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## A REMARK ON WHITTAKER FUNCTIONS ON $SL(n, \mathbb{R})$

by Taku ISHII

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### 1. Preliminaries.

We first recall the definition and some basic facts on class one Whittaker functions on  $G = SL(n, \mathbb{R})$ . Let  $N$  be the subgroup of  $G$  consisting of upper triangular unipotent matrices. Fix a nondegenerate unitary character  $\eta$  of  $N$  by

$$\eta(n) = \exp\left(2\pi\sqrt{-1} \sum_{i=1}^{n-1} n_{i,i+1}\right)$$

for  $n = (n_{i,j}) \in N$ . Also let  $A = \{\text{diag}(a_1, \dots, a_n) \mid a_i > 0, \prod_{i=1}^n a_i = 1\}$  and  $K = SO(n)$ . Then the Iwasawa decomposition  $G = NAK$  holds. We denote by  $\mathfrak{g}, \mathfrak{n}, \mathfrak{a}$  and  $\mathfrak{k}$  the Lie algebras of  $G, N, A$  and  $K$  respectively.

Let  $P_0 = MAN$  be the standard minimal parabolic subgroup of  $G$  with  $M = Z_K(A)$ . For a linear form  $\nu = (\nu_1, \dots, \nu_n) \in \mathfrak{a}_{\mathbb{C}}^* = \text{Hom}_{\mathbb{R}}(\mathfrak{a}, \mathbb{C}) \cong \mathbb{C}^{n-1}$  ( $\sum_{i=1}^n \nu_i = 0$ ), define a character  $e^\nu$  of  $A$  by  $e^\nu(\text{diag}(a_1, \dots, a_n)) = \exp(\nu_1(\log a_1) + \dots + \nu_n(\log a_n))$ . We call the induced representation

$$\pi_\nu = L^2 - \text{Ind}_{P_0}^G(1_M \otimes e^{\nu+\rho} \otimes 1_N)$$

the *class one principal series representation* of  $G$ . Here  $\rho = (n-1, n-2, \dots, 0)$  is the half sum of positive roots.

Let  $U(\mathfrak{g}_{\mathbb{C}})$  and  $U(\mathfrak{a}_{\mathbb{C}})$  be the universal enveloping algebras of the complexifications of  $\mathfrak{g}$  and  $\mathfrak{a}$  respectively. Let  $p$  be the projection  $U(\mathfrak{g}_{\mathbb{C}}) \rightarrow U(\mathfrak{a}_{\mathbb{C}})$

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corresponding to the decomposition  $U(\mathfrak{g}_{\mathbb{C}}) = U(\mathfrak{a}_{\mathbb{C}}) \oplus (\mathfrak{n}U(\mathfrak{g}_{\mathbb{C}}) + U(\mathfrak{g}_{\mathbb{C}})\mathfrak{k})$ . Set

$$U(\mathfrak{g}_{\mathbb{C}})^K = \{X \in U(\mathfrak{g}_{\mathbb{C}}) \mid \text{Ad}(k)X = X \text{ for all } k \in K\}.$$

Define the automorphism  $\gamma$  of  $U(\mathfrak{a}_{\mathbb{C}})$  by  $\gamma(H) = H + \rho(H)$  for  $H \in \mathfrak{a}_{\mathbb{C}}$ . Then  $\gamma \circ p$  induces an algebra homomorphism from  $U(\mathfrak{g}_{\mathbb{C}})^K$  to  $U(\mathfrak{a}_{\mathbb{C}})$  with the kernel  $U(\mathfrak{g}_{\mathbb{C}})^K \cap U(\mathfrak{g}_{\mathbb{C}})\mathfrak{k}$ . For  $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ , define the algebra homomorphism  $\chi_{\nu} : U(\mathfrak{g}_{\mathbb{C}})^K \rightarrow \mathbb{C}$  by

$$\chi_{\nu}(z) = \nu(\gamma \circ p(z))$$

for  $z \in U(\mathfrak{g}_{\mathbb{C}})^K$ . Note that  $\chi_{\nu}$  is trivial on  $U(\mathfrak{g}_{\mathbb{C}})^K \cap U(\mathfrak{g}_{\mathbb{C}})\mathfrak{k}$  and the restriction of  $\chi_{\nu}$  to the center  $Z(\mathfrak{g}_{\mathbb{C}})$  of  $U(\mathfrak{g}_{\mathbb{C}})$  coincides with the infinitesimal character of the class one principal series representation  $\pi_{\nu}$ .

DEFINITION 1. — *A smooth function  $f$  on  $G$  is called class one Whittaker function if*

- (i)  $f(n g k) = \eta(n) f(g)$  for all  $n \in N, g \in G$  and  $k \in K$ ,
- (ii)  $z f = \chi_{\nu}(z) f$  for all  $z \in U(\mathfrak{g}_{\mathbb{C}})^K$ .

We denote by  $\text{Wh}(\nu)$  the space of class one Whittaker functions.

