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## ON THE JUNG METHOD IN POSITIVE CHARACTERISTIC

by Olivier PILTANT

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### 1. Introduction.

The first proof of resolution of singularities of complex surfaces can be traced back to Jung [18]. Roughly speaking, the argument can be sketched as follows [24]: a *normal* complex surface  $S$  has a finite number of singular points, so the question is local on  $S$ . One then takes a finite projection  $\pi$  of  $S$  to the affine plane. The branch locus is a plane curve  $C$ . By embedded resolution of plane curves, one reduces to the case that the only singular points of  $C$  are normal crossings. The singular points of  $S$  now have a very simple structure, which is the content of Abhyankar's lemma. We state it below in its analytic form and in all dimensions ([6] Proposition 1, [15] 2.3.4).

**PROPOSITION 1.1.** — *Let  $k$  be an algebraically closed field of characteristic zero, and let  $R := k[[u_1, \dots, u_n]]$ . Let  $S$  be a normal domain,  $S \supseteq R$ , which is finite as a  $R$ -module. Assume that the map  $\pi : \text{Spec } S \rightarrow \text{Spec } R$  is unramified above  $\text{Spec } R[\frac{1}{u_1 \dots u_d}]$  for some  $d, 0 \leq d \leq n$ .*

*There exists a nonsingular matrix  $A = (a_{ij})_{1 \leq i, j \leq d}$  with entries in  $\mathbb{Z}$  such that the extension of fields of fractions  $L/K$  of  $S/R$  is equal to  $K(x_1, \dots, x_d)/K$ , where the  $x_j$ 's satisfy the relations*

$$u_i = \prod_{j=1}^d x_j^{a_{ij}}, \quad 1 \leq i \leq d.$$

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For  $d = n = 1$ , Proposition 1.1 is an equivalent formulation of the Puiseux theorem:  $\bigcup_{a \geq 1} k((u^{\frac{1}{a}}))$  is an algebraic closure of  $k((u))$ . For  $n \geq 2$ , Proposition 1.1 implies that  $S$  is generated as a  $R$ -module by a finite number of monomials  $\{M_j\}_{j \in J}$  in  $u_1^{\frac{1}{\Delta}}, \dots, u_d^{\frac{1}{\Delta}}$ , where  $\Delta := |\det(A)| \geq 1$ . The singularities of  $S$  are *toric* [19], and a resolution of singularities can be described explicitly from some combinatorial data associated with the set of exponents of the  $M_j$ 's. This fact is the main feature in the Jung method.

Abhyankar's series of articles in the 1950's ([1] to [6]) was originally motivated by the positive characteristic version of Proposition 1.1. Some of his observations of fundamental importance [1] are

(1) if  $\text{char } k = p > 0$ , Proposition 1.1 holds if  $\pi$  is *tamely ramified* above  $\text{div}(u_1 \cdots u_d)$  but fails in general.

(2) if  $\text{char } k = p > 0$ , the Galois group of the normal closure of  $L/K$  in Proposition 1.1 needs not be solvable.

Abhyankar's resolution of surfaces singularities in characteristic  $p > 0$  proceeds from the valuative version of Jung's method. Although the difficulties caused by wild ramification in (1) persist along a valuation, those in (2) disappear: Krull's theorem [21] states that the inertia group of a finite and separable extension of valuation rings is solvable. This fact allows Abhyankar to reduce local uniformization for surfaces to pulling it up in a cyclic extension of degree  $p$ , which is performed in [2].

In the last few years, de Jong's foundational paper [17] triggered renewed interest in Jung's method; some of Abhyankar's conjectures have been studied and proved recently by Cutkosky [9] [10] (see also [11] and [13]). A valuative version of de Jong's theorem is proved in [22] using ramification theoretic methods (see also [20]). Plugging in a semistable reduction argument, de Jong [17] succeeds in proving by induction on dimension a weak form of the resolution of singularities theorem for algebraic varieties, which is valid in all characteristics. This weak form is not birational, but requires a finite extension of the function field. The gap from being birational comes precisely from fact (1) mentioned above. It was subsequently proved in [7] and [8] that no such extension of the function field is necessary in characteristic zero. In positive characteristic, even in dimension two, it is still an open problem that such an extension is necessary. This article is devoted to stating the *precise* definition of its valuative version (Definition 4.1) and to giving evidence that enlarging the function field should not be necessary (Theorems 5.3, 6.5 and 7.1).

Known facts and definitions are stated in Sections 3 and 4. For convenience, we restate in here Definition 4.1, which is our definition of the valuative Jung problem for an algebraic surface over an algebraically closed field  $k$  of characteristic  $p \geq 0$ : let  $L/K$  be a finite and separable extension of function fields of dimension two over  $k$ , and let  $W/k$  be a valuation ring which is birational to  $L$ . We say that  $(L/K, W)$  has the *strong* (resp. *weak*) Jung property if there exists a (resp. for any) local uniformization  $R$  of  $V := W \cap K$  such that for any (resp. there exists a) local uniformization  $R'$  of  $V$  dominating  $R$ , the integral closure  $\overline{R}'$  of  $R'$  in  $L$  has a *toric* singularity at the center of  $W$ . Basically, this means that the singularity of  $\overline{R}$  at the center of  $W$  is *toric* once  $R$  has been blown up “sufficiently many times” along  $V$ , and that this remains true after performing further blowing ups in case  $(L/K, W)$  has the strong Jung property. The main question addressed in this paper is: does any such pair  $(L/K, W)$  have the weak or even strong Jung property?

In Section 5, we give a simple criterion for  $\overline{R}$  to have a toric singularity at the center of  $W$  for a given local uniformization  $R$  of  $V$ . Theorem 5.3 provides an affirmative answer to the strong form of the above question whenever the value group  $\Gamma$  of  $W$  is finitely generated. Since  $L$  and  $K$  have dimension two over  $k$ , this includes all valuation rings except those whose value group  $\Gamma$  is rational and nondiscrete.

In this remaining case, two main difficulties arise: the extension of valuation rings  $W/V$  may have nontrivial *defect* (Definition 4.2), and  $\Gamma$  may be  $p$ -divisible. In Section 6, it is proved that the strong (resp. weak) Jung property holds in the defectless (resp. not  $p$ -divisible) case.

Finally, we prove in Section 7 that any  $(L/K, W)$  has the weak Jung property if  $L/K$  is a finite and separable (but not necessarily Galois) extension of degree  $p$  (in characteristic  $p > 0$ ). We view this fact as convincing evidence that any pair  $(L/K, W)$  should have the weak Jung property, since fact (1) above is the main difficulty along a valuation in positive characteristic.

## 2. Notations.

From now on,  $k$  denotes an algebraically closed field of characteristic  $p \geq 0$ . Function fields over  $k$  are denoted by  $K, L$ . A model  $X/k$  of  $K/k$  is an integral separated scheme of finite type over  $k$  whose function field  $K(X)$  is  $K$ .

