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## EXTREMAL PROPERTIES OF EIGENVALUES FOR A METRIC GRAPH

by Leonid FRIEDLANDER<sup>(\*)</sup>

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### 1. Introduction.

Let  $\Gamma$  be a connected finite graph; by  $V$  we denote the set of its vertices, and by  $E$  we denote the set of its edges. If each edge  $e$  is considered as a segment of certain length  $l(e) > 0$  then such a graph is called a *metric graph*. One can find a good survey and numerous references in [K]. A metric graph with a given combinatorial structure  $\Gamma$  is determined by a vector of edge lengths  $(l(e)) \in \mathbb{R}_+^{|E|}$ . We will use the notation  $G = (\Gamma, (l(e)))$ . The length of a metric graph,  $l(G)$ , is the sum of the lengths of all its edges. Sometimes, it is convenient to treat each edge as a pair of oriented edges; then, on an oriented edge, one defines a coordinate  $x_e$  that runs from 0 to  $l(e)$ . If  $-e$  is the same edge, with the opposite orientation, then  $x_{-e} = l(e) - x_e$ . If an edge  $e$  emanates from a vertex  $v$ , we express it by writing  $v \prec e$ .

A function  $\phi$  on  $G$  is a collection of functions  $\phi_e(x)$  defined on each edge  $e$ . We say that it belongs to  $L^2(G)$  if each function  $\phi_e$  belongs to  $L^2$

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(\*) The first version of this paper dealt with the smallest positive eigenvalue only; the proof was completely different. Y. Colin de Verdiere and S. Gallot suggested the use of the symmetrization technique. As a result, the theorem became more general and the proof became simpler. My great thanks to them.

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on the corresponding edge; then

$$\|\phi\|^2 = \sum_e \|\phi_e\|^2.$$

The Sobolev space  $H^1(G)$  is defined as the space of continuous functions on  $G$  that belong to  $H^1$  on each edge. The Laplacian on  $G$  is defined via the quadratic form

$$\int_G |\phi'(x)|^2 dx = \sum_{e \in E} \int_0^{l(e)} |\phi'_e(x)|^2 dx$$

considered on the natural domain  $H^1(G)$ . The Laplacian  $\Delta$  is given by the differential expression  $-d^2/dx_e^2$  on each edge. Its domain is the set of continuous functions that belong to the Sobolev space  $H^2$  on each edge and that satisfy the Kirchhoff condition

$$(1.1) \quad \sum_{e \succ v} \frac{d\phi}{dx_e}(v) = 0$$

for every vertex  $v$ . This operator is self-adjoint, and its spectrum consists of eigenvalues

$$0 = \mu_1(G) < \mu_2(G) \leq \mu_3(G) \leq \dots \nearrow \infty$$

of finite multiplicity. The eigenvalues are the numbers for which the problem

$$(1.2) \quad \frac{d^2\phi_e}{dx_e^2} + \lambda\phi_e = 0,$$

subject to the Kirchhoff conditions (1.1), has a non-trivial solution. For the sake of brevity, we will call  $\{\mu_j(G)\}$  the spectrum of the metric graph  $G$ .

In this paper, we study the extremal properties for  $\mu_j(G)$  in the class of metric graphs with a fixed length  $l$ . First, let us make explicit computations for three simple examples.

*Example 1.* —  $\Gamma$  is a cyclic graph with  $k$  vertices  $v_1, \dots, v_k$ . It has  $k$  edges that connect  $v_1$  with  $v_2$ ,  $v_2$  with  $v_3$ ,  $\dots$ ,  $v_k$  with  $v_1$ . Obviously, the spectrum of the Laplacian on such a graph is the same as the spectrum of the Laplacian on a circle of circumference  $l = l(G)$ , so

$$(1.3) \quad \mu_1(G) = 0, \quad \mu_{2k}(G) = \mu_{2k+1}(G) = 4\pi^2 k^2 l(G)^{-2}, \quad k \geq 1.$$

