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A COMPACTIFICATION OF $(\mathbb{C}^*)^4$ WITH NO NON-CONSTANT MEROMORPHIC FUNCTIONS

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1. Introduction.

There is a well-known example, due to J.-P. Serre, of a Zariski open subset of a ruled surface over an elliptic curve, which is Stein, but not affine ([Ha] 6.3). This example plays an interesting role in complex analysis, for example in the theory of local cohomology of analytic sheaves (e.g. [KP]) and the theory of nef vector bundles ([DPS] 1.7). The purpose of this note is to extend this construction to the dimension 4 by interpreting it from the view-point of additive group action. Of course, it may be possible to have more direct generalization of Serre's construction to higher dimensions. But our approach via additive group action reveals a number of interesting features of the resulting 4-dimensional compact complex manifold. This complex manifold is interesting in the following aspects.

A well-known conjecture in the study of compactifications of \mathbb{C}^n is the following:

CONJECTURE. — *Every compactification of \mathbb{C}^n is Moishezon. Namely, a compact complex manifold containing \mathbb{C}^n as a Zariski open subset has n algebraically independent meromorphic functions.*

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Although this is true in dimension 2, it is completely open in higher-dimensions, except for some partial results in dimension 3 (cf. [PS]). Even under the additional assumption that the compactifying divisor is smooth

the assumption that the compactification is Kähler, the problem remains unsolved. One may ask the same question for compactifications of $(\mathbb{C}^*)^n$. But our construction will give a negative answer:

THEOREM 1. — *Let T be any complex torus of dimension 2. Then there exists a compact Kähler 4-fold $X = X(T)$ and a smooth divisor $D \subset X$ such that D is biholomorphic to $\mathbb{P}_1 \times T$ and $X - D$ is biholomorphic to $(\mathbb{C}^*)^4$.*

Since the image of a Moishezon manifold is Moishezon, if T is not an abelian variety, $X(T)$ is a compactification of $(\mathbb{C}^*)^4$ which is not Moishezon.

construction to a compactification of $(\mathbb{C}^*)^{2n}$, which is a \mathbb{P}_n -bundle over an n -dimensional complex torus.

2. An approach to Serre's example via \mathbb{C} -action.

It is instructive first to give a construction of Serre's example from the view-point of \mathbb{C} -action on $\mathbb{C}^* \times \mathbb{C}^*$, to clarify the construction in Theorem 1.

Let $\alpha \in \mathbb{C} - \mathbb{R}$. Consider the \mathbb{C} -action on $\mathbb{C}^* \times \mathbb{C}^*$ given by

$$(1) \quad s \cdot (x, z) = (e^s x, e^{\alpha s} z).$$

Since α and 1 are independent over \mathbb{Z} , the map $s \mapsto s \cdot p$ is injective for any $p \in \mathbb{C}^* \times \mathbb{C}^*$, i.e., the action is faithful. Moreover, since α and 1 are independent over \mathbb{R} the same map is actually proper. Indeed, if $\{s_j\}_{j \in \mathbb{N}}$ is a divergent sequence in \mathbb{C} such that $\{\log |e^{s_j} x|\}_{j \in \mathbb{N}}$ is bounded, then the real part of s_j is confined to a strip of finite width in the s -plane for all j . Thus the imaginary part of s_j diverges. But since α has non-zero imaginary part, $\log |e^{\alpha s} z|$ is unbounded.

It follows from general theory of Lie group actions that the quotient of $\mathbb{C}^* \times \mathbb{C}^*$ by the action (1) is a Riemann surface B . We claim in fact that it is an elliptic curve. Indeed, this action realizes $\mathbb{C}^* \times \mathbb{C}^*$ as a locally trivial \mathbb{C} -bundle over B , and thus in particular, B is homotopy equivalent to $\mathbb{C}^* \times \mathbb{C}^*$. Since every noncompact Riemann surface has no second homology, we see that B must be compact. The homology of B then forces it to be an elliptic curve. In fact, one can check that B is the torus $\mathbb{C}/(\mathbb{Z} + \alpha\mathbb{Z})$.

