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SOLVABILITY NEAR THE CHARACTERISTIC SET FOR A CLASS OF PLANAR VECTOR FIELDS OF INFINITE TYPE

by Adalberto P. BERGAMASCO and Abdelhamid MEZIANI

0. Introduction.

We are concerned with the solvability of a planar complex vector field L. We assume that the characteristic set, Σ, of L is a simple closed curve and that L is of infinite type along Σ. The equations dealt with here are of the form

$$Lu = f, \quad Lu = pu, \quad \text{and} \quad Lu = pu + f. \tag{0.1}$$

These equations are considered in a full neighborhood of Σ.

It should be noted right away that such a vector field L satisfies the Nirenberg-Treves condition (P) (since it is tangent to Σ and elliptic away from Σ). It follows then that the above equations are locally solvable in a neighborhood of each point. Thus the problems are relevant only in a (semi) global setting. Note also that such vector fields do not satisfy the Hörmander condition near Σ (see Theorem 16.11.3 in [H]). New obstructions to solvability appear in this setting. The first obstruction is of a topological nature. Since we are seeking solutions u in an annulus (with a non trivial fundamental group), the periods of p and of f on Σ have to satisfy certain relations. Another obstruction, of number theoretical nature, appears for the C^ω-solvability of a class of vector fields.

Keywords: Characteristic set, complex vector field, infinite type, solvability.
Math. classification: 35F05, 30G20.

The results contained in this paper generalize those contained in [BgM], [BhM1], [BhM2], [M1], [M2], and [M3]. The approach and the motivation for this work are related to those in the papers [BT], [B1], [B2], [BCH], [BCM], [BCP], [BgM], [BHS], [BhM1], [BhM2], [CH], [M1], [M2], [M3], [T1], [T2] and in many others.

The sections are organized as follows. In section 1, we recall the necessary background and the reduction to models. In sections 2 and 3, we consider the solvability of $Lu = f$ in the C^ω and C^∞ categories. In section 4, we study the equation $Lu = pu$, and in section 5, the equation $Lu = pu + f$.

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1. Preliminaries.

In this section, we recall the necessary definitions and the motivation for the model cases. The definitions and background material about local solvability can be found in [T2].

Let

$$L = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y} \tag{1.1}$$

be a vector field in $\mathbb{R}^2$ with a, b smooth (C^∞ or C^ω). We assume that the coefficients a, b are $\mathbb{C}$-valued and that they do not vanish simultaneously (L is free from singularities). Let $\overline{L}$ be the complex conjugate vector field;

$$\overline{L} = \overline{a(x,y)}\frac{\partial}{\partial x} + \overline{b(x,y)}\frac{\partial}{\partial y}. \tag{1.2}$$

The vector field L is elliptic at a point $p \in \mathbb{R}^2$ if L and $\overline{L}$ are independent at p. If L is elliptic at each point of a region Ω, then it is equivalent (in Ω) to the CR operator $\partial/\partial\overline{z}$, and the pde's associated with L are therefore well understood in Ω. Denote by Σ the set of points where L and $\overline{L}$ are dependent:

$$\Sigma = \{p \in \mathbb{R}^2;\ L_p \ \text{ and } \overline{L}_p \ \text{ are dependent}\}. \tag{1.3}$$

Σ is called the characteristic set of L (or of the structure defined by L). This set can be split into those points where L is of finite type and those points where it is of infinite type. L is of finite type at a point $p \in \Sigma$ if the Lie algebra generated by L, $\overline{L}$, and their successive Lie brackets, generates the complexified tangent space $\mathbb{C}T_p\mathbb{R}^2$. Otherwise L is of infinite type. Thus, L is of infinite type at p means that the vector fields L, $\overline{L}$, and Z are dependent at p, where Z is any vector field of the form

$$Z = [X_1, [X_2, [\cdots [X_{n-1}, X_n] \cdots]]$$

where the X_j's are either L or $\overline{L}$, n is any positive integer, and $[\ ,\ ]$ denotes the Lie Bracket.

From now on, we will assume that Σ is a smooth simple closed curve and that L is of infinite type at each point of Σ. It follows that L is tangent to Σ at each point $p \in \Sigma$. It can be proved that in the real analytic category (this follows from the local canonical representation of vector fields, see [T2] for example) that for a given $p \in \Sigma$, there are coordinates (s,t) centered at p, such that in a neighborhood of p, the expression of L with respect to the these coordinates has the form

$$\alpha(s,t)\left(\frac{\partial}{\partial t} + is^j\beta(s,t)\frac{\partial}{\partial s}\right),$$

with α and β nonzero functions, β real valued, and j a positive integer. It follows at once that L satisfies the Nirenberg-Treves condition (P) (see [NT] or [T2]). As a consequence, equation (0.1) is solvable (in the C^∞ or C^ω category) in a neighborhood of each point $p \in \Sigma$. In this paper we consider the solvability of (0.1) in an open neighborhood of the characteristic set Σ. As will be seen in the following sections, there are obstructions to such solvability.

With our hypothesis that Σ is a smooth simple closed curve, we can assume that Σ is a circle in the plane. Equation (0.1), in a tubular neighborhood of the characteristic set, can thus be viewed as an equation in a ring

$$A_\epsilon = (-\epsilon,\ \epsilon) \times S^1 \subset \mathbb{R} \times S^1 \tag{1.4}$$

where the characteristic set is the circle

$$\Sigma = \{0\} \times S^1.$$

We will assume that L satisfies the homogeneity condition $L \wedge \overline{L}$ vanishing to a constant order $1 + n \in Z^+$ along the characteristic set Σ. Normal forms for such vector fields have been obtained in [M3] for the case $n = 0$ and in [M4] for the case $n > 0$ (see also [CG] for the case $n = 0$.)

Now we describe briefly the model vector fields. In view of our assumptions, we can assume that, in a neighborhood of Σ, L has the expression

$$L_n = \frac{\partial}{\partial\theta} - ir^{n+1}a(r,\theta)\frac{\partial}{\partial r} \tag{1.5}$$

where (r,θ) are the canonical coordinates of $\mathbb{R}\times S^1$, $a \in C^\infty$, and $\Re a(0,\theta)\neq 0$ for every $\theta\in S^1$. When $n=0$, a complex number $\lambda\in\mathbb{R}^+ + i\mathbb{R}$ is invariantly associated to L (see [M3]). It is shown in [M3] that L is equivalent to the model vector field

$$T_\lambda = \lambda\frac{\partial}{\partial\theta} - ir\frac{\partial}{\partial r}, \tag{1.6}$$

when $\Im\lambda\neq 0$. This result is generalized in [CG] to include the case $\lambda\in\mathbb{R}$ except for those values of λ that are well approximable by rationals.

When $n>0$ it is shown in [M4] that there exist a unique polynomial $P(r)$ of degree at most $n-1$ with $\Re P(0)<0$ and a unique number $\mu\in\mathbb{C}$ such that L is equivalent to the vector field

$$R_n = \frac{\partial}{\partial\theta} - i\frac{r^{n+1}}{rP'(r)-nP(r)+\mu r^n}\frac{\partial}{\partial r}. \tag{1.7}$$

Note that the function

$$F_n(r,\theta) = exp\left(\epsilon(r)^n\left[\frac{P(r)}{r^n}+\mu\log|r|+i\theta\right]\right), \tag{1.8}$$

where $\epsilon(r)=r/|r|$, is a C^∞ first integral of R_n (see [M4] for details).

The study of equations (0.1) in a neighborhood of Σ is then reduced to corresponding equations for the vector fields T_λ and R_n in a neighborhood of the circle $\{r=0\}$.

By using Fourier series, it is seen that in order for the equations

$$T_\lambda u = f; \tag{1.9}$$

$$R_n u = f, \tag{1.10}$$

to have a C^0 solution in the above ring, the function f must satisfy the compatibility conditions

$$\int_0^{2\pi} f(0,\theta)d\theta = 0 \tag{1.11}$$

in the case of equation (1.9), and

$$\int_0^{2\pi}\left(\frac{\partial}{\partial r}\right)^j f(0,\theta)d\theta = 0,\quad j=0,\dots,n \tag{1.12}$$

in the case of equation (1.10). Thus when dealing with equations (1.9) and (1.10), we will assume that the right-hand side satisfies the compatibility conditions.

Equation (1.9) has been studied in [BhM2], [M3] and [BgM]. In [BhM2], C^0 solutions are considered. In [M3] it is proved, in particular, that if $\lambda \notin \mathbb{Q}$, f is C^∞ and satisfies (1.11), then for every $k \in \mathbb{Z}^+$, equation (1.9) has a C^k solution defined near the characteristic circle. In [M4] the kernel of R_n is completely described. When $n > 0$, the paper [BCP] also considers the C^∞ solvability of the equation $Lu = f$, where L is an operator such as (1.5) for which the coefficient a is assumed to be real-valued and is allowed to vanish on segments of the characteristic circle. The results of [BgM] will be described in sections 2 and 3. In the next section, we study the problem of finding analytic solutions for equations (1.9) and (1.10) when f is analytic.

2. Analytic solvability.

In this section we consider the analytic solvability of the model operators T_λ and R_n. It turns out that R_n ($n \geqslant 1$) is never solvable in the analytic category and that T_λ is solvable in the analytic category when either $\lambda \notin \mathbb{R}$ or else λ is an irrational number not well approximable by rational numbers (see below).

For the operator T_λ it was already proved in [BhM2] that if $\lambda \in \mathbb{R}^+ + i\mathbb{R}$ and $\lambda \notin \mathbb{R}$, then T_λ is analytically solvable, and if $\lambda \in \mathbb{Q}$ then it is not analytically solvable. We recall here a result in [BgM] for the case of irrational λ : analytic solutions exist for every analytic function f satisfying (1.11) if and only if λ satisfies a diophantine condition. We also prove that, for every $n \geqslant 1$, there exist C^ω functions f satisfying (1.12) such that equation (1.10) has no C^ω solution.

We first describe a diophantine condition (DC) for $\alpha \in \mathbb{R}^+$, namely, we say that (DC) holds if the following equivalent conditions hold:

$$(\mathrm{DC})_1 \qquad \exists C > 0 \ \left|\exp\Big(i\frac{2\pi j}{\alpha}\Big) - 1\right| \geqslant C^{j+1} \quad \forall j \in \mathbb{Z}^+;$$

$$(\mathrm{DC})_2 \qquad \exists C > 0 \ \ |j + \alpha k| \geqslant C^{j+1} \quad \forall j \in \mathbb{Z}^+,\ \forall k \in \mathbb{Z}.$$

It can be easily proved that $(\mathrm{DC})_1$ and $(\mathrm{DC})_2$ are equivalent and that if α satisfies (DC) then so does $1/\alpha$ (see [BgM]).

An irrational number α is said to be an exponential Liouville number if, for some $\epsilon > 0$, the inequality

$$\left|\alpha - \frac{p}{q}\right| \leqslant \exp(-\epsilon q) \tag{2.1}$$

has infinitely many rational solutions p/q, with $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^+$; to say that the same is true for every $\epsilon > 0$ is equivalent to saying that the number α does not satisfy the diophantine condition (DC).

We are now ready to state the result about the analytic solvability of T_λ. For the proof of this result see the proof of Theorem 2.1 in [BgM].