*Example 2.* —  $\Gamma$  is a linear graph with  $k$  vertices. It is the same graph as in the previous example, with the edge connecting  $v_k$  and  $v_1$  removed. The spectrum of the Laplacian on such a graph coincides with the spectrum of the Neumann Laplacian on the interval  $[0, l]$ , so

$$(1.4) \quad \mu_k(G) = \pi^2(k - 1)^2 l(G)^{-2}.$$

*Example 3.* —  $\Gamma$  is a star with  $k$  edges. It has  $k + 1$  vertices  $v_0, v_1, \dots, v_k$ , and  $v_0$  is connected with all other vertices. We assume that  $k \geq 2$ ; in the case when  $k = 2$ ,  $\Gamma$  is a linear graph. For a metric graph  $G = H_k$ , we take the lengths of all edges to be equal to  $l/k$ . Let us orient an edge  $e_j$  that connects  $v_j$  with  $v_0$  toward  $v_0$ . Then an eigenfunction of the Laplacian on  $e_j$  must be of the form  $a_j \cos(\sqrt{\lambda}x_j)$  because it satisfies the Neumann condition at  $x_j = 0$ . If  $l\sqrt{\lambda}/k \neq -(\pi/2) + \pi m$ ,  $m \in \mathbb{Z}_+$ , then this function does not vanish at  $v_0$ , all  $a_j$  must be equal to each other, and the Kirchhoff condition (1.1) is satisfied if  $\sin(l\sqrt{\lambda}/k) = 0$ , or  $l\sqrt{\lambda}/k = \pi m$ ,  $m \in \mathbb{Z}_+$ . One gets a family of simple eigenvalues  $\pi^2 k^2 m^2 / l(G)^2$ ,  $m \in \mathbb{Z}_+$ , of the Laplacian. If  $l\sqrt{\lambda}/k = -(\pi/2) + \pi m$  then the function vanishes at  $v_0$ , and it is continuous for all values of  $a_j$ . The Kirchhoff condition at  $v_0$  is equivalent to  $a_1 + \dots + a_k = 0$ . Therefore,

$$\lambda = \pi^2 k^2 (2m - 1)^2 / 4l(G)^2, \quad m \in \mathbb{Z}_+,$$

are also eigenvalues of the Laplacian; their multiplicity equal  $k - 1$ . We see that, for a star,

$$(1.5) \quad \mu_2(H_k) = \mu_k(H_k) = \frac{\pi^2 k^2}{4l(H_k)^2}.$$

The third example shows that, in the class of metric graphs of fixed length,  $\mu_2(G)$ , and, therefore,  $\mu_j(G)$ ,  $j \geq 2$ , does not admit an upper bound. The best lower bound for  $\mu_j(G)$ ,  $j \geq 2$ , can be seen when  $G = H_j$ . The main purpose of this paper is to prove that, in fact, the smallest possible value for  $\mu_j(G)$  is achieved when  $G = H_j$ .

Obviously, one can always remove vertices of degree 2 from the list of vertices. To make some statements simpler, from this point, we assume that there are no vertices of degree 2 in  $G$ .

THEOREM 1. — *Let  $G$  be a connected metric graph. Then*

$$(1.6) \quad \mu_j(G) \geq \frac{\pi^2 j^2}{4l(G)^2}, \quad j \geq 2.$$

Moreover, an equality in (1.6) occurs if and only if  $G$  is a segment when  $j = 2$  and  $G = H_j$  when  $j \geq 3$ .

*Remark.* — It is known that, in the class of bounded, connected planar domains of given area,  $\Omega$ , the first eigenvalue  $\lambda_1(\Omega)$  of the Dirichlet Laplacian in  $\Omega$  is minimized when  $\Omega$  is a circle, and the first positive eigenvalue  $\mu_2(\Omega)$  of the Neumann Laplacian in  $\Omega$  assumes its maximal value when  $\Omega$  is a circle [PS]. Moreover  $\lambda_1(\Omega)$  can be arbitrarily big, and  $\mu_2(\Omega)$  can be arbitrarily close to 0. Though it may look like the eigenvalues of a metric graph should be analogues of the eigenvalues of the Neumann Laplacian: the domain of the Dirichlet functional in the variational formulation is the whole space  $H^1(G)$ , their extremal properties are closer to those of the eigenvalues of the Dirichlet Laplacian.