The maximal ideal of a local ring  $R$  is denoted by  $m_R$  and its residue field by  $\kappa(R)$ . We will denote the quotient field of a domain  $R$  by  $QF(R)$ . Suppose that  $R \subset S$  is an inclusion of local rings. We say that  $R$  dominates  $S$  if  $m_S \cap R = m_R$ . Given a function field  $K/k$ , and a valuation ring  $V$  of  $K$  containing  $k$  such that  $QF(V) = K$ , a model of  $V/k$  (or of  $V$  for simplicity) is a *normal* local domain  $R$ , essentially of finite type over  $k$ , such that  $QF(R) = K$  and which is dominated by  $V$ .

Suppose that  $R$  is a local domain. A *quadratic transform* of  $R$  is a local domain of the form  $R_1 = R[\frac{m_R}{x}]_{m_1}$  where  $0 \neq x \in m_R$  and  $m_1$  is a prime ideal of  $R[\frac{m_R}{x}]$  such that  $m_1 \cap R = m_R$ . If  $R$  is a model of some valuation ring  $V/k$ , the *quadratic transform of  $R$  along  $V$*  is the unique quadratic transform of  $R$  which is dominated by  $V$ . In other terms,  $R_1$  is obtained from  $R$  by taking any  $x \in m_R$  having minimal value w.r.t.  $V$  among all elements of  $m_R$ , and taking  $m_1 := m_V \cap R[\frac{m_R}{x}]$ .

Suppose that  $L/K$  is a finite extension of function fields over  $k$ . If  $X/k$  is a model of  $K/k$ , the normalization of  $X$  in  $L$  is denoted by  $\bar{X}$  whenever there is no risk of confusion about  $L$ . Also, let  $R$  be a local domain with  $QF(R) = K$  and  $S$  a local domain with  $QF(S) = L$ . We say that  $S$  lies over  $R$  if  $S$  is a localization at a maximal ideal of the integral closure of  $R$  in  $L$ .

### 3. The generalized Jung problem.

In this section, we state well-known results on unramified coverings of the complement of a divisor with strict normal crossings in a regular surface. We also include some classical examples, due to Abhyankar, which point out some of the main differences between characteristic zero and positive characteristic.

LEMMA 3.1. — *Let  $L/K$  be a finite and separable extension of function fields of dimension two over  $k$  and let  $X/k$  be a proper and smooth model of  $K/k$ . There exists a commutative diagram*

$$\begin{array}{ccc}
 \bar{X} & \xleftarrow{\bar{\pi}} & \bar{X}' \\
 n \downarrow & & n' \downarrow \\
 X & \xleftarrow{\pi} & X'
 \end{array}$$

*such that  $\pi$  is a composition of point blowing ups and  $n'$  is ramified above a divisor with strict normal crossings.*

*Proof.* — The branch locus  $B \subset X$  of  $n$  is a divisor by purity of branch locus ([14] Theorem X.3.1) since  $X$  is regular. There exists a composition of point blowing ups  $\pi$  such that  $\pi^*B$  is a divisor with strict normal crossings by [16] V.3.9. The branch divisor  $B' \subset X'$  of  $n'$  is obviously contained in the support of  $\pi^*B$ , and therefore has strict normal crossings.  $\square$

When  $k$  has characteristic zero, each singular point of a given surface  $\overline{X}'$  satisfying the conclusion of Lemma 3.1 has a toric singularity (see Proposition 3.3 below). We use the following definition.

**DEFINITION 3.2.** — *Let  $(R, m_R)$  be a normal local domain which is essentially of finite type over  $k$  and such that  $R/m_R \simeq k$ . We will say that  $R$  has a toric singularity if there exists a lattice  $N \simeq \mathbb{Z}^d$  and a strongly convex rational cone  $\sigma \subset N \otimes \mathbb{Q}$  spanning  $N \otimes \mathbb{Q}$  such that the formal completion  $\hat{R}$  of  $(R, m_R)$  is isomorphic to the power series ring  $k[[t^{\sigma^\vee \cap M}]]$  with coefficients in the semigroup  $\sigma^\vee \cap M$ , where  $M = \text{Hom}(N, \mathbb{Z})$ , and  $\sigma^\vee \subset M \otimes \mathbb{Q}$  is the dual cone of  $\sigma$ . Necessarily,  $\dim R = d$ .*

A typical example of toric singularity is obtained as follows: let  $(R, m_R)$  be a *regular* local ring of dimension  $d \geq 2$  which is essentially of finite type over  $k$  and let  $I$  be an ideal which is generated by monomials in a regular system of parameters (r.s.p. for short)  $(u_1, \dots, u_d)$  of  $R$ . Then the local ring of each singular point of the normalized blowing up of  $\text{Spec } R$  along  $I$  has a toric singularity.

**PROPOSITION 3.3.** — *Assume  $k$  has characteristic zero. Let  $L/K$  be a finite extension of function fields of dimension two over  $k$ . There exists a proper and smooth model  $X/k$  of  $K/k$  such that for any composition of point blowing ups  $\pi' : X' \rightarrow X$ , the local ring of each singular point of the normalization  $\overline{X}'$  of  $X'$  in  $L$  has a toric singularity.*

*Proof.* — Lemma 3.1 shows existence of a proper and smooth model  $X/k$  of  $K/k$  whose normalization  $\overline{X}$  in  $L$  is ramified above a divisor with strict normal crossings. Any  $X'$  as in the statement of the proposition shares this same property. By Abhyankar's Lemma ([6] Proposition 1 or [15] 2.3.4),  $\overline{X}'$  has toric singularities.  $\square$

**Examples 3.4.** — 1) The ring  $R = k[x, y, z]/(xy + z^{n+1})$  has an  $\mathbb{A}_n$  singularity at its maximal ideal  $M = (x, y, z)$  which is toric. This singularity is also a Brieskorn-Pham singularity. If  $k$  has characteristic  $p \geq 0$  such that

$p$  does not divide  $n+1$ , the covering  $\text{Spec } R \rightarrow \mathbb{A}_k^2 = \text{Spec } k[x, y]$  is ramified above the curve  $xy = 0$ .

2) It is well-known that Abhyankar's lemma does not hold in general in positive characteristic as soon as wild ramification appears above the branch locus. For example ([1] p. 586), the covering

$$\overline{X} = \text{Spec } k[x, y, z]/(z^p + x^{p-1}z + xy^{p-1}) \rightarrow \mathbb{A}_k^2 = \text{Spec } k[x, y]$$

is ramified above the line  $x = 0$  if  $k$  has characteristic  $p > 0$ . The surface  $\overline{X}$  is a cone over a smooth curve of degree  $p$  which is not rational if  $p \geq 3$ . Consequently, the singular point  $M = (x, y, z)$  of  $\overline{X}$  is not a rational singularity if  $p \geq 3$  and *a fortiori* is not a toric singularity.

3) Similarly ([1] p. 589), the covering

$$\overline{X} = \text{Spec } k[x, y, z]/(z^{p+1} + y^{p-1}z + x^{p+1}) \rightarrow \mathbb{A}_k^2$$

is ramified above the line  $L : x = 0$  if  $\text{char } k = p > 0$ . Note that the origin in  $\overline{X}$  is a singular point and that the inverse image of  $L$  in  $\overline{X}$  splits in a union of two curves. Neither phenomenon occurs in characteristic zero ([6] Proposition 2).

