \\ &= \frac{1}{4\pi i} \oint_\Gamma \Big(\frac{2\omega(z)}{\sqrt{R(z)}} + h(z) \Big) \frac{1}{z-a} \,\mathrm{d}z = \frac{h(a)}{2} \cdot \end{aligned} \tag{8.4}$$

Similarly,

$$\begin{cases} \dfrac{\partial T_0(a, b;t)}{\partial b} = \dfrac{h(b)}{2}, \\ \dfrac{\partial T_1(a, b;t)}{\partial a} = \dfrac{a h(a)}{2}, \quad \dfrac{\partial T_1(a, b;t)}{\partial b} = \dfrac{b h(b)}{2} \cdot \end{cases} \tag{8.5}$$

Thus, the Jacobian,

$$J = \det \begin{pmatrix} \partial T_0/\partial a & \partial T_0/\partial b \\ \partial T_1/\partial a & \partial T_1/\partial b \end{pmatrix} = \frac{h(a)h(b)(b-a)}{4} \neq 0. \tag{8.6}$$

The function $T_0(a, b;t)$ is analytic in a, b, t, hence by the implicit function theorem, $a(t), b(t)$ are analytic on $[0, t_0]$. Theorem 8.1 is proved. □

Corollary 8.2. — *Suppose* $V(z;t)$ *satisfies Hypothesis* (S)*. Then*

1) *the function* $h(x;t)$ *is analytic on* $R^1 \times [0,t_0]$,

2) *the free energy* $F(t)$ *is analytic on* $[0,t_0]$,

3) *the functions* $\gamma(t),\beta(t)$ *are analytic on* $[0,t_0]$.

Proof. — The analyticity of h follows from formula (4.12) and the one of F, from (4.18). Finally, the analyticity of $\gamma(t),\beta(t)$ follows from (5.4). □

Theorem 8.1 and Corollary 8.2 can be extended to multi-cut V. We will say that V is *q-cut* if the support of its equilibrium measure consists of q intervals, $[a_i,b_i]$, $i=1,\dots,q$. We will assume the following hypothesis.

Hypothesis (S_q). — $V(z;t)$, $t\in[0,t_0]$, is a one-parameter family of real analytic functions such that

(a) there exists a domain $\Omega\subset\mathbb{C}$ such that $\mathbb{R}\subset\Omega$ and such that $V(z;t)$ is analytic on $\Omega\times[0,t_0]$,

(b) $V(x,t)$ satisfies the uniform growth condition,

$$\lim_{|x|\to\infty}\frac{\min\{V(x;t):\ 0\le t\le t_0\}}{\log|x|}=\infty, \tag{8.7}$$

(c) $V(z;t)$ is q-cut regular for $0<t\le t_0$,

(d) $V(z;0)$ is q-cut singular (with the same q as in (c)) and $h(a_i)\neq 0$, $h(b_i)\neq 0$, $i=1,\dots,q$.

Theorem 8.3. — *Suppose* $V(z;t)$ *satisfies Hypothesis* (S_q)*. Then the end-points* $a_i(t),b_i(t)$ *of the equilibrium measure for* $V(z;t)$ *are analytic on* $[0,t_0]$.

Proof. — Consider system of equations (4.15) for $V=V(z;t)$. As shown in [KM], the Jacobian of the map

$$f:\{a_i,b_i,\ i=1,\dots,q\}\longmapsto\{T_j,N_k,\ j=0,\dots,q;\ k=1,\dots,q-1\} \tag{8.8}$$

at $\{a_i(t),b_i(t)\}$ is equal to

$$\det\Big(\frac{\partial\{T_j,N_k\}}{\partial\{a_i,b_i\}}\Big)=\Big(\prod_{i=1}^{q}\frac{\partial T_0}{\partial a_i}\,\frac{\partial T_0}{\partial b_i}\Big)\pi^{-q+1}\times\int_{b_1}^{a_2}\sqrt{R_+(x_1)}\,\mathrm{d}x_1\cdots\int_{b_{q-1}}^{a_q}\sqrt{R_+(x_{q-1})}\,\mathrm{d}x_{q-1} \tag{8.9}$$

$$\times \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_1 & b_1 & \dots & b_q \\ \vdots & \vdots & \vdots & \vdots \\ a_1^q & b_1^q & \dots & b_q \\ (x_1-a_1)^{-1} & (x_1-b_1)^{-1} & \dots & (x_1-b_q)^{-1} \\ \vdots & \vdots & \vdots & \vdots \\ (x_{q-1}-a_1)^{-1} & (x_{q-1}-b_1)^{-1} & \dots & (x_{q-1}-b_q)^{-1} \end{pmatrix}.$$

The determinant on the right is a mixture of a Vandermonde determinant and a Cauchy determinant. As shown in [KM], it is equal to

$$(8.10)\qquad \frac{\prod_{j=1}^{q}\prod_{k=1}^{q}(b_k-a_j)\prod_{1\le j<k\le q}(a_k-a_j)(b_k-b_j)\prod_{1\le j<k\le q-1}(x_k-x_j)}{(-1)^{q-1}\prod_{j=1}^{q-1}\prod_{k=1}^{q}(x_j-a_k)(x_j-b_k)},$$

which is nonzero for

$$(8.11)\qquad a_1<b_1<x_1<a_2<\cdots<b_{q-1}<x_{q-1}<a_q<b_q,$$

and therefore has a fixed sign. Hence the multiple integral in (8.9) is nonzero. Now,

$$(8.12)\qquad \begin{aligned} \frac{\partial T_0}{\partial a_i} &= \frac{1}{4\pi i}\oint_\Gamma \frac{V'(z)}{(z-a_i)\sqrt{R(z)}}\,\mathrm{d}z \\ &= \frac{1}{4\pi i}\oint_\Gamma \Big(\frac{2\omega(z)}{\sqrt{R(z)}}+h(z)\Big)\frac{\mathrm{d}z}{z-a_i} = \frac{h(a_i)}{2}, \end{aligned}$$

and a similar formula holds for $\partial T_0/\partial b_i$. Thus, the Jacobian (8.9) is nonzero. The functions $\{T_j,N_k\}$ are analytic in $\{a_i,b_i\},t$, hence, by the implicit function theorem, $\{a_i(t),b_i(t)\}$ are analytic on $[0,t_0]$. Theorem 8.3 is proved. □

As a corollary of Theorem 8.3, we obtain the following results.

COROLLARY 8.4. — *Suppose $V(z;t)$ satisfies Hypothesis (S_q). Then*
1) *the function $h(x;t)$ is analytic on $R^1\times[0,t_0]$,*
2) *the free energy $F(t)$ is analytic on $[0,t_0]$.*

Proof. — The analyticity of h follows from formula (4.12) and the one of F, from (4.18). □

The following extension of Theorem 8.3 will be useful for us. Suppose $V(z;t)$ satisfies Hypothesis (S_q). Then, as shown in [KM], for every $t \in (0,t_0]$, there exists $\varepsilon = \varepsilon(t) > 0$ such that for any $s \in [1-\varepsilon,\varepsilon]$ the function $s^{-1}V(z;t)$ is q-cut regular and the end-points, $a_i(s;t), b_i(s;t)$ are analytic in $s \in [1-\varepsilon,\varepsilon]$.

Proposition 8.5. — *Suppose $V(z;t)$ satisfies Hypothesis (S_q). Then for any $j \geq 0$ the functions*

$$\frac{\partial^j a_i(s;t)}{\partial s^j}\Big|_{s=1}, \quad \frac{\partial^j b_i(s;t)}{\partial s^j}\Big|_{s=1}, \quad i=1,\dots,q, \tag{8.13}$$

are analytic on $[0,t_0]$.

Proof. — By differentiating system (4.15) in s and setting $s=1$, we obtain a linear $2q\times 2q$ system of equations on

$$\frac{\partial a_i(s;t)}{\partial s}\Big|_{s=1}, \quad \frac{\partial b_i(s;t)}{\partial s}\Big|_{s=1}, \quad i=1,\dots,q. \tag{8.14}$$

The determinant of the system is calculated in (8.9), (8.10) and it is nonzero. The coefficients of the system are analytic function in $t \in [0,t_0]$, hence functions (8.14) are analytic in $t \in [0,t_0]$. By differentiating system (4.15) twice in s and setting $s=1$, we obtain a linear $2q\times 2q$ system of equations on

$$\frac{\partial^2 a_i(s;t)}{\partial s^2}\Big|_{s=1}, \quad \frac{\partial^2 b_i(s;t)}{\partial s^2}\Big|_{s=1}, \quad i=1,\dots,q. \tag{8.15}$$

The coefficients of the system are the same as for the first derivatives, but the right hand side changes, and it is expressed in terms of a_i, b_i and its first derivatives in s at $s=1$, which are analytic in $t \in [0,t_0]$. This proves the analyticity of the second derivatives, and so on. Proposition 8.5 is proved. □

Let us consider next the coefficients, $\gamma(s;t), \beta(s;t), f_{2k}(s;t), g_{2k}(s;t)$, $k \geq 1$, of asymptotic expansions (5.5) of the recurrence coefficients for the polynomial $s^{-1}V(z;t)$.

Proposition 8.6. — *Suppose $V(z;t)$ is a one-parameter family of polynomials of degree $2d$, which satisfies Hypothesis (S). Then for any $j \geq 0$ the functions*

$$\begin{cases} \partial^j\gamma(s;t)/\partial s^j{}_{|s=1}, & \partial^j\beta(s;t)/\partial s^j{}_{|s=1}, \\ \partial^j f_{2k}(s;t)/\partial s^j{}_{|s=1}, & \partial^j g_{2k}(s;t)/\partial s^j{}_{|s=1}, \end{cases} \tag{8.16}$$

$(k=1,2,\dots)$ *are analytic on* $[0,t_0]$.

To prove Proposition 8.6 we will need the following lemma.

LEMMA 8.7. — *Suppose $V(z;t)$ is a one-parameter family of polynomials of degree $2d$, which satisfies Hypothesis (S). Consider the functions $A(\gamma,\beta;t)$, $B(\gamma,\beta;t)$ corresponding to $V(z;t)$. Then the Jacobian,*

$$\det\begin{pmatrix} \partial A(\gamma,\beta;t)/\partial\gamma & \partial A(\gamma,\beta;t)/\partial\beta \\ \partial B(\gamma,\beta;t)/\partial\gamma & \partial B(\gamma,\beta;t)/\partial\beta \end{pmatrix}, \tag{8.17}$$

evaluated at $\gamma=\gamma(t),\beta=\beta(t)$, is analytic and nonzero on $[0,t_0]$.