## 2. Proof of Theorem 1.

First, it is sufficient to prove the inequality in Theorem 1 for trees. In fact, let  $G$  be a metric graph, and let  $G'$  be the graph that is obtained from  $G$  by cutting an edge  $e$  at some point  $x_0$ . This point gives rise to two different vertices in  $G'$ . Obviously,  $H^1(G) \subset H^1(G')$ , so  $\mu_j(G) \geq \mu_j(G')$  because  $\mu_j(G)$  is obtained by the min-max principle from the Rayleigh quotient over a smaller space. If  $G$  is not a tree, one can cut several edges of  $G$  to make a connected tree out of it, and the  $j$ -th eigenvalue of that tree does not exceed  $\mu_j(G)$ .

Let  $G$  be a connected metric tree. By  $\phi_1(x) = \text{const}$ ,  $\phi_2(x), \dots$ , we denote the eigenfunctions of the Laplacian on  $G$  that correspond to the eigenvalues  $\mu_1 = 0, \mu_2, \dots$ . Fix an integer  $j \geq 2$ . For any collection of points  $x_1, \dots, x_m \in G$ ,  $m \leq j - 1$ , one can find a non-zero linear combination,  $\phi(x)$ , of  $\phi_1(x), \dots, \phi_j(x)$  that vanishes at all those points. One has

$$(2.1) \quad \int_G |\phi'(x)|^2 dx \leq \mu_j(G) \int_G |\phi(x)|^2 dx.$$

The set  $G \setminus \{x_1, \dots, x_m\}$  consists of a certain number of connected components. By  $G(x_1, \dots, x_m)$  we denote the disjoint union of their closures. Each connected component of  $G(x_1, \dots, x_m)$  is a tree. Let us formulate the first lemma that we need.

LEMMA 2. — *Let  $G$  be a connected metric tree, and let  $j \geq 2$  be an integer. Then there exist points  $x_1, \dots, x_m$ ,  $m \leq j - 1$ , such that the length of each connected component of  $G(x_1, \dots, x_m)$  does not exceed  $l(G)/j$ .*

**2A. Proof of (1.6) from Lemma 2.**

We choose points  $x_1, \dots, x_m$  from Lemma 2. Then, for at least one of the connected components of  $G(x_1, \dots, x_m)$  (we call it  $G_1$ ),  $\phi(x)$  is not identically 0 on  $G_1$ , and

$$(2.2) \quad \int_{G_1} |\phi'(x)|^2 dx \leq \mu_j(G) \int_{G_1} |\phi(x)|^2 dx.$$

When restricted to  $G_1$ , the function  $\phi(x)$  satisfies the Dirichlet boundary condition at one of its leaves. The next lemma gives a lower bound for the ground state of the Laplacian with the Dirichlet condition at a point.

For a metric graph  $G$  and a point  $y \in G$ , we denote by  $H_y^1(G)$  the space of  $H^1(G)$  functions that vanish at  $y$ .

LEMMA 3. — *Let  $G$  be a connected metric graph and  $y \in G$ . Then*

$$(2.3) \quad \int_G |\phi'(x)|^2 dx \geq \frac{\pi^2}{4l(G)^2} \int_G |\phi(x)|^2 dx$$

for all functions  $\phi \in H_y^1(G)$ . For a non-zero function  $\phi \in H_y^1(G)$ , the equality in (2.3) may happen only if  $G$  is a segment,  $y$  is its endpoint, and  $\phi(x)$  is proportional to  $\sin(\pi s/(2l(G)))$  where  $s$  is the distance to  $y$ .