However, example 2) above does not provide a counterexample to the conclusion of Proposition 3.3 in positive characteristic. It shows that a given  $X$  such that the map  $\overline{X} \rightarrow X$  is ramified above a divisor with strict normal crossings does not satisfy in general the conclusion of Proposition 3.3. The following statement which is a simple consequence of resolution of singularities holds in all characteristics  $p \geq 0$ .

**PROPOSITION 3.5.** — *Let  $L/K$  be a finite extension of function fields of dimension two over  $k$ . There exists a proper and smooth model  $X/k$  of  $K/k$  such that for any composition of point blowing ups  $\pi' : X' \rightarrow X$ , each singular point of the normalization  $\overline{X}'$  of  $X'$  in  $L$  has a rational singularity.*

*Proof.* — Let  $X_0/k$  be a proper and smooth model of  $K/k$  and  $Y/k$  be a proper and smooth model of  $L/k$ . By elimination of indeterminacies ([23] Theorem 26.1), it can be assumed that the rational map  $Y \cdots \rightarrow X_0$  is defined everywhere. Let  $E_1, \dots, E_n$  be those irreducible curves in  $Y$  whose image in  $X_0$  is a point. There exists a composition of point blowing ups  $\pi' : X \rightarrow X_0$  such that the rational map  $Y \cdots \rightarrow X$  is finite at the generic point of  $E_1, \dots, E_n$  by [3] Theorem 3.

Let  $\pi' : X' \rightarrow X$  be a composition of point blowing ups and  $\overline{X}'$  be the normalization of  $X'$  in  $L$ . It follows from the construction of  $X$  that the

rational map  $\overline{X}' \cdots \rightarrow Y$  is defined everywhere by Zariski's Main Theorem ([16] III.11.4). Each singular point of the surface  $\overline{X}'$  therefore has a rational singularity by [23] Proposition 1.2 p. 199.  $\square$

*Remark.* — The above proof actually implies the stronger statement that  $X'$  has a *sandwiched* singularity in the sense of [25].

### 4. A local version of the Jung problem.

In this section, we recall some basic facts of valuation theory and ramification theory as can be found in [5] or [26] and [27]. A local version (in the sense of valuations) of Definition 3.2 is given in Definition 4.1.

Let  $L/K$  be a finite and separable extension of function fields of dimension two over  $k$ . Let  $W/k$  be a valuation ring which is birational to  $L$  and let  $V := W \cap K$ . The value group of  $V$  (resp.  $W$ ) is denoted by  $\Delta$  (resp.  $\Gamma$ ). The *rational rank* of  $W$ ,

$$\text{rat.rk}(W) := \dim_{\mathbb{Q}} \Gamma \otimes \mathbb{Q}$$

and the *residue transcendence degree* of  $W$ ,

$$\text{tr.deg}(W) := \text{tr.deg}(\kappa(W)/k)$$

satisfy Abhyankar's inequality ([27] Proposition 1 p. 330)

$$\text{rat.rk}(W) + \text{tr.deg}(W) \leq \text{tr.deg}(L/k) = 2.$$

The case  $\text{tr.deg}(W) = 2$  corresponds to the zero valuation of  $L$  and the case  $\text{tr.deg}(W) = 1$  corresponds to *prime divisors* of  $L$  ([27] p. 88). In all that follows, we assume that  $\text{tr.deg}(W) = 0$ , and therefore that  $\kappa(W) = k$ , since  $k$  is algebraically closed. The possible value groups  $\Gamma$  are

(1)  $\mathbb{Z}^2$  lexicographically ordered or (2) any free Abelian subgroup of rational rank two of  $(\mathbb{R}, \leq)$  if  $\text{rat.rk}(W) = 2$ , or

(3)  $\mathbb{Z}$  or (4) any nondiscrete subgroup of  $\mathbb{Q}$  if  $\text{rat.rk}(W) = 1$ .

Note that, since  $L/K$  is a finite extension, we also have  $\kappa(V) = k$ , and  $\Delta$  is of the same type (1), (2), (3) or (4) as  $\Gamma$ .

The local uniformization theorem ([2] p. 492) implies that there exists a regular model  $R$  (resp.  $S$ ) of  $V/k$  (resp.  $W/k$ ). The set of all regular models of  $V$  (resp.  $W$ ) dominating  $R$  (resp.  $S$ ) forms an infinite chain

$$(1) \quad R = R_0 < R_1 < \cdots < R_r < \cdots \text{ (resp. } S = S_0 < S_1 < \cdots < S_s < \cdots)$$

by [3] Theorem 3, where each  $R_r$  (resp.  $S_s$ ) is the quadratic transform of  $R_{r-1}$  (resp.  $S_{s-1}$ ) along  $V$  (resp.  $W$ ). In addition, we have  $V = \bigcup_{r \geq 0} R_r$  and  $W = \bigcup_{s \geq 0} S_s$  by [2] Lemma 10. In particular, there exists a pair  $(R, S)$  of regular models as above such that  $S$  dominates  $R$ , since  $R$  is essentially of finite type over  $k$ .

The local version of the Jung problem can now be stated as follows.

**DEFINITION 4.1.** — *Let  $L/K$  be a finite and separable extension of function fields of dimension two over  $k$  and let  $W/k$  be a valuation ring which is birational to  $L$ .*

*The pair  $(L/K, W)$  is said to have the strong Jung property if there exists a regular model  $R$  of  $V := W \cap K$  with the following property: for every regular model  $R'$  of  $K$  dominating  $R$ , the unique model  $\tilde{R}'$  of  $W/k$  lying above  $R'$  has a toric singularity (Definition 3.2).*

*The pair  $(L/K, W)$  is said to have the weak Jung property if for every regular model  $R$  of  $V := W \cap K$ , there exists a regular model  $R'$  of  $K$  dominating  $R$ , such that the unique model  $\tilde{R}'$  of  $W/k$  lying above  $R'$  has a toric singularity (Definition 3.2).*

We recall that, by definition,  $\tilde{R}'$  is the unique local ring of the integral closure of  $R$  in  $L$  which is dominated by  $W$ .

Finally, we recall some basic facts of the ramification theory of  $W/V$  that can be found in [27] pp. 50-82. The *ramification index* of  $W/V$  is the positive integer

$$e := [\Gamma : \Delta]$$

and the *residue degree* of  $W/V$  is  $f := [\kappa(W) : \kappa(V)] = 1$ , since the residue extension is assumed to be trivial. When  $L/K$  is Galois, we have the equality

$$(2) \quad [L : K] = e g p^\delta,$$

where  $g$  is the number of conjugates of  $W$  under the action of  $G = \text{Gal}(L/K)$  and  $\delta \geq 0$  ([27] Corollary on p. 78). By convention,  $\delta = 0$ ,  $p^\delta = 1$  if  $p = 0$ , and formula (2) still holds. One extends the definition of  $\delta$  for  $L/K$  not necessarily Galois as follows: let  $L'/K$  be a Galois closure of  $L/K$  and  $W'$  be a valuation ring which is birational to  $L'$  and lies above  $W$ . If  $\delta'$  (resp.  $\bar{\delta}$ ) is the integer which is associated with  $W'/V$  (resp.  $W'/W$ ) as in (2), the *defect* of  $W/V$  is the integer  $p^{\delta(W/V)}$ , where

$$\delta(W/V) := \delta' - \bar{\delta} \geq 0.$$

Let  $W_1, \dots, W_g$  be the set of all valuation rings which are birational to  $L$  and lie above  $V$ . Then

$$(3) \quad [L : K] = \sum_{j=1}^g e(W_j/V) p^{\delta(W_j/V)}$$

extends (2) to the case when  $L/K$  is separable but not necessarily Galois.

