Aleksandra NOWEL

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TOPOLOGICAL INVARIANTS OF ANALYTIC SETS ASSOCIATED WITH NOETHERIAN FAMILIES

by Aleksandra NOWEL

Introduction.

In [12] Parusiński and Szafraniec proved, that for any regular morphism $\phi : X \to W$ of real algebraic sets there exist real polynomials $g_1, g_2, \ldots, g_s$ on $W$ such that for every $w \in W$

$$\chi(\phi^{-1}(w)) = \text{sgn} g_1(w) + \text{sgn} g_2(w) + \ldots + \text{sgn} g_s(w),$$

where $\text{sgn} g(w)$ denotes the sign of $g(w)$, $\chi(A)$ denotes the Euler characteristic of the set $A$ (compare also the result of Coste and Kurdyka [4]).

Let $\Omega \subset \mathbb{R}^n$ be a compact semianalytic set and let $\mathcal{F}$ be a collection of real analytic functions defined in some neighbourhood of $\Omega$. With each $\omega \in \Omega$ we can associate an analytic germ $Y_\omega = \bigcap_{f \in \mathcal{F}} f^{-1}(0)$ at $\omega$ and an analytic germ $X_\omega = \{x \mid x + \omega \in Y_\omega\}$ at 0. Using arguments similar to Parusiński and Szafraniec, and the properties of Noetherian families, we will show (Theorem 4.11) that there exist analytic functions $v_1, v_2, \ldots, v_s$ defined in a neighbourhood of $\Omega$ such that for each $\omega \in \Omega$ there exists $0 < \epsilon_\omega \ll 1$ such that for each $0 < \epsilon < \epsilon_\omega$

$$\frac{1}{2} \chi(S_{\epsilon}^{-1} \cap X_\omega) = \sum_{i=1}^{s} \text{sgn} v_i(\omega),$$

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where $S^n_{\epsilon} - 1$ denotes a sphere in $\mathbb{R}^n$ centered at the origin with the radius $\epsilon$.

This result is proven in section 4. In fact it holds in the more general case, where $\mathcal{F}$ is a family of analytic functions from an $\Omega$–Noetherian algebra satisfying some additional assumptions (see Remark after Theorem 4.11). The $\Omega$–Noetherian algebras were defined by El Khadiri and Tougeron in [5].

Let $\Omega$ be a locally closed subset of $\mathbb{R}^n$, and let $\mathcal{O}(\Omega)$ be a subalgebra of the algebra of analytic functions on $\Omega$ (or on a neighbourhood of $\Omega$) to $\mathbb{R}$. Let us identify $\Omega$ with a subspace of the maximal spectrum $SM(\mathcal{O}(\Omega))$. With each point from $\Omega$ we associate the maximal ideal of $\mathcal{O}(\Omega)$ consisting of the functions which vanish at this point. The subalgebra $\mathcal{O}(\Omega)$ is called $\Omega$–Noetherian if it is closed under derivation, $\mathbb{R}[x] \subset \mathcal{O}(\Omega)$ and $\Omega$, identified as above with a subspace of the maximal spectrum $SM(\mathcal{O}(\Omega))$, is a Noetherian space. El Khadiri and Tougeron have given other examples of $\Omega$–Noetherian algebras, for instance

- the algebra of Nash functions (i.e. analytic semialgebraic functions) on $\Omega$, where $\Omega$ is open semialgebraic in $\mathbb{R}^n$,
- the algebra $\mathbb{R}[x][f_1, \ldots, f_q]$, where $\mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]$ is the ring of polynomials on $\mathbb{R}^n$, $f_i = e^{Q_i}$, $Q_i \in \mathbb{R}[x]$.

In sections 1–3 we recall the definition and properties of Noetherian families, proved by El Khadiri and Tougeron in [5] and prove some useful properties of germs of some special complex analytic sets and of Noetherian families. Finally, in section 5, we show some consequences of the main result.

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1. Preliminaries.

Let $A$ be a commutative algebra with an identity element over a commutative field $k$ of characteristic zero, and let $\Gamma$ be a subset of the maximal spectrum $SM(A)$ of $A$. In $\Gamma$ we have the topology induced by the topology of $SM(A)$, i.e. $F$ is closed in $\Gamma$ if $F = \{ \gamma \in \Gamma \mid B \subset \gamma \}$ for some $B \subset A$. 
Following El Khadiri and Tougeron [5] we assume that $A$ and $\Gamma$ satisfy the following conditions:

(a) for all $\gamma \in \Gamma$ the canonical mapping $k \rightarrow A/\gamma$ is an isomorphism.

(b) $\Gamma$ equipped with the topology of $SM(A)$ is a Noetherian space.

This means that every decreasing sequence of closed sets in $\Gamma$ is stationary. Consequently any closed set in $\Gamma$ is a union of finitely many irreducible closed sets.

If $a \in A$ and $\gamma \in \Gamma$, let $a(\gamma) \in k$ denote the image of $a$ under the mapping $A \rightarrow A/\gamma \cong k$. If $F$ is a subset of $\Gamma$, let $I(F) = \{a \in A \mid a(\gamma) = 0$ for all $\gamma \in F\}$. If $S$ is a subset of $A$, let $V(S) = \{\gamma \in \Gamma \mid a(\gamma) = 0$ for all $a \in S\}$. Then closed sets in $\Gamma$ are the sets $V(S)$, where $S \subset A$. A closed set $F$ in $\Gamma$ is irreducible if and only if $I(F)$ is a prime ideal.

Let $x = (x_1, \ldots, x_n)$, $k = \mathbb{R}$ or $\mathbb{C}$, and denote by $A[[x]]$ (resp. $k[[x]]$) the ring of formal power series in $x$ with coefficients in $A$ (resp. in $k$), and by $k\{x\}$ the ring of formal power series which are convergent in some neighbourhood of the origin. If $\gamma \in \Gamma$ and $f = \sum a_{\beta} x^\beta \in A[[x]]$, let $f_\gamma = \sum a_{\beta}(\gamma) x^\beta \in k[[x]]$. If $f = (f_1, \ldots, f_p) \in A[[x]]^p$, we write $f_\gamma = (f_{1,\gamma}, \ldots, f_{p,\gamma})$. Finally if $N$ is a submodule of $A[[x]]^p$ generated by $f_\alpha$, let $N_\gamma$ be the submodule of $k[[x]]^p$ generated by $f_{\alpha,\gamma}$.

El Khadiri and Tougeron have proved a lot of properties of submodules of $A[[x]]^p$ (see [5]). We recall some of them.

**Theorem 1.1** ([5], Proposition 6.2.1). — Let $N$ be a submodule of $A[[x]]^p$. There exists a submodule $N' \subset N$, generated by finitely many elements, such that $N_\gamma = N'_\gamma$ for all $\gamma \in \Gamma$.

**Theorem 1.2** ([5], Proposition 6.8). — Let $I$ be an ideal in $A[[x]]$. There exists a positive integer $\mu$ such that

$$\forall_{\gamma \in \Gamma} (\text{rad}(I_\gamma))^{\mu} \subset I_\gamma.$$ 

Denote by $A_c[[x]]$ the subring of the ring $A[[x]]$ such that

$$f \in A_c[[x]] \iff \forall_{\gamma \in \Gamma} f_\gamma \in k\{x\}.$$ 

Theorems 1.1 and 1.2 are valid if we replace $A[[x]]$ by $A_c[[x]]$.

**Definition.** — A collection $\mathcal{N}$ of submodules of $k[[x]]^p$ (resp. of $k\{x\}^p$) is called a Noetherian family (parameterized by $(A, \Gamma)$) if there
exists a couple \((A, \Gamma)\) satisfying the conditions (a) and (b) given above, and a submodule \(N\) of \(A[[x]]^p\) (resp. \(A_c[[x]]^p\)) such that \(N = (N_\gamma)_{\gamma \in \Gamma}\).

Each subcollection of a Noetherian family is a Noetherian family, a union of two Noetherian families is a Noetherian family (if \(N_1\) and \(N_2\) are Noetherian families parametrized resp. by \((A_1, \Gamma_1)\) and \((A_2, \Gamma_2)\) then \(N_1 \cup N_2\) is parametrized by \((A_1 \oplus A_2, \Gamma_1 \cup \Gamma_2)\)).

**DEFINITION.** — Let \(I\) be an ideal in \(\mathbb{R}\{x\}\) generated by \(f_1, \ldots, f_p\) and let \(V(I)\) be the germ of the set of zeros of \(I\) at the origin. The Lojasiewicz exponent of \(I\) is the infimum of all the positive real numbers \(\alpha\) for which there exists a constant \(c > 0\) such that

\[
\sum_{i=1}^{p} |f_i(x)| \geq c d(x, V(I))^\alpha
\]

in some neighbourhood of the origin (\(d\) denotes the Euclidean distance and we put \(d(x, \emptyset) = 1\)).

**THEOREM 1.3** ([5], Proposition 8.3). — Let \((I_\gamma)_{\gamma \in \Gamma}\) be a Noetherian family of ideals of \(\mathbb{R}\{x\}\). Then the family of the Lojasiewicz exponents \(\mathcal{L}(I_\gamma)\) of \(I_\gamma\) is bounded.