Proof. — The analyticity follows from Theorem 8.3. Let us prove that the Jacobian is nonzero. Consider the two-parameter family of polynomials,

$$V(z;t,t_1)=V(z;t)+t_1z.$$

Then for every $t\in(0,t_0]$ there exists $\varepsilon=\varepsilon(t)>0$ such that $V(z;t,t_1)$ is q-cut regular for any $t_1\in[-\varepsilon,\varepsilon]$. As in Proposition 8.5, we obtain that the functions

$$\frac{\partial a_i(t_1;t)}{\partial t_1}\Big|_{t_1=0},\qquad \frac{\partial b_i(t_1;t)}{\partial t_1}\Big|_{t_1=0},\qquad i=1,\dots,q, \tag{8.18}$$

are analytic on $[0,t_0]$. By using identity (5.24), we obtain that the Jacobian (8.17) is nonzero. Lemma 8.7 is proved. □

Proof of Proposition 8.6. — Analyticity of γ and β follows from (5.4). To prove the analyticity of

$$\frac{\partial\gamma(s;t)}{\partial s}\Big|_{s=1},\qquad \frac{\partial\beta(s;t)}{\partial s}\Big|_{s=1}, \tag{8.19}$$

let us differentiate string equations (5.17) in s and set $s=1$. This gives a linear analytic in $t\in[0,t_0]$ system of equations, whose determinant is nonzero by Lemma 8.7, hence functions (8.19) are indeed analytic on $[0,t_0]$. By differentiating string equations (5.17) in s twice we obtain the analyticity of the second derivatives, and so on.

Let prove the analyticity of f_2,g_2. By following the proof of Lemmas 5.5, 5.6 we obtain that the functions f_2, g_2 also satisfy a system of linear equations with the same coefficients of partial derivatives of A and B and an analytic right hand side. Hence f_2,g_2 are analytic. By differentiating with respect to s the system of linear equations on f_2,g_2 and setting $s=1$ we obtain a similar linear system for the derivatives of f_2, g_2, and so on. The same argument applies to f_4,g_4 and their derivatives, etc. Proposition 8.6 is proved. □

Now we can prove the one-side analyticity of the coefficients of the asymptotic expansion of the free energy. We will assume the following hypothesis.

Hypothesis (T). — $V(z)$ is a polynomial of degree $2d$ such that

(a) $\tau_t V(z)$ is one-cut regular for $t>1$,

(b) $V(z)$ is one-cut singular and $h(a)\neq 0$, $h(b)\neq 0$, where $[a,b]$ is the support of the equilibrium measure for $V(z)$.

By Proposition 3.1, if V satisfies Hypothesis (T), then $\tau_t V$, $t>1$, satisfies Hypothesis (R), hence by Theorem 6.2, the free energy $F_N(t)$ of $\tau_t V$ admits the asymptotic expansion,

$$F_N(t)-F_N^{\mathrm{Gauss}}\sim F(t)+N^{-2}F^{(2)}(t)+N^{-4}F^{(4)}(t)+\cdots. \tag{8.20}$$

Theorem 8.8. — *Suppose $V(z)$ satisfies Hypothesis (T). Then the functions $F(t)$ and $F^{(2k)}(t)$, $k\geq 1$, are analytic on* $[1,\infty)$.

Proof. — The analyticity of $F(t)$ is proved in Corollary 8.2. Let us prove the analyticity of $F^{(2j)}(t)$, $j\geq 1$. To that end substitute expansions (5.5) into (3.19), and expand the appearing functions $\gamma,\beta,f_{2k},g_{2k}$ in the Taylor series at $n/N=1$. As a result, we obtain asymptotic expansion (8.20), so that the coefficients $F^{(2j)}(t)$ are expressed in terms of functions (8.16). By Proposition 8.6 functions (8.16) are analytic on $[0,t_0]$, hence the ones $F^{(2j)}(t)$ are analytic as well. Theorem 8.8 is proved. □

The asymptotics of the partition function for a singular V is a difficult question. The leading term is defined by the $(N=\infty)$-free energy F, see (4.17), but the subleading terms have a nontrivial scaling. The behavior of the subleading terms depends on the type of the singular V. The entire problem includes the investigation of the scaling behavior of the partition function for a parametric family $V(t)$ passing through V. This is the problem of the double scaling limit. In the next section we discuss the double scaling limit for a singular V of the type I in the terminology of [DKMVZ], when $h(z)=0$ inside of a cut. We consider a family $V(t)$ of even quartic polynomials passing through the singular polynomial V.

9. Double scaling limit of the free energy.

We will consider the asymptotics of the free energy near the critical point of the family $\tau_t V(z)$ generated by the singular quartic polynomial $V(z)=\frac{1}{4}z^4-z^2$,

$$\tau_t V(z) \equiv V(z;t) = \frac{1}{4t^2} z^4 + (1 - \frac{2}{t}) z^2. \tag{9.1}$$

We have that $V(z;1) = V(z) = \frac{1}{4} z^4 - z^2$, and for $t > 1$ the support of the equilibrium measure consists of one interval, while for $t < 1$ it consists of two intervals. We want to analyse the asymptotics of the free energy $F_N(t)$ as $N \to \infty$ and the parameter t is confined near its critical value, i.e. $t = 1$. Specifically, we shall assume the following scaling condition,

$$\left|(t-1)N^{\frac{2}{3}}\right| < C,$$

and will introduce a scaling variable x according to the equation

$$t = 1 + N^{-\frac{2}{3}} 2^{-\frac{2}{3}} x. \tag{9.2}$$

Our aim will be to prove the following theorem.

THEOREM 9.1. — *Let $F_N(t)$ be the partition function corresponding to the family $V(z;t)$ of quartic potentials* (9.1)*. Then, for every $\epsilon > 0$,*

$$F_N(t) - F_N^{\mathrm{Gauss}} = F_N^{\mathrm{reg}}(t) + N^{-2} F_N^{\mathrm{sing}}(t) + O(N^{-\frac{7}{3}+\epsilon}), \tag{9.3}$$

as $N \to \infty$ and $|(t-1)N^{\frac{2}{3}}| < C$. Here,

$$F_N^{\mathrm{reg}}(t) \equiv F(t) + N^{-2} F^{(2)}(t)$$

is the order N^{-2} (regular at $t = 1$) piece of the one-cut expansion (8.20)*, and*

$$F_N^{\mathrm{sing}}(t) = -\log F_{TW}\big((t-1)2^{\frac{2}{3}} N^{\frac{2}{3}}\big).$$

The function $F_{TW}(x)$ is the Tracy-Widom distribution function defined by the formulae [TW]

$$F_{TW}(x) = \exp\Big\{ \int_x^{\infty} (x-y) u^2(y)\, \mathrm{d}y \Big\}, \tag{9.4}$$

where $u(y)$ is the Hastings-McLeod solution to the Painlevé II equation

$$u''(y) = y u(y) + 2u^3(y), \tag{9.5}$$

which is characterized by the conditions at infinity [HM],

$$\lim_{y \to -\infty} \frac{u(y)}{\sqrt{-\frac{1}{2} y}} = 1, \qquad \lim_{y \to \infty} \frac{u(y)}{\mathrm{Ai}(y)} = 1. \tag{9.6}$$

Proof. — The proof of this theorem is based on the integral representation (3.19) of the free energy wich in the case of even potentials $V(z)$ can be rewriten as follows

$$F_N(t) = F_N^{\mathrm{Gauss}} + \int_t^\infty \frac{t-\tau}{\tau^2}\,\Theta_N(\tau)\,\mathrm{d}\tau, \tag{9.7}$$

where

$$\Theta_N(t) := R_N(t)(R_{N+1} + R_{N-1}) - \frac{1}{2}, \tag{9.8}$$

and we have used a standard notation

$$\gamma_n^2 \equiv R_n. \tag{9.9}$$

Note also that for even potentials all the beta recurrence coefficients are zero. Assuming the double scaling substitution (9.2), and making simultaniously the change of the variable of integration,

$$\tau = 1 + N^{-\frac{2}{3}} 2^{-\frac{2}{3}} y,$$

we can, in turn, rewrite (9.7) as

$$F_N(x) = F_N^{\mathrm{Gauss}} + 2^{-\frac{4}{3}} N^{-\frac{4}{3}} \int_x^\infty \frac{x-y}{(1+N^{-\frac{2}{3}}2^{-\frac{2}{3}}y)^2}\,\Theta_N(y)\,\mathrm{d}y, \tag{9.10}$$

where, we use the notations,

$$F_N(x) := F_N(t)\big|_{t=1+N^{-\frac{2}{3}}2^{-\frac{2}{3}}x}, \tag{9.11}$$

$$\Theta_N(y) := \Theta_N(\tau)\big|_{\tau=1+N^{-\frac{2}{3}}2^{-\frac{2}{3}}y}. \tag{9.12}$$

Our next move toward the proof of Theorem 9.1 is to split the integration in (9.10) into the following two pieces.

$$\begin{aligned} 2^{-\frac{4}{3}} N^{-\frac{4}{3}} \int_x^\infty \frac{x-y}{(1+N^{-\frac{2}{3}}2^{-\frac{2}{3}}y)^2}\,\Theta_N(y)\,\mathrm{d}y &= 2^{-\frac{4}{3}} N^{-\frac{4}{3}} \int_x^{N^\epsilon} \frac{x-y}{(1+N^{-\frac{2}{3}}2^{-\frac{2}{3}}y)^2}\,\Theta_N(y)\,\mathrm{d}y \\ &\quad + 2^{-\frac{4}{3}} N^{-\frac{4}{3}} \int_{N^\epsilon}^\infty \frac{x-y}{(1+N^{-\frac{2}{3}}2^{-\frac{2}{3}}y)^2}\,\Theta_N(y)\,\mathrm{d}y. \end{aligned} \tag{9.13}$$

Going back in the second integral to the original variable τ, we have the formula, where $\delta = \frac{2}{3} - \epsilon$:

$$F_N(x) = F_N^{\text{Gauss}} + 2^{-\frac{4}{3}} N^{-\frac{4}{3}} \int_x^{N^\epsilon} \frac{x-y}{(1+N^{-\frac{2}{3}}2^{-\frac{2}{3}}y)^2}\,\Theta_N(y)\,\mathrm{d}y + \int_{1+2^{-\frac{2}{3}}N^{-\delta}}^{\infty} \frac{1+N^{-\frac{2}{3}}2^{-\frac{2}{3}}x-\tau}{\tau^2}\,\Theta_N(\tau)\,\mathrm{d}\tau. \tag{9.14}$$

The main point now is that we can produce the uniform estimates for the recurrence coefficients R_n, and hence for the function Θ_N, on each of the two domains of integration. Indeed, the needed estimates are the extensions to the larger parameter domains of the double-scaling asymptotics obtained in [BI2] (the first integral) and the one-cut asymptotics obtained (in particular) in [DKMVZ] (the second integral). Let us first discuss the double-scaling estimates.