DEFINITION 4.2. — *With notations as above,  $W/V$  is said to be defectless (resp. tamely ramified) if  $\delta(W/V) = 0$  (resp. if  $\delta(W/V) = 0$  and  $p$  does not divide  $e(W/V)$ ). If  $p = 0$ ,  $W/V$  is always tamely ramified.*

### 5. Finitely generated value groups.

In this section, we show that any pair  $(L/K, W)$  such that  $L/K$  is a finite and separable extension of function fields of dimension two over  $k$  has the strong Jung property (Definition 4.1) if the value group  $\Gamma$  of  $W$  is finitely generated, with  $V := W \cap K$ . We keep conventions and notations as in the previous section.

In order to prove that a given pair  $(L/K, W)$  has the strong or weak Jung property, we will use repeatedly the following criterion.

LEMMA 5.1. — *Let  $(R, S)$  be a pair of regular models of  $(V, W)$  such that  $S$  dominates  $R$ , where  $R$  (resp.  $S$ ) has a regular system of parameters  $(u, v)$  (resp.  $(x, y)$ ). Let  $R'$  be a regular model of  $V$  dominating  $R$  which has a r.s.p.  $(u', v')$  given by*

$$\begin{cases} u' = u^{a'} v^{c'} \\ v' = u^{b'} v^{d'}, \end{cases}$$

where

$$A' = \begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix}$$

is a unimodular matrix with integer entries. Assume the following condition holds:

- (1) *The integral closure  $\overline{IS}$  of the ideal  $IS$  is generated by monomials in  $(x, y)$ , where*

$$I := (u^{|a'|}, v^{|c'|})(u^{|b'|}, v^{|d'|}) \subset R.$$

- (2) *The unique model  $\tilde{R}'$  of  $W/k$  lying above  $R'$  dominates  $S$ .*

Then  $\tilde{R}'$  has a toric singularity.

*Proof.* — We claim that  $R'$  is the unique local ring of the blowing up  $X'$  of  $\text{Spec}R$  along  $I$  which is dominated by  $V$ .

First assume that at least one of  $a', b', c', d'$ , say  $c'$  is zero. Necessarily  $a' = 1$  since  $A$  is unimodular and  $Vu' > 0$ . Since  $R \subseteq R'$ ,  $A^{-1}$  has nonnegative entries. Therefore  $d' = 1$  and  $b' \leq 0$ . We get that  $R' = R[v/u^{-b'}]_{(u, v/u^{-b'})}$  and the claim is proved.

Assume now that  $a'b'c'd' \neq 0$ . We have  $a'c', b'd' < 0$  since  $A^{-1}$  has nonnegative entries. The inclusion

$$R[u^{a'}v^{c'}, u^{b'}v^{d'}] \subseteq R'$$

is a birational inclusion of regular rings which is unramified at  $(u', v')$ , since  $u' = u^{a'}v^{c'}, v' = u^{b'}v^{d'}$ . Therefore we have

$$R[u^{a'}v^{c'}, u^{b'}v^{d'}]_{(u^{a'}v^{c'}, u^{b'}v^{d'})} = R',$$

which proves the claim.

Let  $\bar{R}$  (resp.  $\bar{X}'$ ) be the normalization of  $R$  (resp.  $X'$ ) in  $L = QF(S)$ , so that by definition,  $\tilde{R}'$  is the unique local ring of  $\bar{X}'$  which is dominated by  $W$ . There is a commutative diagram with proper arrows

$$\begin{array}{ccc} \text{Spec } \bar{R} & \longleftarrow & \bar{X}' \\ \downarrow & & \downarrow \\ \text{Spec } R & \longleftarrow & X'. \end{array}$$

Now, the universal properties of blowing up ([16] II.7.14) and normalization together imply that  $\bar{X}'$  is the *normalized* blow up  $Y'$  of  $\text{Spec}\bar{R}$  along  $I\bar{R}$ .

By assumption (2), we have  $\bar{R} \subseteq S < \tilde{R}'$ , so that  $\tilde{R}'$  is also the unique local ring of the normalized blow up  $Z'$  of  $\text{Spec}S$  along  $IS$  which is dominated by  $W$ .

By assumption (1),  $\bar{I}\bar{S}$  is generated by monomials in  $x, y$ . Since  $S$  is essentially of finite type over  $k$ ,  $Z'$  is obtained by blowing up the integral closure of  $I^n S$  for some  $n \geq 1$  which is also generated by monomials in  $x, y$  (in fact one can take  $n = 1$  by Zariski's Theorem [27] Corollary 2 p. 380). The ring  $\tilde{R}'$  therefore has a toric singularity. □

The following statement is [13] Theorem 7.3.

PROPOSITION 5.2. — *Let  $W/k$  be a valuation ring of  $L/k$  which is birational to  $L$  and  $V := W \cap K$ . Assume that the value group  $\Gamma$  of  $W$  is*

finitely generated, that is, of type (1), (2) or (3) with notations as in the previous section.

There exists a pair of regular models  $(R_0, S_0)$  of  $(V, W)$  such that  $S_0$  dominates  $R_0$ ,  $R_0$  (resp.  $S_0$ ) has a regular system of parameters  $(u, v)$  (resp.  $(x, y)$ ) and there is a relation

$$\begin{cases} u = \gamma_1 x^a y^c \\ v = \gamma_2 x^b y^d, \end{cases}$$

where  $\gamma_1, \gamma_2 \in S_0$  are units,  $a, b, c, d \geq 0$  and  $ad - bc \neq 0$ . Let  $A$  be the matrix

$$A := \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

If  $\text{rat.rk}(W) = 2$  (type (1) or (2)), we have  $\Gamma = \mathbb{Z}Wx \oplus \mathbb{Z}Wy$ .

If  $\Gamma \simeq \mathbb{Z}$  (type (3)), we have  $\Gamma = \mathbb{Z}Wy$  and  $A$  has the following form:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}.$$

In all cases,  $W/V$  is defectless and

$$(5) \quad \Gamma/\Delta \simeq \mathbb{Z}^2/A\mathbb{Z}^2.$$

Proposition 5.2 is the main ingredient in the proof of the following theorem.

**THEOREM 5.3.** — *Let  $L/K$  be a finite and separable extension of function fields of dimension two over  $k$ . Assume that the value group  $\Gamma$  of  $W/k$  is finitely generated.*

*Then  $(L/K, W)$  has the strong Jung property.*

*Proof.* — Since  $V = \bigcup_{r \geq 0} R_r$  by (1), the unique model  $\tilde{R}_r$  of  $W/k$  lying above  $R_r$  contains the local ring  $S_0$  of Proposition 5.2 for  $r \geq r_0$ . Remark that the special form of  $A$  provided by Proposition 5.2 implies that one can take  $r_0 = 0$  if  $\Gamma \simeq \mathbb{Z}$ . Let  $R := R_{r_0}$  and  $r \geq r_0$ .