Let \((\overline{A}, \overline{\Gamma})\) be a second couple satisfying conditions (a) and (b). A change of parametrization is a morphism of \(k\)-algebras \(\phi : A \rightarrow \overline{A}\) such that \(\phi_* : \text{Spec} \overline{A} \rightarrow \text{Spec} A\) induces a morphism from \(\Gamma\) onto \(\overline{\Gamma}\). If \(N = (N_\gamma)_{\gamma \in \Gamma}\) is a Noetherian family and \(\overline{N}\) is the submodule of \(\overline{A}[[x]]^p\) (resp. \(\overline{A}_c[[x]]^p\)) generated by \(\overline{\phi}(N)\) then \(N = (\overline{N}_\gamma)_{\gamma \in \overline{\Gamma}}\) and \((\overline{A}, \overline{\Gamma})\) is a new parametrization of this family (here \(\overline{\phi} : A[[x]]^p \rightarrow \overline{A}[[x]]^p\) is a natural extension of \(\phi\)). A composition of changes of parametrization is a change of parametrization.

**THEOREM 1.4** ([6], Proposition 6.6). — Let \(N\) be a submodule of \(A[[x]]^p\). There exist a change of parametrization \(\phi : (A, \Gamma) \rightarrow (\overline{A}, \overline{\Gamma})\), a finite partition \((\overline{\Gamma}_i)_{i \in I}\) of \(\overline{\Gamma}\), ideals \(p_1, \ldots, p_s\) of \(\overline{A}[[x]]\), submodules \(N_1, \ldots, N_s\) of \(\overline{A}[[x]]^p\) and constants \(s_i \leq s_i, i \in I\), such that for all \(\overline{\gamma} \in \overline{\Gamma}_i\), if \(\overline{\gamma} = \phi_*(\gamma)\):

1. \(p_{1, \overline{\gamma}}, \ldots, p_{s_i, \overline{\gamma}}\) are prime ideals of \(k[[x]]\) and if \(j > s_i\) then \(p_{j, \overline{\gamma}} = k[[x]]\).
2. \(N_{j, \overline{\gamma}}\) is \(p_{j, \overline{\gamma}}\)-primary if \(1 \leq j \leq s_i\) and \(N_{j, \overline{\gamma}} = k[[x]]^p\) if \(j > s_i\).
3. \(N_\gamma = N_{1, \overline{\gamma}} \cap \ldots \cap N_{s_i, \overline{\gamma}}\) and it is a reduced primary decomposition of \(N_\gamma\).
Theorem 1.5 ([5], Proposition 6.4). — Let $N$, $N'$ be submodules of $A[[x]]^p$. There exist a change of parametrization $\phi : (A, \Gamma) \longrightarrow (\overline{A}, \overline{\Gamma})$ and a submodule $\overline{N}$ of $\overline{A}[[x]]^p$ such that for all $\overline{\gamma} \in \overline{\Gamma}$ if $\gamma = \phi_*(\overline{\gamma})$:

$$\overline{N}_{\overline{\gamma}} = N_\gamma \cap N'_\gamma.$$ 

2. Germs of analytic sets.

Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ be a germ of a holomorphic function at the origin and let $r(z) = z_1^2 + \ldots + z_n^2$ for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Denote by $G$ the germ at the origin of the analytic set

$$\bigcap_{i<j} \left\{ z \in \mathbb{C}^n \mid \det \begin{bmatrix} \frac{\partial r}{\partial z_i} & \frac{\partial r}{\partial z_j} \\ \frac{\partial f}{\partial z_i} & \frac{\partial f}{\partial z_j} \end{bmatrix} = 0 \right\},$$

i.e. $z \in G$ if and only if $\nabla r(z) = \left( \frac{\partial r}{\partial z_1}(z), \ldots, \frac{\partial r}{\partial z_n}(z) \right)$ and $\nabla f(z) = \left( \frac{\partial f}{\partial z_1}(z), \ldots, \frac{\partial f}{\partial z_n}(z) \right)$ are linearly dependent.

Denote by $G'$ the germ of the set $G \setminus f^{-1}(0)$ at the origin. We will show, that $G' \cap f^{-1}(0) = G' \cap r^{-1}(0)$.

Lemma 2.1. — $G \cap r^{-1}(0) \subset f^{-1}(0)$.

Proof. — Assume that $(G \cap r^{-1}(0)) \setminus (G \cap f^{-1}(0)) \neq \emptyset$. According to the curve selection lemma there exists an analytic curve $\gamma = (\gamma_1, \ldots, \gamma_n)$ such that $\gamma(0) = 0$ and $\gamma \setminus \{0\} \subset (G \cap r^{-1}(0)) \setminus (G \cap f^{-1}(0))$. Then we have $r(\gamma(t)) \equiv 0$. Hence

$$\frac{d}{dt} r(\gamma(t)) = \frac{\partial r}{\partial z_1} (\gamma(t)) \frac{d\gamma_1}{dt}(t) + \ldots + \frac{\partial r}{\partial z_n} (\gamma(t)) \frac{d\gamma_n}{dt}(t) \equiv 0.$$  

Since $\nabla r(z) \neq 0$ for $z \neq 0$ and $\gamma(t) \in G$,

$$\forall t \exists c(t) \nabla f(\gamma(t)) = c(t) \nabla r(\gamma(t)).$$

Thus by (1) we have $\frac{d}{dt} f(\gamma(t)) \equiv 0$, so $f \circ \gamma = const$. Since $(f \circ \gamma)(0) = 0$, $\gamma \subset f^{-1}(0)$ — a contradiction. \qed

Lemma 2.2. — $G'$ is a germ of an analytic set.

Proof. — Germs of sets $G$, $G \cap f^{-1}(0)$ are analytic, so the representative of $G \setminus f^{-1}(0)$ is an analytically constructible set. The complex closure
of an analytically constructible set is analytic, so the representative of the germ $G'$ is an analytic set ([8], Proposition IV 8.3.5).

**Lemma 2.3.** — $G' = G_1 \cup \ldots \cup G_p$, where $G_1, \ldots, G_p$ are the irreducible components of $G$ such that $G_i \setminus f^{-1}(0) \neq \emptyset$ for $i = 1, \ldots, p$. Moreover, $G_i \setminus f^{-1}(0)$ is dense in $G_i$.

**Proof.** — According to [8], Theorem IV 2.10.5, $G' = G_1 \cup \ldots \cup G_p$. Since each germ $G_i$ is irreducible, [8], Proposition IV 2.8.3, implies that $G_i \cap f^{-1}(0)$ is nowhere dense in $G_i$, so $G_i \setminus f^{-1}(0) = G_i \setminus (G_i \cap f^{-1}(0))$ is dense in $G_i$. □

**Lemma 2.4.** — Let $G_1, \ldots, G_p$ be defined as in Lemma 2.3. Let $G_i \setminus r^{-1}(0) = \bigcup A_{i,k}$ be a decomposition into finitely many disjoint analytic submanifolds. Then for each $i, k$ the restriction of $r$ to the set $A_{i,k}$ has no critical points in some neighbourhood of the origin.

**Proof.** — Fix $i, k$ and assume that the set of critical points of $r|_{A_{i,k}}$ is nonempty. Then it is analytically constructible. According to the curve selection lemma there is a curve $\gamma$ such that $\gamma(0) = 0$ and $\gamma \setminus \{0\}$ is contained in the set of critical points of $r|_{A_{i,k}}$. Then the function $r|_{A_{i,k}} \circ \gamma$ is constant. We have $r(\gamma(0)) = r(0) = 0$, so $r|_{A_{i,k}} \circ \gamma \equiv 0$. But it contradicts $\gamma \cap r^{-1}(0) = \emptyset$. So the set of critical points of $r|_{A_{i,k}}$ is empty. □

We will say that an analytic set has a Whitney stratification, if it has such a stratification whose every two strata satisfy Whitney conditions $a$ and $b$.

**Theorem 2.5** (see e.g. [18] Theorem 19.2, [1] Theorem 9.7.11). Any analytic set has a Whitney stratification. Any stratification $(E_i)_{i \in I}$ of this set has a Whitney refinement, i.e. there exists a Whitney stratification $(F_j)_{j \in J}$ such that each stratum $E_i$ is a union of some strata of $(F_j)_{j \in J}$.

**Lemma 2.6.** — $G' \cap f^{-1}(0) \setminus r^{-1}(0) = \emptyset$.

**Proof.** — Fix $i \in \{1, \ldots, p\}$. We will show that $G_i \cap f^{-1}(0) \setminus r^{-1}(0) = \emptyset$. The set $G_i$ admits a Whitney stratification such that $G_i \cap f^{-1}(0)$, as well as $G_i \setminus r^{-1}(0)$ is a union of strata. According to Lemma 2.4 the restriction $r|_{A_{i,k}}$ is a submersion for each $k$. 

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Assume that $z_0 \in G_i \cap f^{-1}(0) \setminus r^{-1}(0)$. Let $A$ be the stratum such that $z_0 \in A$ and let $\bigcup_j B_j$ be the union of all strata $B_j \subset G_i \setminus f^{-1}(0)$ such that $A \subset \overline{B_j}$. According to Lemma 2.3 there is at least one nonempty stratum satisfying this condition. Denote $Z = A \cup \bigcup_j B_j$.