Set

$$g_0 = \frac{1}{t^2}, \quad \kappa = 2 - \frac{4}{t}, \tag{9.15}$$

so that the potential (9.1) is written as

$$V(z;t) = \frac{g_0}{4} z^4 + \frac{\kappa}{2} z^2. \tag{9.16}$$

Following [BI2], define $\widehat{y}$ as

$$\widehat{y} = c_0^{-1} N^{\frac{2}{3}} \Big(\frac{n}{N} - \frac{\kappa^2}{4g_0}\Big), \quad c_0 = \Big(\frac{\kappa^2}{2g_0}\Big)^{\frac{1}{3}}. \tag{9.17}$$

Then, as shown in [BI2],

$$R_n(t) = -\frac{\kappa}{2g_0} + N^{-\frac{1}{3}} c_1 (-1)^{n+1} u(\widehat{y}) + N^{-\frac{2}{3}} c_2 v(\widehat{y}) + O(N^{-1}),$$
$$c_1 = \Big(\frac{2(-\kappa)}{g_0^2}\Big)^{\frac{1}{3}}, \quad c_2 = \frac{1}{2}\Big(\frac{1}{2(-\kappa)g_0}\Big)^{\frac{1}{3}}, \tag{9.18}$$

as $N \to \infty$ and as long as the values of t and n are such that $\widehat{y}$ stays bounded,

$$|\widehat{y}| < C. \tag{9.19}$$

In (9.18), $u(y)$ is the Hastings-McLeod solution to the Painlevé II equation defined in (9.5)–(9.6), and

$$v(y) = y + 2u^2(y). \tag{9.20}$$

Assume that $t = 1 + N^{-\frac{2}{3}} 2^{-\frac{2}{3}} y$. Then, by simple calculations, we have

$$\frac{\kappa^2}{4g_0} = 1 - 2^{\frac{1}{3}} N^{-\frac{2}{3}} y + O(N^{-\frac{4}{3}} y^2), \quad c_0^{-1} = 2^{-\frac{1}{3}} + O(N^{-\frac{2}{3}} y). \tag{9.21}$$

Therefore,

$$\widehat{y} = y + O(N^{-\frac{2}{3}} y^2), \quad \text{and} \quad \widehat{y} = y \pm 2^{-\frac{1}{3}} N^{-\frac{1}{3}} + O(N^{-\frac{2}{3}} y^2), \tag{9.22}$$

if $n = N$ and $n = N \pm 1$, respectively. Simultaneously,

$$\frac{\kappa}{2g_0} = -1 + O(N^{-\frac{4}{3}} y^2), \tag{9.23}$$

$$c_1 = 2^{\frac{2}{3}} + O(N^{-\frac{2}{3}} y), \quad \text{and} \quad c_2 = 2^{-\frac{5}{3}} + O(N^{-\frac{2}{3}} y). \tag{9.24}$$

Let (cf. (9.11), (9.12))

$$R_n(y) := R_n(t)_{|t=1+N^{-\frac{2}{3}} 2^{-\frac{2}{3}} y}, \tag{9.25}$$

and assume that

$$|y| < C. \tag{9.26}$$

Then, we conclude from (9.18)–(9.24) that, as $N \to \infty$, the recurrence coefficients $R_n(y)$, $n = N-1, N, N+1$, have the following asymptotics:

$$R_N(y) = 1 - N^{-\frac{1}{3}} 2^{\frac{2}{3}} (-1)^N u(y) + N^{-\frac{2}{3}} 2^{-\frac{5}{3}} v(y) + O(N^{-1}), \tag{9.27}$$

$$R_{N\pm1}(y) = R_N(y) \mp N^{-\frac{2}{3}} 2^{\frac{1}{3}} (-1)^N u'(y) + O(N^{-1}). \tag{9.28}$$

To be able to use the estimates (9.27)–(9.28) in the first integral in (9.14) we need them on the expanding domain, i.e. we want to be able to replace the inequality (9.26) by the inequality $|y| < N^{\epsilon}$.

Proposition 9.2. — *For every* $0 < \epsilon < \frac{1}{6}$ *there exists a positive constant* $C \equiv C(\epsilon)$ *such that the error terms in* (9.27)–(9.28), *which we will denote* $r_n(y)$, $n = N, N+1, N-1$, *satisfy the uniform estimates,*

$$\left|r_n(y)\right| \le C N^{-1+2\epsilon}, \quad n = N, N+1, N-1, \tag{9.29}$$

$$\textit{for all } N \ge 1 \textit{ and } |y| < N^{\epsilon} \tag{9.30}$$

Proof. — A simple examination of the proofs of [BI2] shows that the error term in (9.18) can be specified as $O(N^{-1}\widehat{y}^{\frac{3}{2}})$. This means that, under condition

$$|\widehat{y}| \le N^{\epsilon}, \quad 0 < \epsilon < \frac{1}{6},$$

we have from (9.18) the estimate

$$R_n(t) = -\frac{\kappa}{2g_0} + N^{-\frac{1}{3}} c_1 (-1)^{n+1} u(\widehat{y}) + N^{-\frac{2}{3}} c_2 v(\widehat{y}) + O(N^{-1+\frac{3}{2}\epsilon}). \tag{9.31}$$

Estimate (9.31) together with (9.21)–(9.24) yield the following modification of (9.27) and (9.28).

$$R_N(y) = 1 - N^{-\frac{1}{3}} 2^{\frac{2}{3}} (-1)^N u(y) + N^{-\frac{2}{3}} 2^{-\frac{5}{3}} v(y) + O(N^{-1+2\epsilon}), \tag{9.32}$$

$$R_{N\pm 1}(y) = R_N(y) \mp N^{-\frac{2}{3}} 2^{\frac{1}{3}} (-1)^N u'(y) + O(N^{-1+2\epsilon}). \tag{9.33}$$

The error $O(N^{-1+2\epsilon})$ is produced by the second term of (9.31). For instance, if $n = N$, we have

$$\begin{aligned} N^{-\frac{1}{3}} c_1 u(\widehat{y}) &= N^{-\frac{1}{3}} (2^{\frac{2}{3}} + O\big(N^{-\frac{2}{3}} y)\big)\big(u(y) + O(N^{-\frac{2}{3}} y^2)\big) \\ &= N^{-\frac{1}{3}} u(y) + O(N^{-1} y^2) = N^{-\frac{1}{3}} u(y) + O(N^{-1+2\epsilon}). \end{aligned}$$

Similar arguments lead to (9.33). Asymptotics (9.32) and (9.33) complete the proof of the proposition. Note that the restriction $\epsilon < \frac{1}{6}$ is needed to ensure that the droped terms are of higher order that $N^{-\frac{2}{3}}$. □

Let us now turn to the analysis of the one-cut estimates of $R_n(t)$ which are needed in the second integral in (9.14). These estimates can be extracted from the general one-cut expansion (6.13). In the case of the quartic potential (9.1), the first two terms of (6.13) can be specified as

$$\begin{gathered} R_n(t) = R(\frac{n}{N};t) + N^{-2} R^{(2)}(\frac{n}{N};t) + O(t^{-1}N^{-4}), \\ n = N-1, N, N+1, \quad N \to \infty, \quad t \equiv t_0 > 1, \end{gathered} \tag{9.34}$$

where the coefficient functions $R(\lambda;t)$ and $R^{(2)}(\lambda;t)$ can be found with the help of the string equation (5.9) which in the case under consideration takes the form of the single recurrence relation,

$$\frac{n}{N} = \Big(2 - \frac{4}{t}\Big) R_n + \frac{1}{t^2} R_n (R_{n+1} + R_n + R_{n-1}). \tag{9.35}$$

We also note that the change $t^{-\frac{1}{2}} \to t^{-1}$ in the error estimate is due to the evenness of potential (9.1). Substituting (9.34) into equation (9.35) we arrive to the following explicit formulae for $R(\lambda;t)$ and $R^{(2)}(\lambda;t)$.

$$R(\lambda;t) = \frac{t}{3}\big(2 - t + \sqrt{(2-t)^2 + 3\lambda}\big), \tag{9.36}$$

$$R^{(2)}(\lambda;t) = -\frac{t}{8}\,\frac{2 - t + \sqrt{(2-t)^2 + 3\lambda}}{((2-t)^2 + 3\lambda)^2}. \tag{9.37}$$

We of course need an extension of the validity of the asymptotics (9.34) to the large domain of the parameter t.

Proposition 9.3. — *For every $0 < \delta < \frac{2}{3}$ there exists a positive constant $C \equiv C(\delta)$ such that the error terms in* (9.34), *which we will denote* $r_n(t)$, $n = N, N+1, N-1$, *satisfy the uniform estimates,*

$$\big|r_n(t)\big| \le Ct^{-1}N^{-4}, \quad n = N, N+1, N-1, \tag{9.38}$$

$$\textit{for all } N \ge 1 \textit{ and } t \ge 1 + 2^{-\frac{2}{3}}N^{-\delta}. \tag{9.39}$$

The proof of the proposition is given in Appendix A.

We are now ready to proceed with the asymptotic evaluation of the integrals in the right hand side of (9.14). We shall start with the first integral,

$$2^{-\frac{4}{3}}N^{-\frac{4}{3}}\int_x^{N^\epsilon} \frac{x-y}{(1 + N^{-\frac{2}{3}}2^{-\frac{2}{3}}y)^2}\,\Theta_N(y)\,\mathrm{d}y \equiv I_1.$$

Fisrt we notice that, in virtue of Proposition 9.2,

$$\Theta_N(y) = \frac{3}{2} - 2^{\frac{4}{3}}N^{-\frac{2}{3}}u^2(y) + 2^{\frac{1}{3}}N^{-\frac{2}{3}}y + O(N^{-1+2\epsilon}). \tag{9.40}$$

Therefore, the I_1 can be represented as

$$I_1 = 2^{-\frac{4}{3}}N^{-\frac{4}{3}}\int_x^{N^\epsilon} \frac{x-y}{(1 + N^{-\frac{2}{3}}2^{-\frac{2}{3}}y)^2}\,\Theta_N^0(y)\,\mathrm{d}y + O(N^{-\frac{7}{3}+4\epsilon}). \tag{9.41}$$

where

$$\Theta_N^0(y) := \frac{3}{2} - 2^{\frac{4}{3}}N^{-\frac{2}{3}}u^2(y) + 2^{\frac{1}{3}}N^{-\frac{2}{3}}y. \tag{9.42}$$

Assuming that

$$0 < \epsilon < \frac{1}{12}, \tag{9.43}$$

we make the error term in (9.41) of order $o(N^{-2})$.