First assume that  $\Gamma$  has rational rank two. Then  $Wx$  and  $Wy$  are rationally independent since they generate  $\Gamma$ . Equation (4) and the condition  $ad - bc \neq 0$  together imply that  $Vu$  and  $Vv$  are also rationally independent. Therefore the regular local ring  $R' := R_r$  has a regular system of parameters  $(u', v')$  satisfying the relation

$$\begin{cases} u' = u^{a'} v^{c'} \\ v' = u^{b'} v^{d'}, \end{cases}$$

where

$$A' = \begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix}$$

is a unimodular matrix with integer entries. Then  $R'$  clearly satisfies the hypotheses of Lemma 5.1 from which the conclusion follows.

In case  $\Gamma \simeq \mathbb{Z}$  we now have  $\Gamma = \mathbb{Z}Wy$ . Equation (5) and the special form of the matrix  $A$  in this case show that  $\Delta = \mathbb{Z}Vv$ . Let  $m := Vu/Vv \in \mathbb{Z}$ . There exists  $\mu \in k$  such that  $V(u - \mu v^m) > Vu$ . After replacing  $u$  with  $u_1 := u - \mu v^m \in R$  and  $x$  with  $x_1 := u - \mu v^m \in S$ , one gets a new value  $m_1 := Vu_1/Vv > m$ . After iterating a finite number of times, it can be assumed that  $m > r$ . Therefore the regular local ring  $R' := R_r$  has a r.s.p.  $(u', v')$  satisfying the relation

$$\begin{cases} u' = wv^{-r} \\ v' = v. \end{cases}$$

One then concludes as in the case when  $\Gamma$  has rational rank two. □

## 6. Nondiscrete subgroups of $\mathbb{Q}$ .

In this section, we consider any pair  $(L/K, W)$  such that  $L/K$  is a finite and separable extension of function fields of dimension two over  $k$  and  $\Gamma$  is of type (4), that is a nondiscrete subgroup of  $\mathbb{Q}$ . We prove that  $(L/K, W)$  has the strong (resp. weak) Jung property when  $W/V$  is defectless, where  $V := W \cap K$  (resp. when  $\Gamma$  is not  $p$ -divisible).

Let  $R$  (resp.  $S$ ) be a regular model of  $V$  (resp.  $W$ ) such that  $S$  dominates  $R$ . We restrict our attention to certain such pairs  $R \subset S$  which are called prepared pairs.

DEFINITION 6.1. — *With notations as before,*

(1) *Given an index  $r \geq 1$ ,  $R_r$  (resp.  $S_s$ ) is said to be a 1-point if the reduced exceptional locus  $E_r$  (resp.  $F_s$ ) of  $\text{Spec } R_r \rightarrow \text{Spec } R$  (resp.  $\text{Spec } S_s \rightarrow \text{Spec } S$ ) has precisely one irreducible component.*

(2) *Given a pair  $(r, s)$  of positive integers, the pair  $(R_r, S_s)$  is said to be prepared if the following properties hold:*

- (i)  $S_s$  dominates  $R_r$ .
- (ii) Both of  $R_r$  and  $S_s$  are 1-points.

(iii) The singular locus of  $\text{Spec } S_s \rightarrow \text{Spec } R_r$  is contained in  $F_s$ .

(iv) We have  $u = \gamma x^a$  for some  $a > 0$ , where  $u$  (resp.  $x$ ) is a regular parameter of  $R_r$  (resp.  $S_s$ ) whose support is  $E_r$  (resp.  $F_s$ ), and  $\gamma$  is a unit in  $S_s$ .

PROPOSITION 6.2. — Assume that the value group of  $W$  is a nondiscrete subgroup of  $\mathbb{Q}$ .

The set of prepared pairs is cofinal in the set of all pairs  $(R_r, S_s)$ . Given a prepared pair  $(R_r, S_s)$ , any pair  $(R_{r'}, S_{s'})$  with  $r' \geq r$ ,  $s' \geq s$ , such that both of  $R_{r'}$  and  $S_{s'}$  are 1-points and  $S_{s'}$  dominates  $R_{r'}$  is also prepared.

Proof. — Since the value group of  $W$  is a nondiscrete subgroup of  $\mathbb{Q}$  it is true that the set of 1-points  $R_r$  (resp.  $S_s$ ) is cofinal in the set of all  $R_r$  (resp.  $S_s$ ) ([5] Theorem 4.7(A)).

Since  $L/K$  is separable, the singular locus of the map  $\text{Spec } S_s \rightarrow \text{Spec } R_r$  is a (possibly empty) curve  $C_{r,s}$  in  $\text{Spec } S_s$ .

If  $C$  is a curve in  $\text{Spec } S_s$ , its total transform in  $\text{Spec } S_{s'}$  is contained in  $F_{s'}$  for all large enough  $s'$  ([2] Proposition 3). Applying this statement to  $E_r$  and  $C_{r,s}$  for a given pair  $(R_r, S_s)$  satisfying (i) and (ii) of Definition 6.1, one gets that (iii) and (iv) automatically hold for any pair  $(R_{r'}, S_{s'})$  satisfying (i) and (ii) provided  $r'$  and  $s'$  are large enough. This proves the proposition.  $\square$

DEFINITION 6.3. — Let  $(R_r, S_s)$  be prepared. A r.s.p.  $(u, v)$  (resp.  $(x, y)$ ) of  $R_r$  (resp.  $S_s$ ) is said to be prepared if the support of  $u$  (resp.  $x$ ) is equal to  $E_r$  (resp.  $F_s$ ) and if  $Vv$  (resp.  $Wy$ ) is maximal among all such r.s.p.'s containing  $u$  (resp.  $x$ ).

The following result is [13] Theorems 7.33 and 7.35 and gives a satisfactory analogue of Proposition 5.2 in the defectless or not  $p$ -divisible case. Equations (7) and (8) are a rephrasing of the statement “ $g_n = 1$  for  $n \gg 0$ ” in *loc.cit.* For  $\delta = 0$ , the inequality  $n_i > 1$  follows from the definitions in [13] and for  $\delta > 0$ , the inequality  $n_i > p^\delta$  follows from *ibid.* Lemma 7.29 (2).

PROPOSITION 6.4. — Let  $W/k$  be a valuation ring of  $L/k$  whose

value group  $\Gamma$  is a nondiscrete subgroup of  $\mathbb{Q}$ . Assume that  $\Gamma$  is not  $p$ -divisible, or that  $W/V$  is defectless (Definition 4.2), where  $V := W \cap K$ .