We will show, that $z_0$ is not isolated in $\bigcup_j B_j \cap r^{-1}(r(z_0))$, using the following Thom-Mather theorem:

\textbf{Theorem 2.7} ([16] Theorem 4.3.1). — Let $X = \bigcup X_\alpha$ be an analytic space admitting a Whitney stratification. For each $x \in X_\alpha$, each local embedding $X \subset \mathbb{C}^n$ in a neighborhood of $x$, and each local retraction $\rho : \mathbb{C}^n \to X_\alpha$ there exist an open neighborhood $U$ of $x$ in $\mathbb{C}^n$ and a homeomorphism compatible with $\rho$ such that, denoting $V = U \cap X_\alpha$ and $\Pi_2 : (\rho^{-1}(x) \cap X \cap U) \times V \to V$ — the projection on the second variable, we have

$$X \cap U \xrightarrow{\rho \mid X \cap U} (\rho^{-1}(x) \cap X \cap U) \times V \xrightarrow{\Pi_2} V$$

inducing for each $X_\beta$ containing $X_\alpha$ the analogous homeomorphism

$$X_\beta \cap U \xrightarrow{\rho \mid X_\beta \cap U} (\rho^{-1}(x) \cap X_\beta \cap U) \times V \xrightarrow{\Pi_2} V$$

The set $Z$ satisfies the assumptions of the theorem. Fix $B_j \neq \emptyset$ and denote $k = \dim_\mathbb{C} A$. Since $\tilde{r} := r \mid A$ has no critical points, there exist $r_2, \ldots, r_k : \mathbb{C}^n \to \mathbb{C}$ defined in some neighborhood of $z_0$ such that, denoting $\tilde{r}_i = r_i \mid A$, $d \tilde{r}(z_0), d \tilde{r}_2(z_0), \ldots, d \tilde{r}_k(z_0)$ are linearly independent. Take $R = (r, r_2, \ldots, r_k) : \mathbb{C}^n \to \mathbb{C}^k$. $A$ is transversal to $R^{-1}(R(z_0))$ and crosses it at $z_0$. Denote $\tilde{R} = R \mid A$, then $\operatorname{rank} D \tilde{R}(z_0) = k$. So $\tilde{R} : (A, z_0) \to (\mathbb{C}^k, R(z_0))$ is an analytic diffeomorphism. Denote by $S : (\mathbb{C}^k, R(z_0)) \to (A, z_0)$ the inverse of $\tilde{R}$.

Let define a local retraction $\rho : \mathbb{C}^n \to A$, $\rho(z) = (S \circ R)(z)$. According to Theorem 2.7 there exist a neighborhood $U$ of $z_0$ and a homeomorphism $h$ such that, for $V = U \cap A$

$$\overline{B_j} \cap U \xrightarrow{\rho \mid \overline{B_j} \cap U} (\rho^{-1}(z_0) \cap \overline{B_j} \cap U) \times V \xrightarrow{\Pi_2} V$$

We have $(\rho^{-1}(z_0) \cap \overline{B_j} \cap U) \times V = (R^{-1}(R(z_0)) \cap \overline{B_j} \cap U) \times V \subset (r^{-1}(r(z_0)) \cap \overline{B_j} \cap U) \times V$. Since $A \subset \overline{B_j}$, there exist a sequence $(z_n) \subset B_j$.
such that $z_n \to z_0$. Let $(y_n) \subset (R^{-1}(R(z_0)) \cap B_j \cap U)$ be such that $z_n = h^{-1}(y_n, \rho(z_n))$. Then $y_n \to z_0$ and $(y_n) \subset r^{-1}(r(z_0))$.

Hence $z_0$ is not isolated in $\bigcup_j B_j \cap r^{-1}(r(z_0))$, so by the curve selection lemma there is a curve $\gamma$ such that $\gamma(0) = z_0$ and $\gamma \setminus \{z_0\} \subset \bigcup_j B_j \cap r^{-1}(r(z_0))$.

Because $\gamma \subset G_i \subset G$ and $r|_{A_i,k}$ are submersions, we can deduce as above, using arguments from the proof of Lemma 2.1, that $f$ is constant along $\gamma$ and $f(\gamma(0)) = f(z_0) = 0$, so $f \equiv 0$ along $\gamma$. But $\gamma \setminus \{z_0\} \subset G_i \setminus f^{-1}(0)$, a contradiction. Then $G' \cap f^{-1}(0) \setminus r^{-1}(0) = \emptyset$.

Hence we obtain

**Corollary 2.8.** — $G' \cap f^{-1}(0) = G' \cap r^{-1}(0)$.

### 3. Properties of Noetherian families.

Assume that $\Omega \subset \mathbb{R}^n$ is a semianalytic compact subset and denote by $A(\Omega)$ the algebra of real analytic functions defined in a neighbourhood of $\Omega$. We can treat $\mathbb{R}^n$ as a subspace of $\mathbb{C}^n$, so $\Omega \subset \mathbb{C}^n$ and we denote by $H(\Omega)$ the algebra of complex analytic functions defined in a neighbourhood of $\Omega$.

El Khadiri and Tougeron have proven (see [5]), that if $O(\Omega) = A(\Omega)$ or $H(\Omega)$, then $O(\Omega)$ is an $\Omega$-Noetherian algebra, so $\Omega$ is a Noetherian space with the topology induced from $SM(O(\Omega))$ (by identifying $\omega \in \Omega$ with the ideal $p_\omega = \{f \in O(\Omega) \mid f(\omega) = 0\}$, $\{\bigcap_{f \in B} f^{-1}(0) \cap \Omega\}_{B \subset O(\Omega)}$ is the family of closed sets in $\Omega$), and the pair $(O(\Omega), \Omega)$ satisfies conditions (a) and (b) from the section 1. Notice that since $\Omega$ is a Noetherian space, for every closed (with respect to the topology induced by the topology on the maximal spectrum) subset $D$ of $\Omega$ there exist $f_1, \ldots, f_p \in O(\Omega)$ such that $D = \bigcap_{i=1}^p f_i^{-1}(0) \cap \Omega$, so $D$ is an intersection of $\Omega$ and an analytic set.

The result of Frisch [7] says, that $A(\Omega)$ is Noetherian and if $\Omega$ admits a fundamental system of Stein neighborhoods, then $H(\Omega)$ is also Noetherian.

If $f \in A(\Omega)$ and $\omega \in \Omega$, we denote $\tilde{f} = \sum_\alpha \frac{1}{\alpha!} D^\alpha f x^\alpha$, $\tilde{f}_\omega = \sum_\alpha \frac{1}{\alpha!} D^\alpha f(\omega)x^\alpha$. Of course $\tilde{f} \in A(\Omega)_{\mathbb{C}}[[x]]$.

Define $\tilde{f}_\omega^C : (\mathbb{C}^n, 0) \to \mathbb{C}$ as $\tilde{f}_\omega^C = \sum_\alpha \frac{1}{\alpha!} D^\alpha f(\omega)z^\alpha$, then $\tilde{f}^C = \sum_\alpha \frac{1}{\alpha!} D^\alpha f z^\alpha \in H(\Omega)_{\mathbb{C}}[[x]]$. 

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Theorem 3.1. — Let \( f \in \mathcal{A}(\Omega) \). There is \( N_0 > 0 \) such that for each \( N \geq N_0, \omega \in \Omega \) there exist \( \epsilon_\omega > 0 \) and \( c_\omega > 0 \) such that if \( \epsilon \in (0; \epsilon_\omega) \) and \( x \in S^m_\epsilon \setminus f^{-1}_\omega(0) \) is a critical point of \( f_\omega|_{S^m_\epsilon} \), then

\[
|f_\omega(x)| \geq \frac{1}{c_\omega} ||x||^{2N}.
\]

Proof. — Let \( r(z) = z_1^2 + \ldots + z_n^2 \) for \( z \in \mathbb{C}^n \). Let define \( M^{ij} = \det \begin{bmatrix} \frac{\partial r}{\partial z_i} & \frac{\partial r}{\partial z_j} \\ \frac{\partial \tilde{f}_c^i}{\partial z_i} & \frac{\partial \tilde{f}_c^i}{\partial z_j} \end{bmatrix} \). Then \( M^{ij} \in \mathcal{H}(\Omega)_c[[x]] \), \( M^{ij}_\omega \) are germs of complex analytic functions at the origin. Let \( \mathcal{G}_\omega = V((M^{ij}_\omega)_{i < j}) \) for \( \omega \in \Omega \). According to Theorem 1.2 for all \( \bar{\phi} \in \mathfrak{P}_\omega \) there exist \( J(\omega) \in \mathcal{H}(\Omega)_c[[x]] \) and a decomposition into irreducible components \( \mathcal{G}_\omega = \mathcal{G}_1(\omega) \cup \ldots \cup \mathcal{G}_p(\omega) \). According to Theorem 1.4, there exist a composition of changes of parametrization \( \mathcal{G}_\omega = \mathcal{G}_1(\omega) \cup \ldots \cup \mathcal{G}_p(\omega) \). Denote \( J_{j,\omega} = I(\mathcal{G}_{j,\omega}) \). \( \mathcal{G}_{j,\omega} \) are irreducible components of a complex analytic germ \( \mathcal{G}_\omega \), so \( J_{j,\omega} \) are prime and \( I(\mathcal{G}_\omega) = J_1(\omega) \cup \ldots \cup J_l(\omega) \) is a reduced prime decomposition.