The integral in the right hand side of (9.41) can be, in accordance with (9.42), splited into the three integrals,

$$2^{-\frac{4}{3}}N^{-\frac{4}{3}}\int_x^{N^\epsilon}\frac{x-y}{(1+N^{-\frac{2}{3}}2^{-\frac{2}{3}}y)^2}\,\Theta_N^0(y)\,\mathrm{d}y \equiv I_{11}+I_{12}+I_{13}, \tag{9.44}$$

where

$$\begin{aligned}
I_{11} &= 2^{-\frac{4}{3}}N^{-\frac{4}{3}}\,\frac{3}{2}\int_x^{N^\epsilon}\frac{x-y}{(1+N^{-\frac{2}{3}}2^{-\frac{2}{3}}y)^2}\,\mathrm{d}y,\\
I_{12} &= -N^{-2}\int_x^{N^\epsilon}\frac{x-y}{(1+N^{-\frac{2}{3}}2^{-\frac{2}{3}}y)^2}\,u^2(y)\,\mathrm{d}y,\\
I_{13} &= \frac{1}{2}N^{-2}\int_x^{N^\epsilon}\frac{x-y}{(1+N^{-\frac{2}{3}}2^{-\frac{2}{3}}y)^2}\,y\,\mathrm{d}y.
\end{aligned}$$

The integral I_{11} can be estimated, up to the terms of order N^{-2}, as follows,

$$\begin{aligned}
I_{11} &= 2^{-\frac{4}{3}}N^{-\frac{4}{3}}\,\frac{3}{2}\int_x^{N^\epsilon}(x-y)\big(1-2N^{-\frac{2}{3}}2^{-\frac{2}{3}}y+O(N^{-\frac{4}{3}}y^2)\big)\,\mathrm{d}y\\
&= 2^{-\frac{4}{3}}N^{-\frac{4}{3}}\,\frac{3}{2}\int_x^{N^\epsilon}(x-y)(1-2N^{-\frac{2}{3}}2^{-\frac{2}{3}}y)\,\mathrm{d}y+O(N^{-\frac{8}{3}+4\epsilon})\\
&= -\frac{3x^2}{2^{\frac{10}{3}}}N^{-\frac{4}{3}}+\frac{x^3}{8}N^{-2}+\frac{3x}{2^{\frac{7}{3}}}N^{-\frac{4}{3}+\epsilon}-\frac{3}{2^{\frac{10}{3}}}N^{-\frac{4}{3}+2\epsilon}\\
&\qquad+\frac{1}{4}N^{-2+3\epsilon}-\frac{3x}{8}N^{-2+2\epsilon}+O(N^{-\frac{8}{3}+4\epsilon}).
\end{aligned} \tag{9.45}$$

For the second integral in (9.44), i.e. for the integral I_{12}, we have

$$\begin{aligned}
I_{12} &= -N^{-2}\int_x^{N^\epsilon}(x-y)\big(1+O(N^{-\frac{2}{3}}y)\big)u^2(y)\,\mathrm{d}y\\
&= N^{-2}\int_x^{N^\epsilon}(y-x)u^2(y)\,\mathrm{d}y+O(N^{-\frac{8}{3}+3\epsilon})\\
&= N^{-2}\int_x^{\infty}(y-x)u^2(y)\,\mathrm{d}y+O(N^{-\frac{8}{3}+3\epsilon})
\end{aligned} \tag{9.46}$$

and similarly, for the third integral,

$$\begin{aligned}
I_{13} &= \frac{1}{2}N^{-2}\int_x^{N^\epsilon}(x-y)\big(1+O(N^{-\frac{2}{3}}y)\big)y\,\mathrm{d}y\\
&= \frac{1}{2}N^{-2}\int_x^{N^\epsilon}(x-y)y\,\mathrm{d}y+O(N^{-\frac{8}{3}+4\epsilon})\\
&= -\frac{x^3}{12}N^{-2}-\frac{1}{6}N^{-2+3\epsilon}+\frac{x}{4}N^{-2+2\epsilon}+O(N^{-\frac{8}{3}+4\epsilon}).
\end{aligned} \tag{9.47}$$

Adding the estimates (9.45)–(9.47), we arrive to the following, up to the order N^{-2}, asymptotic formula for the first integral in our basic equation (9.14),

$$(9.48)\quad I_1 = -\frac{3x^2}{2^{\frac{10}{3}}} N^{-\frac{4}{3}} + \Big(\int_x^\infty (y-x)u^2(y)\,\mathrm{d}y + \frac{x^3}{24} \Big) N^{-2} + I_1(N,\epsilon) + O(N^{-\frac{7}{3}+4\epsilon}),$$

where

$$(9.49)\quad I_1(N,\epsilon) := -\frac{3}{2^{\frac{10}{3}}} N^{-\frac{4}{3}+2\epsilon} + \frac{3x}{2^{\frac{7}{3}}} N^{-\frac{4}{3}+\epsilon} + \frac{1}{12} N^{-2+3\epsilon} - \frac{x}{8} N^{-2+2\epsilon}.$$

Consider now the second integral in the right hand side of (9.14),

$$\int_{1+2^{-\frac{2}{3}}N^{-\delta}}^\infty \frac{1+N^{-\frac{2}{3}}2^{-\frac{2}{3}}x-\tau}{\tau^2}\,\Theta_N(\tau)\,\mathrm{d}\tau \equiv I_2.$$

With the help of Proposition 9.3, we can specify the general expansion (6.15) for our case as follows

$$(9.50)\qquad \Theta_N(\tau) = \Theta(\tau) + N^{-2}\Theta^{(2)}(\tau) + O(t^{-1}N^{-4}),$$

where

$$(9.51)\qquad \Theta(\tau) = 2R^2(1;\tau) - \frac{1}{2},$$

$$(9.52)\qquad \Theta^{(2)}(\tau) = 4R(1;\tau)R^{(2)}(1;\tau) + R(1;\tau)R_{\lambda\lambda}(1;\tau).$$

Therefore, similar to the integral I_1, we can represent the integral I_2, up to the terms of order N^{-2}, as the sum of the following three integrals,

$$(9.53)\qquad I_2 = I_{21} + I_{22} + I_{23} + O(N^{-\frac{8}{3}}),$$

where

$$(9.54)\qquad I_{21} = \int_{1+2^{-\frac{2}{3}}N^{-\delta}}^\infty \frac{1-\tau}{\tau^2}\,\Theta(\tau)\,\mathrm{d}\tau,$$

$$(9.55)\qquad I_{22} = N^{-\frac{2}{3}}2^{-\frac{2}{3}}x \int_{1+2^{-\frac{2}{3}}N^{-\delta}}^\infty \frac{1}{\tau^2}\,\Theta(\tau)\,\mathrm{d}\tau,$$

$$(9.56)\qquad I_{23} = N^{-2} \int_{1+2^{-\frac{2}{3}}N^{-\delta}}^\infty \frac{1-\tau}{\tau^2}\,\Theta^{(2)}(\tau)\,\mathrm{d}\tau.$$

In analysing each of the integrals I_{2k} we shall recall that $\delta = \frac{2}{3} - \epsilon$, and make use of the following elementary estimate,

$$(9.57)\quad \int_{1+s}^{\infty} f(\tau)\,\mathrm{d}\tau = \int_{1}^{\infty} f(\tau)\,\mathrm{d}\tau - f(1)s - \frac{s^2}{2} f'(1) - \frac{s^3}{6} f''(1) + O(s^4)$$

which is true under the natural conditions fulfiled in the case of each of the integrals I_{2k}. Applying (9.57) to the integral I_{21} we obtain the asymptotic relation,

$$(9.58)\qquad I_{21} = \int_{1}^{\infty} \frac{1-\tau}{\tau^2}\Theta(\tau)\,\mathrm{d}\tau - \frac{1}{2}\Big(\frac{1-\tau}{\tau^2}\Theta(\tau)\Big)'\Big|_{\tau=1} 2^{-\frac{4}{3}} N^{-\frac{4}{3}+2\epsilon} - \frac{1}{6}\Big(\frac{1-\tau}{\tau^2}\Theta(\tau)\Big)''\Big|_{\tau=1} 2^{-2} N^{-2+3\epsilon} + O(N^{-\frac{8}{3}+4\epsilon}).$$

Observe that

$$\Big(\frac{1-\tau}{\tau^2}\Theta(\tau)\Big)'\Big|_{\tau=1} = -\Theta(1) \text{ and } \Big(\frac{1-\tau}{\tau^2}\Theta(\tau)\Big)''\Big|_{\tau=1} = 4\Theta(1) - 2\Theta'(1).$$

This, together with equations (9.51) and (9.36) allows us to evaluate the coefficients of expansion (9.58). Indeed we have

$$(9.59)\qquad \Theta(1) = \tfrac{3}{2}, \quad \Theta'(1) = 2,$$

and hence

$$(9.60)\quad I_{21} = \int_{1}^{\infty} \frac{1-\tau}{\tau^2}\Theta(\tau)\,\mathrm{d}\tau + \frac{3}{2^{\frac{10}{3}}} N^{-\frac{4}{3}+2\epsilon} - \frac{1}{12} N^{-2+3\epsilon} + O(N^{-\frac{8}{3}+4\epsilon}).$$

Similarly, for the integral I_{22} we have

$$\begin{aligned} I_{22} &= N^{-\frac{2}{3}} 2^{-\frac{2}{3}} x \int_{1}^{\infty} \frac{1}{\tau^2}\Theta(\tau)\,\mathrm{d}\tau - x\Theta(1) 2^{-\frac{4}{3}} N^{-\frac{4}{3}+\epsilon} \\ &\quad - \frac{x}{2}\Big(\frac{1}{\tau^2}\Theta(\tau)\Big)'\Big|_{\tau=1} 2^{-2} N^{-2+2\epsilon} + O(N^{-\frac{8}{3}+3\epsilon}) \\ &= N^{-\frac{2}{3}} 2^{-\frac{2}{3}} x \int_{1}^{\infty} \frac{1}{\tau^2}\Theta(\tau)\,\mathrm{d}\tau - x\Theta(1) 2^{-\frac{4}{3}} N^{-\frac{4}{3}+\epsilon} \\ &\quad + \frac{x}{8}(2\Theta(1) - \Theta'(1)) N^{-2+2\epsilon} + O(N^{-\frac{8}{3}+3\epsilon}), \end{aligned}$$

and, taking into account (9.59),

$$(9.61)\qquad I_{22} = N^{-\frac{2}{3}} 2^{-\frac{2}{3}} x \int_{1}^{\infty} \frac{1}{\tau^2}\Theta(\tau)\,\mathrm{d}\tau - \frac{3x}{2^{\frac{7}{3}}} N^{-\frac{4}{3}+\epsilon} + \frac{x}{8} N^{-2+2\epsilon} + O(N^{-\frac{8}{3}+3\epsilon}).$$

The estimation of the integral I_{23} up to the order N^{-2} is very simple– we only need to use the first term of (9.57):

$$I_{23} = N^{-2} \int_1^{\infty} \frac{1-\tau}{\tau^2}\, \Theta^{(2)}(\tau)\, \mathrm{d}\tau + O(N^{-\frac{8}{3}+\epsilon}). \tag{9.62}$$

Adding the estimates (9.60), (9.61) and (9.62) we conclude that

$$\begin{aligned} I_2 = \int_1^{\infty} \frac{1-\tau}{\tau^2}\, \Theta(\tau)\, \mathrm{d}\tau + N^{-\frac{2}{3}} 2^{-\frac{2}{3}} x \int_1^{\infty} \frac{1}{\tau^2}\, \Theta(\tau)\, \mathrm{d}\tau \\ + N^{-2} \int_1^{\infty} \frac{1-\tau}{\tau^2}\, \Theta^{(2)}(\tau)\, \mathrm{d}\tau - I_1(N,\epsilon) + O(N^{-\frac{8}{3}+4\epsilon}), \end{aligned} \tag{9.63}$$

where $I_1(N,\epsilon)$ is exactly the same collection of the epsilon-depending terms as the one which has appeared in formula (9.48) evaluating the integral I_1, and which is defined in (9.49).