There exists an infinite sequence of prepared pairs of models  $(R_{r_i}, S_{s_i})$ ,  $i \geq 0$ , such that for each  $i \geq 1$ ,  $R_{r_i}$  (resp.  $S_{s_i}$ ) dominates  $R_{r_{i-1}}$  (resp.  $S_{s_{i-1}}$ ) and having the following properties: each  $R_{r_i}$  (resp.  $S_{s_i}$ ) has a r.s.p.  $(u_i, v'_i)$  (resp.  $(x_i, y'_i)$ ) and a prepared r.s.p.  $(u_i, v_i)$  (resp.  $(x_i, y_i)$ ). There are relations

$$(6) \quad \begin{cases} u_i = \gamma_i x_i^e \\ v_i = f_i, \end{cases}$$

where  $\gamma_i \in S_{s_i}$  is a unit,  $e$  is the ramification index of  $W/V$ , and  $\text{ord}_{y_i}(f_i \bmod x_i) = p^\delta$  is the defect of  $W/V$ .

There are relations

$$(7) \quad \begin{cases} x_{i-1} = x_i^{n_i} (y'_i + \mu_i)^{c_i} \\ y_{i-1} = x_i^{e l_i / p^\delta} (y'_i + \mu_i)^{d_i}, \end{cases}$$

and

$$(8) \quad \begin{cases} u_{i-1} = u_i^{n_i} (v'_i + \lambda_i)^{a_i} \\ v_{i-1} = u_i^{l_i} (v'_i + \lambda_i)^{b_i}, \end{cases}$$

where  $\lambda_i, \mu_i \in k$  are nonzero,  $n_i > p^\delta$ , and  $n_i d_i - e c_i l_i / p^\delta = n_i b_i - a_i l_i = 1$ .

**THEOREM 6.5.** — *Let  $L/K$  be a finite and separable extension of function fields of dimension two over  $k$  and  $W/k$  be a valuation ring of  $L/k$  whose value group  $\Gamma$  is a nondiscrete subgroup of  $\mathbb{Q}$ .*

(1) *Assume that  $W/V$  is defectless, where  $V := W \cap K$ . Then  $(L/K, W)$  has the strong Jung property.*

(2) *Assume that  $\Gamma$  is not  $p$ -divisible. For each  $r \gg 0$  such that  $R_r$  is a 1-point, the unique model  $\tilde{R}_r$  of  $W/k$  lying above  $R_r$  has a toric singularity. In particular,  $(L/K, W)$  has the weak Jung property.*

*Proof.* — Let  $R := R_{r_0}$  and let  $r \geq r_0$ . Pick the unique integer  $i \geq 0$  such that  $r_i \leq r < r_{i+1}$ . The unique model  $\tilde{R}_r$  of  $W/k$  lying above  $R_r$  contains  $S_{s_i}$ . We clearly have  $\tilde{R}_{r_i} = S_{s_i}$  which is a regular local ring so it can be assumed that  $r > r_i$ .

Since  $r < r_{i+1}$ , it follows from (8) that the regular local ring  $R' := R_r$  has a r.s.p.  $(u', v')$  satisfying a relation

$$\begin{cases} u' = u_i^{a'} v_i^{c'} \\ v' = u_i^{b'} v_i^{d'}, \end{cases}$$

where

$$A' := \begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix}$$

is a unimodular matrix with integer entries.

First assume that  $W/V$  is defectless. Then  $R'$  clearly satisfies the hypotheses of Lemma 5.1 from which the conclusion follows.

Assume now that  $W/V$  has defect  $p^\delta > 1$ , so that  $p > 0$  and we are in case 2 in the Theorem, i.e.,  $\Gamma$  is not  $p$ -divisible. We need only consider the case when  $R_r$  is a 1-point, so that  $a' = d' = 1$ ,  $c' = 0$ ,  $b' < 0$ . More generally, let  $\alpha, \beta \in \mathbb{Z}$  be such that  $\alpha < 0$ ,  $\beta > 0$  and  $W(u_i^\alpha v_i^\beta) > 0$ . We claim that the integral closure of  $IS$ , where

$$I := (u_i^{-\alpha}, v_i^\beta) \subset R,$$

is generated by monomials in  $x, y$ . By (6), (7) and (8), we have

$$(9) \quad Wv_i = l_{i+1}Wu_{i+1} = el_{i+1}Wx_{i+1} = p^\delta Wy_i.$$

Since  $\text{ord}_{y_i}(f_i \bmod x_i) = p^\delta$ , the Weierstrass preparation theorem allows to write  $f_i$  as

$$(10) \quad f_i = \theta(x_i, y_i) \sum_{k=0}^{p^\delta} y_i^k \sigma_k(x_i) \in k[[x_i, y_i]],$$

where  $\theta(x_i, y_i)$  is an invertible power series,  $\sigma_{p^\delta}(x_i) := 1$  and  $\sigma_k(x_i) \in k[[x_i]]$ . We have

$$\Gamma = \sum_{i \geq 0} \mathbb{Z}Wx_i$$

by [2] Proposition 3. Computing from (7), we get

$$\Gamma = \bigcup_{i \geq 1} \mathbb{Z} \frac{Wx_0}{n_1 \cdots n_i}.$$

Since  $\Gamma$  is not  $p$ -divisible,  $p$  does not divide  $n_i$  for large enough  $i$ , so to begin with, it can be assumed that  $p$  does not divide  $n_i$  for all  $i \geq 1$ . We have  $\text{g.c.d.}(n_i, el_i/p^\delta) = 1$  and  $\text{g.c.d.}(n_i, l_i) = 1$ , so that we deduce that  $\text{g.c.d.}(e, n_i) = 1$  for each  $i \geq 1$ . Therefore by (7), the image of  $Wy_i$  has order  $n_{i+1}$  in the quotient group  $(\mathbb{Z}Wy_i + \mathbb{Z}Wx_i)/\mathbb{Z}Wx_i$ . Using (9), (10) and the fact that  $n_{i+1} > p^\delta$ , we deduce that each  $\sigma_k$ ,  $0 \leq k < p^\delta$ , satisfies the condition

$$W(y_i^k \sigma_k(x_i)) > p^\delta Wy_i.$$

Using once again (9), we hence get for each  $k$ ,  $0 \leq k \leq p^\delta$ ,

$$\beta W(y_i^k \sigma_k(x_i)) \geq \beta p^\delta Wy_i = W(u_i^\alpha v_i^\beta) - \alpha Wu_i.$$

Since  $W(u_i^\alpha v_i^\beta) > 0$ , this shows that for each  $k$ ,  $0 \leq k \leq p^\delta$ , we have

$$\beta W(y_i^k \sigma_k(x_i)) \geq \beta W y_i^{p^\delta} > -\alpha W u_i = -e\alpha W x_i.$$

Consequently,

$$(11) \quad (u_i^{-\alpha}, v_i^\beta) S_{s_i} \subseteq \text{integral closure of } (y_i^{p^\delta \beta}, x_i^{-e\alpha}).$$

Since

$$\text{mult}_{S_{s_i}}(u_i^{-\alpha}, v_i^\beta) = -ep^\delta \alpha \beta = \text{mult}_{S_{s_i}}(y_i^{p^\delta \beta}, x_i^{-e\alpha}),$$

equation (11) is an equality of integral closures

$$\overline{(u_i^{-\alpha}, v_i^\beta) S_{s_i}} = \overline{(y_i^{p^\delta \beta}, x_i^{-e\alpha})},$$

which proves the claim.