Denote by \( J_\omega \) the ideal in \( \mathcal{H}(\Omega)_c[[x]] \) generated by \( M^{ij}, i < j \), so \( J_\omega = (M^{ij}_\omega)_{i < j} \). Then, by the local Hilbert Nullstellensatz, \( \text{rad}(J_\omega) = I(\mathcal{G}_\omega) \) and then \( J_1(\omega), \ldots, J_l(\omega) \) are minimal prime ideals associated with the ideal \( J_\omega \). According to Theorem 1.4, there exist a change of parametrization \( \phi : (\mathcal{H}(\Omega), \Omega) \rightarrow (A, \Gamma) \), a finite partition \( (\Gamma_i)_{i \in I} \) of \( \Gamma \), ideals \( p_1, \ldots, p_s \) of \( A_c[[x]] \) and constants \( s_i \leq s, i \in I \), such that for all \( \gamma \in \Gamma_i \), if \( \omega = \phi_\ast(\gamma) \) then \( p_{1,\gamma}, \ldots, p_{s_i, \gamma} \) are minimal prime ideals associated with \( J_\omega \). Because of uniqueness of such ideals, for each \( j \in \{1, \ldots, l(\omega)\} \) there exists \( q \in \{1, \ldots, s_i\} \) such that \( J_{j,\omega} = p_{q, \gamma} \).

According to Theorem 1.5 there exist a change of parametrization \( \phi : (A, \Gamma) \rightarrow (\overline{A}, \overline{\Gamma}) \) and ideals \( N_\overline{\gamma}^Q \) of \( \overline{A}_c[[x]] \), \( Q \subset \{1, \ldots, s\} \), such that for all \( \overline{\gamma} \in \overline{\Gamma}_i \), if \( \gamma = \phi_\ast(\overline{\gamma}) \) then \( N_\overline{\gamma}^Q = \bigcap_{j \in Q} p_{j, \gamma} \).

A finite union of Noetherian families is a Noetherian family, so let \( \mathcal{K} = (K_\gamma)_{\gamma \in \Gamma} \) be a Noetherian family containing all families \( (N_\overline{\gamma}^Q)_{\gamma \in \overline{\Gamma}} \), \( Q \subset \{1, \ldots, s\} \). Then \( \mathcal{K} \) contains all \( I(G'_\omega) \) for \( \omega \in \Omega \). Let \( (M_\gamma)_{\gamma \in \Gamma} \) denote the Noetherian family \( (f'_\omega)_{\omega \in \Omega} \) after the change of parametrization \( \phi' : (\mathcal{H}(\Omega), \Omega) \rightarrow (\overline{A}, \overline{\Gamma}) \) which is a composition of changes of parametrization. According to Theorem 1.2

\[
\exists N_0 > 0 \ \forall N \geq N_0 \ \forall \gamma \in \Gamma \ (\text{rad}(K_\gamma + M_\gamma))^N \subset (K_\gamma + M_\gamma).
\]
According to Corollary 2.8, for each $\omega \in \Omega$ we have $V(I(G'_\omega) + (r)) = G'_\omega \cap r^{-1}(0) = G'_\omega \cap (r^{-1}(0)) = \gamma(G'_\omega + (r)).$ By the local Hilbert Nullstellensatz, $\text{rad}(I(G'_\omega) + (r)) = \text{rad}(I(G'_\omega) + (\tilde{f}_C)).$ For each $\omega \in \Omega$ there exists $\gamma \in \Omega$ such that $I(G'_\omega) = K_\gamma,$ and then

$$(I(G'_\omega) + (r))^{N_0} \subset (\text{rad}(I(G'_\omega) + (r)))^{N_0} = (\text{rad}(I(G'_\omega) + (\tilde{f}_C)))^{N_0} = (\text{rad}(K_\gamma + M_\gamma))^{N_0} \subset (K_\gamma + M_\gamma) = (I(G'_\omega) + (\tilde{f}_C))).$$

Let $g_{i,\omega}$ be the generators of $I(G'_\omega).$ Then $r^{N_0} = a_\omega \tilde{f}_C + \sum_i c_{i,\omega} g_{i,\omega}$ for some germs of complex analytic functions $a_\omega, c_{i,\omega}.$

Let $0 < \epsilon_\omega \ll 1$ be such that representatives of the germs $\tilde{f}_C, a_\omega$ and all $c_{i,\omega}, g_{i,\omega}$ are defined on $\{z \in C^m \mid ||z|| < \epsilon_\omega\}.$ If $0 < \epsilon < \epsilon_\omega$ and $x$ is a critical point of $\tilde{f}_C|_{S^m_{\epsilon}} = \sum_{i=1}^{p} g_i \frac{|x|^i}{i!}$ such that $x \not\in \tilde{f}_C^{-1}(0)$ then $x \in G'_\omega$ and for each $i$ we have $g_{i,\omega}(x) = 0.$ Then $r^{N_0}(x) = a_\omega(x) \tilde{f}_C(x),$ so

$$\exists_{\epsilon_\omega > 0} \forall_{N > N_0} r^N(x) \leq r^{N_0}(x) = |a_\omega(x)||\tilde{f}_C(x)| \leq c_\omega|\tilde{f}_C(x)|,$$

Thus

$$|\tilde{f}_C(x)| \geq \frac{1}{c_\omega} r^N(x) = \frac{1}{c_\omega} ||x||^{2N}.$$

\[\square\]

**Corollary 3.2.** — Let $f \in A(\Omega).$ Then there is $\alpha = 2N_0 + 1$ such that for each $\omega \in \Omega$ there exists $0 < \epsilon_\omega \ll 1$ such that if $0 < \epsilon < \epsilon_\omega$ and $x \in S^m_{\epsilon} \setminus \tilde{f}_C^{-1}(0)$ is a critical point of $\tilde{f}_C|_{S^m_{\epsilon}}$ then

$$|\tilde{f}_C(x)| \geq ||x||^\alpha.$$

**4. Families of germs of real analytic functions.**

Let $k = \mathbb{R}$ or $k = \mathbb{C}$ and let $m$ be the maximal ideal of $k[[x]] = k[[x_1, \ldots, x_n]].$ Let $\mathcal{F}_p = \bigoplus_p m \subset k[[x]]^p.$ If $g \in \mathcal{F}_p,$ then $g = (g_1, \ldots, g_p),$ where

$$g_j = \sum_{|\alpha| \geq 1} \frac{a_j^\alpha}{\alpha!} x^\alpha \quad \text{(i.e. } a_j^\alpha = D^\alpha g_j(0)).$$

Let $\Psi_1, \ldots, \Psi_s$ be formal power series in $x$ with coefficients which depend polynomially on $a_j^\alpha,$ where $|\alpha| \geq 1$ and $1 \leq j \leq p.$ If $g = (g_1, \ldots, g_p) \in \mathcal{F}_p,$ we denote by $\Psi_{i,g}$ the formal power series obtained by
putting $a_j^\alpha = D^\alpha g_j(0)$ in $\Psi_i$. Let $I_g$ be the ideal of $k[[x]]$ generated by $\Psi_{1,g}, \ldots, \Psi_{s,g}$.

Denote by $W_h$ the set \{ $g \in \mathcal{F}_p \mid \dim_k(k[[x]]/I_g) > h$ \}. Then, by [17], Corollary II.5.2, we have $W_h = \{ g \in \mathcal{F}_p \mid \dim_k(I_g + m^{h+1}/m^{h+1}) < \binom{n+h}{n} - h \}$. We consider $k[[x]] + m^{h+1}/m^{h+1}$ which is an affine space of finite dimension. The space $I_g + m^{h+1}/m^{h+1}$ is its linear subspace generated by $x^\alpha \Psi_{i,g}$, where $\alpha \in \mathbb{N}^n$, $0 \leq |\alpha| \leq h$. The above description of $W_h$ involves only finitely many coefficients of the series.

Let $\Psi_{i,g}^{\alpha,\beta}$, $|\beta| \leq h, |\alpha| \leq h$ be the coefficients at $x^\beta$ in the series $x^\alpha \Psi_{i,g}$. Then the set $W_h$ is the set of such $g \in \mathcal{F}_p$, for which all the minors of the matrix $(\Psi_{i,g}^{\alpha,\beta})$ of degree $(n+h)n - h$ vanish ($i, \alpha$ is a row index, $\beta$ is a column index).

Theorem 4.1 ([17], Lemma VII.5.3]). — The sets $W_h$ are algebraic and
\[ \{ g \in \mathcal{F}_p \mid \dim_k(k[[x]]/I_g) < \infty \} = \mathcal{F}_p \setminus \bigcap_{h=0}^{\infty} W_h. \]

We will say that a germ of an analytic mapping $F = (F^1, \ldots, F^n) : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ has an algebraically isolated zero at the origin if $\dim_{\mathbb{R}} \mathbb{R}[[x]]/(P_1, \ldots, P_n) < \infty$, where $P_i = \sum_{\alpha} \frac{1}{\alpha!} D^\alpha F^i(0)x^\alpha$. If $0 \in \mathbb{C}^n$ is isolated in the inverse image of 0 for the complexification of $F$ then the origin is an algebraically isolated zero of $F$.

By $\deg_0 F$ we denote the local topological degree at the origin of the mapping $F$ which has an isolated zero at the origin.

Recall that a closed subset of $\Omega$ has to be understood with respect to the topology induced from $SM(A(\Omega))$. We will say that a closed subset of $\Omega$ is irreducible if it is not a union of two its proper closed subsets. Every closed subset $D$ of a Noetherian space $\Omega$ has a decomposition into finitely many irreducible components, i.e. $D = \bigcup_{i=1}^k D_i$, where every $D_i$ is a closed irreducible subset of $D$ and $D_i \not\subset \bigcup_{j \neq i} D_j$.

Let $D \subset \Omega$ be a closed subset. Denote $J = \{ f \in \mathcal{A}(\Omega) \mid f|_D \equiv 0 \}$, and define $A(D) := \mathcal{A}(\Omega)/J$.

If $D$ is irreducible then $J$ is a prime ideal and $A(D)$ is an integral domain.