Theorem 2.1([BgM]). — *Let $\lambda = a \in \mathbb{R}^+\backslash\mathbb{Q}$. Equation* (1.9) *has a real analytic solution u defined near Σ for every real analytic function f satisfying the compatibility condition* (1.11) *if and only if the invariant a satisfies the diophantine condition (DC).*

Our next result concerns the analytic solvability of the operator R_n.

Theorem 2.2. — *For every $n \geqslant 1$, there exists $f \in C^\omega$ satisfying the compatibility conditions* (1.12) *such that the equation* (1.10) *has no C^ω solution in any neighborhood A_ϵ of Σ.*

Before we proceed with the proof of the theorem, we prove a lemma about an o.d.e. with an irregular singular point.

Lemma 2.1. — *Given $n \in \mathbb{Z}^+$, $a_0 \in \mathbb{C}^*$ and $a_n \in \mathbb{C}$, there exist holomorphic functions f defined in a neighborhood of $0 \in \mathbb{C}$ such that the differential equation*

$$z^{n+1}\frac{dv}{dz} - (a_0 + a_n z^n)v = f(z) \tag{2.2}$$

has no holomorphic solution in a neighborhood of 0.

Proof. — We prove the lemma for $f(z) = z^p$ with p a nonnegative integer to be chosen later. We prove that any power series solution has radius of convergence equal to 0. Suppose that

$$\sum_{j\geqslant 0} v_j z^j \tag{2.3}$$

is a series solution of (2.2). Then

$$-\sum_{k=0}^{n-1} a_0 v_k z^k + \sum_{k=n}^{\infty} \left[(k-n-a_n)v_{k-n} - a_0 v_k\right] z^k = z^p. \tag{2.4}$$

We obtain

$$v_k = 0, \quad \text{for} \quad k \notin p + n\mathbb{Z}^+ \tag{2.5}$$

and

$$v_p = -\frac{1}{a_0} \quad \text{and} \quad v_{p+mn} = -\frac{\Pi_{\ell=0}^{m-1}(p - a_n + \ell n)}{a_0^{m+1}}, \quad m \geqslant 1. \tag{2.6}$$

We now choose p so that $p - a_n + \ell n \neq 0, \forall \ell \in \mathbb{Z}^+$; for instance, it suffices to require $p \in \mathbb{Z}^+$ with $p > \Re a_n$. Then the series

$$\sum_{j=0}^{\infty} v_j z^j = \sum_{m=0}^{\infty} v_{p+mn} z^{p+mn} = z^p \sum_{m=0}^{\infty} b_m w^m, \tag{2.7}$$

where $w = z^n$, and $b_m = v_{p+mn}$, has radius of convergence equal to zero since the ratio

$$\left| \frac{b_{m+1}}{b_m} \right| = \left| \frac{v_{p+(m+1)n}}{v_p + mn} \right| = \frac{|p - a_n + mn|}{|a_0|} \to \infty \quad \text{as} \quad m \to \infty. \tag{2.8}$$

□

Proof of Theorem 2.2. — Consider the equation

$$R_n u = \alpha(r) e^{i\theta} \tag{2.9}$$

with α real analytic in a neighborhood of $0 \in \mathbb{R}$. The function $\alpha(r)e^{i\theta}$ satisfies the compatibility condition (1.12). We are going to show that, for an appropriate choice of the function α, equation (2.9) has no real analytic solution in a neighborhood of Σ. In order for a function u to solve (2.9), its Fourier coefficient

$$u_1(r) = \frac{1}{2\pi} \int_0^{2\pi} u(r,\theta) e^{-i\theta}\, d\theta \tag{2.10}$$

needs to satisfy the o.d.e.

$$iu_1(r) - i \frac{r^{n+1}}{rP'(r) - nP(r) + \mu r^n} \frac{du_1}{dr} = \alpha(r). \tag{2.11}$$

We rewrite this o.d.e. as

$$r^{n+1} \frac{du_1}{dr} - c(r) u_1(r) = ic(r)\alpha(r) \tag{2.12}$$

where $c(r)$ is the polynomial

$$c(r) = rP'(r) - nP(r) + \mu r^n = c_0 + c_1 r + \cdots + c_n r^n \tag{2.13}$$

with $c(0) = -nP(0) \neq 0$. We study the complexification of (2.12):

$$\zeta^{n+1} \frac{d\hat{u}_1}{d\zeta} - \hat{c}(\zeta)\hat{u}_1(\zeta) = i\hat{c}(\zeta)\hat{\alpha}(\zeta), \tag{2.14}$$

where $\zeta = r + i\tilde{r} \in \mathbb{C}$ and $\hat{u}, \hat{c}, \hat{\alpha}$ are the complexifications of u, c, α. By means of a holomorphic change of variable, we transform (2.14) into an equation as in lemma 2.1. For this we consider the holomorphic function $F(\zeta, X)$ defined in a neighborhood of $0 \in \mathbb{C}^2$ by

$$\begin{aligned} F(\zeta, X) = \frac{-c_0}{n(1+X)^n} + c_n\zeta^n \log(1+X) + \frac{c_0}{n} \\ + \frac{c_1}{n-1}\zeta + \cdots + \frac{c_{n-2}}{2}\zeta^{n-2} + c_{n-1}\zeta^{n-1}. \end{aligned} \tag{2.15}$$

The function F satisfies

$$F(0,0) = 0 \quad \text{and} \quad \frac{\partial F}{\partial X}(0,0) = c_0 \neq 0. \tag{2.16}$$

By the implicit function theorem, there exists a holomorphic function $X(\zeta)$ with $X(0) = 0$ defined in a disk $|\zeta| < \rho$ such that

$$F(\zeta, X(\zeta)) = 0, \quad \forall \zeta, \quad |\zeta| < \rho. \tag{2.17}$$

Now consider the new variable

$$z = \zeta(1 + X(\zeta)). \tag{2.18}$$

A direct calculation shows that

$$\frac{c_0 + c_n z^n}{z^{n+1}} \frac{dz}{d\zeta} = \frac{c_0 + c_1\zeta + \cdots + c_n\zeta^n}{\zeta^{n+1}}. \tag{2.19}$$

Hence it follows that, with respect to the variable z, the o.d.e. (2.14) takes the form

$$z^{n+1}\frac{dw}{dz} - (c_0 + c_n z^n)w = f(z) \tag{2.20}$$

where

$$f(z) = i\hat{\alpha}(\zeta(z))(c_0 + c_n z^n). \tag{2.21}$$

For the choice of the function α

$$\alpha(r) = -i\frac{r^p(1 + X(r))^p}{c_0 + c_n r^n(1 + X(r))^p} \tag{2.22}$$

with $p \in \mathbb{Z}^+$, we obtain

$$f(z) = z^p. \tag{2.23}$$

It follows from lemma 2.1 that the corresponding o.d.e. (2.20) — and consequently (2.14) — has no holomorphic solution. This implies that (2.11) has no real analytic solution $u_1(r)$ hence (2.9) has no real analytic solution $u(r, \theta)$. □

3. C^∞ solvability.

In this section, we consider the C^∞ solvability of the vector fields T_λ and R_n. It turns out that T_λ is not solvable in the C^∞ category while R_n is always solvable.

For T_λ we have the following result the proof of which can be found in [BgM].

THEOREM 3.1([BgM]). — *Let $\lambda = a + ib \in \mathbb{R}^+ + i\mathbb{R}$. Then there exist C^∞ functions f satisfying* (1.11) *so that equation* (1.9) *does not have C^∞ solutions in any neighborhood of Σ.*

For the vector field R_n we have the following theorem.

THEOREM 3.2. — *If ϵ is small enough then, for every $f \in C^\infty(A_\epsilon)$ satisfying* (1.12), *there exists $u \in C^\infty(A_\epsilon)$ such that* (1.10) *holds.*

To prove this theorem, we need two propositions.

PROPOSITION 3.1. — *Let $D(0, R)$ be the disc with center 0 and radius R in $\mathbb{C}$ and let $F(z) \in C^0(\bar{D}(0, R))$ be such that*

$$\lim_{z\to 0}\left(\log \frac{1}{|z|}\right)^q F(z) = 0, \quad \forall q > 0. \tag{3.1}$$

Then the inhomogeneous CR equation

$$\frac{\partial U}{\partial \bar{z}} = \frac{F(z)}{z} \tag{3.2}$$

has a solution $U \in C^0(\bar{D}(0, R))$ such that

$$\lim_{z\to 0}\left(\log \frac{1}{|z|}\right)^q U(z) = 0, \quad \forall q > 0. \tag{3.3}$$

Proof. — Let

$$V(z) = \frac{-1}{\pi}\iint_{D(0,R)} \frac{F(\zeta)}{\zeta - z} d\xi d\eta \qquad (\zeta = \xi + i\eta). \tag{3.4}$$

Since $F \in C^0(\bar{D}(0, R))$, V is in the Hölder class $C^\alpha(\bar{D}(0, R))$, for every $0 < \alpha < 1$ (see [V]) and V solves the CR equation

$$V_{\bar{z}} = F.$$

It follows from (3.1) that there is $C > 0$ such that

$$\left|\frac{F(\zeta)}{\zeta^2}\right| \leqslant C \frac{1}{|\zeta|^2 (\log(1/|\zeta|))^2}, \quad \forall \zeta \in D(0, R). \tag{3.5}$$

Hence

$$\left|\iint_{D(0,R)} \frac{F(\zeta)}{\zeta^2} d\xi d\eta\right| \leqslant C \int_0^{2\pi} \int_0^R \frac{r dr d\theta}{r^2 (\log \frac{1}{r})^2} = \frac{2\pi C}{\log \frac{1}{R}}. \tag{3.6}$$

Define the function W by

$$W(z) = V(z) - V(0) - az, \tag{3.7}$$

where

$$a = \frac{-1}{\pi} \iint_{D(0,R)} \frac{F(\zeta)}{\zeta^2} d\xi d\eta. \tag{3.8}$$

Note that a is well defined because of (3.6); in fact, $a = \frac{\partial V}{\partial z}(0)$. Note also that W satisfies

$$W_{\bar{z}} = F. \tag{3.9}$$

Now we need an estimate for the behavior of W at 0. We have

$$\begin{aligned} W(z) &= \frac{-1}{\pi} \iint_{D(0,R)} F(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta} - \frac{z}{\zeta^2} \right) d\xi d\eta \\ &= \frac{-z^2}{\pi} \iint_{D(0,R)} \frac{F(\zeta)}{\zeta^2 (\zeta - z)} d\xi d\eta. \end{aligned} \tag{3.10}$$

Hence

$$|W(z)| \leqslant \frac{|z|^2}{\pi} I, \tag{3.11}$$

where

$$I = \iint_{D(0,R)} \frac{|F(\zeta)|}{|\zeta|^2 |\zeta - z|} d\xi d\eta. \tag{3.12}$$

Given $q > 0$, let $p \in \mathbb{R}^+$ such that $p > 2q + 1$. Define the function $G_p \in C^0(\bar{D}(0,R))$ by

$$G_p(z) = \left(\log \frac{1}{|z|} \right)^p F(z). \tag{3.13}$$

Let

$$M_p = \max_{|z| \leqslant R} |G_p(z)|.$$

We have then

$$\begin{aligned} I &\leqslant \iint_{D(0,R)} \frac{|G_p(\zeta)|}{|\zeta|^2 |\zeta - z| \left(\log \frac{1}{|\zeta|} \right)^p} d\xi d\eta \\ &\leqslant M_p \iint_{D(0,R)} \frac{d\xi d\eta}{|\zeta|^2 |\zeta - z| \left(\log \frac{1}{|\zeta|} \right)^p}. \end{aligned} \tag{3.14}$$