One obtains the inequality in Theorem 1 by applying Lemma 3 to  $G_1$  and comparing (2.2) and (2.3).

*Proof of Lemma 3.* — We use the symmetrization technique (see [B], [BG], [G1], [G2], [PS].) First, one can assume that  $\phi \geq 0$ : replacing  $\phi(x)$  by  $|\phi(x)|$  does not result in the change of either the right hand side or

the left hand side in (2.3). For  $t \geq 0$ , let  $m_\phi(t)$  be the measure of the set  $\{x \in G : \phi(x) < t\}$ ; this is a lower semi-continuous function that increases from 0 to  $M = \max \phi(x)$ . One can uniquely define a continuous, non-decreasing function  $\phi^*(s)$  on the interval  $[0, l(G)]$  such that  $\phi^*(0) = 0$  and  $m_{\phi^*}(t) = m_\phi(t)$ . Then

$$(2.4) \quad \int_G |\phi(x)|^2 dx = \int_0^M t^2 dm_\phi(t) = \int_0^{l(G)} |\phi^*(s)|^2 ds.$$

The set of  $H_y^1(G)$  functions that are continuously differentiable on closed edges is dense in  $H_y^1(G)$ ; therefore, for the proof of (2.3), one can assume that  $\phi(x)$  is continuously differentiable on closed edges. A critical point of  $\phi(x)$  is either a critical point on an open edge or a vertex. By Sard's theorem the set of critical values have measure 0. Let  $t$  be a regular value of  $\phi(x)$ . The number of pre-images of  $t$  under  $\phi(x)$  is finite; we denote this number by  $n(t)$ . The co-area formula (e.g., see [B]) implies

$$\int_G |\phi'(x)|^2 dx = \int_0^M dt \sum_{x:\phi(x)=t} |\phi'(x)|.$$

By the Cauchy–Schwarz inequality,

$$(2.5) \quad \begin{aligned} \sum_{x:\phi(x)=t} |\phi'(x)| &\geq n(t)^2 \left( \sum_{x:\phi(x)=t} \frac{1}{|\phi'(x)|} \right)^{-1} \\ &\geq \left( \sum_{x:\phi(x)=t} \frac{1}{|\phi'(x)|} \right)^{-1} = \frac{1}{m'_\phi(t)}. \end{aligned}$$

Therefore,

$$(2.6) \quad \int_G |\phi'(x)|^2 dx \geq \int_0^M \frac{dt}{m'_\phi(t)}.$$

The same argument applies to the function  $\phi^*(s)$ ; that function takes every regular value once, and all inequalities become exact equalities. One concludes that

$$\int_G |\phi'(x)|^2 dx \geq \int_0^{l(G)} |(\phi^*)'(s)|^2 ds.$$

Function  $\phi^*(s)$  belongs to  $H^1([0, l(G)])$  and  $\phi^*(0) = 0$ . Therefore,

$$(2.7) \quad \int_0^{l(G)} |(\phi^*)'(s)|^2 ds \geq \frac{\pi^2}{4l(G)^2} \int_0^{l(G)} |\phi^*(s)|^2 ds$$

because  $\pi/(2l(G))$  is the first eigenvalue of the operator  $-d^2/ds^2$  on the interval  $[0, l(G)]$ , with the Dirichlet condition at 0 and the Neumann condition at  $l(G)$ .

This finishes the proof of the inequality (2.3). Now, suppose that an equality in (2.3) takes place for a non-zero function  $\phi(x)$ . Then

- (1) the function  $\phi(x)$  minimizes the Rayleigh quotient

$$\int_G |\phi'(x)|^2 dx \Big/ \int_G |\phi(x)|^2 dx$$

on the space  $H_y^1(G)$ ;

- (2) the equality in (2.5) holds;
- (3) the equality in (2.7) holds.