It is well known ([5]) that for almost all  $\nu$ , the dimension of  $\text{Wh}(\nu)$  is equal to the order of the Weyl group  $W \cong \mathfrak{S}_n$  of  $G$ . Hashizume ([3]) constructed the basis function  $M_n(\nu; g)$  of  $\text{Wh}(\nu)$ , which will be explained below. Shalika ([7]) proved the uniqueness of the element in  $\text{Wh}(\nu)$  which has the moderate growth property and contributes to the Fourier expansions of automorphic forms. Further, this unique element can be written as Jacquet integral ([4]):

$$W_n(\nu; g) = \int_N a(s_0^{-1} n g)^{\nu + \rho} \eta(n)^{-1} dn.$$

Here  $s_0$  is the longest element in the Weyl group  $W$  and  $g = n(g)a(g)k(g)$  the Iwasawa decomposition of  $g \in G$ . In case of  $G = \text{SL}(n, \mathbb{R})$ , Stade ([9]) found a remarkable recursive integral formula of  $W_n(\nu; g)$  involving  $K$ -Bessel functions by evaluating the above integral and applied it for the computation of the gamma factors of certain automorphic  $L$ -functions.

Since  $W_n(\nu; g)$  can be explicitly written by the linear combination of  $M_n(\nu; g)$ 's ([3, Theorem 7.8]), the basis function is related to the theory of automorphic forms. On the other hand, one of the possible direct applications of the explicit formula of  $M_n(\nu; g)$  is the construction of Poincaré series (cf. [6], [8]).

Let us review the construction of  $M_n(\nu; g)$ . For  $\mathbf{k} = (k_1, \dots, k_{n-1}) \in (\mathbb{Z}_{\geq 0})^{n-1}$ , define the function  $G_{n,\mathbf{k}}(\nu)$  by the following recurrence relation.

$$(1) \quad \begin{aligned} &G_{n,(0,\dots,0)}(\nu) = 1, \\ &\left[ \sum_{i=1}^{n-1} k_i^2 - \sum_{i=1}^{n-2} k_i k_{i+1} + \frac{1}{2} \sum_{i=1}^{n-1} (\nu_i - \nu_{i+1}) k_i \right] G_{n,\mathbf{k}}(\nu) = \sum_{i=1}^{n-1} G_{n,\mathbf{k}-\mathbf{e}_i}(\nu). \end{aligned}$$

Here  $\mathbf{e}_i = (0, \dots, 1, \dots, 0)$  (1 is at  $i$ -th entry) and  $G_{n,\mathbf{k}}(\nu) = 0$  if  $k_i < 0$  for some  $i$ . Let

$$\begin{aligned} \mathfrak{a}_{\mathbb{C}}^* = \left\{ \nu \in \mathfrak{a}_{\mathbb{C}}^* \mid \sum_{i=1}^{n-1} k_i^2 - \sum_{i=1}^{n-2} k_i k_{i+1} + \frac{1}{2} \sum_{i=1}^{n-1} (\nu_i - \nu_{i+1}) k_i \neq 0 \right. \\ \left. \text{for all } \mathbf{k} \in (\mathbb{Z}_{\geq 0})^{n-1} \setminus \{0\} \right\}. \end{aligned}$$

Then if  $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ , we can determine  $G_{n,\mathbf{k}}(\nu)$  inductively and define a series on  $A$  by

$$M_n(\nu; a) = a^{\nu+\rho} \sum_{\mathbf{k} \in (\mathbb{Z}_{\geq 0})^{n-1}} G_{n,\mathbf{k}}(\nu) \left( \pi \frac{a_1}{a_2} \right)^{2k_1} \cdots \left( \pi \frac{a_{n-1}}{a_n} \right)^{2k_{n-1}}$$

( $a = \text{diag}(a_1, \dots, a_n) \in A$ ). Here  $a^{\nu+\rho} = \prod_{i=1}^n a_i^{\nu_i + (n-i)}$ . We extend it to a function on  $G$  by

$$M_n(\nu; g) = \eta(n(g)) M_n(\nu; a(g)).$$

**THEOREM 2 ([3]).** — *Let  $\Omega_n$  be the set of  $\nu \in \mathfrak{a}_{\mathbb{C}}^*$  satisfying the conditions*

- (i)  $s\nu \in \mathfrak{a}_{\mathbb{C}}^*$  for all  $s \in W$ ,
- (ii)  $s\nu - t\nu \notin (\mathbb{Z}_{\geq 0})^n$  for all distinct pair  $s, t$  in  $W$ . If  $\nu \in \Omega_n$ , then the set  $\{M_n(s\nu; g) \mid s \in W\}$  spans  $\text{Wh}(\nu)$ .

The explicit formulas of  $G_{n,\mathbf{k}}(\nu)$  are obtained by Bump ([2]) for  $n = 3$  and Stade ([10]) for  $n = 4$ . By using them, Stade ([11]) found integral formulas for  $M_3(\nu; a)$  and  $M_4(\nu; a)$ . He also conjectured a recursive integral formula for general  $n$ . The proof of Stade’s conjecture is the main result of this paper.