Let $D(0,\frac{|z|}{2})$ be the disc with center 0 and radius $\frac{|z|}{2}$ and let

$$\begin{aligned} J_1 &= \iint_{D(0,\frac{|z|}{2})} \frac{d\xi d\eta}{|\zeta|^2|\zeta-z|\left(\log\frac{1}{|\zeta|}\right)^p}, \\ J_2 &= \iint_{D(0,R)\setminus D(0,\frac{|z|}{2})} \frac{d\xi d\eta}{|\zeta|^2|\zeta-z|\left(\log\frac{1}{|\zeta|}\right)^p}. \end{aligned} \tag{3.15}$$

In $D(0,\frac{|z|}{2})$, we have $|\zeta-z|\geqslant\frac{|z|}{2}$ and so

$$J_1(z)\leqslant\frac{2}{|z|}\iint_{D(0,\frac{|z|}{2})}\frac{d\xi d\eta}{|\zeta|^2\left(\log\frac{1}{|\zeta|}\right)^p}=\frac{2}{|z|}\int_0^{2\pi}\int_0^{\frac{|z|}{2}}\frac{drd\theta}{r\left(\log\frac{1}{r}\right)^p}. \tag{3.16}$$

Thus

$$J_1(z)\leqslant\frac{2}{|z|}(2\pi)\frac{1}{(p-1)\left(\log\frac{2}{|z|}\right)^{p-1}}=\frac{4\pi}{(p-1)|z|\left(\log\frac{2}{|z|}\right)^{p-1}}. \tag{3.17}$$

To estimate $J_2(z)$, we let

$$\alpha(z)=|z|\left(\log\frac{1}{|z|}\right)^{q+1},\quad \Delta=D(z,\alpha(z))$$

and

$$\Delta^+=\Delta\cap\left(D(0,R)\setminus D(0,\frac{|z|}{2})\right). \tag{3.18}$$

We write

$$J_2(z)=J_2^1(z)+J_2^2(z), \tag{3.19}$$

where

$$\begin{aligned} J_2^1(z) &= \iint_{\Delta^+}\frac{d\xi d\eta}{|\zeta|^2|\zeta-z|\left(\log\frac{1}{|\zeta|}\right)^p}; \\ J_2^2(z) &= \iint_{D(0,R)\setminus\left(D(0,\frac{|z|}{2})\cup\Delta^+\right)}\frac{d\xi d\eta}{|\zeta|^2|\zeta-z|\left(\log\frac{1}{|\zeta|}\right)^p}. \end{aligned} \tag{3.20}$$

In Δ^+, we have $\frac{|z|}{2}\leqslant|\zeta|\leqslant|z|+\alpha(z)$, hence

$$\frac{1}{|\zeta|}\leqslant\frac{2}{|z|}\quad\text{and}\quad\log\frac{1}{|\zeta|}\geqslant\log\frac{1}{|z|+\alpha(z)}.$$

We have then

$$\frac{1}{|\zeta|^2|\zeta-z|\left(\log\frac{1}{|\zeta|}\right)^p}\leqslant\frac{2}{|z|^2\left(\log\frac{1}{|z|+\alpha(z)}\right)^p|\zeta-z|}. \tag{3.21}$$

Therefore

$$
\begin{aligned}
J_2^1(z) &\leqslant \frac{2}{|z|^2\left(\log\frac{1}{|z|+\alpha(z)}\right)^p}\iint_{\Delta^+}\frac{d\xi d\eta}{|\zeta-z|} \\
&\leqslant \frac{2}{|z|^2\left(\log\frac{1}{|z|+\alpha(z)}\right)^p}\iint_{\Delta}\frac{d\xi d\eta}{|\zeta-z|}=\frac{4\pi\alpha(z)}{|z|^2\left(\log\frac{1}{|z|+\alpha(z)}\right)^p}.
\end{aligned}
\tag{3.22}
$$

In $D(0,R)\setminus(D(0,\frac{|z|}{2})\cup\Delta^+)$, we have $|\zeta-z|\geqslant\alpha(z)$. Hence

$$
\begin{aligned}
J_2^2(z) &\leqslant \frac{1}{\alpha(z)}\iint_{D(0,R)\setminus\left(D(0,\frac{|z|}{2})\cup\Delta^+\right)}\frac{d\xi d\eta}{|\zeta|^2\left(\log\frac{1}{|\zeta|}\right)^p} \\
&\leqslant \frac{1}{\alpha(z)}\iint_{D(0,R)}\frac{d\xi d\eta}{|\zeta|^2\left(\log\frac{1}{|\zeta|}\right)^p}=\frac{C(p,R)}{\alpha(z)},
\end{aligned}
\tag{3.23}
$$

where $C(p,R)$ is a positive constant depending only on p and R. It follows from (3.22) and (3.23) and from

$$
|I|\leqslant M_p(J_1+J_2)\leqslant M_p\left(J_1(z)+J_2^1(z)+J_2^2(z)\right). \tag{3.24}
$$

that

$$
|I(z)|\leqslant\frac{A}{|z|\left(\log\frac{2}{|z|}\right)^{p-1}}+\frac{B\alpha(z)}{|z|^2\left(\log\frac{1}{|z|+\alpha(z)}\right)^p}+\frac{C}{\alpha(z)}, \tag{3.25}
$$

where A,B,C are constants independent of z. We use $\alpha(z)=|z|\left(\log\frac{1}{|z|}\right)^{q+1}$ to obtain

$$
|I(z)|\leqslant\frac{A}{|z|\left(\log\frac{2}{|z|}\right)^{p-1}}+\frac{B\left(\log\frac{1}{|z|}\right)^{q+1}}{|z|\left(\log\frac{1}{|z|\left(1+\left(\log\frac{1}{|z|}\right)^{q+1}\right)}\right)^p}+\frac{C}{|z|\left(\log\frac{1}{|z|}\right)^{q+1}}. \tag{3.26}
$$

We have then

$$
|W(z)|\leqslant\frac{A|z|}{\pi\left(\log\frac{2}{|z|}\right)^{p-1}}+\frac{B|z|\left(\log\frac{1}{|z|}\right)^{q+1}}{\pi\left(\log\frac{1}{|z|\left(1+\left(\log\frac{1}{|z|}\right)^{q+1}\right)}\right)^p}+\frac{C|z|}{\pi\left(\log\frac{1}{|z|}\right)^{q+1}}. \tag{3.27}
$$

Set

$$
U(z)=\frac{W(z)}{z}. \tag{3.28}
$$

Then $U \in C^0(\bar{D}(0,R))$ and satisfies

$$\frac{\partial U}{\partial \bar{z}} = \frac{F(z)}{z}. \tag{3.29}$$

Furthermore,

$$\left(\log\frac{1}{|z|}\right)^q |U| \leqslant \frac{\left(A\log\frac{1}{|z|}\right)^q}{\pi\left(\log\frac{2}{|z|}\right)^{p-1}} + \frac{B\left(\log\frac{1}{|z|}\right)^{2q+1}}{\pi\left(\log\frac{1}{|z|}+\log\frac{1}{1+\left(\log\frac{1}{|z|}\right)^{q+1}}\right)^p} + \frac{C}{\pi\log\frac{1}{|z|}}. \tag{3.30}$$

Since

$$\log\frac{1}{|z|}+\log\frac{1}{1+\left(\log\frac{1}{|z|}\right)^{q+1}} \geqslant \frac{1}{2}\log\frac{1}{|z|}, \tag{3.31}$$

for $|z|$ small, and since $p > 2q+1$, then it follows from the above inequalities that

$$\lim_{z\to 0}\left(\log\frac{1}{|z|}\right)^q U(z) = 0. \tag{3.32}$$

The proof of the proposition is complete. □

PROPOSITION 3.2. — *Let $g(r,\theta)$ be a C^∞ function defined in the annulus $A_\epsilon = \{(r,\theta); -\epsilon < r < \epsilon\}$ such that g is flat along the circle $\Sigma = \{r=0\}$. Then, for ϵ small enough, the equation*

$$R_n v = g \tag{3.33}$$

has a solution $v \in C^\infty(A_\epsilon)$. Furthermore, v is flat along Σ.

Proof. — Consider the diffeomorphism determined by the first integral of R_n

$$\Phi:\ A_\epsilon^+ \to D^*(0,\delta(\epsilon));\quad \Phi(r,\theta) = \exp\left[\frac{P(r)}{r^n}+\mu\log r + i\theta\right], \tag{3.34}$$

where $A_\epsilon^+ = (0,\epsilon)\times S^1$ and $D^*(0,\delta(\epsilon)) = \{z\in\mathbb{C};\quad 0<|z|<\delta(\epsilon)\}$, with

$$\delta(\epsilon) = \exp\left(\frac{\Re P(\epsilon)}{\epsilon^n}+\Re\mu\log\epsilon\right).$$

Note that if we set $z = \Phi(r,\theta)$, then

$$|z| = \exp\left(\frac{\Re P(r)}{r^n}+\Re\mu\log r\right). \tag{3.35}$$

By solving this equation for r, we find

$$r = A(|z|) \tag{3.36}$$

for some function $A \in C^\omega((0, \delta(\epsilon))$ and satisfying

$$A(|z|) \sim [-\Re P(0)]^{1/n} \left(\log \frac{1}{|z|}\right)^{-\frac{1}{n}} \quad \text{as} \quad |z| \to 0. \tag{3.37}$$

The expression for θ is

$$\theta = \arg z - \frac{\Im P(A(|z|))}{A(|z|)^n} - \Im\mu \log A(|z|). \tag{3.38}$$

The pushforward of the equation $R_n u = g$ in A_ϵ^+ gives rise to the CR equation

$$\frac{\partial U}{\partial \overline{z}} = \frac{G(z)}{2i\overline{z}} B(A(|z|)), \tag{3.39}$$

where

$$G(z) = g(\Phi^{-1}(z)) \quad \text{and} \quad B(t) = \frac{tP'(t) - nP(t) + \mu t^n}{\Re(tP'(t) - nP(t) + \mu t^n)}. \tag{3.40}$$

Note that $B(A(|z|))$ is C^ω in the interval $(0, \delta(\epsilon))$ and it is continuous up to 0. It follows from the flatness of g along $r = 0$ and from the continuity of $B(A(|z|))$ that

$$\lim_{z \to 0} G(z) B\left(A(|z|)\right) \left(\log \frac{1}{|z|}\right)^p = 0, \quad \forall p > 0. \tag{3.41}$$

We rewrite the above equation as

$$\frac{\partial U}{\partial \overline{z}} = \frac{H(z)}{z} \tag{3.42}$$

where the function

$$H(z) = \frac{z}{2i\overline{z}} G(z) B(A(|z|)) \in C^\infty\left(D^*(0, \delta(\epsilon))\right) \cap C^0\left(\overline{D^*(0, \delta(\epsilon))}\right) \tag{3.43}$$

and

$$\lim_{z \to 0} \left(\log \frac{1}{|z|}\right)^q H(z) = 0, \quad \forall q > 0. \tag{3.44}$$

It follows from Proposition 3.1 that equation (3.42) has a solution

$$U \in C^\infty\left(D^*(0, \delta(\epsilon))\right) \cap C^0\left(\overline{D^*(0, \delta(\epsilon))}\right) \tag{3.45}$$

such that

$$\lim_{z \to 0} \left(\log \frac{1}{|z|}\right)^q U(z) = 0, \quad \forall q > 0. \tag{3.46}$$

Now define the function $u^+(r,\theta) \in C^\infty(A_\epsilon^+)$ by

$$u^+(r,\theta) = U(\Phi(r,\theta)). \tag{3.47}$$

Then u^+ satisfies

$$R_n u^+ = g \quad \text{in } A_\epsilon^+, \tag{3.48}$$

and furthermore

$$\lim_{r\to 0} \frac{u^+(r,\theta)}{r^q} = 0, \quad \forall q > 0,\ \forall\theta \in S^1. \tag{3.49}$$

Thus $u^+ \in C^\infty(A_\epsilon^+ \cup \Sigma)$ and u^+ vanishes to infinite order along Σ.