The first condition implies that  $\phi(x)$  is an eigenfunction of the Laplacian on  $G$ , with the Dirichlet condition at the point  $y$ . Therefore, on each edge of  $G \setminus y$ , it is a trigonometric function. The same is true for  $|\phi(x)|$  because, for that function an equality in (2.3) also holds. The second condition implies that  $n(t) = 1$  for all regular values  $t$ . We conclude that  $y$  is a vertex of  $G$  of degree 1 (a leaf.) In fact, the derivative of  $|\phi(x)|$  at  $y$  in each direction emanating from  $y$  is positive (it can not vanish), so if there is more than one direction then small positive values are taken at least twice. In the same way,  $G$  does not have vertices of degree greater than 2. If  $v$  is a vertex of degree at least 3, then, close to  $v$ , the function  $\phi(x)$  either increases or decreases along each edge; so either  $\phi(x)$  or  $-\phi(x)$  increases in a neighborhood of  $v$  along two different edges emanating from  $v$ . Therefore the values that are close to  $\phi(v)$  either from above or from below are taken at least twice.

We have agreed to disregard vertices of degree 2. Finally,  $G$  is a connected graph, and all its vertices are leaves. There is at least one vertex ( $y$ .) Therefore,  $G$  is a segment  $[0, l(G)]$ , and  $\phi(x)$  is a monotone function on that segment. That implies  $\phi = \phi^*$ . The third condition tells us that  $\phi^*$  is the first eigenfunction of the Laplacian on  $[0, l(G)]$ , with the Dirichlet condition at 0 and the Neumann condition at  $l(G)$ , so it is proportional to  $\sin(\pi s/(2l(G)))$ . □

## 2B. Proof of Lemma 2.

The proof of Lemma 2 is based on the following lemma.

LEMMA 4. — *Let  $G$  be a connected metric tree of length  $L$ . For every  $l$ ,  $0 < l \leq L$ , there exists a point  $x \in G$  such that*

$$G(x) = G_0 \sqcup G_1 \sqcup \cdots \sqcup G_p,$$

and  $l(G_0) \leq L - l$ ,  $l(G_k) \leq l$ ,  $1 \leq k \leq p$ .

One applies Lemma 4 ( $j - 1$ ) times. Fix  $l = L/j$ . First, one finds a point  $x_1$  such that  $G(x_1) = G_1 \sqcup G^{(1)}$  where  $G_1$  is a connected tree of length  $\leq (j - 1)L/j$ , and all connected components of  $G^{(1)}$  have length  $\leq L/j$ . Then one finds  $x_2 \in G_1$  such that  $G_1 = G_2 \sqcup G^{(2)}$ , with  $G_2$  being a connected tree of length  $\leq (j - 2)L/j$ , and all connected components of  $G^{(2)}$  having length  $\leq L/j$ . One keeps going, and, after not more than  $(j - 1)$  steps, one gets the desired decomposition.

*Proof of Lemma 4.* — We fix a leaf  $y_0$  of  $G$ . For a point  $x \in G$  that is not a vertex, we denote by  $G_x$  the connected component of  $G(x)$  that does not contain  $y_0$ . Note that, if  $x$  is not a vertex, then  $G(x)$  consists of exactly two connected components. If  $l(G_x) = l$  for some  $x \in G \setminus V$  (here  $V$  is the set of vertices) then such a point will do the job. Otherwise, on each edge  $e$  of  $G$ , either  $l(G_x) < l$ ,  $x \in e$ , (we call them edges of the first type) or  $l(G_x) > l$ ,  $x \in e$ ; they will be called edges of the second type. Denote by  $G^1$  the closure of the union of all edges of the first type;  $G^2$  is the closure of the union of edges of the second type. All connected components of both  $G^1$  and  $G^2$  are metric trees. Notice that the edge incident to  $y_0$  belongs to  $G^2$ , and the edges that are incident to all other leaves of  $G$  belong to  $G^1$ . Let  $y \neq y_0$  be a leaf of  $G^2$ . By  $G_0$  we denote the component of  $G(y)$  that contains  $y_0$ , and let  $G_1, \dots, G_p$  be other components of  $G(y)$ .