**THEOREM 3 (Stade’s Conjecture).** — *Let us use the coordinate  $y = (y_1, \dots, y_{n-1})$  with  $y_i = a_i/a_{i+1}$  for  $A$  and put*

$$(b) \quad \mu_i = \nu_{i+1} + \frac{\nu_1 + \nu_n}{n - 2}$$

for  $1 \leq i \leq n-2$  and  $\mu = (\mu_1, \dots, \mu_{n-2})$ . Formally define  $u_0 = 1/u_{n-1} = 0$ . If  $\nu \in \Omega_n$ ,  $\mu \in \Omega_{n-2}$  and  $\text{Re}((\nu_1 - \nu_n)/2 + 1) > 0$ , then

$$\begin{aligned}
 M_n(\nu; a) &= \pi \sum_{i=1}^{n-2} (-n+i)\nu_i \prod_{i=1}^{n-1} \Gamma\left(\frac{\nu_1 - \nu_{i+1}}{2} + 1\right) \Gamma\left(\frac{\nu_{i+1} - \nu_n}{2} + 1\right) \\
 &\cdot \prod_{i=1}^{n-1} y_i^{(n/2-i)(\nu_1+\nu_n)/(n-2)+(n-1)/2} \frac{1}{(2\pi\sqrt{-1})^{n-2}} \int_{|u_1|=1} \dots \\
 &\dots \int_{|u_{n-2}|=1} \prod_{i=1}^{n-1} I_{(\nu_1-\nu_n)/2}\left(2\pi y_i \sqrt{(1+u_{i-1})(1+1/u_i)}\right) \\
 &\cdot M_{n-2}\left(\mu; \left(y_2 \sqrt{u_1/u_2}, \dots, y_{n-2} \sqrt{u_{n-3}/u_{n-2}}\right)\right) \\
 &\cdot \prod_{i=1}^{n-2} u_i^{(-n+2i+1)/4-n(\nu_1+\nu_n)/4(n-2)} \frac{du_i}{u_i},
 \end{aligned}$$

(each integral is taken counterclockwise in the complex plane), with  $I_\nu$  the modified Bessel function.

### 2. Relation between $G_{n,k}(\nu)$ and $G_{n-2,k}(\nu)$ .

THEOREM 4. — Let  $\mathbf{m} = (m_1, \dots, m_{n-3}) \in (\mathbb{Z}_{\geq 0})^{n-3}$ . For  $\mathbf{k} = (k_1, \dots, k_{n-1}) \in (\mathbb{Z}_{\geq 0})^{n-1}$  set

$$S(\mathbf{k}) = \{\mathbf{m} \in (\mathbb{Z}_{\geq 0})^{n-3} \mid 0 \leq m_1 \leq k_2, \dots, 0 \leq m_{n-3} \leq k_{n-2}\}.$$

Then for  $\nu \in \Omega_n$  and  $\mu \in \Omega_{n-2}$  satisfying (4),

$$G_{n,\mathbf{k}}(\nu) = P_{n,\mathbf{k}}(\nu) \sum_{\mathbf{m} \in S(\mathbf{k})} G_{n-2,\mathbf{m}}(\mu) Q_{n,\mathbf{k},\mathbf{m}}(\nu),$$

where

$$P_{n,\mathbf{k}}(\nu) = \prod_{i=1}^{n-1} \frac{\left(\frac{\nu_1-\nu_n}{2} + 1\right)_{k_i+k_{i+1}}}{\left(\frac{\nu_1-\nu_n}{2} + 1\right)_{k_i} \left(\frac{\nu_1-\nu_{i+1}}{2} + 1\right)_{k_i} \left(\frac{\nu_{i+1}-\nu_n}{2} + 1\right)_{k_{i+1}}}$$

and

$$\begin{aligned}
 &Q_{n,\mathbf{k},\mathbf{m}}(\nu) \\
 &= \prod_{i=1}^{n-1} \frac{(-1)^{m_{i-1}} \left(\frac{-\nu_{i+1}+\nu_n}{2} - k_{i+1}\right)_{m_{i-1}} \left(\frac{-\nu_1+\nu_n}{2} - k_i\right)_{m_{i-1}} \left(\frac{-\nu_1+\nu_{i+1}}{2} - k_i\right)_{m_i}}{(k_i - m_{i-1})! \left(\frac{-\nu_1+\nu_n}{2} - k_i - k_{i+1}\right)_{m_{i-1}+m_i}}.
 \end{aligned}$$

Here  $(a)_n = \Gamma(a+n)/\Gamma(a)$ . Moreover, we set  $m_j = 0$  for  $j \neq 1, \dots, n-3$  and  $k_j = 0$  for  $j \neq 1, \dots, n-1$ .

*Proof.* — We have only to check that

$$F_{\mathbf{k}}(\nu) := P_{n,\mathbf{k}}(\nu) \sum_{\mathbf{m} \in S(\mathbf{k})} G_{n-2,\mathbf{m}}(\mu) Q_{n,\mathbf{k},\mathbf{m}}(\nu)$$

satisfies the recurrence relation (1). Let us compute  $F_{\mathbf{k}-\mathbf{e}_i}(\nu)$  for  $1 \leq i \leq n-1$ . Since

$$\frac{P_{n,\mathbf{k}-\mathbf{e}_i}(\nu)}{P_{n,\mathbf{k}}(\nu)} = \frac{\left(\frac{\nu_1-\nu_n}{2} + k_i\right)\left(\frac{\nu_1-\nu_{i+1}}{2} + k_i\right)\left(\frac{\nu_i-\nu_n}{2} + k_i\right)}{\left(\frac{\nu_1-\nu_n}{2} + k_{i-1} + k_i\right)\left(\frac{\nu_1-\nu_n}{2} + k_i + k_{i+1}\right)}$$