An analogous argument gives a solution u^- to the equation

$$R_n u^- = g \quad \text{in } A_\epsilon^- = A_\epsilon \cap \{r < 0\} \tag{3.50}$$

with $u^- \in C^\infty(A_\epsilon^- \cup \Sigma)$ and u^- flat along Σ.

Define $u \in C^\infty(A_\epsilon)$ by

$$u = u^+ \quad \text{in } A_\epsilon^+, \quad \text{and} \quad u = u^- \quad \text{in } A_\epsilon^-. \tag{3.51}$$

Then u is flat along the circle Σ and solves (3.33). □

Proof of Theorem 3.2. — We write our operator R_n from (1.7) as

$$R_n = \frac{\partial}{\partial\theta} - ir^{n+1}b(r)\frac{\partial}{\partial r},$$

where

$$b(r) = \frac{1}{rP'(r) - nP(r) + \mu r^n}; \tag{3.52}$$

in particular,

$$b(0) = \frac{1}{(-nP(0))}, \tag{3.53}$$

hence $\Re b(0) > 0$.

Let $f(r,\theta) \in C^\infty(A_\epsilon)$ and assume that f satisfies the compatibility conditions (1.12). Let $\sum_{j\geqslant 0} f_j(\theta)r^j$ be the Taylor expansion of f with respect to r; that is,

$$f_j(\theta) = \frac{1}{j!}\frac{\partial^j f}{\partial r^j}(0,\theta), \quad j = 0, 1, \dots.$$

Let $\sum_{k\geqslant 0} b_k r^k$ be the Taylor expansion of b; thus $b_0 \neq 0$. We solve the formal equation

$$R_n\Big(\sum_{j\geqslant 0} u_j(\theta)r^j\Big) \sim \sum_{j\geqslant 0} f_j(\theta)r^j, \tag{3.54}$$

or, equivalently,

$$\sum_{j\geqslant 0} u_j' r^j - \sum_{j\geqslant n+1}\left\{\sum_{\ell=1}^{j-n} ib_{j-n-\ell}\ell u_\ell\right\}r^j \sim \sum_{j\geqslant 0} f_j r^j. \tag{3.55}$$

We are led to

$$u_j'(\theta) = f_j(\theta), \quad j = 0,\dots,n, \tag{3.56}$$

and

$$u_j'(\theta) = i\sum_{\ell=1}^{j-n} b_{j-n-\ell}\ell u_\ell(\theta) + f_j(\theta), \quad j\geqslant n+1. \tag{3.57}$$

We use (1.12) and obtain the general solution of (3.56) in the form

$$u_j(\theta) = v_j(\theta) + C_j, \quad j = 0,\dots,n, \tag{3.58}$$

where C_j is a constant and

$$v_j(\theta) = \int_0^\theta f_j(\sigma)d\sigma, \quad j = 0,\dots,n. \tag{3.59}$$

The first equation in (3.57), namely,

$$u_{n+1}' = ib_0u_1 + f_{n+1} = ib_0v_1 + ib_0C_1 + f_{n+1}, \tag{3.60}$$

has a solution $u_{n+1}\in C^\infty(S^1)$ if and only if

$$\int_0^{2\pi}\{ib_0v_1(\theta) + ib_0C_1 + f_{n+1}(\theta)\}d\theta = 0, \tag{3.61}$$

and this holds for a unique value of C_1, namely,

$$C_1 = \frac{1}{2\pi b_0}\int_0^{2\pi}\{-v_1(\theta) + ib_0f_{n+1}(\theta)\}d\theta. \tag{3.62}$$

We obtain

$$u_{n+1}(\theta) = v_{n+1}(\theta) + C_{n+1}, \tag{3.63}$$

where C_{n+1} is a constant and

$$v_{n+1}(\theta) = \int_0^\theta h_{n+1}(\sigma)d\sigma, \tag{3.64}$$

with

$$h_{n+1} = ib_0v_1 + ib_0C_1 + f_{n+1}. \tag{3.65}$$

We prove by induction on $j\geqslant n+1$ that there is a unique choice of the constant C_{j-n} in $u_{j-n} = v_{j-n} + C_{j-n}$ such that the equation for u_j' in

(3.57) has a solution $u_j \in C^\infty(S^1)$. Note that the equation for u_j' is of the form

$$u_j'(\theta) = h_j(\theta), \tag{3.66}$$

where

$$h_j(\theta) = ib_0(j-n)(v_{j-n}+C_{j-n})+i\sum_{\ell=1}^{j-n-1} b_{j-n-\ell}\ell(v_\ell(\theta)+C_\ell)+f_j(\theta), \tag{3.67}$$

and $v_1, \ldots, v_{j-1}$ and $C_1, \ldots, C_{j-n-1}$ have already been uniquely determined. It suffices then to choose

$$C_{j-n} = \frac{-1}{2\pi b_0(j-n)}\int_0^{2\pi}\Big[b_0(j-n)v_{j-n}(\theta)+\sum_{\ell=1}^{j-n-1} b_{j-n-\ell}\ell u_\ell(\theta)-i\int_0^{2\pi} f_j(\theta)\Big]d\theta.$$

We obtain

$$u_j(\theta) = v_j(\theta) + C_j, \tag{3.68}$$

where C_j is a constant and

$$v_j(\theta) = \int_0^\theta h_j(\sigma)d\sigma, \tag{3.69}$$

with h_j given by (3.67). This concludes the proof of the solvability of the formal equation (3.54). We use Borel's theorem to obtain a function $u \in C^\infty(A_\epsilon)$ having the expansion $\sum_{j\geqslant 0} u_j(\theta)r^j$; it follows that $g \doteq R_n u - f$ is flat at $r = 0$. Now Proposition 3.2 yields w such that $R_n w = g$; we obtain $R_n(u - w) = f$. The proof of the theorem is complete. □

4. The equation $R_n u = pu$.

In this section we seek nonzero solutions to the equation

$$R_n u = pu \quad \text{in } A_\epsilon, \tag{4.1}$$

where R_n is the vector field defined in (1.7), A_ϵ is the ring $(-\epsilon,\ \epsilon) \times S^1$, and $p \in C^\infty(A_\epsilon)$ is a given function. The corresponding problem for the vector field T_λ was studied in [M3]. Our main result for this equation is the following:

THEOREM 4.1. — *If ϵ is small enough, then equation* (4.1) *has a solution $u \in C^\infty(A_\epsilon)$ with $u \neq 0$ on $\{r = 0\}$ if and only if p satisfies*

$$\frac{1}{2\pi i}\int_0^{2\pi} p(0,\theta)d\theta \in \mathbb{Z}, \tag{4.2}$$

and

$$\int_0^{2\pi} \left(\frac{\partial}{\partial r}\right)^j p(0,\theta)d\theta = 0, \quad j = 1, \ldots, n. \tag{4.3}$$

Before proving this theorem, we fix some notations that will be used throughout the remainder of the article. We write the Taylor expansion

$$p(r,\theta) \sim \sum_{j\geqslant 0} p_j(\theta) r^j, \tag{4.4}$$

and define, for $j \geqslant 0$,

$$P_j(\theta) = \int_0^\theta p_j(\sigma)d\sigma, \tag{4.5}$$

$$\lambda_j = \frac{1}{2\pi}\int_0^{2\pi} p_j(\theta)d\theta, \tag{4.6}$$

and

$$Q_j(r,\theta) = \sum_{k=0}^{j} P_k(\theta) r^k. \tag{4.7}$$

We remark, for further use, that P_j is periodic if and only if $\lambda_j = 0$, and that e^{P_0} is periodic if and only if $\lambda_0 \in i\mathbb{Z}$, which is the same as (4.2). We also remark that (4.3) is the same as saying that $\lambda_1 = \cdots = \lambda_n = 0$.

Proof of Theorem 4.1. — We first prove the necessity. Let $u \in C^\infty(A_\epsilon)$ be a solution to (4.1) with $u \neq 0$ over $\{r = 0\}$. Then $u_0(\theta) \doteq u(0,\theta)$ satisfies

$$u_0' = p_0 u_0. \tag{4.8}$$

Clearly, (4.8) has a nontrivial solution if and only if (4.2) holds. We proceed to prove (4.3) by contradiction. If (4.3) does not hold then there is a unique integer N with $1 \leqslant N \leqslant n$ such that $\lambda_N \neq 0$, and, if $N > 1$, then $\lambda_1 = \cdots = \lambda_{N-1} = 0$. Set

Then the functions $e^{P_0(\theta)}, P_1(\theta), \ldots, P_{N-1}(\theta)$ are 2π-periodic. Set

$$w(r,\theta) = u(r,\theta)e^{-Q_{N-1}(r,\theta)},$$

where Q_{N-1} is defined in (4.7). Then $u = we^{Q_{N-1}}$, and (4.1) becomes

$$\frac{\partial w}{\partial \theta} + \left(\sum_{k=0}^{N-1} p_k(\theta) r^k\right) w - i r^{n+1} b(r)\left(\frac{\partial w}{\partial r} + \left(\sum_{\nu=1}^{N-1} \nu P_\nu(\theta) r^{\nu-1}\right) w\right) = pw. \tag{4.9}$$

From (4.9), we obtain

$$\frac{\partial w}{\partial \theta} \simeq_{(n)} \Big(\sum_{k=N}^{n} p_k(\theta) r^k\Big) w, \tag{4.10}$$

where the notation $f \simeq_{(\ell)} g$ means that the Taylor expansions, in the variable r, of f and g agree to order $\geqslant \ell$. Write

$$w(r,\theta) \sim \sum_{j \geqslant 0} w_j(\theta) r^j,$$

and obtain from (4.10)

$$w_0' + \cdots + w_n' r^n \simeq_{(n)} (p_N r^N + \cdots + p_n r^n)(w_0 + w_1 r + \cdots + w_n r^n).$$

Hence

$$w_0' + w_1' r + \cdots + w_N' r^N \simeq_{(N-1)} w_0 p_N r^N.$$

Thus

$$0 = w_0'(\theta) = \cdots = w_{N-1}'(\theta);$$

in particular, $w_0(\theta) = C_0$, with $C_0 \neq 0$ by assumption. We also have

$$w_N' = p_N w_0 = C_0 p_N. \tag{4.11}$$

Now (4.11) has a periodic solution if and only if $\lambda_N = 0$, a contradiction which completes the proof of necessity.