Applying this fact to the pair  $(\alpha, \beta) = (b', d') = (b', 1)$ , we get that  $\overline{I S_{s_i}}$  is generated by monomials in  $x, y$ , where  $I := (u_i^{-b'}, v_i)$ . Then  $R'$  satisfies the hypotheses of Lemma 5.1 from which the conclusion follows.  $\square$

### 7. Extensions of degree $p$ .

In this section, we consider the case of a (not necessarily Galois) degree  $p$  separable extension  $L/K$ . Let  $W/k$  be a valuation ring which is birational to  $L$  and let  $V := W \cap K$ . The goal of this section is to prove the following theorem.

**THEOREM 7.1.** — *Assume that  $\text{char } k = p > 0$ . Let  $L/K$  be a finite and separable extension of degree  $p$  of function fields of dimension two over  $k$  and let  $W/k$  be a valuation ring which is birational to  $L$ . Then  $(L/K, W)$  has the weak Jung property.*

Recall that  $(L/K, W)$  has the strong Jung property if  $\Gamma$  is finitely generated by Theorem 5.3. In case  $\Gamma$  is a nondiscrete subgroup of  $\mathbb{Q}$ ,  $(L/K, W)$  has the strong Jung property if  $W/V$  is defectless by Proposition 6.5.

By equation (3), the remaining case is when  $W/V$  has defect  $p$  which we assume from now on. Let  $(R, S)$  be a prepared pair, where  $R$  (resp.  $S$ ) has a r.s.p.  $(u, v)$  (resp.  $(x, y)$ ). There is an expression

$$(12) \quad \begin{cases} u = \gamma x^a \\ v = x^b f, \end{cases}$$

where  $\gamma \in S$  is a unit and  $x$  does not divide  $f$ . The following statement follows from [13] Theorems 7.20 (1) and 7.33.

PROPOSITION 7.2. — *Assume that  $f \in S$  is a nonunit. Then*

$$a \operatorname{ord}_y(f \bmod x) = p.$$

LEMMA 7.3. — *There exists a prepared pair  $(R, S)$  with  $b = 0$  in (12).*

*Proof.* — Pick any prepared pair. If  $f$  is a unit in (12), the ideal  $(u, v)S$  is a principal ideal. Therefore,  $R \subset S$  factors through the quadratic transform  $R \subset R_1$  of  $R$  along  $V$  and we replace  $(R, S)$  with  $(R_1, S)$ . After a finite number of iterations, we may assume that  $f$  is a nonunit to begin with. Proposition 7.2 allows to distinguish two cases.

Case 1.  $a = 1, \operatorname{ord}_y(f \bmod x) = p$ .

Let  $R'$  be the iterated quadratic transform of  $R$  along  $V$  with regular parameters  $(u' = u, v' = vu^{-b})$ . We get the following expression for the prepared pair  $(R', S)$ :

$$\begin{cases} u' = \gamma x \\ v' = \gamma^{-b} f, \end{cases}$$

which satisfies the conclusion of the lemma.

Case 2.  $a = p, \operatorname{ord}_y(f \bmod x) = 1$ .

After replacing  $x$  with  $\lambda x$  for some  $\lambda \in k, \lambda \neq 0$ , it can be assumed that  $\gamma \equiv 1 \pmod{m_R}$ . Let  $y := f$ . Note that  $(x, y)$  is a (not necessarily prepared) r.s.p. of  $S$ . Let  $q_1/n_1 := Wy/Wx \in \mathbb{Q}$ , with  $n_1, q_1 > 0$  and  $\operatorname{g.c.d.}(n_1, q_1) = 1$ . Since  $n_1Wy = q_1Wx$ , there exists  $\mu \in k, \mu \neq 0$ , such that  $W(y^{n_1} - \mu x^{q_1}) > W(y^{n_1})$ . Write  $c_1n_1 - d_1q_1 = 1$ , with  $c_1, d_1 > 0$  and let  $S_{s_1}$  be the iterated quadratic transform of  $S$  along  $W$  whose regular parameters  $(x_1, y_1)$  satisfy

$$(13) \quad \begin{cases} x = x_1^{n_1}(y_1 + \mu)^{d_1} \\ y = x_1^{q_1}(y_1 + \mu)^{c_1}. \end{cases}$$

By (12), there is an expression

$$\begin{cases} u = \gamma x_1^{pn_1}(y_1 + \mu)^{d_1p} \\ v = x_1^{bn_1+q_1}(y_1 + \mu)^{c_1+bd_1}. \end{cases}$$

Let  $l_1 := bn_1 + q_1$ . Clearly,  $\text{g.c.d.}(l_1, n_1) = 1$  so that  $e_1 := \text{g.c.d.}(l_1, pn_1) = 1$  or  $p$ . There exists  $\lambda \in k, \lambda \neq 0$ , such that  $V(v^{pn_1/e_1} - \lambda u^{l_1/e_1}) > V(v^{pn_1/e_1})$ . Write  $a_1pn_1/e_1 - b_1l_1/e_1 = 1$ , with  $a_1, b_1 > 0$ . Let  $R_{r_1}$  be the iterated quadratic transform of  $R$  along  $V$  whose regular parameters  $(u_1, v_1)$  satisfy

$$(14) \quad \begin{cases} u = u_1^{pn_1/e_1} (v_1 + \lambda)^{b_1} \\ v = u_1^{l_1/e_1} (v_1 + \lambda)^{a_1}. \end{cases}$$

We get an expression

$$(15) \quad \begin{cases} u_1 &= u^{a_1} v^{-b_1} &= \gamma^{a_1} x_1^{e_1} (y_1 + \mu)^{pd_1 a_1 - b_1(c_1 + bd_1)} \\ v_1 + \lambda &= u^{-l_1/e_1} v^{pn_1/e_1} &= \gamma^{-l_1/e_1} (y_1 + \mu)^{p/e_1}. \end{cases}$$

Since  $\gamma \equiv 1 \pmod{x_1}$ , we get  $\lambda = \mu^{p/e_1}$  and  $v_1 \equiv y_1^{p/e_1} \pmod{x_1}$ . Therefore the prepared pair  $(R_{r_1}, S_{s_1})$  satisfies the conclusion of the lemma.  $\square$

*Proof of Theorem 7.1.* — Pick any prepared pair  $(R, S)$  satisfying the conclusion of Lemma 7.3. We will build up another pair  $(R_{r_1}, S_{s_1})$  also satisfying the conclusion of Lemma 7.3 such that  $R \subset R_{r_1}, S \subset S_{s_1}$ , and any regular model  $R'$  of  $V$  with  $R \subseteq R' \subset R_{r_1}$  satisfies the hypotheses of Lemma 5.1. Since  $V = \cup_{r \geq 0} R_r$  by (1), the theorem then follows from

Lemma 5.1 by induction on  $r$ . We distinguish two cases as in the proof of Lemma 7.3.

*Case 1.*  $a = 1, \text{ord}_y(f \pmod{x}) = p$ .