and

$$\begin{aligned} & \frac{Q_{n,\mathbf{k}-\mathbf{e}_i,\mathbf{m}}(\nu)}{Q_{n,\mathbf{k},\mathbf{m}}(\nu)} \\ &= \frac{(k_i - m_{i-1})\left(\frac{-\nu_1+\nu_n}{2} - k_{i-1} - k_i\right)\left(\frac{-\nu_1+\nu_n}{2} - k_i - k_{i+1}\right)}{\left(\frac{-\nu_1+\nu_n}{2} - k_{i-1} - k_i + m_{i-1} + m_i\right)\left(\frac{-\nu_1+\nu_n}{2} - k_i - k_{i+1} + m_{i-1} + m_i\right)} \\ & \cdot \frac{\left(\frac{-\nu_i+\nu_n}{2} - k_i + m_{i-2}\right)\left(\frac{-\nu_1+\nu_n}{2} - k_i + m_{i-1}\right)\left(\frac{-\nu_1+\nu_{i+1}}{2} - k_i + m_i\right)}{\left(\frac{-\nu_1+\nu_n}{2} - k_i\right)\left(\frac{-\nu_1+\nu_n}{2} - k_i\right)\left(\frac{-\nu_1+\nu_{i+1}}{2} - k_i\right)}, \end{aligned}$$

we have

$$\begin{aligned} F_{\mathbf{k}-\mathbf{e}_i}(\nu) &= P_{n,\mathbf{k}}(\nu) \sum_{\mathbf{m} \in S(\mathbf{k}-\mathbf{e}_i)} \\ & \cdot G_{n-2,\mathbf{m}}(\mu) Q_{n,\mathbf{k},\mathbf{m}}(\nu) \frac{(-1)(k_i - m_{i-1})\left(\frac{-\nu_i+\nu_n}{2} - k_i + m_{i-2}\right)}{\left(\frac{-\nu_1+\nu_n}{2} - k_{i-1} - k_i + m_{i-2} + m_{i-1}\right)} \\ & \cdot \frac{\left(\frac{-\nu_i+\nu_n}{2} - k_i + m_{i-1}\right)\left(\frac{-\nu_1+\nu_{i+1}}{2} - k_i + m_i\right)}{\left(\frac{-\nu_1+\nu_n}{2} - k_i - k_{i+1} + m_{i-1} + m_i\right)}. \end{aligned}$$

Suppose that  $2 \leq i \leq n-2$ . If we decompose this summation by means of

$$\begin{aligned} & \left(\frac{-\nu_i + \nu_n}{2} - k_i + m_{i-2}\right)\left(\frac{-\nu_1 + \nu_{i+1}}{2} - k_i + m_i\right) \\ &= \left(\frac{-\nu_1 + \nu_i}{2} - k_{i-1} + m_{i-1}\right)\left(\frac{-\nu_{i+1} + \nu_n}{2} - k_{i+1} + m_{i-1}\right) \\ &+ \left\{ \left(\frac{-\nu_i + \nu_n}{2} - k_i + m_{i-2}\right)\left(\frac{-\nu_1 + \nu_{i+1}}{2} - k_i + m_i\right) \right. \\ & \left. - \left(\frac{-\nu_1 + \nu_i}{2} - k_{i-1} + m_{i-1}\right)\left(\frac{-\nu_{i+1} + \nu_n}{2} - k_{i+1} + m_{i-1}\right) \right\}, \end{aligned}$$

$F_{\mathbf{k}-\mathbf{e}_i}(\nu)$  becomes the sum of following two terms:

$$\begin{aligned} & P_{n,\mathbf{k}}(\nu) \sum_{\mathbf{m} + \tilde{\mathbf{e}}_i \in S(\mathbf{k})} G_{n-2,\mathbf{m}}(\mu) Q_{n,\mathbf{k},\mathbf{m}}(\nu) \\ & \cdot \left(\frac{-\nu_1 + \nu_i}{2} - k_{i-1} + m_{i-1}\right)\left(\frac{-\nu_{i+1} + \nu_n}{2} - k_{i+1} + m_{i-1}\right) \end{aligned}$$