We now prove the sufficiency. Look for a solution of the form

$$u(r,\theta) = v(r,\theta) e^{P_0(\theta)}.$$

Then (4.1) becomes

$$R_n v = \tilde{p} v, \tag{4.12}$$

where $\tilde{p}(r,\theta) = p(r,\theta) - p_0(\theta)$. Note that $\tilde{p}$ satisfies $\int_0^{2\pi} \tilde{p}_j(\theta) d\theta = 0, \ \ j = 0, \ldots, n$. Now Theorem 3.2 implies the existence of $\varphi \in C^\infty(A_\epsilon)$ such that $R_n \varphi = \tilde{p}$. It follows that $v \doteq e^{\varphi}$ is a solution to (4.12) and $u \doteq e^{\varphi + P_0}$ is a solution to (4.1); clearly, $u \neq 0$. The proof is complete. □

5. The equation $R_n u = pu + f$.

In this section, we consider the C^∞ solvability of the equation

$$R_n u = pu + f \quad \text{ in } A_\epsilon, \tag{5.1}$$

with R_n and the ring A_ϵ as in the previous section and where $p,\ f \in C^\infty(A_\epsilon)$. The corresponding equation for the vector field T_λ is considered in [M3]. We will use the notation established in section 4. Our first result is

THEOREM 5.1. — *Assume that p satisfies* (4.2) *and* (4.3). *If ϵ is small enough then, given $f \in C^\infty(A_\epsilon)$, equation* (5.1) *has a solution $u \in C^\infty(A_\epsilon)$ if and only if f satisfies*

$$\int_0^{2\pi} \left(\frac{\partial}{\partial r}\right)^j (fe^{-Q_n})(0,\theta)d\theta = 0, \quad j = 0,\dots,n, \tag{5.2}$$

where Q_n is as in (4.7).

Proof. — As in the proof of Theorem 3.2, there is $\varphi \in C^\infty(A_\epsilon)$ such that $R_n\varphi(r,\theta) = p(r,\theta) - p(0,\theta)$. Moreover, we have $\varphi(0,\theta) = C$, for some constant C, since

$$\frac{\partial\varphi}{\partial\theta}(r,\theta) - ir^{n+1}b(r)\frac{\partial\varphi}{\partial r}(r,\theta) = p(r,\theta) - p(0,\theta),$$

hence at $r = 0$ we obtain $\frac{\partial\varphi}{\partial\theta}(0,\theta) = 0$. Thus we may assume that $\varphi(0,\theta) = 0$ (replace, if necessary, $\varphi(r,\theta)$ by $\varphi(r,\theta) - \varphi(0,\theta)$).

Set $h(r,\theta) = e^{P_0(\theta)+\varphi(r,\theta)}$, and $u = hv$. Then equation (5.1) becomes

$$R_n v = \frac{f}{h}. \tag{5.3}$$

Now (5.3) has a solution $v \in C^\infty(A_\epsilon)$ if and only if $\frac{f}{h}$ satisfies the following conditions:

$$\int_0^{2\pi} \left(\frac{\partial}{\partial r}\right)^j \left(\frac{f}{h}\right)(0,\theta)d\theta = 0, \quad j = 0,\dots,n. \tag{5.4}$$

It remains to show that (5.4) is the same as (5.2). For this it suffices to prove that

$$(\frac{\partial}{\partial r})^j(P_0 + \varphi - Q_n)(0,\theta) = 0, \quad j = 0,\dots,n. \tag{5.5}$$

For $j = 0$, this is immediate. Straightforward computations show that

$$R_n(P_0 + \varphi - Q_n)(r,\theta) = O(r^{n+1}),$$

hence

$$\frac{\partial}{\partial\theta}(P_0 + \varphi - Q_n)(r,\theta) = O(r^{n+1});$$

now an integration yields

$$P_0(\theta) + \varphi(r,\theta) - Q_n(r,\theta) = O(r^{n+1}),$$

from which (5.5) follows immediately. The proof is complete. □

Remark 5.1. — The conditions in (5.2) involve only the derivatives $(\frac{\partial}{\partial r})^k f(0,\theta)$ and $(\frac{\partial}{\partial r})^\ell p(0,\theta), \quad k,\ell = 0,\dots,n.$

From now on, we restrict ourselves to the situation where either (4.2) or (4.3) fails; hence, if $\lambda_0 \in i\mathbb{Z}$, then there is an integer j with $1 \leqslant j \leqslant n$ such that $\lambda_j \neq 0$. Our next goal is to study the solvability, modulo flat functions, of equation (5.1).

Given $f \in C^\infty(A_\epsilon)$, we say that (5.1) is solvable modulo flat functions if there is $u \in C^\infty(A_\epsilon)$ such that

$$R_n u - pu - f \quad \text{is flat at } r = 0. \tag{5.6}$$

When this happens we write

$$R_n u \sim pu + f. \tag{5.7}$$

Proposition 5.1. — *Assume that either* (4.2) *or* (4.3) *fails. With the above notations we have, for* ϵ *small enough:*

(1) *If* $\lambda_0 \notin i\mathbb{Z}$, *then* (5.1) *is solvable modulo flat functions for every* $f \in C^\infty(A_\epsilon)$.

(2) *Assume that* $\lambda_0 \in i\mathbb{Z}$. *Let* N *be the smallest integer with* $1 \leqslant N \leqslant n$ *such that* $\lambda_N \neq 0$. *Assume, furthermore, that either*

$$1 \leqslant N < n,$$

or else

$$N = n, \quad \textit{and} \quad \lambda_n + i\ell b_0 \neq 0, \quad \forall \ell \in \{1, 2, \ldots\}.$$

Then (5.1) *is solvable modulo flat functions if and only if* f *satisfies*

$$\int_0^{2\pi} \left(\frac{\partial}{\partial r}\right)^j (fe^{-Q_{N-1}})(0,\theta)d\theta = 0, \quad j = 0, \ldots, N-1. \tag{5.8}$$

(3) *Assume that* $\lambda_0 \in i\mathbb{Z}$. *Let* N *be the smallest integer with* $1 \leqslant N \leqslant n$ *such that* $\lambda_N \neq 0$. *Assume, furthermore, that*

$$N = n, \quad \textit{and} \quad \lambda_n + i\ell_0 b_0 = 0, \quad \textit{for some} \quad \ell_0 \in \{1, 2, \ldots\}.$$

Then the set of all $f \in C^\infty(A_\epsilon)$ *for which* (5.1) *is solvable modulo flat functions make up a subspace having finite codimension equal to* $n+1$. *More precisely,* f *must satisfy, in addition to*

$$\int_0^{2\pi} \left(\frac{\partial}{\partial r}\right)^j (fe^{-Q_{n-1}})(0,\theta)d\theta = 0, \quad j = 0, \ldots, n-1, \tag{5.9}$$

a condition bearing on the derivatives of f *of order up to* $n + \ell_0$.

Example 5.1. — In the special case when $b(r) \equiv 1$ and $p(r,\theta) = -ir^n$, we have $\ell_0 = b_0 = 1$ and the additional condition is simply $\int_0^{2\pi} f_{n+1}(\theta)\, d\theta = 0$.

Proof of Proposition 5.1. — Assume first that $\lambda_0 \notin i\mathbb{Z}$. Write $u(r,\theta) \sim \sum_{j\geqslant 0} u_j(\theta) r^j$, and $f(r,\theta) \sim \sum_{j\geqslant 0} f_j(\theta) r^j$. Then

$$R_n u \sim pu + f \tag{5.10}$$

means

$$\sum_{j\geqslant 0} u_j' r^j - \sum_{j\geqslant n+1} \left\{ \sum_{\ell=1}^{j-n} i b_{j-n-\ell} \ell u_\ell \right\} r^j \sim \sum_{j\geqslant 0} \left(\sum_{\ell=0}^{j} p_\ell u_{j-\ell} \right) r^j + \sum_{j\geqslant 0} f_j r^j. \tag{5.11}$$

We obtain

$$u_j' = \sum_{\ell=0}^{j} p_\ell u_{j-\ell} + f_j, \quad j = 0, \ldots, n, \tag{5.12}$$

and

$$u_j' = \sum_{\ell=0}^{j} p_\ell u_{j-\ell} + i \sum_{\ell=1}^{j-n} b_{j-n-\ell} \ell u_\ell + f_j, \quad j \geqslant n+1. \tag{5.13}$$

Since $\lambda_0 \notin i\mathbb{Z}$, the first equation in (5.12), namely, $u_0' = p_0 u_0 + f_0$, has a unique periodic solution for any given f_0. Let now $j \geqslant 1$ and assume that $u_0, \ldots, u_{j-1}$ have been found. The equation for u_j in (5.12) or (5.13) is of the form $u_j' = p_0 u_j + g_j$, where $g_j \doteq \sum_{\ell=1}^{j} p_\ell u_{j-\ell} + i \sum_{\ell=1}^{j-n} b_{j-n-\ell} \ell u_\ell + f_j$ is a known function in $C^\infty(S^1)$. Since $\lambda_0 \notin i\mathbb{Z}$, there is a unique solution $u_j \in C^\infty(S^1)$, and the proof is complete in the first case.

Assume now that $\lambda_0 \in i\mathbb{Z}$, and let N be as in the statement. Set $w(r,\theta) = u(r,\theta) e^{-Q_{N-1}(r,\theta)}$. Note that $e^{P_0}, P_1, \ldots, P_{N-1}$ are in $C^\infty(S^1)$. Now equation (5.1) is the same as

$$\frac{\partial w}{\partial \theta} + \left(\sum_{k=0}^{N-1} p_k(\theta) r^k \right) w - i r^{n+1} b(r) \left(\frac{\partial w}{\partial r} + \left(\sum_{\nu=1}^{N-1} \nu P_\nu(\theta) r^{\nu-1} \right) w \right) = pw + f e^{-Q_{N-1}},$$

or, equivalently,

$$R_n w = \hat{p} w + g, \tag{5.14}$$

where $g = f e^{-Q_{N-1}}$, and

$$\hat{p}(r,\theta) = p(r,\theta) - \sum_{k=0}^{N-1} p_k(\theta) r^k + i r^{n+1} b(r) \sum_{\nu=1}^{N-1} \nu P_\nu(\theta) r^{\nu-1}.$$

Note that

$$\hat{p}(r,\theta) \sim \sum_{k\geqslant N} p_k(\theta) r^k + i r^{n+1} b(r) \sum_{\nu=1}^{N-1} \nu P_\nu(\theta) r^{\nu-1}.$$

Now w solves formally (5.14) if and only if

$$(5.15)\quad \sum_{j\geqslant 0} w_j' r^j - \sum_{j\geqslant n+1}\Big\{\sum_{\ell=1}^{j-n} i b_{j-n-\ell}\ell w_\ell\Big\} r^j \sim \Big\{\sum_{k\geqslant N} p_k r^k + i r^{n+1}\Big(\sum_{j\geqslant 0} b_j r^j\Big)\sum_{\nu=1}^{N-1}\nu P_\nu r^{\nu-1}\Big\}\Big(\sum_{j\geqslant 0} w_j r^j\Big) + \sum_{j\geqslant 0} g_j r^j.$$

For each $j = 0, 1, \ldots$, we obtain an equation for w_j' which must have a solution in $C^\infty(S^1)$ if we are to have formal solvability of (5.1). The first N equations are

$$(5.16)\qquad w_j' = g_j, \quad j = 0, \ldots, N-1,$$

and they have periodic solutions if and only if

$$\int_0^{2\pi} g_j(\theta)d\theta = 0, \quad j = 0, \ldots, N-1.$$

This completes the proof of the necessity of (5.8) in case (2) and of (5.9) in case (3).