We claim that  $l(G_0) \leq L - l$  and  $l(G_k) \leq l$ ,  $1 \leq k \leq p$ . In fact, let  $e_k$ ,  $0 \leq k \leq p$ , be the edge of  $G_k$  incident to  $y$  (notice that  $y$  is a leaf for all  $G_j$ s.) For  $x \in e_0$ , one has  $l(G_x) \geq l$ , and

$$l(G_0) = \lim_{e_0 \ni x \rightarrow y} l(G \setminus G_x) \leq L - l.$$

Because  $y$  is a leaf of  $G^2$ , the edges  $e_1, \dots, e_p$  belong to  $G^1$ ; therefore, for  $1 \leq k \leq p$ , one has

$$l(G_k) = \lim_{e_k \ni x \rightarrow y} l(G_x) \leq l.$$

□

**2C. The case of equality in (1.6).**

To finish the proof of Theorem 1 we have to analyze, under what conditions the equality in (1.6) takes place. First, we consider the case when  $G$  is a connected tree. Then, for any linear combination  $\phi(x)$  of  $\phi_1(x), \dots, \phi_j(x)$  that vanishes at the points  $x_1, \dots, x_m$  from Lemma 2, the inequality (2.1) becomes an exact equality. Therefore,  $\phi(x)$  is an eigenfunction of the Laplacian on  $G$  that corresponds to the eigenvalue  $\mu_j(G) = \pi^2 j^2 / (4l(G)^2)$ . Let  $G_1, \dots, G_p$  be the connected components of  $G(x_1, \dots, x_m)$ . The restriction of  $\phi(x)$  to  $G_k$ ,  $k = 1, \dots, p$ , if not identically zero, is an eigenfunction of the Laplacian on  $G_k$ , with the Dirichlet condition at those points  $x_i$  that belong to  $G_k$ . From Lemma 3 (notice that the length of each  $G_k$  does not exceed  $l(G)/j$ ) we conclude that those components  $G_k$ , on which the function  $\phi(x)$  does not vanish identically, are segments of length  $l(G)/j$ , one endpoint of each segment is one of the points  $x_1, \dots, x_m$ , and the restriction of  $\phi(x)$  to such a segment is proportional to  $\sin(\pi j s / (2l(G)))$  where  $s$  is the distance to the endpoint of the segment where  $\phi(x)$  vanishes. The function  $\phi(x)$  does not vanish at the second end of the segment, so the second end of the segment is a leaf of the tree  $G$  because this segment is a connected component of  $G(x_1, \dots, x_m)$ .

A certain complication arises from the fact that  $\phi(x)$  may vanish on some of the components  $G_k$ : an eigenfunction of the Laplacian on a metric graph may well vanish on some edges of the graph. Now, we do induction in  $j$ . If  $j = 2$  then  $m = 1$ , and one has only one point  $x_1$ . The function  $\phi(x)$  does not vanish on at least two connected components of  $G(x_1)$ : otherwise  $\phi(x)$  would not satisfy the Kirchhoff condition at the point  $x_1$  (notice that  $x_1$  is not a leaf of  $G$ ; if  $x_1$  is not a vertex then the Kirchhoff condition is the same as the differentiability at  $x_1$  condition.) Each connected component of  $G(x_1)$  on which  $\phi(x)$  does not vanish is of length  $l(G)/2$ , so there are exactly two of them, and these are the only connected components of  $G(x_1)$ . We conclude that  $G$  consists of two segments of length  $l(G)/2$  emanating from  $x_1$ , so  $G$  is a segment, and  $x_1$  is its midpoint.