Substituting estimates (9.48) and (9.63) into the basic equation (9.14) we obtain the following asymptotic representation of the free energy $F_N(x)$,

$$\begin{aligned} F_N(x) = F_N^{\mathrm{Gauss}} + \int_1^{\infty} \frac{1-\tau}{\tau^2}\, \Theta(\tau)\, \mathrm{d}\tau + N^{-\frac{2}{3}} 2^{-\frac{2}{3}} x \int_1^{\infty} \frac{1}{\tau^2}\, \Theta(\tau)\, \mathrm{d}\tau \\ - \frac{3x^2}{2^{\frac{10}{3}}} N^{-\frac{4}{3}} + N^{-2} \Big\{ \int_x^{\infty} (y-x) u^2(y)\, \mathrm{d}y \\ + \frac{x^3}{24} + \int_1^{\infty} \frac{1-\tau}{\tau^2}\, \Theta^{(2)}(\tau)\, \mathrm{d}\tau \Big\} + O(N^{-\frac{7}{3}+4\epsilon}) \end{aligned} \tag{9.64}$$

Put (cf. (8.20))

$$\begin{aligned} F_N^{\mathrm{reg}}(t) &\equiv F(t) + N^{-2} F^{(2)}(t) \\ &= \int_t^{\infty} \frac{t-\tau}{\tau^2}\, \Theta(\tau)\, \mathrm{d}\tau + N^{-2} \int_t^{\infty} \frac{t-\tau}{\tau^2}\, \Theta^{(2)}(\tau)\, \mathrm{d}\tau, \end{aligned} \tag{9.65}$$

and consider $F_N^{\mathrm{reg}}(1 + 2^{-\frac{2}{3}} N^{-\frac{2}{3}} x)$. It is easy to see that this object coinside with the sum $I_{21} + I_{22} + I_{23}$ (see (9.53)) up to the following formal replacement:

$$N^{\epsilon} \longmapsto x.$$

Therefore, we can apply (9.63) and see that

$$\begin{aligned} F_N^{\mathrm{reg}}(1 + 2^{-\frac{2}{3}} N^{-\frac{2}{3}} x) = \int_1^{\infty} \frac{1-\tau}{\tau^2}\, \Theta(\tau)\, \mathrm{d}\tau \\ + N^{-\frac{2}{3}} 2^{-\frac{2}{3}} x \int_1^{\infty} \frac{1}{\tau^2}\, \Theta(\tau)\, \mathrm{d}\tau + N^{-2} \int_1^{\infty} \frac{1-\tau}{\tau^2}\, \Theta^{(2)}(\tau)\, \mathrm{d}\tau \\ - \frac{3x^2}{2^{\frac{10}{3}}} N^{-\frac{4}{3}} + \frac{x^3}{24} + O(N^{-\frac{8}{3}}). \end{aligned} \tag{9.66}$$

This allows us to rewrite the final equation (9.64) as

$$F_N(x) = F_N^{\mathrm{Gauss}} + F_N^{\mathrm{reg}}(1 + 2^{-\frac{2}{3}} N^{-\frac{2}{3}} x) - N^{-2} \log F_{TW}(x) + O(N^{-\frac{7}{3}+4\epsilon}), \tag{9.67}$$

which concludes the proof of Theorem 9.1. □

Remark . — In terms of the partition function equation (9.67) reads

$$\frac{Z_N(t)}{Z_N^{\mathrm{Gauss}}} = F_{TW}\big((t-1)2^{\frac{2}{3}} N^{\frac{2}{3}}\big) Z_N^{\mathrm{reg}}(t)\big(1 + O(N^{-\frac{1}{3}+\epsilon})\big), \tag{9.68}$$

where ϵ is an arbitrary positive number.

Appendix A. The proof of Proposition 9.3.

Let us remind the basic steps of the Riemann-Hilbert approach to the asymptotic analysis of orthogonal polynomial following the scheme of [DKMVZ].

The principal observation ([FIK]; see also [BI1] and [DKMVZ]) is that the orthogonal polynomials $P_n(z)$ admit the representation,

$$P_n(z) = Y_{n11}(z), \tag{A.1}$$

where the 2×2 matrix function $Y_n(z)$ is the (unique) solution of the following Riemann-Hilbert (RH) problem.

1) $Y(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$, and it has continuous limits, $Y_{n+}(z)$ and $Y_{n-}(z)$ from above and below the real line,

$$Y_{n\pm}(z) = \lim_{\substack{z' \to z \\ \pm \mathrm{Im}\, z' > 0}} Y_n(z').$$

2) $Y_n(z)$ satisfies the jump condition on the real line,

$$Y_{n+}(z) = Y_{n-}(z) G(z), \tag{A.2}$$

where

$$G(z) = \begin{pmatrix} 1 & \mathrm{e}^{-NV(z)} \\ 0 & 1 \end{pmatrix}. \tag{A.3}$$

3) As $z \to \infty$, the function $Y_n(z)$ has the following uniform asymptotics expansion

$$Y_n(z) \sim \Big(I + \sum_{k=1}^{\infty} \frac{m_k^{(n)}}{z^k}\Big) z^{n\sigma_3}, \quad z \to \infty, \tag{A.4}$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

In addition to equation (A.1), the recurrence coefficients R_n can be also evaluated directly via $Y_n(z)$. In fact, we have that

$$R_n = \big(m_1^{(n)}\big)_{12} \big(m_1^{(n)}\big)_{21}, \tag{A.5}$$

where the matrix $m_1^{(n)}$ is the first coefficient of the asymptotic series (A.4). Equation (A.5) reduces the question of the asymptotic investigation of the recurrence coefficients $R_n(t)$ to the question of the asymptotic solution of the RH problem (1-3). In the case of a fixed $t > 1$, this analysis is performed in [DKMVZ]. In fact, in [DKMVZ] the asymptotics is evaluate for a generic fixed real analytic potential $V(z)$. The approach of [DKMVZ] consists of a succession of steps which, in the end, yields a reduced RH problem in which all the jumps are of the order N^{-1} (cf. the proof of Theorem 5.2). In the relevant for our analysis one-cut situation, these steps are described in detail in [EM]. In what follows we will repeat the construction of [EM] specifying its principal ingredients for the case of the potential (9.1) and showing how it can be modified in order to cover the extanded range of parameter t, i.e. assuming $t \geq 1 + 2^{-\frac{2}{3}} N^{-\delta}$.

Step 1 (*g-function deformation*). — Define

$$g(z) := \int_{-z_0}^{z_0} \ln(z-s)\rho(s)\, \mathrm{d}s, \tag{A.6}$$

where $[-z_0, z_0]$ and $\rho(s)$ are the support and the density of the equilibrium measure (5.1), respectively. More precisely, $[-z_0, z_0]$ and $\rho(s)$ minimise the functional (4.2) where $V(x)$ is replaced by $V(x)/\lambda$, and $\lambda = n/N$, $n = N-1, N, N+1$. In the case of the potential (9.1), the point z_0 and the function $\rho(s)$ are given by the equations (see, e.g. [BPS]),

$$z_0 = 2\Big(\frac{t}{3}\big(2 - t + \sqrt{(t-2)^2 + 3\lambda}\,\big)\Big)^{\frac{1}{2}} \equiv 2R^{\frac{1}{2}}(\lambda;t), \tag{A.7}$$

$$\rho(s) = \frac{1}{\pi\lambda}(b_0 + b_2 s^2)\sqrt{z_0^2 - s^2} \equiv \frac{1}{\pi i \lambda}(b_0 + b_2 s^2)\Big(\sqrt{s^2 - z_0^2}\,\Big)_+, \tag{A.8}$$

where

$$b_0 = \frac{1}{3t}\left(2t - 4 + \sqrt{(t-2)^2 + 3\lambda}\,\right), \tag{A.9}$$

$$b_2 = \frac{1}{2t^2}\cdot \tag{A.10}$$

The branch of $\sqrt{z^2 - z_0^2}$ is defined on $\mathbb{C} \setminus [-z_0, z_0]$ and is fixed by the condition $\sqrt{z^2 - z_0^2} > 0$ if $z > z_0$. The branch of $\ln(z - s)$ is defined on $\mathbb{C} \setminus (-\infty, s]$ and is fixed by the condition $\arg(z - s) = 0$ if $z > s$.

Assume that $t \geq t_0 > 1$ and denote,

$$V_\lambda(x) \equiv \frac{1}{\lambda} V(x).$$

Then the function $g(z)$ satisfies the following characteristic properties (cf. (4.7)–(4.8)) which underline the importance of $g(z)$ for the asymptotic analysis of the RH problem 1)–3).

• The function $g(z)$ is analytic for $z \in \mathbb{C} \setminus (-\infty, z_0]$ with continuous boundary values $g_\pm(z)$ on $(-\infty, z_0]$.