After replacing  $x$  with  $\gamma x$ , it can be assumed that  $\gamma = 1$ . Pick prepared r.s.p.  $(u, v)$  of  $R$  and  $(x, y)$  of  $S$ . By the Weierstrass preparation theorem, we have

$$v = \theta(x, y) \sum_{k=0}^p \sigma_k(x) y^k \in \hat{S} = k[[x, y]],$$

where  $\theta$  is an invertible power series and  $\sigma_p(x) := 1$ . After replacing  $y$  with  $\lambda y$  for some  $\lambda \in k, \lambda \neq 0$ , it can also be assumed that  $\theta(0, 0) = 1$ . Let  $q_1/n_1 := Wy/Wx$ , with  $n_1, q_1 > 0$  and  $\text{g.c.d.}(n_1, q_1) = 1$ . Note that  $n_1 \geq 2$  since  $(x, y)$  is prepared. There exists a unique  $\mu \in k, \mu \neq 0$ , such that  $W(y^{n_1} - \mu x^{q_1}) > W(y^{n_1})$ . Let  $S \subseteq \{0, \dots, p\}$  be defined by

$$k \in S \iff Wv = W(\sigma_k(x)y^k).$$

Given  $k \in \{0, \dots, p\}$ , let

$$\sigma_k(x) =: \lambda_k x^{m_k} + \text{higher order terms.}$$

Finally, note that  $\text{g.c.d.}(n_1, q_1) = 1$  implies that

$$(16) \quad W(\sigma_k(x)y^k) = W(\sigma_{k'}(x)y^{k'}) \iff \exists \kappa \in \mathbb{Z} \mid k' - k = \kappa n_1, m_k - m_{k'} = \kappa q_1.$$

In particular,  $|S| \neq 1$  implies that  $n_1 \leq p$ .

First assume that  $S \neq \emptyset$ . Let  $k_0 := \min\{S\}$ . After possibly replacing  $v$  with  $v - \lambda_0 u^{m_0}$ , it can be assumed that  $k_0 \geq 1$ . Define

$$(17) \quad I(x, y) := \sum_{k \in S} \lambda_k x^{m_k} y^k.$$

Write  $c_1 n_1 - d_1 q_1 = 1$ , with  $c_1, d_1 > 0$  and let  $S_{s_1}$  be the iterated quadratic transform of  $S$  along  $W$  whose regular parameters  $(x_1, y_1)$  satisfy

$$(18) \quad \begin{cases} x = x_1^{n_1} (y_1 + \mu)^{d_1} \\ y = x_1^{q_1} (y_1 + \mu)^{c_1}. \end{cases}$$

Also let  $l_1 := m_{k_0} n_1 + k_0 q_1 = Wv/Wx_1$  and let

$$e_1 := \text{g.c.d.}(n_1, k_0) = \text{g.c.d.}(n_1, l_1).$$

Write  $a_1 n_1/e_1 - b_1 l_1/e_1 = 1$ , with  $a_1, b_1 > 0$ . Let  $\lambda \in k, \lambda \neq 0$ , be such that  $W(v^{n_1/e_1} - \lambda u^{l_1/e_1}) > W(v^{n_1/e_1})$  and  $R_{r_1}$  be the iterated quadratic transform along  $V$  whose regular parameters  $(u_1, v_1)$  satisfy

$$(19) \quad \begin{cases} u = u_1^{n_1/e_1} (v_1 + \lambda)^{b_1} \\ v = u_1^{l_1/e_1} (v_1 + \lambda)^{a_1}. \end{cases}$$

A computation analogous to that in (15) gives an expression

$$(20) \quad \begin{cases} u_1 = x_1^{e_1} (y_1 + \mu)^{a_1 d_1} I((y_1 + \mu)^{d_1}, (y_1 + \mu)^{c_1})^{-b_1} (1 + x_1 g) \\ v_1 + \lambda = I((y_1 + \mu)^{d_1}, (y_1 + \mu)^{c_1})^{n_1/e_1} (1 + x_1 h), \end{cases}$$

with  $g, h \in S_{s_1}$ . Modding out  $v_1$  by  $(x_1, y_1)$ , we get

$$\lambda = I(\mu^{d_1}, \mu^{c_1})^{n_1/e_1}.$$

After performing consecutive Euclidian divisions of  $I(x, y)^{n_1/e_1}$  in  $y$ , we obtain an expression

$$(21) \quad I(x, y)^{n_1/e_1} = \sum_{\mathbb{A}} \mu_{\mathbb{A}} (y^{n_1} - \mu x^{q_1})^{A_2} y^{A_1} x^{A_0}$$

with  $\nu_{\mathbb{A}} \in k$  and  $\mathbb{A} := (A_0, A_1, A_2)$  subject to the conditions

$$0 \leq A_1 < n_1, \quad 0 \leq A_2 n_1 + A_1 \leq p n_1 / e_1.$$

By Proposition 7.2, we have

$$(22) \quad e_1 \text{ord}_{y_1}(v_1 \bmod x_1) = p.$$

After noticing that

$$\text{ord}_{y_1}((y_1 + \mu)^{n_1 c_1} - \mu (y_1 + \mu)^{q_1 d_1}) = 1,$$

we conclude from (20), (21) and (22) that  $e_1$  divides  $p$  and that

$$I(x, y)^{n_1/e_1} = \mu_0(y^{n_1} - \mu x^{q_1})^{p/e_1} + \mu_1 x^{pq_1/e_1}$$

for some  $\mu_0, \mu_1 \in k$ . Necessarily  $\mu_0 = 1$  since  $\sigma_p = 1$ , and  $\mu_1 = (-\mu)^{p/e_1}$  since  $k_0 \geq 1$  so that we had to begin with

$$(23) \quad I(x, y) = y^p.$$

The argument following (9) in the proof of Theorem 6.5 now applies directly by replacing (9) with (23), and the pair  $(R_{r_1}, S_{s_1})$  is as required.

We now sketch the argument in the case  $S = \emptyset$ . Necessarily  $n_1 = p$  and  $v$  can be written in Weierstrass form

$$v = \theta(x, y)(y^p - \mu x^{q_1} + \text{higher value terms}),$$

where  $\theta$  is an invertible power series. Equation (18) is unchanged and equation (19) is replaced with

$$\begin{cases} u = u_1 \\ v = u_1^{q_1} v_1. \end{cases}$$

Finally any regular model  $R'$  of  $V$  with  $R \subseteq R' \subset R_{r_1}$  has a r.s.p.  $(u', v')$  given by

$$\begin{cases} u' = u \\ v' = u^{-k} v \end{cases}$$

for some  $k$  with  $0 \leq k < q_1$ . Therefore there is an equality of integral closures

$$\overline{(u^k, v)S} = \overline{(y^p, x^k)}$$

and  $R'$  satisfies the hypotheses of Lemma 5.1 from which the conclusion follows.

*Case 2.*  $a = p$ ,  $\text{ord}_y(v \bmod x) = 1$ .

The pair  $(R_{r_1}, S_{s_1})$  is defined by equations (13) and (14). That any regular model  $R'$  of  $V$  with  $R \subseteq R' \subset R_{r_1}$  satisfies the hypotheses of Lemma 5.1 is trivial since  $\text{ord}_y(v \bmod x) = 1$ .

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