$$\begin{aligned}
 & \cdot \frac{(-1)(k_i - m_{i-1})\left(\frac{-\nu_1 + \nu_n}{2} - k_i + m_{i-1}\right)}{\left(\frac{-\nu_1 + \nu_n}{2} - k_{i-1} - k_i + m_{i-2} + m_{i-1}\right)\left(\frac{-\nu_1 + \nu_n}{2} - k_i - k_{i+1} + m_{i-1} + m_i\right)} \\
 = & P_{n,\mathbf{k}}(\nu) \sum_{\mathbf{m} \in S(\mathbf{k})} G_{n-2,\mathbf{m}-\tilde{\mathbf{e}}_i}(\mu) Q_{n,\mathbf{k},\mathbf{m}-\tilde{\mathbf{e}}_i}(\nu) \\
 & \cdot \left(\frac{-\nu_1 + \nu_i}{2} - k_{i-1} + m_{i-1} - 1\right) \left(\frac{-\nu_{i+1} + \nu_n}{2} - k_{i+1} + m_{i-1} - 1\right) \\
 & \cdot \frac{(-1)(k_i - m_{i-1} + 1)\left(\frac{-\nu_1 + \nu_n}{2} - k_i + m_{i-1} - 1\right)}{\left(\frac{-\nu_1 + \nu_n}{2} - k_{i-1} - k_i + m_{i-2} + m_{i-1} - 1\right)\left(\frac{-\nu_1 + \nu_n}{2} - k_i - k_{i+1} + m_{i-1} + m_i - 1\right)} \\
 = & P_{n,\mathbf{k}}(\nu) \sum_{\mathbf{m} \in S(\mathbf{k})} G_{n-2,\mathbf{m}-\tilde{\mathbf{e}}_i}(\mu) Q_{n,\mathbf{k},\mathbf{m}}(\nu)
 \end{aligned}$$

( $\tilde{\mathbf{e}}_i$  is  $(n - 3)$ -dimensional row vector with 1 at  $(i - 1)$ -th entry and 0 otherwise) and

$$\begin{aligned}
 & P_{n,\mathbf{k}}(\nu) \sum_{\mathbf{m} \in S(\mathbf{k}-\mathbf{e}_i)} G_{n-2,\mathbf{m}}(\mu) Q_{n,\mathbf{k},\mathbf{m}}(\nu) \\
 & \cdot \left\{ \left(\frac{-\nu_i + \nu_n}{2} - k_i + m_{i-2}\right) \left(\frac{-\nu_1 + \nu_{i+1}}{2} - k_i + m_i\right) \right. \\
 & \quad \left. - \left(\frac{-\nu_1 + \nu_i}{2} - k_{i-1} + m_{i-1}\right) \left(\frac{-\nu_{i+1} + \nu_n}{2} - k_{i+1} + m_{i-1}\right) \right\} \\
 & \cdot \frac{(-1)(k_i - m_{i-1})\left(\frac{-\nu_1 + \nu_n}{2} - k_i + m_{i-1}\right)}{\left(\frac{-\nu_1 + \nu_n}{2} - k_{i-1} - k_i + m_{i-2} + m_{i-1}\right)\left(\frac{-\nu_1 + \nu_n}{2} - k_i - k_{i+1} + m_{i-1} + m_i\right)} \\
 = & P_{n,\mathbf{k}}(\nu) \sum_{\mathbf{m} \in S(\mathbf{k})} G_{n-2,\mathbf{m}}(\mu) Q_{n,\mathbf{k},\mathbf{m}}(\nu) \\
 & \cdot \left\{ \left(\frac{-\nu_i + \nu_n}{2} - k_i + m_{i-2}\right) \left(\frac{-\nu_1 + \nu_{i+1}}{2} - k_i + m_i\right) \right. \\
 & \quad \left. - \left(\frac{-\nu_1 + \nu_i}{2} - k_{i-1} + m_{i-1}\right) \left(\frac{-\nu_{i+1} + \nu_n}{2} - k_{i+1} + m_{i-1}\right) \right\} \\
 & \cdot \frac{(-1)(k_i - m_{i-1})\left(\frac{-\nu_1 + \nu_n}{2} - k_i + m_{i-1}\right)}{\left(\frac{-\nu_1 + \nu_n}{2} - k_{i-1} - k_i + m_{i-2} + m_{i-1}\right)\left(\frac{-\nu_1 + \nu_n}{2} - k_i - k_{i+1} + m_{i-1} + m_i\right)}.
 \end{aligned}$$

Thus  $\sum_{i=1}^{n-1} F_{\mathbf{k}-\mathbf{e}_i}(\nu)$  is equal to

$$\begin{aligned}
 & P_{n,\mathbf{k}}(\nu) \sum_{\mathbf{m} \in S(\mathbf{k})} \left[ \sum_{i=2}^{n-2} G_{n-2,\mathbf{m}-\tilde{\mathbf{e}}_i}(\mu) Q_{n,\mathbf{k},\mathbf{m}}(\nu) + \sum_{i=1}^{n-1} G_{n-2,\mathbf{m}}(\mu) Q_{n,\mathbf{k},\mathbf{m}}(\nu) \right] \\
 & \cdot \left\{ \left(\frac{-\nu_i + \nu_n}{2} - k_i + m_{i-2}\right) \left(\frac{-\nu_1 + \nu_{i+1}}{2} - k_i + m_i\right) \right. \\
 & \quad \left. - \left(\frac{-\nu_1 + \nu_i}{2} - k_{i-1} + m_{i-1}\right) \left(\frac{-\nu_{i+1} + \nu_n}{2} - k_{i+1} + m_{i-1}\right) \right\}
 \end{aligned}$$