• There is a constant ℓ such that for $z \in [-z_0, z_0]$,

$$g_+(z) + g_-(z) - V_\lambda(z) = \ell, \tag{A.11}$$

and for $z \in \mathbb{R} \setminus [-z_0, z_0]$,

$$g_+(z) + g_-(z) - V_\lambda(z) < \ell. \tag{A.12}$$

• Denote

$$p(z) := g_+(z) - g_-(z). \tag{A.13}$$

Then, for $z \in [-z_0, z_0]$,

$$p(z) = 2\pi i \int_z^{z_0} \rho(s)\,\mathrm{d}s, \tag{A.14}$$

and this function possesses an analytic continuation to a neighborhood of $(-z_0, z_0)$. Moreover, for every $0 < d < \frac{1}{2} z_0$ there is a positive number p_0 such that

$$\frac{\mathrm{d}}{\mathrm{d}\sigma} \operatorname{Re} p(s + i\sigma)\Big|_{\sigma=0} = \frac{2}{\lambda}(b_0 + b_2 s^2)\sqrt{z_0^2 - s^2} \geq p_0 > 0, \tag{A.15}$$

for all $s \in [-z_0 + d, z_0 - d]$ and $t \geq t_0 > 1$.

• For $z > z_0$,

$$p(z) = 0, \tag{A.16}$$

and for $z < -z_0$,

$$p(z) = 2\pi i. \tag{A.17}$$

• As $z \to \infty$,

$$g(z) = \ln z + O\Big(\frac{1}{z^2}\Big). \tag{A.18}$$

We also notice that there is the following alternative representation of the function $g(z)$,

$$g(z) = -\frac{1}{\lambda}\int_{z_0}^{z} (b_0 + b_2 s^2)\sqrt{s^2 - z_0^2}\,\mathrm{d}s + \frac{1}{2}V_\lambda(z) + \frac{\ell}{2}\cdot \tag{A.19}$$

Having introduced the function $g(z)$ and the constant ℓ, we define the first transformation, $Y(z) \mapsto \Phi(z)$ of the original RH problem, by the equation,

$$Y(z) = \mathrm{e}^{\frac{1}{2}n\ell\sigma_3}\Phi(z)\,\mathrm{e}^{n(g(z)-\frac{1}{2}\ell)\sigma_3}. \tag{A.20}$$

In terms of the function $\Phi(z)$ the RH problem (1-3) reads as follows.

1′) $\Phi(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$.

2′) $\Phi(z)$ satisfies the jump condition on the real line,

$$\Phi_+(z) = \Phi_-(z)G_\Phi(z), \tag{A.21}$$

where

$$G_\Phi(z) = \begin{pmatrix} \mathrm{e}^{-np(z)} & \mathrm{e}^{n(g_+(z)+g_-(z)-V_\lambda-\ell)} \\ 0 & \mathrm{e}^{np(z)} \end{pmatrix} \tag{A.22}$$

3′) as $z \to \infty$, the function $\Phi(z)$ has the following uniform asymptotics:

$$\Phi(z) = I + O\Big(\frac{1}{z}\Big), \quad z \to \infty \tag{A.23}$$

(which can be extended to the whole asymptotic series).

Observe that, in virtue of (A.16) and (A.17), we have

$$G_\Phi(z) = \begin{pmatrix} 1 & \mathrm{e}^{n(g_+(z)+g_-(z)-V_\lambda-\ell)} \\ 0 & 1 \end{pmatrix}, \quad \text{for } z \in \mathbb{R} \setminus [-z_0, z_0], \tag{A.24}$$

and in virtue of (A.11),

$$G_\Phi(z) = \begin{pmatrix} \mathrm{e}^{-np(z)} & 1 \\ 0 & \mathrm{e}^{np(z)} \end{pmatrix}, \quad \text{for } z \in [-z_0, z_0]. \tag{A.25}$$

Step 2 (*second transformation* $\Phi \mapsto \Phi^{(1)}$). — Next we introduce the lens-shaped region $\Omega = \Omega^{(u)} \cup \Omega^{(\ell)}$ around $(-z_0, z_0)$ as indicated in Figure 1 and define $\Phi^{(1)}(z)$ as follows

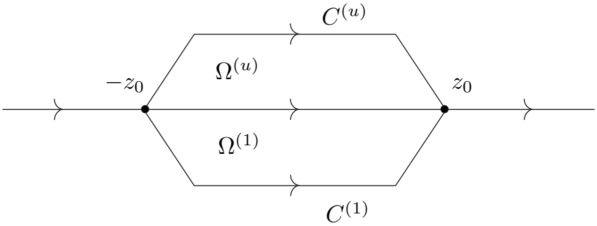

Figure 1. *The contour* Γ

(i) for z outside the domain Ω,

$$\Phi^{(1)}(z) = \Phi(z); \tag{A.26}$$

(ii) for z within the domain $\Omega^{(u)}$ (the upper lens),

$$\Phi^{(1)}(z) = \Phi(z)\begin{pmatrix} 1 & 0 \\ -\,\mathrm{e}^{-np(z)} & 1 \end{pmatrix}; \tag{A.27}$$

(iii) for z within the domain $\Omega^{(\ell)}$ (the lower lens),

$$\Phi^{(1)}(z) = \Phi(z)\begin{pmatrix} 1 & 0 \\ \mathrm{e}^{np(z)} & 1 \end{pmatrix}. \tag{A.28}$$

(We note that the function $p(z)$ admits the analytic continuation to the domain Ω.)

With the passing to $\Phi^{(1)}(z)$, the RH problem $1')$–$3')$ transforms to the RH problem posed on the contour Γ consisting of the real axes and the curves $C^{(u)}$ and $C^{(\ell)}$ which form the boundary of the domain Ω,

$$\Omega = C^{(\ell)} - C^{(u)}$$

(see Figure 1).

We have:

1″) $\Phi^{(1)}(z)$ is analytic for $z \in \mathbb{C} \setminus \Gamma$.

2″) $\Phi^{(1)}(z)$ satisfies the jump condition on the real line,

$$\Phi_+^{(1)}(z) = \Phi_-^{(1)}(z) G_{\Phi^{(1)}}(z), \tag{A.29}$$

where

$$G_{\Phi^{(1)}}(z) = \begin{cases} \begin{pmatrix} 1 & \mathrm{e}^{n(g_+(z)+g_-(z)-V_\lambda-\ell)} \\ 0 & 1 \end{pmatrix} & \text{for } z \in \mathbb{R} \setminus [-z_0, z_0], \\ \begin{pmatrix} 1 & 0 \\ \mathrm{e}^{-np(z)} & 1 \end{pmatrix} & \text{for } z \in C^{(u)}, \\ \begin{pmatrix} 1 & 0 \\ \mathrm{e}^{np(z)} & 1 \end{pmatrix} & \text{for } z \in C^{(\ell)}, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } z \in [-z_0, z_0]. \end{cases} \tag{A.30}$$

3″) As $z \to \infty$, the function $\Phi^{(1)}(z)$ has the following uniform asymptotics:

$$\Phi^{(1)}(z) = I + O\Big(\frac{1}{z}\Big), \quad z \to \infty \tag{A.31}$$

(which can be extended to the whole asymptotic series). Indeed, in view of the equations (A.26)–(A.28) defining the function $\Phi^{(1)}(z)$, the properties 1′)–3′) of the function $\Phi(z)$ and equation (A.24), we only need to explane the last line of equation (A.30). The latter is a direct consequance of equation (A.25) and the elementary algebraic identity,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\mathrm{e}^{np} & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-np} & 1 \\ 0 & \mathrm{e}^{np} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\mathrm{e}^{-np} & 1 \end{pmatrix}.$$

Step 3 (*The construction of a global aproximation to* $\Phi^{(1)}(z)$). — The point of the transformation of the original Y-RH problem 1)–3) to the Φ-RH problem 1″)–3″) is that in virtue of the inequalities (A.12) and (A.15), the jump matrix $G_{\Phi^{(1)}}(z)$, for $z \neq \pm z_0$, is exponentially close to the identity matrix on the part $\Gamma \setminus [-z_0, z_0]$ of the jump contour Γ, so that one can expect that, as $N \to \infty$, $n = N-1, N, N+1$, and $|z \pm z_0| > \delta$,

$$\Phi^{(1)}(z) \sim \Phi^{(\infty)}(z), \tag{A.32}$$

where $\Phi^{(\infty)}(z)$ is the solution of the following model RH problem.

1‴) $\Phi^{(\infty)}(z)$ is analytic for $z \in \mathbb{C} \setminus [-z_0, z_0]$.

2‴) $\Phi^{(\infty)}(z)$ satisfies the jump condition on $(-z_0, z_0)$

$$\Phi^{(\infty)}_+(z) = \Phi^{(\infty)}_-(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.33}$$

3‴) as $z \to \infty$, the function $\Phi^{(\infty)}(z)$ has the following uniform asymptotics:

$$\Phi^{(\infty)}(z) = I + O\Big(\frac{1}{z}\Big), \quad z \to \infty \tag{A.34}$$

(which can be extended to the convergent Laurent series at $z = \infty$). The important fact is that this Riemann-Hilbert problem admits an explicit solution:

$$\Phi^{(\infty)}(z) = \begin{pmatrix} \frac{1}{2}(\alpha + \alpha^{-1}) & \frac{1}{2i}(\alpha - \alpha^{-1}) \\ -\frac{1}{2i}(\alpha - \alpha^{-1}) & \frac{1}{2}(\alpha + \alpha^{-1}) \end{pmatrix}, \tag{A.35}$$

$$\alpha(z) = \Big(\frac{z - z_0}{z + z_0}\Big)^{1/4}, \quad \alpha(\infty) = 1. \tag{A.36}$$

In order to prove and specify the error term in estimation (A.32) we need to construct the parametrix of the solution $\Psi^{(1)}(z)$ near the end points $\pm z_0$. Let B_d denote a disc of radius d centered at z_0, and let us introduce the change-of-the-variable function $w(z)$ on B_d by the formula,

$$w(z) = \Big(\frac{3}{4}\Big)^{\frac{2}{3}} \big(-2g(z) + V_\lambda(z) + \ell\big)^{\frac{2}{3}}. \tag{A.37}$$

In view of equation (A.19), the function $w(z)$ can be also written as,

$$w(z) = \Big(\frac{3}{4}\Big)^{\frac{2}{3}} \Big(\frac{2}{\lambda}\int_{z_0}^{z} (b_0 + b_2 s^2)\sqrt{s^2 - z_0^2}\, \mathrm{d}s\Big)^{\frac{2}{3}}, \tag{A.38}$$

which, taking into account that $|b_0 + b_2 z_0^2| > c_0 > 0$ for all $t \geq 1$, implies that, for sufficiently small d, the function $w(z)$ is holomorphic and in fact conformal in the disc B_d,

$$w(z) = \sum_{k=1}^{\infty} w_k (z - z_0)^k, \quad z \in B_d. \tag{A.39}$$

We shall assume that the branch of the root $(\,)^{\frac{2}{3}}$ is choosen in such a way that

$$w_1 \geq c_0 > 0 \quad \text{for all } t \geq 1 \text{ and } N \geq 1. \tag{A.40}$$

We also note that, for sufficiently small d, the following inequality takes place,

$$\big|w(z)\big| \geq c_0, \quad \text{for all } z \in S_d,\ t \geq 1 \text{ and } N \geq 1, \tag{A.41}$$

where S_d denote the boundary of B_d, i.e the circle of radius d centered at z_0.