Denote by $S_n(D)$ the set of families $\{ F_\omega = (F_\omega^1, \ldots, F_\omega^n) : (\mathbb{R}^n, 0) \rightarrow \}$.
\((\mathbb{R}^n, 0)\) of analytic germs at the origin such that
\[
\forall 1 \leq i \leq n \exists f_i \in \mathcal{A}(\Omega)_c[[x]] \forall \omega \in D F^i_\omega(x) = f_i(\omega, x).
\]
In particular if
\[
\forall 1 \leq i \leq n \exists h_i \in \mathcal{A}(\Omega) \forall \omega \in D F^i_\omega(x) = h_i(x + \omega),
\]
then \(\{F_\omega\}_{\omega \in D} \in \mathcal{S}_n(D)\).

**Lemma 4.2.** Assume that a closed subset \(D \subset \Omega\) is irreducible, \(\{F_\omega\}_{\omega \in D} \in \mathcal{S}_n(D)\) and \(0 \in \mathbb{R}^n\) is isolated in \(F_\omega^{-1}(0)\) for all \(\omega \in D\). Then there exist a proper closed subset \(\Sigma \subset D\), and a family \(\{G_\omega\}_{\omega \in D} \in \mathcal{S}_n(D)\) such that

(i) \(\forall \omega \in D \setminus \Sigma\) \(G_\omega\) has an algebraically isolated zero at the origin,

(ii) \(\forall \omega \in D\) \(\deg_0 F_\omega = \deg_0 G_\omega\).

**Proof.** For \(\omega \in D\) we define the germ \(G_\omega\):
\[
G_\omega(x) = F_\omega(x) + a(x_1^k, \ldots, x_n^k),
\]
where \(k\) is a positive integer, \(a \neq 0\). We have \(G^i_\omega(x) = f_i(\omega, x) + ax_i^k\), so \(G^i_\omega\) is a real analytic germ. Let \(c_i^\alpha \in \mathcal{A}(D)\) be residue classes of \(\frac{1}{\alpha!} D^\alpha G^i_\omega(0) \in \mathcal{A}(\Omega)\), and let associate with \(G^i_\omega\) the formal power series
\[
P_i(\omega, x) = \sum_\alpha c_i^\alpha(\omega) x^\alpha \in \mathcal{A}(D)_c[[x]].
\]

According to Theorem 4.1 the set \(\{\omega \in D \mid \dim_{\mathbb{R}}(\mathbb{R}[[x]]/(P_1(\omega, \cdot), \ldots, P_n(\omega, \cdot))) < \infty\} = D \setminus \bigcap_{h=0}^\infty \Sigma_h\), where \(\Sigma_h = \{\omega \in D \mid \dim_{\mathbb{R}}(\mathbb{R}[[x]]/(P_1(\omega, \cdot), \ldots, P_n(\omega, \cdot))) > h\}\) are closed in \(D\). Indeed, \(\Sigma_h\) is the intersection of the zero sets of some compositions of \(c_i^\alpha\) and polynomials. So \(\Sigma = \bigcap_{h=0}^\infty \Sigma_h\) is a closed subset of \(D\) such that the origin is algebraically isolated in \(G_\omega^{-1}(0) \subset \mathbb{R}^n\) for \(\omega \in D \setminus \Sigma\).

Using arguments similar as in the proof of [15], Lemma 1.3, we can show, that \(\Sigma\) is a proper subset of \(D\). We have
\[
P_1(\omega, x) = G^i_\omega(x) = F^i_\omega(x) + ax_i^k = f_i(\omega, x) + ax_i^k
\]
for \(x\) sufficiently close to the origin. Fix \(\omega_0 \in D\). The set
\[
A = \{a \in \mathbb{R} \setminus \{0\} \mid \dim_{\mathbb{R}}(\mathbb{R}[[x]]/(f_1(\omega_0, x) + ax_1^k, \ldots, f_n(\omega_0, x) + ax_n^k)) > h\}
\]
is finite for \(h\) sufficiently large.
Indeed, denote $H_i^a(x) = a f_i(\omega_0, x) + x_k^i$ for $a \in \mathbb{R}$. Then $H_0^i = x_k^i$ and we have $\dim_{\mathbb{R}}(\mathbb{R}[x]/(x_1^k, \ldots, x_n^k)) = k^n$. Then according to Theorem 4.1 the set

$$A' = \{ a \in \mathbb{R} \mid \dim_{\mathbb{R}}(\mathbb{R}[x]/(H_1^a, \ldots, H_n^a)) > h \}$$

is algebraic and $0 \not\in A'$ for $h > k^n$, so $A'$ is finite for $h > k^n$. If $a \neq 0$ then we have $H_n^a(x) = \frac{1}{a} P_i(\omega_0, x)$, so $A$ is also finite for $h > k^n$.

Take $a \not\in A$ in the definition of $G_\omega$, then

$$\omega_0 \not\in \Sigma_h = \{ \omega \in D \mid \dim_{\mathbb{R}}(\mathbb{R}[x]/(P_1(\omega, \cdot), \ldots, P_n(\omega, \cdot))) > h \},$$

so $\Sigma_h \neq D$ for $h$ sufficiently large and $\Sigma$ is a proper subset of $D$.

Let $I_\omega \subset \mathbb{R}\{x\}$ be the ideal generated by germs $F_\omega^1, \ldots, F_\omega^n$. Theorem 1.3 implies that the Łojasiewicz exponent of $I_\omega$ is bounded:

$$\exists_M \forall \omega \in D \alpha_\omega = \inf \{ \alpha > 0 \mid \exists_{\omega > 0} \sum_{i=1}^n |F_\omega^i(x)| \geq c d(x, V_0(I_\omega))^\alpha \} \leq M.$$

The origin is isolated in the zero set of $F_\omega$, so

$$\exists_M \forall \omega \in D \exists_{\alpha > 0} \sum_{i=1}^n |F_\omega^i(x)| \geq c_\omega d(x, V_0(I_\omega))^\alpha = c_\omega d(x, \{0\})^\alpha \geq c_\omega ||x||^M$$

for $x$ near 0. Hence if we take $k > M$ in the definition of $G_\omega$ then there exists $c_\omega > 0$ such that

$$||t G_\omega(x) + (1-t) F_\omega(x)|| = ||F_\omega(x) + at(x_1^k, \ldots, x_n^k)||$$

$$\geq c_\omega ||x||^M - at||(x_1^k, \ldots, x_n^k)|| \geq \frac{c_\omega}{2} ||x||^M,$$

where $0 \leq t \leq 1$, $x$ near 0 (see [12]).

Then $\deg_0 F_\omega = \deg_0 G_\omega$. \hfill \Box

**Lemma 4.3.** — Under the assumptions of Lemma 4.2 there exist $q_1, \ldots, q_t \in \mathcal{A}(\Omega)$ and a proper closed subset $\Sigma \subset D$ such that for $\omega \in D \setminus \Sigma$

$$\deg_0 F_\omega = \sgn q_1(\omega) + \ldots + \sgn q_t(\omega).$$

**Proof.** — According to Lemma 4.2 we can assume that $\{F_\omega\}_{\omega \in D}$ is a family in $\mathcal{S}_n(D)$ for which there exists a proper closed subset $\Sigma' \subset D$ such that $F_\omega$ has an algebraically isolated zero at the origin for $\omega \in D \setminus \Sigma'$.

Taking $\mathcal{A} = \mathcal{A}(D)$ (an integral domain) we can follow the arguments of [12], Lemma 3.3 (in particular studying $\deg_0 F_\omega$ in the context of
Eisenbud and Levine Theorem). They imply that there exist a proper closed subset $\Sigma \subset D$ such that $\Sigma' \subset \Sigma$, and a symmetric matrix $T$ whose entries belong to $A(D)$ such that for every $\omega \in D \setminus \Sigma$ the matrix $T(\omega)$ is non-degenerate and $\deg_0 F_\omega = \text{signature } T(\omega)$. Let $\tilde{q}_1, \ldots, \tilde{q}_t \in A(D)$ be the elements of the diagonal of $T$ after making $T$ diagonal by a change of variables over the rational fractions on $A(D)$ and multiplying by the squares of the denominators of the entries. Then, if we enlarge $\Sigma$ in such a way, that the zeros of the denominators belong to $\Sigma$, and take $q_i \in A(\Omega)$ such that $\tilde{q}_i$ is the residue class of $q_i$, $i = 1, \ldots, t$, we have

$$\deg_0 F_\omega = \text{sgn } q_1(\omega) + \ldots + \text{sgn } q_t(\omega)$$

for $\omega \in D \setminus \Sigma$.

Lemma 4.4. — Assume that $\tilde{\Omega} \subset \Omega$ is a closed subset and $0 \in \mathbb{R}^n$ is isolated in $F_\omega^{-1}(0)$ for $\omega \in \tilde{\Omega}$. Then there exist $v_1, \ldots, v_s \in A(\Omega)$ and a proper closed subset $\Sigma \subset \tilde{\Omega}$ such that for $\omega \in \tilde{\Omega} \setminus \Sigma$ we have

$$\deg_0 F_\omega = \text{sgn } v_1(\omega) + \ldots + \text{sgn } v_s(\omega).$$

Proof. — Induction on the number of irreducible components of $\tilde{\Omega}$.

If $\tilde{\Omega}$ is irreducible then Lemma 4.3 implies the result.