Now, let us do the inductive step. Let  $j \geq 3$ . Let  $G_1$  be a connected component of  $G(x_1, \dots, x_m)$  on which  $\phi(x)$  does not vanish. Suppose that  $x_1$  is an endpoint of  $G_1$ . As we have already seen,  $G_1$  is a segment of length  $l(G)/j$  than connects  $x_1$  with a leaf of the graph  $G$ . Therefore,  $G' = G \setminus G_1$  is a connected tree,  $x_1$  is one of its vertices, and  $l(G') = (j-1)l(G)/j$ . By  $\mathcal{L}$  we denote the space of all linear combinations of  $\phi_1(x), \dots, \phi_j(x)$  that vanish at  $x_1$ . Clearly,  $\dim \mathcal{L} = j-1$ . A non-zero function  $\psi(x) \in \mathcal{L}$  can not vanish identically on  $G'$ . In fact, if it vanishes on  $G'$ , then

$$\int_{G_1} |\psi'|^2 dx \leq \frac{\pi^2}{4l(G_1)^2} \int_{G_1} |\psi(x)|^2 dx,$$

so the restriction of  $\psi(x)$  to  $G_1$  is proportional to  $\sin(\pi s/(2l(G_1)))$ , and the Kirchhoff condition breaks at the point  $x_1$ . Denote by  $\mathcal{L}'$  the space of restrictions of functions from  $\mathcal{L}$  to  $G'$ . Then

$$(2.8) \quad \dim \mathcal{L}' = j-1.$$

For every  $\psi \in \mathcal{L}$ , one has

$$\int_G |\psi'(x)|^2 dx \leq \frac{\pi^2 j^2}{4l(G)^2} \int_G |\psi(x)|^2 dx$$

and

$$\int_{G_1} |\psi'(x)|^2 dx \geq \frac{\pi^2 j^2}{4l(G)^2} \int_{G_1} |\psi(x)|^2 dx.$$

Therefore,

$$(2.9) \quad \begin{aligned} \int_{G'} |\psi'(x)|^2 dx &\leq \frac{\pi^2 j^2}{4l(G)^2} \int_{G'} |\psi(x)|^2 dx \\ &= \frac{\pi^2 (j-1)^2}{4l(G')^2} \int_{G'} |\psi(x)|^2 dx. \end{aligned}$$

From (2.8) and (2.9), one concludes that

$$\mu_{j-1}(G') \leq \frac{\pi^2 (j-1)^2}{4l(G')^2},$$

and, by the induction assumption,  $G' = H_{j-1}$ . In the case  $j = 3$ , we treat a segment as  $H_2$  by inserting a vertex at the midpoint of the segment.

Denote by  $y$  the center of  $G' = H_{j-1}$ . The question is, how the segment  $G_1$  is attached to  $G'$ . There are three possibilities:

- (1)  $x_1 = y$ ;
- (2)  $x_1$  lies inside of an edge  $e = (y, z)$  of  $G'$ ;
- (3)  $x_1$  coincides with a leaf  $z$  of  $G'$ .

In the first case,  $G = H_j$ , so we have to rule out two remaining possibilities.

Suppose that  $x_1$  lies inside of  $(y, z)$ . Let  $G'' = G' \setminus (x_1, z]$ . Every function  $\psi \in \mathcal{L}'$  satisfies

$$(2.10) \quad \int_{G''} |\psi'(x)|^2 dx \leq \frac{\pi^2(j-1)^2}{4l(G')^2} \int_{G''} |\psi(x)|^2 dx$$

because

$$(2.11) \quad \int_{(x_1, z)} |\psi'(x)|^2 dx \geq \frac{\pi^2(j-1)^2}{4l(G')^2} \int_{(x_1, z)} |\psi(x)|^2 dx$$

(notice that the length of  $(x_1, z)$  is smaller than  $l(G)/j = l(G')/(j-1)$ .) A function  $\psi \in \mathcal{L}'$  can not vanish on  $G''$  because, otherwise, a strict inequality would hold in (2.11), and that would contradict (2.9). Therefore, the inequality (2.10) holds for functions from a  $(j-1)$ -dimensional subspace of  $H^1(G'')$ . Hence,

$$\mu_{j-1}(G'') \leq \frac{\pi^2(j-1)^2}{4l(G')^2} < \frac{\pi^2(j-1)^2}{4l(G'')^2}.$$

The last inequality contradicts (1.6).