By using the recurrence relation of  $G_{n-2,\mathbf{m}}(\mu)$  for  $\sum_{i=2}^{n-2}$  and after some rearrangement for  $\sum_{i=1}^{n-1}$ , the sum  $\sum_{i=1}^{n-1} F_{\mathbf{k}-\mathbf{e}_i}(\nu)$  becomes

$$\begin{aligned}
 P_{n,\mathbf{k}}(\nu) & \sum_{\mathbf{m} \in S(\mathbf{k})} G_{n-2,\mathbf{m}}(\mu) Q_{n,\mathbf{k},\mathbf{m}}(\nu) \left[ \frac{(-1)(k_i - m_{i-1})(\frac{-\nu_1 + \nu_n}{2} - k_i + m_{i-1})}{(\frac{-\nu_1 + \nu_n}{2} - k_{i-1} - k_i + m_{i-2} + m_{i-1})(\frac{-\nu_1 + \nu_n}{2} - k_i - k_{i+1} + m_{i-1} + m_i)} \right] \\
 & + \frac{1}{2} \sum_{i=1}^{n-3} (\nu_{i+1} - \nu_{i+2}) m_i \\
 & + \sum_{i=1}^{n-1} \left\{ k_i^2 + \frac{1}{2} (\nu_i - \nu_{i+1}) k_i - m_{i-1}^2 - \frac{1}{2} (\nu_i - \nu_{i+1}) m_{i-1} \right. \\
 & + \frac{(\frac{\nu_i - \nu_n}{2} + k_i - m_{i-1}) k_{i-1} k_i + (\frac{-\nu_1 + \nu_i}{2} - k_i + m_{i-1}) m_{i-2} m_{i-1}}{\frac{-\nu_1 + \nu_n}{2} - k_{i-1} - k_i + m_{i-2} + m_{i-1}} \\
 & + \frac{\frac{1}{2} (-\nu_i + \nu_n) k_{i-1} m_{i-1} + \frac{1}{2} (\nu_1 - \nu_i) k_i m_{i-2}}{\frac{-\nu_1 + \nu_n}{2} - k_{i-1} - k_i + m_{i-2} + m_{i-1}} \\
 & + \frac{(\frac{\nu_1 - \nu_{i+1}}{2} + k_i - m_{i-1}) k_i k_{i+1} + (\frac{-\nu_{i+1} + \nu_n}{2} - k_i + m_{i-1}) m_{i-1} m_i}{\frac{-\nu_1 + \nu_n}{2} - k_i - k_{i+1} + m_{i-1} + m_i} \\
 & \left. + \frac{\frac{1}{2} (\nu_{i+1} - \nu_n) k_i m_i + \frac{1}{2} (-\nu_1 + \nu_{i+1}) k_{i+1} m_{i-1}}{\frac{-\nu_1 + \nu_n}{2} - k_i - k_{i+1} + m_{i-1} + m_i} \right\} \\
 & = \left\{ \sum_{i=1}^{n-1} k_i^2 - \sum_{i=1}^{n-2} k_i k_{i+1} + \frac{1}{2} \sum_{i=1}^{n-1} (\nu_i - \nu_{i+1}) k_i \right\} P_{n,\mathbf{k}}(\nu) \\
 & \cdot \sum_{\mathbf{m} \in S(\mathbf{k})} G_{n-2,\mathbf{m}}(\mu) Q_{n,\mathbf{k},\mathbf{m}}(\nu).
 \end{aligned}$$

This completes the proof. □

*Remark.* — Let  $n = 4$  in this theorem. Since  $G_{2,\mathbf{k}}(\nu) = 1 / (\frac{\nu_1 - \nu_2}{2} + 1)_{k_1}$ ,

$$\begin{aligned}
 G_{4,\mathbf{k}}(\nu) & = \frac{1}{(\frac{\nu_1 - \nu_4}{2} + 1)_{k_1} (\frac{\nu_1 - \nu_4}{2} + 1)_{k_2} (\frac{\nu_1 - \nu_4}{2} + 1)_{k_3}} \\
 & \cdot \frac{(\frac{\nu_1 - \nu_4}{2} + 1)_{k_1 + k_2} (\frac{\nu_1 - \nu_4}{2} + 1)_{k_2 + k_3}}{(\frac{\nu_1 - \nu_2}{2} + 1)_{k_1} (\frac{\nu_2 - \nu_4}{2} + 1)_{k_2} (\frac{\nu_1 - \nu_3}{2} + 1)_{k_2} (\frac{\nu_3 - \nu_4}{2} + 1)_{k_3}} \\
 & \cdot \frac{1}{k_1! k_2! k_3!} {}_4F_3 \left( \begin{matrix} -k_2, \frac{-\nu_1 + \nu_2}{2} - k_1, \frac{-\nu_1 + \nu_4}{2} - k_2, \frac{-\nu_3 + \nu_4}{2} - k_3 \\ \frac{\nu_2 - \nu_3}{2} + 1, \frac{-\nu_1 + \nu_4}{2} - k_1 - k_2, \frac{-\nu_1 + \nu_4}{2} - k_2 - k_3 \end{matrix} \middle| 1 \right).
 \end{aligned}$$

Here  ${}_4F_3$  is the generalized hypergeometric series,

$${}_4F_3 \left( \begin{matrix} a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3 \end{matrix} \middle| z \right) = \sum_{n \geq 0} \frac{(a_1)_n (a_2)_n (a_3)_n (a_4)_n}{(b_1)_n (b_2)_n (b_3)_n} \frac{z^n}{n!}.$$



Stade ([10, Theorem 3.1]) found the similar formula including  ${}_4F_3(1)$ . The equivalence of these two formulas can be checked by using Saalschützian formula for  ${}_4F_3(1)$  ([1, 7.2(1)]).