Assume that $\tilde{\Omega} = D_1 \cup D_2 \cup \ldots \cup D_m$ is a decomposition of $\tilde{\Omega}$ into irreducible components. Denote $\Omega' = D_2 \cup \ldots \cup D_m$. Let $h_1 \in A(\Omega)$, $h_2 \in A(\Omega)$ be non-negative and such that

$$h_1 \equiv 0 \text{ on } D_1, \ h_1 \not\equiv 0 \text{ on } \Omega',$$

$$h_2 \equiv 0 \text{ on } \Omega', \ h_2 \not\equiv 0 \text{ on } D_1.$$  

According to Lemma 4.3 and the inductive assumption, there exist $q_1, \ldots, q_t, p_1, \ldots, p_{t'} \in A(\Omega)$ and proper closed subsets $\Sigma_1 \subset D_1$, $\Sigma_2 \subset \Omega'$ such that for $\omega \in D_1 \setminus \Sigma_1$ we have

$$\deg_0 F_\omega = \text{sgn } q_1(\omega) + \ldots + \text{sgn } q_t(\omega)$$

and for $\omega \in \Omega' \setminus \Sigma_2$ we have

$$\deg_0 F_\omega = \text{sgn } p_1(\omega) + \ldots + \text{sgn } p_{t'}(\omega).$$

Let $\Sigma = \Sigma_1 \cup \Sigma_2 \cup (h_1^{-1}(0) \cap \Omega') \cup (h_2^{-1}(0) \cap D_1)$, then

$$\deg_0 F_\omega = \sum_{i=1}^t \text{sgn } h_2(\omega)q_i(\omega) + \sum_{j=1}^{t'} \text{sgn } h_1(\omega)p_j(\omega)$$

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for $\omega \in \tilde{\Omega} \setminus \Sigma$. We take $s = t + t'$, $v_i(\omega) = h_2(\omega)q_i(\omega)$ for $i = 1, \ldots t$ and $v_i(\omega) = h_1(\omega)p_{t-i}(\omega)$ for $i = t + 1, \ldots s$.

We will use below the following fact (see [12]):

Let $h \in A(\Omega)$ be non-negative and such that $h(0) \cap \Omega = \Sigma$ (such $h$ exists because $\Omega$ is a Noetherian space). Then

$$\sum \text{sgn } h(\omega)v_i(\omega) = \sum \text{sgn } v_i(\omega)$$

for $\omega \in \Omega \setminus \Sigma$ and

$$\sum \text{sgn } h(\omega)v_i(\omega) = 0$$

for $\omega \in \Sigma$.

Similarly, let $p_1, \ldots, p_r \in A(\Omega)$, then

$$\sum \text{sgn } p_j(\omega) + \sum \text{sgn } (-h(\omega)p_j(\omega)) = 0$$

for $\omega \in \Omega \setminus \Sigma$ and

$$\sum \text{sgn } p_j(\omega) + \sum \text{sgn } (-h(\omega)p_j(\omega)) = \sum \text{sgn } p_j(\omega)$$

for $\omega \in \Sigma$.

So we have

$$\sum \text{sgn } h(\omega)v_i(\omega) + \sum \text{sgn } p_j(\omega) + \sum \text{sgn } (-h(\omega)p_j(\omega))$$

$$= \begin{cases} \sum \text{sgn } v_i(\omega), & \omega \in \Omega \setminus \Sigma \\ \sum \text{sgn } p_j(\omega), & \omega \in \Sigma. \end{cases}$$

**Theorem 4.5.** — Let $\{F_\omega\}_{\omega \in \Omega} \in S_n(\Omega)$ and let $0 \in \mathbb{R}^n$ be isolated in $F_\omega^{-1}(0)$ for each $\omega \in \Omega$. Then there exist $v_1, \ldots, v_s \in A(\Omega)$ such that for $\omega \in \Omega$

$$\deg_0 F_\omega = \text{sgn } v_1(\omega) + \ldots + \text{sgn } v_s(\omega).$$

**Proof.** — According to Lemma 4.4 there exist a proper closed subset $\Sigma_1 \subset \Omega$ and $u_1, \ldots, u_{s(1)} \in A(\Omega)$ such that for $\omega \in \Omega \setminus \Sigma_1$

$$\deg_0 F_\omega = \text{sgn } u_1(\omega) + \ldots + \text{sgn } u_{s(1)}(\omega).$$

Let $\Omega_1 = \Sigma_1$; using Lemma 4.4 again, we obtain $\Sigma_2 \subset \Sigma_1$ and $w_1, \ldots, w_{s(2)} \in A(\Omega)$ such that for $\omega \in \Omega_1 \setminus \Sigma_2$

$$\deg_0 F_\omega = \text{sgn } w_1(\omega) + \ldots + \text{sgn } w_{s(2)}(\omega).$$

Continuing this construction we obtain a descending family of proper closed subsets

$$\Omega \supset \Sigma_1 \supset \Sigma_2 \supset \ldots$$
Ω is a Noetherian space, so this family has to be finite and for some positive integer \( k \) we have \( \Sigma_k = \emptyset \).

Now we apply the above fact and the proof is complete. \( \square \)

Let us recall that if \( f \in \mathcal{A}(\Omega) \) then we denote \( \tilde{f} = \sum_{\alpha} \frac{1}{\alpha!} D^\alpha f x^\alpha \in \mathcal{A}(\Omega)_c[[x]] \), and if \( h = \sum_{\alpha} h_{\alpha} x^\alpha \in \mathcal{A}(\Omega)[[x]] \) then we denote \( h_\omega = \sum_{\alpha} h_{\alpha}(\omega) x^\alpha \).

Let \( F \subset \mathcal{A}(\Omega) \). For each \( \omega \in \Omega \) let \( I_\omega \subset \mathbb{R}\{x\} = \mathbb{R}\{x_1, \ldots, x_n\} \) denote the ideal generated by \( \{\tilde{f}_\omega \mid f \in F\} \), and let \( X_\omega \) denote a representative of \( V_0(I_\omega) \). We will show, that there exist \( v_1, v_2, \ldots, v_s \in \mathcal{A}(\Omega) \) such that

\[
\forall \omega \in \Omega \exists \varepsilon_\omega > 0 \forall \varepsilon < \varepsilon_\omega \frac{1}{2} \chi(S^{n-1}_\varepsilon \cap X_\omega) = \sum_{i=1}^s \text{sgn} v_i(\omega),
\]

where \( S^{n-1}_\varepsilon = \{x \in \mathbb{R}^n \mid ||x|| = \varepsilon\} \) and \( \chi(A) \) is the Euler characteristic of the set \( A \).

**Lemma 4.6.** — There exist \( h_1, h_2, \ldots, h_q \in \mathcal{A}(\Omega)_c[[x]] \) such that for \( \omega \in \Omega \)

\[
X_\omega = V_0(h_1, \omega, \ldots, h_q, \omega).
\]

**Proof.** — Denote by \( I \) the ideal in \( \mathcal{A}(\Omega)_c[[x]] \) generated by the set \( \{\tilde{h} \mid h \in F\} \). Theorem 1.1 implies that there is an ideal \( I' = (h_1, \ldots, h_q) \subset \mathcal{A}(\Omega)_c[[x]] \) generated by finitely many elements such that

\[
\forall \omega \in \Omega \quad I_\omega = I'_\omega,
\]

where \( I'_\omega = (h_{1, \omega}, \ldots, h_{q, \omega}) \). We have

\[
X_\omega = V_0(I_\omega) = V_0(I'_\omega) = V_0(h_{1, \omega}, \ldots, h_{q, \omega}).
\]

\( \square \)

**Remark.** — Since \( \mathcal{A}(\Omega) \) is Noetherian, this lemma is clear for \( \mathcal{A}(\Omega) \), but it is valid for any \( \Omega \)-Noetherian algebra instead of \( \mathcal{A}(\Omega) \).

**Corollary 4.7.** — There exists \( h = h_1^2 + \ldots + h_q^2 \in \mathcal{A}(\Omega)_c[[x]] \) such that \( X_\omega = V_0(h_\omega) \) for each \( \omega \in \Omega \).

Now we will show that for any \( h \in \mathcal{A}(\Omega)_c[[x]] \) such that \( h(0) = 0 \) there exists such \( k > 0 \) that for all \( \omega \in \Omega \) there exists \( \epsilon_\omega > 0 \) such that

\[
g_\omega(x) = h_\omega(x) - (x_1^2 + \ldots + x_n^2)^k
\]
has an isolated critical point at the origin and for $0 < \epsilon < \epsilon_\omega$
\[ \chi(S_{\epsilon}^{n-1} \cap \{ h_{\omega} \leq 0 \}) = 1 - \deg_0 \nabla g_{\omega}, \]
where $\nabla g_{\omega} = \left( \frac{\partial g_{\omega}}{\partial x_1}, \ldots, \frac{\partial g_{\omega}}{\partial x_n} \right) : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$.

We will strictly follow the proof of [14], Theorem 1.

Denote $r(x) = x_1^2 + \ldots + x_n^2$. Assume that $h_{\omega}$, $r$ are the representatives of germs defined on an open neighbourhood $U$ of the origin. Define
\[ V_{\omega} = \{(x, \epsilon, y) \in U \times \mathbb{R} \times \mathbb{R} \mid r(x) = \epsilon^2, \text{rank}(d r(x), d h_{\omega}(x)) \leq 1, y = h_{\omega}(x) \}. \]
Let $\pi : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ be the projection. $V_{\omega}$ is an analytic set and $\pi : V_{\omega} \rightarrow \pi(V_{\omega})$ is proper in some neighbourhood of the origin. Hence $\pi(V_{\omega})$ is closed and subanalytic in some neighbourhood of the origin.