Now, since $\text{Aut}(\mathbb{C})$ is an affine group, the bundle $\mathbb{C}^* \times \mathbb{C}^* \rightarrow B$ is an affine bundle with fibers \mathbb{C} . Thus we can attach ∞ to each fiber and obtain a \mathbb{P}_1 -bundle over the elliptic curve B . Equivalently, the affine transition functions of the bundle $\mathbb{C}^* \times \mathbb{C}^* \rightarrow B$ can be homogenized so as to define a rank 2 vector bundle $E \rightarrow B$ whose projectivization $\mathbb{P}(E) \rightarrow B$ has a distinguished section, and the complement of this section is $\mathbb{C}^* \times \mathbb{C}^*$.

The construction outlined above can be carried out quite explicitly, and the reader is invited to do so and obtain in particular the following additional facts:

- The bundle $\mathbb{P}(E) \rightarrow B$ is real analytically isomorphic to $\mathbb{P}_1 \times B$. Thus the section of this bundle has self intersection 0. However, since the complement of this section is Stein, the section is holomorphically rigid.

- The vector bundle $E \rightarrow B$ is flat, i.e., it can be given transition functions which are locally constant.
- In fact, $E \rightarrow B$ is a non-split extension of \mathcal{O} by \mathcal{O} , and thus the algebraic structure inherited by $\mathbb{C}^* \times \mathbb{C}^*$ from $\mathbb{P}(E)$ is not affine.

Remark. — A famous problem in complex analysis is to determine whether or not $\mathbb{C}^* \times \mathbb{C}^*$ contains an open subset biholomorphic to \mathbb{C}^2 ([RR] Appendix). Perhaps one can show that no open subset of $\mathbb{P}(E)$ is biholomorphic to \mathbb{C}^2 .

3. Proof of Theorem 1.

Every complex torus T is biholomorphic to one of the form $\mathbb{C}^2/(\mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}\lambda + \mathbb{Z}\mu)$, where e_1, e_2 are the standard unit vectors in \mathbb{C}^2 and $\{e_1, e_2, \lambda, \mu\}$ are independent over \mathbb{R} . We call $\{\lambda, \mu\}$ *normalized lattice vectors* for T . In terms of normalized lattice vectors $\lambda = (\lambda^1, \lambda^2)$ and $\mu = (\mu^1, \mu^2)$, T can be obtained as a quotient of $\mathbb{C}^* \times \mathbb{C}^*$ by the \mathbb{Z}^2 action

$$(m, n) \cdot (z, w) = (ze^{2\pi\sqrt{-1}(m\lambda^1+n\mu^1)}, we^{2\pi\sqrt{-1}(m\lambda^2+n\mu^2)}).$$

We denote the quotient map by $(z, w) \mapsto [z, w]$.

Fix a torus $T = \mathbb{C}^2/\Gamma = \mathbb{C}^2/(\mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}\lambda + \mathbb{Z}\mu)$ and consider the following \mathbb{C}^2 action on $(\mathbb{C}^*)^4$:

$$(2) \quad (s, t) * (x, y, z, w) := (e^s x, e^t y, e^{\lambda^1 s + \mu^1 t} z, e^{\lambda^2 s + \mu^2 t} w),$$

where $\lambda = \lambda^1 e_1 + \lambda^2 e_2$, and similarly for μ .

First, notice that this is a faithful action. Indeed, for fixed $p \in (\mathbb{C}^*)^4$, if $(s, t) * p = (s', t') * p$ then

$$\begin{aligned} s - s' &= 2\pi\sqrt{-1}m_1 \\ t - t' &= 2\pi\sqrt{-1}m_2 \\ (\lambda^1 s + \mu^1 t) - (\lambda^1 s' + \mu^1 t') &= 2\pi\sqrt{-1}k_1 \\ (\lambda^2 s + \mu^2 t) - (\lambda^2 s' + \mu^2 t') &= 2\pi\sqrt{-1}k_2 \end{aligned}$$

for some integers m_1, m_2, k_1, k_2 . Thus $m_1\lambda + m_2\mu - k_1e_1 - k_2e_2 = 0$ and so $m_1 = m_2 = k_1 = k_2 = 0$.