### 3. Proof of Stade’s Conjecture.

In the same way as the proof for  $n = 3, 4$ , we use the following lemma.

LEMMA 5 ([11, Theorem 1]). — *If  $\operatorname{Re}(x + y - 1) > 0$ , then*

$$\frac{\Gamma(x + y - 1)}{\Gamma(x)\Gamma(y)} = \frac{1}{2\pi\sqrt{-1}} \int_{|u|=1} (1 + 1/u)^{x-1} (1 + u)^{y-1} \frac{du}{u}.$$

Here the path is taken counterclockwise in the complex plane.

If we denote  $a^{\nu+\rho} = \prod_{i=1}^{n-1} y_i^{i(n-i)/2 + \sum_{k=1}^i \nu_k}$  by  $y^{\nu+\rho}$ , Theorem 3 implies that

$$\begin{aligned} M_n(\nu; a) &= y^{\nu+\rho} \sum_{\mathbf{k} \in (\mathbb{Z}_{\geq 0})^{n-1}} G_{n,\mathbf{k}}(\nu) (\pi y_1)^{2k_1} \cdots (\pi y_{n-1})^{2k_{n-1}} \\ &= y^{\nu+\rho} \prod_{i=1}^{n-1} \Gamma\left(\frac{\nu_1 - \nu_{i+1}}{2} + 1\right) \Gamma\left(\frac{\nu_{i+1} - \nu_n}{2} + 1\right) \\ &\quad \cdot \sum_{\mathbf{k} \in (\mathbb{Z}_{\geq 0})^{n-1}} (\pi y_1)^{2k_1} \cdots (\pi y_{n-1})^{2k_{n-1}} \sum_{\mathbf{m} \in S(\mathbf{k})} G_{n-2,\mathbf{m}}(\mu) \\ &\quad \cdot \prod_{i=1}^{n-1} \frac{1}{(k_i - m_{i-1})! \Gamma\left(\frac{\nu_1 - \nu_n}{2} + k_i - m_{i-1} + 1\right)} \\ &\quad \cdot \prod_{i=1}^{n-2} \frac{\Gamma\left(\frac{\nu_1 - \nu_n}{2} + k_i + k_{i+1} - m_{i-1} - m_i + 1\right)}{\Gamma\left(\frac{\nu_{i+1} - \nu_n}{2} + k_{i+1} - m_{i-1} + 1\right) \Gamma\left(\frac{\nu_1 - \nu_{i+1}}{2} + k_i - m_i + 1\right)} \end{aligned}$$

If we put  $k_i = m_{i-1} + l_i$  ( $1 \leq i \leq n - 1$ ) and  $\mathbf{l} = (l_1, \dots, l_{n-1})$ , then  $M_n(\nu; a)$  becomes  $y^{\nu+\rho} \prod_{i=1}^{n-1} \Gamma\left(\frac{\nu_1 - \nu_{i+1}}{2} + 1\right) \Gamma\left(\frac{\nu_{i+1} - \nu_n}{2} + 1\right)$  times

$$\begin{aligned} &\sum_{\substack{\mathbf{l} \in (\mathbb{Z}_{\geq 0})^{n-1} \\ \mathbf{m} \in (\mathbb{Z}_{\geq 0})^{n-3}}} (\pi y_1)^{2l_1} (\pi y_2)^{2(l_2+m_1)} \cdots (\pi y_{n-2})^{2(l_{n-2}+m_{n-3})} (\pi y_{n-1})^{2l_{n-1}} \\ (2) \cdot &G_{n-2,\mathbf{m}}(\mu) \prod_{i=1}^{n-1} \frac{1}{l_i! \Gamma\left(\frac{\nu_1 - \nu_n}{2} + l_i + 1\right)} \\ &\cdot \prod_{i=1}^{n-2} \frac{\Gamma\left(\frac{\nu_1 - \nu_n}{2} + l_i + l_{i+1} + 1\right)}{\Gamma\left(\frac{\nu_{i+1} - \nu_n}{2} + l_{i+1} - m_{i-1} + m_i + 1\right) \Gamma\left(\frac{\nu_1 - \nu_{i+1}}{2} + l_i + m_{i-1} - m_i + 1\right)}. \end{aligned}$$

Suppose  $\text{Re}(\frac{\nu_1 - \nu_n}{2} + 1) > 0$ . By Lemma 5, the last term  $\prod_{i=1}^{n-2}$  in (2) can be written as

$$\begin{aligned} & \prod_{i=1}^{n-2} \frac{1}{2\pi\sqrt{-1}} \int_{|u_i|=1} (1 + 1/u_i)^{(\nu_1 - \nu_{i+1})/2 + l_i + m_{i-1} - m_i} \\ & \quad \cdot (1 + u_i)^{(\nu_{i+1} - \nu_n)/2 + l_{i+1} - m_{i-1} + m_i} \frac{du_i}{u_i} \\ & = \prod_{i=1}^{n-2} \frac{1}{2\pi\sqrt{-1}} \int_{|u_i|=1} (1 + 1/u_i)^{(\nu_1 - \nu_n)/4 + l_i} (1 + u_i)^{(\nu_1 - \nu_n)/4 + l_{i+1}} \\ & \quad \cdot u_i^{(\nu_{i+1})/2 - (\nu_1 + \nu_n)/4 - m_{i-1} + m_i} \frac{du_i}{u_i}. \end{aligned}$$