Let us decompose B_d into four regions (see Figure 2),

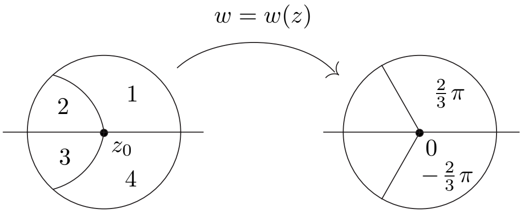

Figure 2. Decomposition of B_d

$$B_d = B_d^{(1)} \cup B_d^{(2)} \cup B_d^{(3)} \cup B_d^{(4)}, \tag{A.42}$$

where

$$B_d^{(1)} = \big\{z \in B_d : 0 \leq \arg w(z) \leq \tfrac{2}{3}\pi\big\},$$
$$B_d^{(2)} = \big\{z \in B_d : \tfrac{2}{3}\pi \leq \arg w(z) \leq \pi\big\},$$
$$B_d^{(3)} = \big\{z \in B_d : -\pi \leq \arg w(z) \leq -\tfrac{2}{3}\pi\big\},$$
$$B_d^{(4)} = \big\{z \in B_d : -\tfrac{2}{3}\pi \leq \arg w(z) \leq 0\big\}.$$

We shall assume that the parts of the curves $C^{(u,\ell)}$ which are inside B_d coincide with the relevant partys of the boundaries of the domains $B_d^{(k)}$. Let us also introduce the standard collection of the Airy functions,

$$\begin{cases} y_0(z) := \mathrm{Ai}(z), \\ y_1(z) := \mathrm{e}^{-\frac{1}{6}\pi i}\,\mathrm{Ai}(\mathrm{e}^{-\frac{2}{3}\pi i} z), \\ y_2(z) := \mathrm{e}^{\frac{1}{6}\pi i}\,\mathrm{Ai}(\mathrm{e}^{\frac{2}{3}\pi i} z). \end{cases} \tag{A.43}$$

We will now define the approximation (parametrix) $\Phi^{(z_0)}(z)$ within B_δ by the following equation,
(A.44)

$$\Phi^{(z_0)}(z) = E(z) n^{\frac{1}{6}\sigma_3} \begin{cases} \Psi^{(u)}_{\mathrm{Ai}}\big(n^{\frac{2}{3}} w(z)\big)\, \mathrm{e}^{\frac{2n}{3} w^{\frac{3}{2}}(z)\sigma_3} & \text{for } z \in B_d^{(1)}, \\ \Psi^{(u)}_{\mathrm{Ai}}\big(n^{\frac{2}{3}} w(z)\big)\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right) \mathrm{e}^{\frac{2n}{3} w^{\frac{3}{2}}(z)\sigma_3} & \text{for } z \in B_d^{(2)}, \\ \Psi^{(\ell)}_{\mathrm{Ai}}\big(n^{\frac{2}{3}} w(z)\big)\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \mathrm{e}^{\frac{2n}{3} w^{\frac{3}{2}}(z)\sigma_3} & \text{for } z \in B_d^{(3)}, \\ \Psi^{(\ell)}_{\mathrm{Ai}}\big(n^{\frac{2}{3}} w(z)\big)\, \mathrm{e}^{\frac{2n}{3} w^{\frac{3}{2}}(z)\sigma_3} & \text{for } z \in B_d^{(4)}. \end{cases}$$

where the model functions $\Psi^{(u,d)}_{\mathrm{Ai}}(z)$ are the matrices,

$$\Psi^{(u)}_{\mathrm{Ai}}(z) = \begin{pmatrix} y_0(z) & iy_1(z) \\ y_0'(z) & iy_1'(z) \end{pmatrix}, \tag{A.45}$$

$$\Psi^{(\ell)}_{\mathrm{Ai}}(z) = \begin{pmatrix} y_0(z) & iy_2(z) \\ y_0'(z) & iy_2'(z) \end{pmatrix}, \tag{A.46}$$

and the gauge matrix multiplier $E(z)$ is

$$E(z) = \sqrt{\pi} \begin{pmatrix} \alpha^{-1} & -\alpha \\ -i\alpha^{-1} & -i\alpha \end{pmatrix} w^{\frac{1}{4}\sigma_3}(z). \tag{A.47}$$

We note that, as it follows from (A.36) and (A.39), the matrix-valued function $E(z)$ is analytic in the disc B_d.

We are now ready to define an explicit global approximation, $\Phi^{(A)}(z)$, to the solution $\Phi^{(1)}(z)$ of the RH problem $1''')$–$3''')$. We take

$$\Phi^{(A)}(z) = \begin{cases} \Phi^{(\infty)}(z) & \text{for } z \notin B_d \cup (-B_d), \\ \Phi^{(z_0)}(z) & \text{for } z \in B_d, \\ \sigma_3 \Phi^{(z_0)}(-z)\sigma_3 & \text{for } z \in (-B_d). \end{cases} \tag{A.48}$$

To see that these formulae indeed provide an approximation to the solution $\Phi^{(1)}(z)$ we consider the matrix ratio,

$$X(z) := \Phi^{(1)}(z)\big(\Phi^{(A)}(z)\big)^{-1}. \tag{A.49}$$

Due to equation (A.33) and the definitions (A.44) of the parametrix $\Phi^{(z_0)}(z)$, the function $X(z)$ has no jumps across the interval $(-z_0+d, z_0-d)$ and inside the discs B_d and $(-B_d)$. It is still have jumps across the contour

$$\Gamma_0 = (-\infty, -z_0-d] \cup (-S_d) \cup C_0^{(u)} \cup C_0^{(\ell)} \cup S_d \cup [z_0+d, +\infty), \tag{A.50}$$

where $C_0^{(u,\ell)}$ are the parts of the curves $C^{(u,\ell)}$ which lie outside of the discs B_d and $(-B_d)$. The curves $C_0^{(u,\ell)}$ can be taken as straight lines. The contour Γ_0 is shown in Figure 3. The matrix-valued function $X(z)$ solves the following RH problem posed on the contour Γ_0.

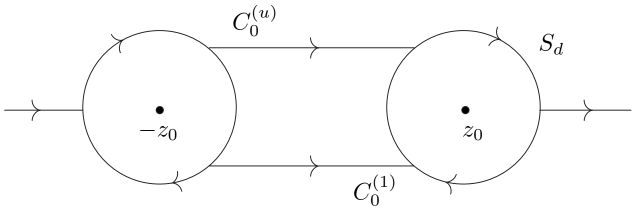

Figure 3. *The contour* Γ_0

1^0) $X(z)$ is analytic for $z \in \mathbb{C} \setminus \Gamma_0$, and it has continuous limits, $X_+(z)$ and $X_-(z)$ from the left and the right of Γ_0.

2^0) $X(z)$ satisfies the jump condition on Γ_0

$$X_+(z) = X_-(z) G_X(z), \tag{A.51}$$

where

$$G_X(z) = \begin{cases} \Phi^{(\infty)}(z) \begin{pmatrix} 1 & \mathrm{e}^{n(g_+(z)+g_-(z)-V_\lambda-\ell)} \\ 0 & 1 \end{pmatrix} \left(\Phi^{(\infty)}(z)\right)^{-1} \\ \qquad\qquad \text{for } z \in \mathbb{R} \setminus (-z_0 - d, z_0 + d), \\ \Phi^{(\infty)}(z) \begin{pmatrix} 1 & 0 \\ \mathrm{e}^{-np(z)} & 1 \end{pmatrix} \left(\Phi^{(\infty)}(z)\right)^{-1} & \text{for } z \in C_0^{(u)}, \\ \Phi^{(\infty)}(z) \begin{pmatrix} 1 & 0 \\ \mathrm{e}^{np(z)} & 1 \end{pmatrix} \left(\Phi^{(\infty)}(z)\right)^{-1} & \text{for } z \in C_0^{(\ell)}, \\ \Phi^{(z_0)}(z) \left(\Phi^{(\infty)}(z)\right)^{-1} & \text{for } z \in S_d, \\ \sigma_3 \Phi^{(z_0)}(-z) \left(\Phi^{(\infty)}(-z)\right)^{-1} \sigma_3, & \text{for } z \in (-S_d). \end{cases} \tag{A.52}$$

3^0) As $z \to \infty$, the function $X(z)$ has the following uniform asymptotics:

$$X(z) = I + O\left(\frac{1}{z}\right), \quad z \to \infty \tag{A.53}$$

The important feature of this RH problem is that the jump matrix $G_X(z)$ is uniformly close to the identity matrix as $N \to \infty$. Indeed, using the known asymptotics of the Airy functions and inequality (A.41) one can check directly that the functions $\Phi^{(z_0)}(z)$ and $\Phi^{(\infty)}(z)$ match on the circle S_d, and the uniform estimate,

$$\text{(A.54)}\qquad \big|G_X(z) - I\big| \le \frac{C}{N}, \quad \text{for all } z \in S_d \cup (-S_d),\ t \ge 1 \text{ and } N \ge 1,$$

takes place. Simultaneously, as z runs over $\mathbb{R} \setminus (-z_0 - d, z_0 + d)$, we observe that we have

$$\text{(A.55)}\qquad \begin{aligned} 0 < \mathrm{e}^{n(g_+(z)+g_-(z)-V_\lambda-\ell)} &\equiv \mathrm{e}^{-N\left(\int_{z_0}^{z}(b_0+b_2s^2)\sqrt{s^2-z_0^2}\,\mathrm{d}s\right)} \\ &< \mathrm{e}^{-Nc_0z^2}, \end{aligned}$$

where the positive constant c_0 can be choosen the same for all $t \ge 1$ and $N \ge 1$. Therefore, we conclude that

$$\text{(A.56)}\qquad \begin{aligned} &\big|G_X(z) - I\big| \le C\,\mathrm{e}^{-Nc_0z^2}, \\ &\qquad\qquad \text{for all } z \in \mathbb{R} \setminus (-z_0 - d, z_0 + d),\ t \ge 1 \text{ and } N \ge 1. \end{aligned}$$