It is possible to show directly, as in Section 2, that the map $(s, t) \mapsto (s, t) * p$ is an embedding of \mathbb{C}^2 into $(\mathbb{C}^*)^4$. This also follows from the next proposition. Let $\pi : (\mathbb{C}^*)^4 \rightarrow T$ be the holomorphic map defined by

$$\pi(x, y, z, w) = \left[ze^{-(\lambda^1 \log x + \mu^1 \log y)}, we^{-(\lambda^2 \log x + \mu^2 \log y)} \right].$$

PROPOSITION 3.1. — *The map π is the quotient map for the action $*$. That is to say,*

$$\pi(x, y, z, w) = \pi(x', y', z', w') \iff (s, t) * (x, y, z, w) = (x', y', z', w')$$

for some $(s, t) \in \mathbb{C}^2$.

Proof. — Suppose $\pi(x, y, z, w) = \pi(x', y', z', w')$. Then

$$\frac{z'}{z} = e^{[\lambda^1(\log(x'/x) + 2\pi\sqrt{-1}c) + \mu^1(\log(y'/y) + 2\pi\sqrt{-1}d)]}$$

and

$$\frac{w'}{w} = e^{[\lambda^2(\log(x'/x) + 2\pi\sqrt{-1}c) + \mu^2(\log(y'/y) + 2\pi\sqrt{-1}d)]}$$

for some integers c and d . The reader can observe that it is possible to choose a single branch of the logarithm so that all the numbers appearing in these equations make sense. Now let $s = \log(x'/x) + 2\pi\sqrt{-1}c$ and $t = \log(y'/y) + 2\pi\sqrt{-1}d$. Then

$$\frac{x'}{x} = e^s \quad \text{and} \quad \frac{y'}{y} = e^t,$$

and so $(s, t) * (x, y, z, w) = (x', y', z', w')$. □

From general theory of Lie group actions, it follows that the bundle $\pi : (\mathbb{C}^*)^4 \rightarrow T$ is a locally trivial \mathbb{C}^2 bundle. However, we will show this more directly.

To this end, let $D = D_1$ be a fundamental domain of T in \mathbb{C}^2 , e.g., D is the convex hull of the 16 vertices $\{b_1e_1 + b_2e_2 + b_3\lambda + b_4\mu \mid b_1, b_2, b_3, b_4 \in \{0, 1\}\}$, and let D_2, \dots, D_N be translates of D in \mathbb{C}^2 such that

$$\overline{D} \subset \bigcup_{j=1}^N D_j.$$

Let \mathcal{D}_j be the image of D_j in $\mathbb{C}^* \times \mathbb{C}^*$ under the map $(z, w) \mapsto (e^{2\pi\sqrt{-1}z}, e^{2\pi\sqrt{-1}w})$. The restriction to \mathcal{D}_j of the projection $p : \mathbb{C}^* \times \mathbb{C}^* \rightarrow T$

is biholomorphic onto its image Δ_i . The bundle structure of $(\mathbb{C}^*)^4 \rightarrow T$ is

now defined as follows. Let $Y_j = \pi^{-1}(\Delta_j) \subset (\mathbb{C}^*)^4$ and let $\varphi_j : \Delta_j \times \mathbb{C}^2 \rightarrow Y_j$ be given as follows. Suppose $(\zeta, \eta) \in D_j$. Then

$$(3) \quad \varphi_j([\zeta, \eta], (s, t)) = (e^s, e^t, e^{\lambda^1 s + \mu^1 t} \zeta, e^{\lambda^2 s + \mu^2 t} \eta).$$

This map is well defined because D_j is a fundamental domain, and thus \mathcal{D}_j contains a unique (ζ, η) projecting onto $[\zeta, \eta]$.