In view of

$$\left| \int_{|u_i|=1} (1 + 1/u_i)^{(\nu_1 - \nu_n)/4 + l_i} (1 + u_i)^{(\nu_1 - \nu_n)/4 + l_{i+1}} \cdot u_i^{(\nu_{i+1})/2 - (\nu_1 + \nu_n)/4 - m_{i-1} + m_i} \frac{du_i}{u_i} \right| \ll 2^{l_i + l_{i+1}}$$

([11, pp. 102-103]) and the estimate of  $|G_{n-2, \mathbf{m}}(\mu)|$  in [3, Lemma 4.5], we can verify the change of order of integration and summation. Then (2) is equal to

$$\begin{aligned} & \frac{1}{(2\pi\sqrt{-1})^{n-2}} \int_{|u_1|=1} \cdots \int_{|u_{n-2}|=1} \\ & \quad \cdot \prod_{i=1}^{n-1} \left[ \sum_{l_i \geq 0} \frac{(\pi y_i)^{2l_i} \{(1 + u_{i-1})(1 + 1/u_i)\}^{(\nu_1 - \nu_n)/4 + l_i}}{l_i! \Gamma(\frac{\nu_1 - \nu_n}{2} + l_i + 1)} \right] \\ & \quad \cdot \left[ \sum_{\mathbf{m} \in (\mathbb{Z}_{\geq 0})^{n-3}} G_{n-2, \mathbf{m}}(\mu) \prod_{i=1}^{n-3} (\pi y_{i+1} \sqrt{u_i/u_{i+1}})^{2m_i} \right] \\ & \quad \cdot \prod_{i=1}^{n-2} u_i^{\nu_{i+1}/2 - (\nu_1 + \nu_n)/4} \frac{du_i}{u_i} \\ & = \frac{1}{(2\pi\sqrt{-1})^{n-2}} \int_{|u_1|=1} \cdots \int_{|u_{n-2}|=1} \prod_{i=1}^{n-1} \left[ y_i^{-(\nu_1 - \nu_n)/2} \right. \\ & \quad \cdot I_{(\nu_1 - \nu_n)/2} \left( 2\pi y_i \sqrt{(1 + u_{i-1})(1 + 1/u_i)} \right) \Big] \\ & \quad \cdot M_{n-2} \left( \mu; (y_2 \sqrt{u_1/u_2}, \dots, y_{n-2} \sqrt{u_{n-3}/u_{n-2}}) \right) \\ & \quad \cdot \prod_{i=1}^{n-3} (\pi y_{i+1} \sqrt{u_i/u_{i+1}})^{-i(n-2-i)/2 - \sum_{k=1}^i \mu_k} \prod_{i=1}^{n-2} u_i^{\nu_{i+1}/2 - (\nu_1 + \nu_n)/4} \frac{du_i}{u_i}. \end{aligned}$$

Collecting the exponents of  $y_i$  and  $u_i$ , we complete the proof.

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## BIBLIOGRAPHY

- [1] W. N. BAILEY, *Generalized Hypergeometric Series*, Cambridge, 1935.
- [2] D. BUMP, *Automorphic Forms on  $GL(3, \mathbb{R})$* , *Lect. Notes in Math.*, 1083 (1984).
- [3] M. HASHIZUME, *Whittaker functions on semisimple Lie groups*, *Hiroshima Math. J.*, 12 (1982), 259–293.
- [4] H. JACQUET, *Fonctions de Whittaker associées aux groupes de Chevalley*, *Bull. Soc. Math. France*, 95 (1967), 243–309.
- [5] B. KOSTANT, *On Whittaker vectors and representation theory*, *Invent. Math.*, 48 (1978), 101–184.
- [6] R. MIATELLO and N. WALLACH, *Automorphic Forms Constructed from Whittaker Vectors*, *J. of Funct. Anal.*, 86 (1989), 411–487.
- [7] J. SHALIK, *The multiplicity one theorem for  $GL(n)$* , *Ann. of Math.*, 100 (1974), 171–193.
- [8] E. STADE, *Poincaré Series For  $GL(3, \mathbb{R})$ -Whittaker Functions*, *Duke Math. J.*, 58-3 (1989), 695–729.
- [9] E. STADE, *On Explicit Integral Formulas For  $GL(n, \mathbb{R})$ -Whittaker Functions*, *Duke Math. J.*, 60-2 (1990), 313–362.
- [10] E. STADE,  *$GL(4, \mathbb{R})$ -Whittaker functions and  ${}_4F_3(1)$  hypergeometric series*, *Trans. Amer. Math. Soc.*, 336-1 (1993), 253–264.
- [11] E. STADE, *The reciprocal of the beta function and  $GL(n, \mathbb{R})$  Whittaker functions*, *Ann. Inst. Fourier, Grenoble*, 44-1 (1994), 93–108.

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