Denote $Y_1 = \mathbb{R} \times \{0\}$, $Y_{2\omega} = \pi(V_{\omega}) \setminus Y_1$. Then $Y_{2\omega}$ is subanalytic. If $\epsilon \neq 0$ then
\[ \pi(V_{\omega}) \cap \{\epsilon\} \times \mathbb{R} = \{\epsilon\} \times \{\text{the set of critical values of } h_{\omega}|_{S_{\epsilon}^{n-1}}\}. \]
Since $h_{\omega}$ is analytic, $\pi(V_{\omega}) \cap \{\epsilon\} \times \mathbb{R}$ is finite. Hence $\dim \pi(V_{\omega}) = \dim Y_{2\omega} = 1$, and then 0 is an isolated point of $Y_1 \cap Y_{2\omega}$.

According to Corollary 3.2, there exists a constant $\alpha > 0$ such that for each $(\epsilon, y) \in Y_{2\omega}$ such that $\epsilon < \epsilon_\omega$ and $y$ is sufficiently close to the origin. Let $k > \alpha$ be an integer. Define $g_{\omega}(x) = h_{\omega}(x) - r^k(x)$.

Set
\[ V'_{\omega} = \{(x, \epsilon, y) \in U \times \mathbb{R} \times \mathbb{R} \mid r(x) = \epsilon^2, \text{rank}(d r(x), d g_{\omega}(x)) \leq 1, y = g_{\omega}(x) \}. \]
Because $\text{rank}(d r(x), d g_{\omega}(x)) = \text{rank}(d r(x), d h_{\omega}(x))$, \[ V'_{\omega} = \{(x, \epsilon, y) \in U \times \mathbb{R} \times \mathbb{R} \mid r(x) = \epsilon^2, \text{rank}(d r(x), d h_{\omega}(x)) \leq 1, y = h_{\omega}(x) - \epsilon^{2k}\}. \]
Define $G(\epsilon, y) = (\epsilon, y - \epsilon^{2k})$. Then $\pi(V'_{\omega}) = G(\pi(V_{\omega}))$, so we have
\[ \pi(V'_{\omega}) \cap \mathbb{R} \times \{0\} = \{(0, 0)\} \]
in some neighbourhood of the origin. Hence, if $\epsilon \neq 0$ is sufficiently close to the origin, 0 is a regular value of $g_{\omega}|_{S_{\epsilon}^{n-1}}$ and then $g_{\omega}$ has an isolated critical point at the origin.

According to [14], Lemma 1, we have
\[ \chi(S_{\epsilon}^{n-1} \cap \{ h_{\omega} \leq 0 \}) = 1 - \deg_0 \nabla g_{\omega}. \]
Hence, applying Theorem 4.5 and the fact, that for \( \omega \in \Omega \) and sufficiently small \( \epsilon > 0 \) if \( h(0) \neq 0 \) then \( \chi(S_{\epsilon}^{n-1} \cap \{ h_\omega \leq 0 \}) \) is equal to 0 or 2, we obtain:

**Theorem 4.8.** — If \( f \in \mathcal{A}(\Omega) \) then there exist \( v_1, v_2, \ldots, v_s \in \mathcal{A}(\Omega) \) such that for \( \omega \in \Omega \) there exists \( 0 < \epsilon_{\omega} \ll 1 \) such that for \( 0 < \epsilon < \epsilon_{\omega} \)

\[
\chi(S_{\omega,\epsilon}^{n-1} \cap \{ f \leq 0 \}) = \sum_{i=1}^{s} \text{sgn} v_i(\omega),
\]

where \( S_{\omega,\epsilon}^{n-1} \) denotes a sphere in \( \mathbb{R}^n \) centered at \( \omega \) with the radius \( \epsilon \).

**Lemma 4.9.** — If \( f \in \mathcal{A}(\Omega) \) then there exist \( h_1, h_2, \ldots, h_s \in \mathcal{A}(\Omega) \) such that for \( \omega \in \Omega \) there exists \( 0 < \epsilon_{\omega} \ll 1 \) such that for \( 0 < \epsilon < \epsilon_{\omega} \)

\[
\frac{1}{2} (\chi(S_{\epsilon}^{n-1} \cap \{ f \leq 0 \}) \pm \chi(S_{\epsilon}^{n-1} \cap \{ f \geq 0 \})) = \sum_{i=1}^{s} \text{sgn} h_i(\omega).
\]

**Proof.** — Let define \( g(\omega, t) = tf(\omega) \), where \( \omega \) belongs to some neighbourhood of \( \Omega \), \( t \in [-1;1] \). The set \( \Omega \times [-1;1] \) is compact and semianalytic, so \( g \in \mathcal{A}(\Omega \times [-1,1]) \).

Then \( g \geq 0 \) if \( f \geq 0 \) and \( t \geq 0 \) or if \( f \leq 0 \) and \( t \leq 0 \). Hence for \( t > 0 \)

\[
\chi(S_{\epsilon}^{n-1} \cap \{ f \geq 0 \}) = 2 - \chi(S_{\epsilon}^{n-1} \cap \{ f \leq 0 \})
\]

and

\[
\chi(S_{\epsilon}^{n-1} \cap \{ f \leq 0 \}) = 2 - \chi(S_{\epsilon}^{n-1} \cap \{ g \geq 0 \})
\]

for \( \epsilon \) sufficiently small.

According to Theorem 4.8 there exist \( g_1, g_2, \ldots, g_s \in \mathcal{A}(\Omega \times [-1;1]) \) such that

\[
\forall (\omega, t) \in \Omega \times [-1;1] \exists 0 < \epsilon(\omega, t) \ll 1 \forall 0 < \epsilon < \epsilon(\omega, t) \chi(S_{\epsilon}^{n} \cap \{ g \geq 0 \}) = \sum_{i=1}^{s} \text{sgn} g_i(\omega, t).
\]

For \( 0 < \epsilon < \epsilon(\omega, t) \) we obtain

\[
\frac{1}{2} (\chi(\{ f \geq 0 \} \cap S_{\epsilon}^{n-1}) - \chi(\{ f \leq 0 \} \cap S_{\epsilon}^{n-1}))
\]

\[
= \frac{1}{2} \lim_{t \to 0^+} (2 - \chi(S_{(\omega, t), \epsilon}^{n} \cap \{ g \geq 0 \}) - 2 + \chi(S_{(\omega, -t), \epsilon}^{n} \cap \{ g \geq 0 \}))
\]

\[
= \frac{1}{2} \lim_{t \to 0^+} (\chi(S_{(\omega, t), \epsilon}^{n} \cap \{ g \geq 0 \}) - \chi(S_{(\omega, t), \epsilon}^{n} \cap \{ g \geq 0 \}))
\]

\[
= \frac{1}{2} \lim_{t \to 0^+} \sum_{i=1}^{s} (\text{sgn} g_i(\omega, -t) - \text{sgn} g_i(\omega, t)).
\]
Let $\Omega = D_1 \cup \ldots \cup D_m$ be the decomposition into irreducible components. Fix $j$. We can assume, that $g_i \neq 0$ on $D_j \times [-1;1]$. For all $i = 1, 2, \ldots, s$ there exists $h_i \in \mathcal{A}(\Omega \times [-1;1])$ and a non–negative integer $k_i$ such that $g_i(\omega, t) = t^{k_i}h_i(\omega, t)$, and $h_i \neq 0$ on $D_j \times \{0\}$. Let $\Sigma := \{ \omega \in D_j \mid \exists i = 1, \ldots, s \ h_i(\omega, 0) = 0 \}$, then $\Sigma$ is proper and closed subset of $D_j$. For $\omega \in D_j \setminus \Sigma$

$$\frac{1}{2} \lim_{t \to 0^+} \sum_{i=1}^{s} (\text{sgn} \ g_i(\omega, -t) - \text{sgn} \ g_i(\omega, t)) = \sum_{i=1}^{s} \text{sgn} \ h_i'(\omega),$$

where $h_i'(\omega) = -h_i(\omega, 0)$ if $k_i$ is odd, and $h_i'(\omega) = 0$ if $k_i$ is even. Obviously $h_i' \in \mathcal{A}(\Omega)$.

In the other hand

$$\frac{1}{2} (\chi(\{ f \geq 0 \} \cap S_{\omega,\epsilon}^{n-1}) + \chi(\{ f \leq 0 \} \cap S_{\omega,\epsilon}^{n-1}))$$

$$= \frac{1}{2} \lim_{t \to 0^+} (2 - \chi(S_{(\omega,t),t}^{n} \cap \{ g \geq 0 \}) + 2 - \chi(S_{(\omega,-t),t}^{n} \cap \{ g \geq 0 \}))$$

$$= \frac{1}{2} \lim_{t \to 0^+} (4 - \chi(S_{(\omega,-t),t}^{n} \cap \{ g \geq 0 \}) - \chi(S_{(\omega,t),t}^{n} \cap \{ g \geq 0 \}))$$

$$= 2 - \frac{1}{2} \lim_{t \to 0^+} \sum_{i=1}^{s} (\text{sgn} \ g_i(\omega, -t) + \text{sgn} \ g_i(\omega, t)).$$

As above for $\omega \in D_j \setminus \Sigma$

$$\frac{1}{2} \lim_{t \to 0^+} \sum_{i=1}^{s} (\text{sgn} \ g_i(\omega, -t) + \text{sgn} \ g_i(\omega, t)) = \sum_{i=1}^{s} \text{sgn} \ h_i''(\omega),$$

where $h_i''(\omega) = h_i(\omega, 0)$ if $k_i$ is even, and $h_i''(\omega) = 0$ if $k_i$ is odd.