Let us now treat the case  $x_1 = z$ . Then the graph  $G$  consists of  $(j-2)$  edges,  $e_1, \dots, e_{j-2}$  emanating from  $y$ , of length  $L/j$  each, and one edge,  $f$ , of length  $2L/j$  emanating from  $y$  (here  $L = l(G)$ .) All edges connect  $y$  with leaves of  $G$ . We parametrize each edge by the distance from  $y$ . An eigenfunction of the Laplacian on  $G$  that corresponds to an eigenvalue  $\mu = \lambda^2 \neq 0$  equals  $a_k \cos(\lambda((L/j) - s))$  on an edge  $e_k$ , and it equals  $b \cos(\lambda((2L/j) - s))$  on the edge  $f$ . When  $s = 0$ , all the values must coincide, so

$$(2.12) \quad a_1 \cos(\lambda L/j) = \dots = a_{j-2} \cos(\lambda L/j) = b \cos(2\lambda L/j).$$

The Kirchhoff condition at  $y$  reads

$$(2.13) \quad (a_1 + \cdots + a_{j-2}) \sin(\lambda L/j) + b \sin(2\lambda L/j) = 0.$$

We will count the number of eigenvalues of the Laplacian on  $G$  that do not exceed  $\pi^2 j^2 / (4L^2)$ . There is an eigenvalue 0 of multiplicity 1. In the case  $\cos(\lambda L/j) = 0$ , (2.12) and (2.13) imply  $a_1 + \cdots + a_{j-2} = 0$  and  $b = 0$ ; one gets a  $(j - 3)$ -dimensional space of eigenfunctions that correspond to the eigenvalue  $\pi^2 j^2 / (4L^2)$ . If  $\sin(\lambda L/j) = 0$  then  $\mu = \lambda^2 \geq (\pi^2 j^2 / L^2) > \pi^2 j^2 / (4L^2)$ . In the case when  $\cos(\lambda L/j) \neq 0$  and  $\sin(\lambda L/j) \neq 0$ , (2.12) and (2.13) imply  $a_1 = \cdots = a_{j-2} = a$ ,

$$a \cos(\lambda L/j) = b(2 \cos^2(\lambda L/j) - 1), \quad \text{and} \quad \cos^2(\lambda L/j) = \frac{j-2}{2(j-1)}.$$

Therefore,  $\cos(2\lambda L/j) = -1/(j-1)$ . This case gives rise to one eigenvalue  $\arccos^2(-1/(j-1))j^2/(4L^2)$  of multiplicity one that is smaller than  $\pi^2 j^2 / (4L^2)$ ; all other eigenvalues are bigger than  $\pi^2 j^2 / (4L^2)$ . Finally, in the case  $x_1 = z$ , there are exactly  $(j - 1)$  eigenvalues of  $G$  that are smaller than or equal to  $\pi^2 j^2 / (4l(G)^2)$ , so, for such a graph, an equality in (1.6) does not take place.

We have proved that if an equality in (1.6) takes place, and if  $G$  is a connected tree, then  $G = H_j$ . If  $G$  is a connected graph that is not a tree then one can cut it at points  $x_1, \dots, x_m$  lying on open edges in such a way that  $G' = G(x_1, \dots, x_m)$  is a connected tree. As it was noted earlier,  $\mu_j(G') \leq \mu_j(G)$ . If  $\mu_j(G) = \pi^2 j^2 / (4l(G)^2)$  then the last inequality, in combination with (1.6), imply  $\mu_j(G') = \pi^2 j^2 / (4l(G')^2)$ . Therefore,  $G' = H_j$  for any choice of points  $x_1, \dots, x_m$  that make  $G(x_1, \dots, x_m)$  a connected tree. Clearly, this is impossible.  $\square$

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