We now concentrate on case (2) and prove the sufficiency of (5.8). Assume that (5.8) holds. Then the general solution of (5.16) is

$$w_j(\theta) = \hat{w}_j(\theta) + C_j, \quad j = 0, \ldots, N-1,$$

where C_j is a constant and

$$\hat{w}_j(\theta) = \int_0^\theta g_j(\sigma)d\sigma, \quad j = 0, \ldots, N-1.$$

Let us consider the next equation arising from (5.15), namely, $w_N' = p_N w_0 + g_N$, or, equivalently,

$$(5.17)\qquad w_N' = p_N C_0 + p_N \hat{w}_0 + g_N.$$

Now (5.17) has a solution $w_N \in C^\infty(S^1)$ if and only if

$$(5.18)\qquad \int_0^{2\pi}\{p_N C_0 + p_N\hat{w}_0 + g_N\}(\theta)d\theta = 0.$$

There is a unique value of C_0, namely,

$$C_0 = \frac{-1}{2\pi\lambda_N}\Big(\int_0^{2\pi} p_N\hat{w}_0 d\theta + \int_0^{2\pi} g_N d\theta\Big),$$

such that (5.18) holds. We obtain $w_N(\theta) = \hat{w}_N(\theta) + C_N$, where C_N is a constant and $\hat{w}_N(\theta) = \int_0^\theta h_N(\sigma)d\sigma$, with $h_N = p_N C_0 + p_N\hat{w}_0 + g_N$.

More generally, for indices j such that $N \leqslant j \leqslant n$, we prove by induction on j that there is a unique choice of the constant C_{j-N} in

$$w_{j-N} = \hat{w}_{j-N} + C_{j-N} \tag{5.19}$$

such that the equation for w_j' has a solution $w_j \in C^\infty(S^1)$. Now, if $N \leqslant j \leqslant n$, the equation for w_j' arising from (5.15) is of the form $w_j' = \sum_{\ell=N}^{j} p_\ell w_{j-\ell} + g_j$, or

$$w_j' = p_N w_{j-N} + \chi_j, \tag{5.20}$$

where $\chi_j \in C^\infty(S^1)$ depends only on $g_j, w_0, \ldots, w_{j-N-1}$ and has already been uniquely determined.

It suffices then to choose $C_{j-N} = \frac{-1}{2\pi\lambda_N}\{\int_0^{2\pi} p_N \hat{w}_{j-N} d\theta + \int_0^{2\pi} \chi_j d\theta\}$. We obtain $w_j(\theta) = \int_0^\theta h_j(\sigma)d\sigma + C_j$, with $h_j = p_N C_{j-N} + p_N \hat{w}_{j-N} + \chi_j$, which concludes the induction procedure when $N \leqslant j \leqslant n$.

We now consider the indices j such that $j \geqslant n+1$. In this case the equation for w_j' is of the form

$$\begin{aligned} w_j' - i\sum_{\ell=1}^{j-n} b_{j-n-\ell}\ell w_\ell \\ = \sum_{\ell=N}^{j} p_\ell w_{j-\ell} + i\sum_{k=0}^{j-n-1}\sum_{\nu=0}^{N-2}(\nu+1)P_{\nu+1}b_{k-\nu}w_{j-n-1-k} + g_j. \end{aligned} \tag{5.21}$$

Note that some of the summations in the last formula may be vacuous; for instance, if $N \leqslant j \leqslant n$, then (5.21) becomes (5.20), whereas if $0 \leqslant j \leqslant N-1$, then it becomes simply (5.16).

Note also that, up to this point, we have already chosen values for the following constants: $C_0, \ldots, C_{n-N}$ (it may be the case that $N = n$, and hence that we only have chosen C_0).

Let us consider first the case when $N < n$. In this case, the equation for w_j' is, as before, of the form

$$w_j' = p_N w_{j-N} + \chi_j,$$

hence there is no problem in this case.

The remaining case is when $N = n$. In the situation (2) of our proposition we then have that $\lambda_n + i\ell b_0 \neq 0, \quad \forall \ell \in \{1, 2, \ldots\}$.

The equation for w_j', for $j \geqslant n+1$, is of the form

$$w_j' = \big[p_n + i(j-n)b_0\big] w_{j-n} + \chi_j, \tag{5.22}$$

where χ_j has already been uniquely determined. The assumption that $\lambda_n + i\ell b_0 \neq 0, \quad \forall \ell \in \{1, 2, \ldots\}$ is precisely what is needed for the above equation to have a (unique) solution in $C^\infty(S^1)$ The proof is complete in case (2) of our statement.

We finally move on to case (3) of our proposition; then there is $\ell_0 \in \{1, 2, \ldots\}$, such that $\lambda_n + i\ell_0 b_0 = 0$. Now, if the index j satisfies $n+1 \leqslant j \leqslant n+\ell_0-1$, then the equation for w_j' is of the form (5.22) above; since the coefficient of w_{j-n} in (5.22) has nonzero average, we again have unique solutions . In the next equation, which corresponds to $j = n+\ell_0$, such a coefficient vanishes and we are left with

$$w_{n+\ell_0}' = \chi,$$

where χ has already been uniquely determined (in other words, all the constants $C_0, ..., C_{\ell_0-1}$ have already been uniquely chosen.) Thus the only possibility for the existence of a periodic solution is if $\int_0^{2\pi} \chi = 0$, which is precisely the condition alluded to in our statement of Proposition 5.2. After this, there will be no further trouble, for in the remaining equations the average of the coefficient of w_{j-n} in the right-hand side does not vanish if $j \geqslant n+\ell_0+1$. The proof is complete. □

We have now reduced our problem of solvability of (5.1) to the case where f is flat at $r = 0$, and, furthermore, either $\lambda_0 \notin i\mathbb{Z}$, or $\lambda_j \neq 0$ for some $j = 1, \ldots, n$. Our next goal is to reduce to the case of a simpler term of order zero.

Proposition 5.2. — *Assume that either* (4.2) *or* (4.3) *fails. Define the integer N as follows:*

(1) *if $\lambda_0 \notin i\mathbb{Z}$, set $N = 0$;*

(2) *if $\lambda_0 \in i\mathbb{Z}$, let N be the smallest integer with $1 \leqslant N \leqslant n$ such that $\lambda_N \neq 0$. If ϵ is small enough and if the equation*

$$R_n w = \left\{ \sum_{k=N}^{n} \lambda_k r^k \right\} w + h \tag{5.23}$$

has a solution $w \in C^\infty(A_\epsilon)$ for every flat h, then (5.1) *has a solution $u \in C^\infty(A_\epsilon)$ for every flat f.*

Proof. — Write $p(r,\theta) = \sum_{k=0}^{n} p_k(\theta) r^k + r^{n+1}\chi(r,\theta)$, with $\chi \in C^\infty(A_\epsilon)$. Set

$$\tilde{p}_k(\theta) = p_k(\theta) - \lambda_k, \quad \widetilde{P}_k(\theta) = \int_0^\theta \tilde{p}_k(\sigma) d\sigma, \quad k = 0, \ldots, n,$$

$$\widetilde{Q}_n(r,\theta)=\sum_{k=0}^{n}\widetilde{P}_k(\theta)r^k,\quad v(r,\theta)=u(r,\theta)e^{-\widetilde{Q}_n(r,\theta)},\text{and}\quad g=fe^{-\widetilde{Q}_n}.$$

Note that g is flat and $e^{\widetilde{P}_0(\theta)},\widetilde{P}_1(\theta),\ldots,\widetilde{P}_n(\theta)$ are 2π-periodic. Equation (5.1) is equivalent to

$$R_n v=\hat{p}v+g, \tag{5.24}$$

where $\hat{p}(r,\theta)=p(r,\theta)-\sum_{k=0}^{n}\tilde{p}_k(\theta)r^k+ir^{n+1}b(r)\sum_{\nu=1}^{n}\nu\widetilde{P}_\nu(\theta)r^{\nu-1}$. We may write $\hat{p}(r,\theta)=\sum_{k=N}^{n}\lambda_k r^k+p^\sharp(r,\theta)$, where $p^\sharp(r,\theta)=r^{n+1}\{\chi(r,\theta)+i\sum_{\nu=1}^{n}\nu\widetilde{P}_\nu(\theta)r^{\nu-1}\}$. Now Theorem 3.2 applies to yield $\varphi\in C^\infty(A_\epsilon)$ such that $R_n\varphi=p^\sharp$ in A_ϵ. Set $w=ve^{-\varphi}$ and $h=ge^{-\varphi}$. Then (5.24) becomes

$$R_n w=\left\{\sum_{k=N}^{n}\lambda_k r^k\right\}w+h.$$

Note that the function $h=fe^{-\varphi-\widetilde{Q}_n}$ is flat. The reduction is complete. □

We now deal with the solvability of (5.23). A few words about the proof are in order. The proof uses Fourier series in the variable θ. One obtains $w_k(r)$ in terms of $h_k(r)$. One is then led to estimate the Fourier coefficients $w_k(r)$ and their derivatives. The key ideas for the estimates are inspired in the paper [BCP]; note that the main concern in [BCP] was with operators of the form $L=\frac{\partial}{\partial\theta}-ir^{n+1}b(r,\theta)\frac{\partial}{\partial r}$, with b real-valued, and with the equation $Lu=f$, that is, no term of order zero was present, and therefore the results there cannot be applied to our operators in a straightforward manner. Furthermore, at this point of the present paper, our operators have been simplified considerably along the way and, therefore, it is now possible to use partial Fourier series and to compute explicitly the solutions to the corresponding ode's.

Proposition 5.3. — *Assume that either* (4.2) *or* (4.3) *fails. Define the integer N as follows:*

(1) if $\lambda_0\notin i\mathbb{Z}$, set $N=0$;

(2) if $\lambda_0\in i\mathbb{Z}$, let N be the smallest integer with $1\leqslant N\leqslant n$ such that $\lambda_N\neq 0$. Then the equation (5.23) *has a solution $w\in C^\infty(A_\epsilon)$ for every flat h.*

Proof. — We will write the term of order zero as $\sum_{k=0}^{n}\lambda_k r^k$ where, if $N\geqslant 1$, it is understood that $0=\lambda_0=\cdots=\lambda_{N-1}$. We write the Fourier series

$$w(r,\theta)=\sum_{k\in\mathbb{Z}}w_k(r)e^{ik\theta},\quad\text{and}\quad h(r,\theta)=\sum_{k\in\mathbb{Z}}h_k(r)e^{ik\theta}. \tag{5.25}$$

In order to solve (5.23) we must solve, for each $k \in \mathbb{Z}$, the ordinary differential equation $ikw_k - ir^{n+1}b(r)w_k' = (\lambda_0 + \lambda_1 r + \cdots + \lambda_n r^n)w_k + h_k$, or

$$w_k' + q_k w_k = g_k, \quad k \in \mathbb{Z}, \tag{5.26}$$

where

$$q_k(r) = -r^{-n-1}(rP'(r) - nP(r) + \mu r^n)\big[(k + i\lambda_0) + i\lambda_1 r + \cdots + i\lambda_n r^n\big],$$

and

$$g_k(r) = ir^{-n-1}(rP'(r) - nP(r) + \mu r^n)h_k(r).$$

Notice that the function $g(r,\theta) \doteq ir^{-n-1}(rP'(r) - nP(r) + \mu r^n)h(r,\theta)$ is also C^∞ and flat at $r = 0$.