We have proven that $\frac{1}{2} (\chi(\{ f \geq 0 \} \cap S_{\omega,\epsilon}^{n-1}) \pm \chi(\{ f \leq 0 \} \cap S_{\omega,\epsilon}^{n-1}))$ is a sum of signs of analytic functions on $D_j \setminus \Sigma$. As in proofs of Lemma 4.4 and Theorem 4.5, proceeding by induction we can complete the proof. □

**Corollary 4.10.** — If $f \in \mathcal{A}(\Omega)$ then there exist $g_1, g_2, \ldots, g_q \in \mathcal{A}(\Omega)$ such that for $\omega \in \Omega$ there exists $0 < \epsilon_\omega \ll 1$ such that for each $0 < \epsilon < \epsilon_\omega$

$$\frac{1}{2} \chi(S_{\omega,\epsilon}^{n-1} \cap V_0(f)) = \frac{1}{2} \chi(S_{\epsilon}^{n-1} \cap V_0(\tilde{f}_\omega)) = \sum_{i=1}^{q} \text{sgn} \ g_i(\omega).$$

**Proof.** — We have

$$\chi(S_{\epsilon}^{n-1} \cap V_0(\tilde{f}_\omega)) = \chi(S_{\epsilon}^{n-1} \cap \{ \tilde{f}_\omega \leq 0 \}) + \chi(S_{\epsilon}^{n-1} \cap \{ \tilde{f}_\omega \geq 0 \}) - \chi(S_{\epsilon}^{n-1}),$$

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so according to Lemma 4.9

\[
\frac{1}{2} \chi(S_n^{-1} \cap V_0(\tilde{f}_\omega)) = \sum_{i=1}^{s} h_i(\omega) - \frac{1 + (-1)^{n-1}}{2}.
\]

\[\square\]

Corollary 4.7 and Corollary 4.10 imply:

**Theorem 4.11.** — There exist \(v_1, v_2, \ldots, v_q \in A(\Omega)\) such that

\[
\forall \omega \in \Omega \exists 0 < \epsilon_\omega < 1 \forall 0 < \epsilon < \epsilon_\omega \frac{1}{2} \chi(S_n^{-1} \cap X_\omega) = \sum_{i=1}^{q} \text{sgn} \ v_i(\omega).
\]

**Remark.** — Following the proof of the Lemma 4.9 one can check that this result is true also if instead of \(A(\Omega)\) we take any \(\Omega\)-Noetherian algebra \(O(\Omega)\) (\(\Omega\) is a locally closed subset of \(\mathbb{R}^n\)) such that:

1) there exists a subset \(I \subset \mathbb{R}\) containing a neighbourhood of 0 such that \(O(\Omega \times I)\) is \(\Omega \times I\)-Noetherian and there is a natural inclusion \(O(\Omega) \subset O(\Omega \times I)\).

2) For \(g \in O(\Omega \times I)\) and an irreducible component \(D\) of \(\Omega\) if \(g \not\equiv 0\) on \(D \times I\) then there exist \(h \in O(\Omega \times I)\) and a non-negative integer \(k\) that \(g(\omega, t) = t^k h(\omega, t)\) for \(\omega \in D\) and \(t\) sufficiently close to 0, \(h(\cdot, 0) \in O(\Omega)\), and \(h \not\equiv 0\) on \(D \times \{0\}\).

The algebra of Nash functions on an open semialgebraic set \(\Omega \subset \mathbb{R}^n\) satisfies these assumptions.

For the algebra \(\mathbb{R}[x][f_1, \ldots, f_q]\) defined in the Introduction we can define the algebra \(\mathbb{R}[x, t][F_1, \ldots, F_q]\), where \(F_i : \mathbb{R}^n \times [-1; 1] \rightarrow \mathbb{R}\), \(F_i(x, t) = f_i(x)\). It is \(\mathbb{R}^n \times [-1; 1]\)-Noetherian and \(F_1, \ldots, F_q\) do not depend on the last variable, so it has the property 2).

## 5. Sums of signs of real analytic functions.

Let \(Y \subset \mathbb{R}^n\) be a real compact semianalytic set. Suppose that a function \(\phi : Y \rightarrow \mathbb{Z}\) admits a presentation as a finite sum

\[
\phi = \sum_i m_i \mathbf{1}_{Y_i},
\]

where the \(m_i\)'s are integers, the \(Y_i\)'s are semianalytic subsets of \(Y\) and where \(\mathbf{1}_{Y_i}\) denotes the characteristic function of the subset \(Y_i\).
We can choose $Y_i$ such that they are compact semianalytic subsets of $Y$. Following [9] and [2] we define the Euler integral, the link of $\phi$, and the duality operator $D$ on $\phi$:

$$\int_Y \phi = \sum_i m_i \chi(Y_i),$$

$$\Lambda \phi(y) = \int_Y \phi 1_{S_{y,\epsilon}^{n-1}},$$

where $\epsilon$ is sufficiently small,

$$D \phi(y) = \phi(y) - \Lambda \phi(y).$$

Let $\Omega$, as above, be a compact semianalytic subset of $\mathbb{R}^n$. We will say, that a function $g : \Omega \rightarrow \mathbb{Z}$ is a sum of signs of analytic functions if there exist $v_1, v_2, \ldots, v_s \in A(\Omega)$ such that $g(\omega) = \sum_{i=1}^s \text{sgn} \ v_i(\omega)$. Then in fact $g$ is defined on a compact semianalytic neighbourhood $Y$ of $\Omega$. In that case, for $\omega \in \text{int} \ Y \supset \Omega$ we have:

$$\Lambda g(\omega) = \int_Y g 1_{S_{\omega,\epsilon}^{n-1}} = \int_{S_{\omega,\epsilon}^{n-1}} g = \sum_{i=1}^s \left( \chi(A_i \cap S_{\omega,\epsilon}^{n,k-1}) - \chi(B_i \cap S_{\omega,\epsilon}^{n,k-1}) \right)$$

where $A_i = \{ v_i \geq 0 \}$, $B_i = \{ v_i \leq 0 \}$, $\epsilon$ is sufficiently small.

Using Theorem 4.11, Lemma 4.9, and arguments like in [12], Corollary 6.3 and Theorem 6.4, we can show similar results as the main result of [4].

Suppose that $f$ is an analytic function defined in a neighbourhood of $\Omega$. Then $X = f^{-1}(0)$ is an analytic set defined in a neighbourhood of $\Omega$. According to Theorem 4.11, there exist $v_1, v_2, \ldots, v_q \in A(\Omega)$ such that for each $\omega \in \Omega$ there exists $0 < \epsilon_\omega \ll 1$ such that for each $0 < \epsilon < \epsilon_\omega$, $\frac{1}{2} \chi(S_{\epsilon}^{n-1} \cap X_\omega) = \sum_{i=1}^q \text{sgn} \ v_i(\omega)$. Let $\Omega = \Omega_1 \cup \ldots \Omega_m$ be a decomposition into irreducible components. Assume that $v_i$ does not vanish identically on $\Omega_1$ for $i = 1, \ldots, l \leq q$. Taking $v = v_1 v_2 \ldots v_l$ and $\Sigma = \{ \omega \in \Omega_1 \mid v(\omega) = 0 \} \cup \bigcup_{i=2}^m \Omega_i$ we obtain:

**Corollary 5.1.** — There exist a proper closed subset $\Sigma \subset \Omega$, an integer $\mu = l - 1$, and an analytic function $v \in A(\Omega)$, such that $v$ does not vanish on $\Omega \setminus \Sigma$ and

$$\forall \omega \in \Omega \setminus \Sigma \exists 0 < \epsilon_\omega \ll 1 \forall 0 < \epsilon < \epsilon_\omega \frac{1}{2} \chi(S_{\epsilon}^{n-1} \cap X_\omega) = \mu + \text{sgn} \ v(\omega) \ (\text{mod} \ 4).$$

In particular, for such $\omega$, $\frac{1}{2} \chi(S_{\epsilon}^{n-1} \cap X_\omega) = \mu + 1 \ (\text{mod} \ 2)$.

**Theorem 5.2.** — If $g : \Omega \rightarrow \mathbb{Z}$ is a sum of signs of analytic functions $v_1, v_2, \ldots, v_s \in A(\Omega)$ (in particular if $g(\omega) = \frac{1}{2} \chi(S_{\epsilon}^{n-1} \cap X_\omega)$),
then the function $\frac{1}{2} \Lambda g$, as well as $\frac{1}{2} (g + D g)$, is integer–valued and it is a sum of signs of analytic functions.

**Proof.** — We have

$$
\Lambda g(\omega) = \sum_{i=1}^{s} \left( \chi(\{v_i(\omega) \geq 0\} \cap S_{\omega, \epsilon}^{n-1}) - \chi(\{v_i(\omega) \leq 0\} \cap S_{\omega, \epsilon}^{n-1}) \right)
$$

for $\epsilon$ sufficiently small, so the theorem is implied by Lemma 4.9.

So, proceeding the same way as McCrory and Parusiński in [10] one may get a large family of topological invariants associated with $\Omega \subset X$.

**BIBLIOGRAPHY**


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Aleksandra NOWEL,
University of Gdańsk
Institute of Mathematics
Wita Stwosza 57
80–952 Gdańsk (Poland)
Aleksandra.Nowel@math.univ.gda.pl