Finally, inequality (A.15) indicates that on the segments $C_0^{(u)}$ and $C_0^{(\ell)}$, if they are choosen close enough to the real line, the following estimate holds:

$$\text{(A.57)}\qquad \begin{aligned} &|G_X(z) - I| \le C\,\mathrm{e}^{-Nc_0}, \\ &\qquad\qquad \text{for all } z \in C_0^{(u)} \cup C_0^{(\ell)},\ t \ge t_0 > 1 \text{ and } N \ge 1. \end{aligned}$$

Unlike the estimates (A.54) and (A.56), estimate (A.57) can not be extended to $t \ge 1$. However, a slightly weaker version of it is valied for $t \ge 1 + 2^{-\frac{2}{3}}N^{-\delta}$ with $\delta < \frac{2}{3}$. To see this, let us analyse more carefully the behavior of the function $\operatorname{Re} p(z)$ near the real line. To this end let us notice that, in addition to (A.15) we have

$$\frac{\mathrm{d}^2}{\mathrm{d}\sigma^2}\operatorname{Re} p(s + i\sigma)\Big|_{\sigma=0} = 0, \quad \forall z \in (-z_0, z_0),$$

and hence

$$\text{(A.58)}\qquad \begin{aligned} \operatorname{Re} p(z) &= \sigma\Big(\frac{\mathrm{d}}{\mathrm{d}\sigma}\operatorname{Re} p(s + i\sigma)\Big|_{\sigma=0}\Big) + O(\sigma^3) \\ &= \sigma\big(\frac{2}{\lambda}(b_0 + b_2s^2)\sqrt{z_0^2 - s^2}\big) + O(\sigma^3), \quad z \equiv s + i\sigma \in C_0^{(u)} \cup C_0^{(\ell)}. \end{aligned}$$

By a straightforward calculation one can check that

$$\frac{7}{24}N^{-\delta} \le b_0 \le 1, \quad \forall t \ge 1 + 2^{-\frac{2}{3}}N^{-\delta}.$$

Therefore, equation (A.58) yields the estimates

$$n\,\mathrm{Re}\,p(z) \ge c_0\sigma N^{1-\delta}\big(1+O(\sigma^2 N^\delta)\big), \quad z\equiv s+i\sigma \in C_0^{(u)}, \tag{A.59}$$

$$n\,\mathrm{Re}\,p(z) \le c_0\sigma N^{1-\delta}\big(1+O(\sigma^2 N^\delta)\big), \quad z\equiv s+i\sigma \in C_0^{(\ell)}, \tag{A.60}$$

with some positive constant c_0.

If we now choose $C^{(u,\ell)}$ so that

$$|\,\mathrm{Im}\,z| \equiv |\sigma| = N^{-\frac{1}{3}}, \quad z\in C_0^{(u)}\cup C_0^{(\ell)}, \tag{A.61}$$

and assume $0<\delta<\frac{2}{3}$, then (A.59) and (A.60) would imply

$$n\,\mathrm{Re}\,p(z) \ge c_0 N^{\frac{2}{3}-\delta}, \quad z\equiv s+i\sigma\in C_0^{(u)}, \tag{A.62}$$

$$n\,\mathrm{Re}\,p(z) \le -c_0 N^{\frac{2}{3}-\delta}, \quad z\equiv s+i\sigma\in C_0^{(\ell)}. \tag{A.63}$$

(We follow the usual convention to use the same symbol for perhaps different positive constants whose exact value is not important to us.) These inequalities in turn yield the following modification of estimate (A.57):

$$\big|G_X(z)-I\big| \le C\,\mathrm{e}^{-c_0 N^{\frac{2}{3}-\delta}}, \quad \text{for all } z\in C_0^{(u)}\cup C_0^{(\ell)}, \quad t\ge 1+2^{-\frac{2}{3}}N^{-\delta} \text{ and } N\ge 1, \tag{A.64}$$

which together with (A.54) and (A.56) lead to the conclusion that

$$\|G_X-I\|_{L^\infty(\Gamma_0)}, \quad \|G_X-I\|_{L^2(\Gamma_0)} \le \frac{C}{N}, \quad \text{for all } t\ge 1+2^{-\frac{2}{3}}N^{-\delta} \text{ and } N\ge 1. \tag{A.65}$$

This means that the needed extention of the basic uniform estimate of the jump matrix has been almost obtained. What is left is the control of the t -dependence of the estimate. This can be achieved as follows.

Let us attach the subscript "Gauss" to all the relevant objects, i.e. the equilibrium measure, the model solutions, etc., which correspond to the gaussian potential, $V_{\mathrm{Gauss}}(z)=z^2$. By the very nature of our approach, as $t\to\infty$, all the main ingredients of the above scheme, i.e.

$$g(z),\ \Phi^{(\infty)}(z),\ \Phi^{(z_0)}(z),$$

converge to the respective Gauss-quantities, i.e. to

$$g_{\mathrm{Gauss}}(z),\ \Phi^{(\infty)}_{\mathrm{Gauss}}(z),\ \Phi^{(z_0)}_{\mathrm{Gauss}}(z).$$

Moreover, the following inequalities for the jump matrix of the X-RH problem can be established by a straightforward calculations.

$$\text{(A.66)} \qquad \left|G_X(z)G^{-1}_{X_{\text{Gauss}}}(z) - I\right| \leq \frac{CN\,\mathrm{e}^{-c_0 N z^2} z^4}{t}, \quad \text{for all} \quad z \in \mathbb{R} \setminus (-z_0 - d, x_0 + d), \quad t \geq 1 + 2^{-\frac{2}{3}} N^{-\delta} \text{ and } N \geq 1,$$

$$\text{(A.67)} \qquad \left|G_X(z)G^{-1}_{X_{\text{Gauss}}}(z) - I\right| \leq \frac{CN\,\mathrm{e}^{-c_0 N^{\frac{2}{3}-\delta}}}{t}, \quad \text{for all} \quad z \in C_0^{(u)} \cup C_0^{(\ell)} \quad t \geq 1 + 2^{-\frac{2}{3}} N^{-\delta} \text{ and } N \geq 1,$$

$$\text{(A.68)} \qquad |G_X(z)G^{-1}_{X_{\text{Gauss}}}(z) - I| \leq \frac{C}{tN}, \quad \text{for all} \quad z \in S_d \cup (-S_d), \quad t \geq 1 + 2^{-\frac{2}{3}} N^{-\delta} \text{ and } N \geq 1.$$

Put

$$\text{(A.69)} \qquad \widetilde{X}(z) = X(z)X^{-1}_{\text{Gauss}}(z).$$

The function $\widetilde{X}(z)$ solves the RH problem on the same contour Γ_0 as the function $X(z)$ and with the jump matrix,

$$G_{\widetilde{X}}(z) \equiv G_X(z)G^{-1}_{X_{\text{Gauss}}}(z).$$

The inequalities (A.66)–(A.68) yield then the following modification of estimate (A.65),

$$\text{(A.70)} \quad \|G_{\widetilde{X}} - I\|_{L^\infty(\Gamma_0)}, \quad \|G_{\widetilde{X}} - I\|_{L^2(\Gamma_0)} \leq \frac{C}{tN}, \quad \text{for all} \quad t \geq 1 + 2^{-\frac{2}{3}} N^{-\delta} \text{ and } N \geq 1.$$

The proof of Proposition 9.3 can be now completed in the usual way, by iterating the $\widetilde{X}$-RH problem (cf. [DKMVZ] and [EM]). Indeed, by iterative arguments, we can see that for any $K \geq 0$,

$$\text{(A.71)} \qquad \left|\widetilde{X}(z) - I - \sum_{k=1}^{K} N^{-k} f_k\Big(\frac{n}{N}; t, z\Big)\right| \leq \frac{C(K)}{tN^{K+1}(1 + |z|)},$$

and also

$$\left|f_k\left(\frac{n}{N};t,z\right)\right| \le \frac{C(k)}{t(1+|z|)}. \tag{A.72}$$

Denote m_1^∞, $m_{1,\mathrm{Gauss}}$ and $\widetilde{m}_1$ the matrix coefficients of the terms $1/z$ in the asymptotic series at $z=\infty$ of the functions $\Phi^{(\infty)}(z)$, $X_{\mathrm{Gauss}}(z)$ and $\widetilde{X}(z)$, respectively. Then, for the coefficient $m_1^{(n)}$ of series (A.4) we will have from (A.20), (A.49), and (A.69) that

$$\mathrm{e}^{-\frac{1}{2}n\ell\sigma_3} m_1^{(n)}\, \mathrm{e}^{\frac{1}{2}n\ell\sigma_3} = m_1^\infty + m_{1,\mathrm{Gauss}} + \widetilde{m}_1. \tag{A.73}$$

In virtue of estimates (A.71) and (A.72) we have that

$$\left|\widetilde{m}_1 - I - \sum_{k=1}^{K} N^{-k} r_k\left(\frac{n}{N};t\right)\right| \le C(K)N^{-K-1}t^{-1}, \tag{A.74}$$

$$\left|r_k\left(\frac{n}{N};t\right)\right| \le C(k)t^{-1}, \tag{A.75}$$

while

$$m_{1,\mathrm{Gauss}} = \mathrm{e}^{-\frac{1}{2}n\ell_{\mathrm{Gauss}}\sigma_3} m_{1,\mathrm{Gauss}}^{(n)}\, \mathrm{e}^{\frac{1}{2}n\ell_{\mathrm{Gauss}}\sigma_3} - m_{1,\mathrm{Gauss}}^\infty. \tag{A.76}$$

Observe now that the matrices m_1^∞, $m_{1,\mathrm{Gauss}}^\infty$, and $m_{1,\mathrm{Gauss}}^{(n)}$ can be evaluated explicitly. Indeed, the first two can be obtained from (A.35), taking into account that $z_{0,\mathrm{Gauss}} = \sqrt{2\lambda}$, and the third one follows from the fact that the normalizing constants $h_{n,\mathrm{Gauss}}$ are known$-$see (3.22). Therefore, performing the calculations indicated, we derive from equations (A.74), (A.73), and (A.5) the following estimates for the recurrence coefficients R_n,

$$\left|R_n(t) - \frac{z_0^2}{4} - \sum_{k=1}^{K} N^{-k} f_k\left(\frac{n}{N};t\right)\right| \le C(K)N^{-K-1}t^{-1}, \tag{A.77}$$

$$\left|f_k\left(\frac{n}{N};t\right)\right| \le C(k)t^{-1}, \quad \text{for all } t \ge 1 + 2^{-\frac{2}{3}}N^{-\delta}, \quad n = N-1, N, N+1 \text{ and } N \ge 1. \tag{A.78}$$

Finally, repeating the arguments we used in the proof of Theorem 5.2, we conclude that the odd coefficients f_k in the series from (A.77) are actually absent. The Proposition 9.3 follows.

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