We may write

$$q_k(r) = \sum_{\ell=-n-1}^{n-1} q_{k,\ell} r^\ell,$$

where $q_{k,-n-1} = nP(0)(k + i\lambda_0)$. Take $\Phi_k(r)$ to be the following primitive of q_k :

$$\Phi_k(r) = \ln(r^{q_{k,-1}}) + \left\{\sum_{\ell=-n}^{-1} + \sum_{\ell=1}^{n}\right\} \frac{q_{k,\ell-1} r^\ell}{\ell}.$$

The coefficient of r^{-n} in $\Phi_k(r)$ is $-P(0)(k+i\lambda_0)$. Now equation (5.26) is equivalent to

$$\frac{d}{dr}(w_k e^{\Phi_k}) = g_k e^{\Phi_k}.$$

We divide the analysis in two regions, namely,

REGION I: $0 < r < R$,
REGION II: $-R < r < 0$.

We further divide the study in three cases, depending on the value of

$$\Re(-P(0)(k + i\lambda_0)),$$

namely,

CASE I. $\Re(-P(0)(k + i\lambda_0)) < 0$,
CASE II. $\Re(-P(0)(k + i\lambda_0)) > 0$,
CASE III. $\Re(-P(0)(k + i\lambda_0)) = 0$.

In cases I and II, we may assume that R is so small that the term $-P(0)(k+i\lambda_0)r^{-n}$ dominates the other terms in $\Phi_k(r)$; more precisely, we have

$$\Phi_k(r) = -P(0)(k+i\lambda_0)r^{-n}(1+\psi_k(r)), \tag{5.27}$$

and the following holds: given any $\sigma > 0$, there is $R > 0$ such that $|\psi_k(r)| < \sigma$, for all $0 < r < R$, and for all k in cases I and II. We may also assume that R is so small that $\Re\Phi_k(r)$ is monotonic over the interval $(0, R)$.

Notice that, since $\Re P(0) < 0$, there exist $k_1, k_2 \in \mathbb{Z}$ such that case I occurs for all $k \leqslant k_1$, while case II occurs for all $k \geqslant k_2$; also, case III occurs for at most one value of k.

We first concentrate on the region I and analyze each of the three cases there. In each case, we will write an explicit formula for the solution w_k to equation (5.26).

Case I. $0 < r < R, \quad \Re(-P(0)(k+i\lambda_0)) < 0 \quad (k \leqslant k_1).$

Define

$$w_k(r) = \int_0^r g_k(s)e^{\Phi_k(s)-\Phi_k(r)}ds, \quad 0 < r < R, \quad k \leqslant k_1. \tag{5.28}$$

We will prove that $\forall \nu, j \in \mathbb{Z}_+$, $\forall k \leqslant k_1$, one has

$$\sup\Big\{|r^{-\nu}w_k^{(j)}(r)|;\ 0 < r < R\Big\} < \infty. \tag{5.29}$$

and that $\forall N, \nu, j \in \mathbb{Z}_+$, one has

$$\sup\Big\{|k^N r^{-\nu}w_k^{(j)}(r)|;\ 0 < r < R,\ k \leqslant k_1,\ k \neq 0\Big\} < \infty. \tag{5.30}$$

Recall that we have assumed that R is so small that $\Re\Phi_k(r)$ is monotonic, hence we have

$$\Re\Phi_k(s) - \Re\Phi_k(r) \leqslant 0, \quad 0 < s \leqslant r \leqslant R.$$

It is not difficult to see that w_k, defined by (5.28), is C^∞ in the open interval $(0, R)$, and that it satisfies equation (5.26) there.

Let us prove the estimates for $j = 0$. We have

$$\begin{aligned}
|k^N r^{-\nu}w_k(r)| &\leqslant \int_0^r r^{-\nu}e^{\Re\Phi_k(s)-\Re\Phi_k(r)}|k^N g_k(s)|ds \\
&\leqslant \int_0^r s^\nu r^{-\nu}|k^N s^{-\nu}g_k(s)|ds \\
&\leqslant r\sup\{|k^N s^{-\nu}g_k(s)|;\ 0 < s < r\} \\
&\leqslant R\sup\{|k^N s^{-\nu}g_k(s)|;\ 0 < s < R,\ k \leqslant k_1\} \leqslant M < \infty,
\end{aligned}$$

where we have used the fact that g is C^∞ on $[0,R)$ and is flat at $r=0$.

Note that, if $k_0 \geqslant 0$, then we must also take care of the case $k=0$ in (5.29) here. Clearly, a computation similar to the one above shows that for every $\nu \in \mathbb{Z}_+$, there exists $M = M(\nu) > 0$, such that $|r^{-\nu} w_0(r)| \leqslant M, \quad 0<r<R$. This completes the proof of estimates (5.29) and (5.30) in the case $j=0$.

We now prove the estimates in the case of $j=1$. We have, for each $0<r<R$, and each $k \leqslant k_1$,

$$w_k'(r) = \int_0^r (-q_k(r)) g_k(s) e^{\Phi_k(s)-\Phi_k(r)} ds + g_k(r). \tag{5.31}$$

We have, for some constant $C>0$, and for all $k \neq 0$,

$$|r^{n+1} q_k(r)| \leqslant C|k|. \tag{5.32}$$

Therefore, we obtain, for each $0<r<R$, and each $k \leqslant k_1$, with $k \neq 0$,

$$\begin{aligned}
&|k^N r^{-\nu} w_k'(r)| \\
&\quad \leqslant C \int_0^r s^{\nu+n+1} r^{-\nu-n-1} |k^{N+1} s^{-\nu-n-1} g_k(s)| ds + |k^N r^{-\nu} g_k(r)| \\
&\quad \leqslant rC \sup\{|k^{N+1} s^{-\nu-n-1} g_k(s)|;\ 0<s<r,\ k \leqslant k_1\} + |k^N r^{-\nu} g_k(r)| \\
&\quad \leqslant M < \infty,
\end{aligned}$$

where we have used again the fact that g is C^∞ on $[0,R)$ and is flat at $r=0$.

A similar proof works for the other values of j.

When $k=0$ a similar computation shows that $\forall \nu, j \in \mathbb{Z}_+$, there exists $M = M(\nu, j) > 0$, such that

$$|r^{-\nu} w_0^{(j)}(r)| \leqslant M, \quad \forall 0<r<R.$$

We have concluded the proof of (5.29) and (5.30).

It then follows from (5.29) that, for each $k \leqslant k_1$ and each $j = 0, 1, \ldots$, the limit $\lim_{r\to 0^+} w_k^{(j)}(r)$ exists and is equal to 0, which in turn implies that each such w_k defines a function which is C^∞ in $[0,R)$ and is flat at $r=0$. In fact, the full strength of the estimates (5.29) and (5.30) imply that the function $\sum_{k\leqslant k_1} w_k(r) e^{i\theta}$ is in $C^\infty([0,R) \times S^1)$ and is flat along $r=0$.

This concludes the analysis of case I.

Case II. $0<r<R, \quad \Re(-P(0)(k+i\lambda_0)) > 0 \quad (k \geqslant k_2)$.

Define

$$w_k(r) = -\int_r^R g_k(s)e^{\Phi_k(s)-\Phi_k(r)}ds \quad 0 < r < R, \quad k \geqslant k_2. \tag{5.33}$$

It is clear that w_k, defined by this formula, is C^∞ in the open interval $(0, R)$, and that it satisfies the ode there. In fact, w_k is a product of two smooth functions on $(0, R)$, that is,

$$w_k(r) = -e^{-\Phi_k(r)} \times \int_r^R g_k(s)e^{\Phi_k(s)}ds.$$

The relevant estimates in case II are

$$\sup\{|k^N r^{-\nu} w_k^{(j)}(r)|; 0 < r < R, k \geqslant k_3\} < \infty, \tag{5.34}$$

where $k_3 \geqslant k_2$ will be chosen later.

As in [BCP], it will be useful to decompose w_k into two integrals, namely,

$$w_k(r) = w_{k,1} + w_{k,2},$$

where

$$w_{k,1}(r) = -\int_r^{2r} g_k(s)e^{\Phi_k(s)-\Phi_k(r)}ds \tag{5.35}$$

and

$$w_{k,2}(r) = -\int_{2r}^R g_k(s)e^{\Phi_k(s)-\Phi_k(r)}ds. \tag{5.36}$$

Let us first prove the estimates for $w_{k,1}(r)$ and for $j = 0$. Let us observe right away that in the integral defining $w_{k,1}$, we have $r \leqslant s$, and $\Re\Phi_k(s) - \Re\Phi_k(r) \leqslant 0$, hence

$$\begin{aligned} |k^N r^{-\nu} w_{k,1}(r)| &\leqslant \int_r^{2r} s^\nu r^{-\nu} e^{\Re\Phi_k(s)-\Re\Phi_k(r)} |k^N s^{-\nu} g_k(s)|ds \\ &\leqslant \int_r^{2r} |k^N s^{-\nu} g_k(s)|ds \\ &\leqslant r \sup\big\{|k^N s^{-\nu} g_k(s)|;\ r < s < 2r\big\} \\ &\leqslant R \sup\big\{|k^N s^{-\nu} g_k(s)|;\ 0 < s < R,\ k \geqslant k_2\big\} \\ &\leqslant M < \infty, \end{aligned} \tag{5.37}$$

where we have used the fact that g is C^∞ on $[0, R)$ and is flat at $r = 0$.

Before we begin studying the component $w_{k,2}(r)$, we study the function $e^{\Psi_k(r)}$, where

$$\Psi_k(r) = \Re\Phi_k(2r) - \Re\Phi_k(r).$$

We claim that the function $e^{\Psi_k(r)}$ is flat at $r = 0$.

To prove this, let $\sigma > 0$ be a small number to be chosen later. We take $R > 0$ as in the discussion following (5.27). From (5.27) we obtain the estimate

$$|\Re\Phi_k(r) - \alpha_1 r^{-n}| < \alpha_1 \sigma r^{-n},$$

where $\alpha_1 \doteq \Re(-P(0)(k + i\lambda_0)) > 0$.

It follows that $\Re\Phi_k(2r) < \alpha_1(1 + \sigma)2^{-n}r^{-n}$, hence $\Re\Phi_k(2r) - \Re\Phi_k(r) < -\alpha_1 r^{-n}\{1 - \sigma - (1 + \sigma)2^{-n}\}$. Now choose σ so that $0 < \sigma < (2^n - 1)/(2^n + 1)$ and take the corresponding R. We obtain, for every $0 < r < R$, and for every k in case II, $\Psi_k(r) \leqslant -\alpha_1\tilde{\sigma}r^{-n}$, or

$$\Psi_k(r) \leqslant -\Re(-P(0)(k + i\lambda_0))\tilde{\sigma}r^{-n},$$

where $\tilde{\sigma} \doteq 1 - \sigma - (1 + \sigma)2^{-n} > 0$. This completes the proof of our claim.

We now claim that there exist $\alpha > 0$ and $k_3 \geqslant 1$ such that

$$\Psi_k(r) \leqslant -\alpha k r^{-n}, \quad 0 < r < R, \quad k \geqslant k_3. \tag{5.38}$$

The proof is straightforward; indeed, we have

$$\Re(-P(0)(k + i\lambda_0)) = \Re(-P(0))k + \Re P(0)\Im\lambda_0 + \Im P(0)\Re\lambda_0.$$

Now we require $k_3 \geqslant 1$, $k_3 \geqslant k_2$, and

$$-\Re P(0)k_3 \geqslant 2|\Re P(0)\Im\lambda_0 + \Im P(0)\Re\lambda_0|$$

to obtain,

$$\Psi_k(r) \leqslant 2^{-1}\Re P(0)kr^{-n}, \quad 0 < r < R, \quad k \geqslant k_3,$$

which is (5.38), with $\alpha = -\Re P(0)/2$.

We are now ready to estimate the term $w_{k,2}$, still in the case $j = 0$. We have

$$\begin{aligned}
|r^{-\nu}w_{k,2}(r)| &\leqslant \int_{2r}^{R} r^{-\nu}|g_k(s)|e^{\Re\Phi_k(s) - \Re\Phi_k(r)}ds \\
&\leqslant \int_{2r}^{R} r^{-\nu}|g_k(s)|e^{\Re\Phi_k(2r) - \Re\Phi_k(r)}ds \\
&\leqslant \int_{2r}^{R} |g_k(s)|r^{-\nu}e^{\Psi_k(r)}ds \\
&\leqslant (R - 2r) \times \sup\big\{|g_k(s)|; s < R\big\} \times r^{-\nu}e^{\Psi_k(r)} \\
&\leqslant R \times \sup\big\{|g_k(s)|; s < R\big\} \times \sup\big\{r^{-\nu}e^{\Psi_k(r)};\ 0 < r < R\big\}.
\end{aligned}$$

hence

$$|r^{-\nu}w_{k,2}(r)| \leqslant M < \infty.$$

Furthermore, for $k \geqslant k_3$, we have

$$
\begin{aligned}
|k^N r^{-\nu} w_{k,2}(r)| &\leqslant \int_{2r}^{R} k^N r^{-\nu} |g_k(s)| e^{\Re\Phi_k(s)-\Re\Phi_k(r)} ds \\
&\leqslant \int_{2r}^{R} k^N |g_k(s)| r^{-\nu} e^{\Psi_k(r)} ds \\
&\leqslant (R-2r) \times \sup\{|g_k(s)|; s < R\} \times k^N r^{-\nu} e^{-\alpha k r^{-n}} \\
&\leqslant R \times \sup\{|g_k(s)|; s < R\} \\
&\qquad \times \sup\{k^N r^{-\nu} e^{-\alpha k r^{-n}}; 0 < r < R, k \geqslant k_3\}.
\end{aligned} \tag{5.39}
$$

We may (and will) assume that $R < 1$; since $k_3 \geqslant 1$, we have

$$k^N r^{-\nu} \leqslant (kr^{-n})^{N'},$$

where $N' \geqslant \max\{N, \nu/n\}$, and therefore we obtain the estimates

$$k^N r^{-\nu} e^{-\alpha k r^{-n}} \leqslant (kr^{-n})^{N'} e^{-\alpha k r^{-n}},$$

$$\sup\{k^N r^{-\nu} e^{-\alpha k r^{-n}}; 0 < r < R, k \geqslant k_3\} \leqslant \sup\{s^{N'} e^{-\alpha s}; s > 0\}.$$

It follows that

$$|k^N r^{-\nu} w_{k,2}(r)| \leqslant M < \infty, \quad 0 < r < R,\ k \geqslant k_3. \tag{5.40}$$

The conjunction of (5.37) and (5.40) yield, in the case $j = 0$, the estimates in (5.34), namely

$$\sup\{|k^N r^{-\nu} w_k(r)|; 0 < r < R, k \geqslant k_3\} < \infty.$$

We now address the estimates (5.34) in the case $j = 1$, namely we claim that

$$\sup\{|k^N r^{-\nu} w'_k(r)|; 0 < r < R, k \geqslant k_3\} < \infty.$$

It is easy to see that

$$w'_k(r) = -\int_r^R (-q_k(r)) g_k(s) e^{\Phi_k(s)-\Phi_k(r)} ds + g_k(r),$$

which we may write as

$$w'_k = u_{k,1} + u_{k,2} + g_k,$$

where

$$
\begin{aligned}
u_{k,1}(r) &= \int_r^{2r} q_k(r) g_k(s) e^{\Phi_k(s)-\Phi_k(r)} ds, \\
u_{k,2}(r) &= \int_{2r}^{R} q_k(r) g_k(s) e^{\Phi_k(s)-\Phi_k(r)} ds.
\end{aligned}
$$

Let us comment briefly on how one proves these estimates. First of all, we recall from (5.32) that for some constant $C > 0$, and for all $k \neq 0$, we have $|r^{n+1}q_k(r)| \leqslant C|k|$. The term $q_k(r)$ in $u_{k,1}(r)$ can be absorbed in the estimates, in the same way as it was taken care of in the estimate of (5.31). More precisely, we have $0 < s \leqslant 2r$ in the integral, hence $r^{-\ell} \leqslant 2^\ell s^{-\ell}$, and so

$$\begin{aligned}
|r^{-\nu}q_k(r)| &\leqslant C|k|r^{-\nu-n-1} \leqslant 2^{\nu+n+1}C|k|s^{-\nu-n-1}\\
|k^N r^{-\nu}u_{k,1}(r)| &\leqslant C2^{\nu+n+1}\int_r^{2r} s^{\nu+n+1}r^{-\nu-n-1}|k^{N+1}s^{-\nu-n-1}g_k(s)|ds\\
&\leqslant rC2^{\nu+n+1}\sup\{|k^{N+1}s^{-\nu-n-1}g_k(s)|; 0 < s < r, k \geqslant k_3\}\\
&\leqslant M < \infty.
\end{aligned}$$

Similarly, we have

$$|k^N r^{-\nu}u_{k,2}(r)| \leqslant C\int_{2r}^{R} r^{-\nu-n-1}k^{N+1}e^{\Psi_k(r)}|g_k(s)|ds$$

and, from this point on, the proof continues as in (5.39).

Finally, the third term has already been estimated before.

The proof for smaller values of k and for larger values of j is similar and will be omitted.

We reach the conclusion that the function $\sum_{k\geqslant k_2} w_k(r)e^{i\theta}$ is in $C^\infty([0,R)\times S^1)$ and is flat along $r = 0$.

Case III. $0 < r < R, \quad \Re(-P(0)(k + i\lambda_0)) = 0.$

Recall that this case occurs for either one or no value of k. If there is no k such that $\Re(-P(0)(k+i\lambda_0)) = 0$, then nothing needs to be done. From now on, we assume that there is then a (unique) value of k, say, $k = k_4$, such that $\Re(-P(0)(k + i\lambda_0)) = 0$. What we must prove is that, for this value of k, the estimate (5.29) holds. In other words, we must prove that for $k = k_4$, and $\forall \nu, j \in \mathbb{Z}_+$, one has

$$\sup\{|r^{-\nu}w_k^{(j)}(r)|; 0 < r < R\} < \infty.$$

In other words, we have to prove that this w_k is C^∞ on $[0,R)$ and is flat at 0.

In this case, we have to look at the other terms appearing in the expression of q_k.

Sub-case III.1. There is $\ell \in \{-n, -n+1, \ldots, -2\}$ such that $\Re q_{k,\ell} \neq 0$, and $\Re q_{k,m} = 0$, for $m < \ell$.

In this case, a procedure similar to that used in cases I and II works to produce a flat solution; the term $\exp[\Re q_{k,\ell} r^{\ell+1}/(\ell+1)]$ dominates the others in a neighborhood of 0 and it absorbs all the powers of r^{-1}that appear when we differentiate the function $\phi_k(r)$.

Sub-case III.2. $\ell = -1$, that is, $\Re q_{k,-1} \neq 0$, and $\Re q_{k,m} = 0$, for $m < -1$.

In this case, the procedure is also similar to the previous ones, one difference being that we do not have a flat function arising from the exponential, but rather, we obtain a finite (in general, non-integral) power of r. More precisely, we have

$$\begin{aligned}
&\Re q_k(r) = (\Re q_{k,-1}) r^{-1}(1 + O(r))\\
&\Re \phi_k(r) = (\Re q_{k,-1}) \ln(r) + \gamma_1 r + \cdots + \gamma_n r^n\\
&e^{\Re\phi_k(r)} = r^{\Re q_{k,-1}} e^{\gamma_1 r + \cdots + \gamma_n r^n}
\end{aligned}$$

Another difference is that one formula suffices for the solutions, namely,

$$w_k(r) = \int_0^r g_k(s) e^{\Phi_k(s) - \Phi_k(r)} ds, \quad 0 < r < R.$$

Since g_k is C^∞ and flat and since e^{Φ_k} has at most a finite power singularity, it follows that the product $g_k e^{\Phi_k}$ is likewise C^∞ and flat; the same is true of the primitive $\int_0^r g_k(s) e^{\Phi_k(s)} ds$ and also of its product with $e^{-\Phi_k(r)}$, which is equal to w_k.

Sub-case III.3. $\ell \geqslant 0$, that is, $\Re q_{k,m} = 0$, for $m < 0$.

The proof in this case is omitted, since it is very similar to that of case III.2.

Summing up, we conclude that each function $w_k(r)$ is in $C^\infty[0, R)$ and is flat at $r = 0$; furthermore , the function $w(r, \theta) \doteq \sum_{k\in\mathbb{Z}} w_k(r) e^{i\theta}$ is in $C^\infty([0, R) \times S^1)$ and is flat along $r = 0$.

By similar arguments, one obtains a function $w(r, \theta)$ which is in $C^\infty([-R, 0) \times S^1)$ and is flat along $r = 0$. The functions obtained then glue smoothly to produce a function $w(r, \theta)$ which is in $C^\infty(-R, R) \times S^1)$, is flat along $r = 0$, and solves equation (5.23). The proof of Proposition 5.3 is complete. □

By combining Propositions 5.1, 5.2 and 5.3, we easily obtain our second result about equation (5.1).

Theorem 5.2. — *Assume that either* (4.2) *or* (4.3) *fails. Then, for* ϵ *small enough, the following statements hold true.*

(1) *If* $\lambda_0 \notin i\mathbb{Z}$, *then equation* (5.1) *has a solution* $u \in C^\infty(A_\epsilon)$ *for every* $f \in C^\infty(A_\epsilon)$.

(2) *Assume that* $\lambda_0 \in i\mathbb{Z}$. *Let* N *be the smallest integer with* $1 \leqslant N \leqslant n$ *such that* $\lambda_N \neq 0$. *Assume, furthermore, that either*

$$1 \leqslant N < n, \quad \text{or else} \quad N = n, \quad \text{and} \quad \lambda_n + i\ell b_0 \neq 0, \quad \forall \ell \in \{1, 2, \ldots\}.$$

Then, given $f \in C^\infty(A_\epsilon)$, *equation* (5.1) *has a solution* $u \in C^\infty(A_\epsilon)$ *if and only if* f *satisfies* (5.8).

(3) *Assume that* $\lambda_0 \in i\mathbb{Z}$. *Let* N *be the smallest integer with* $1 \leqslant N \leqslant n$ *such that* $\lambda_N \neq 0$. *Assume, furthermore, that*

$$N = n, \quad \text{and} \quad \lambda_n + i\ell_0 b_0 = 0, \quad \text{for some} \quad \ell_0 \in \{1, 2, \ldots\}.$$

Then the set of all $f \in C^\infty(A_\epsilon)$ *for which* (5.1) *is solvable make up a subspace having finite codimension equal to* $n + 1$. *More precisely,* f *must satisfy, in addition to* (5.9), *a condition bearing on the derivatives of* f *of order up to* ℓ_0.

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