It can be verified that the map φ_j is biholomorphic, but we will actually write down the inverse. To this end, fix $\xi = (x, y, z, w) \in \pi^{-1}(\Delta_j)$, and choose a branch of \log such that $\log x$ and $\log y$ are well defined. Define the integers $m = m_j(\xi)$ and $n = n_j(\xi)$ to be those integers such that

$$(e^{-(\lambda^1 \log x + \mu^1 \log y + 2\pi\sqrt{-1}(\lambda^1 m + \mu^1 n))} z, e^{-(\lambda^2 \log x + \mu^2 \log y + 2\pi\sqrt{-1}(\lambda^2 m + \mu^2 n))} w) \in D_j.$$

Then

$$\varphi_j^{-1}(\xi) = ([e^{-(\lambda^1 \log x + \mu^1 \log y + 2\pi\sqrt{-1}(\lambda^1 m + \mu^1 n))} z, e^{-(\lambda^2 \log x + \mu^2 \log y + 2\pi\sqrt{-1}(\lambda^2 m + \mu^2 n))} w], \log x + 2\pi\sqrt{-1}m_j(\xi), \log y + 2\pi\sqrt{-1}n_j(\xi)).$$

We leave it to the reader to verify that φ_j^{-1} is well defined. The main thing

Evidently the projectivization $X := \mathbb{P}(E)$ of E is a \mathbb{P}_2 bundle over T . Moreover, even though the coordinate functions r, s, t are not globally defined, the divisor $D = (r = 0) \subset X$ is well defined, i.e.,

$$G_{ij}([z, w]) \begin{pmatrix} 0 \\ s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ s \\ t \end{pmatrix}.$$

From this, we also see that D is a trivial \mathbb{P}_1 -bundle over T . Since the projectivization of a vector bundle over a compact Kähler manifold is itself Kähler, X is Kähler. The proof of Theorem 1 is complete.

4. Proof of Theorem 2.

Let us return to the affine bundle $\pi : (\mathbb{C}^*)^4 \rightarrow T$ in Proposition 3.2. Though somewhat abusive, we will also denote the \mathbb{P}_2 -bundle $X \rightarrow T$ by π .

LEMMA 4.1. — *The affine bundle π does not have a section or an affine subbundle of rank 1.*

Proof. — A section of π gives a compact complex torus in $(\mathbb{C}^*)^4$, which contradicts the maximum principle. Under the local trivialization $\varphi_j : \Delta_j \times \mathbb{C}^2 \rightarrow Y_j \subset (\mathbb{C}^*)^4$ in (3), an affine subbundle is given by a linear equation

$$a_j + b_j s + c_j t = 0$$

where a_j, b_j, c_j are holomorphic functions on Δ_j . The transition functions (4) give the relations

$$a_i + b_i s + c_i t = a_j + b_j(s + 2\pi\sqrt{-1}m_{ij}) + c_j(t + 2\pi\sqrt{-1}n_{ij}).$$

We then have that b_j and c_j define global holomorphic functions on T . Thus they are constant and the functions a_j on Δ_j satisfy

$$a_i - a_j = bm_{ij} + cn_{ij}.$$

Thus the \mathbb{Z} -valued cocycles $\{m_{ij}\}$ and $\{n_{ij}\}$ become linearly dependent in $H^1(T, \mathcal{O})$.

But the \mathbb{C}^2 -bundle π is precisely the quotient of the trivial \mathbb{C}^2 -bundle on the universal cover \mathbb{C}^2 of T where $\gamma \in \Gamma$ acts by $(p, q) \mapsto (p + \gamma, q + \gamma)$. Thus the two cocycles are linearly independent. \square

For the rest of this section, we assume that T has no nonconstant meromorphic functions or curves, and that every line bundle on T is flat. This is true for a general choice of T .

LEMMA 4.2. — *There cannot be two algebraically independent meromorphic functions on X .*

Proof. — To obtain a contradiction, suppose that f and g are two independent meromorphic functions on X . Since T has no nonconstant meromorphic function, possibly after perturbing f and g , we can assume that there is an irreducible component Z of the variety $(f = g = 0)$ whose intersection with the generic fiber of π is a finite set disjoint from D , the compactifying divisor. Let $\mathcal{A} \subset T$ be the set of points t such that either $\pi^{-1}(t) \cap Z$ is not finite, or else $\pi^{-1}(t) \cap Z \cap D \neq \emptyset$. Since \mathcal{A} is a proper analytic subvariety of T , it must be finite.

For $t \in T - \mathcal{A}$, let ζ_t be the center of mass of the set-with-multiplicity $\pi^{-1}(t) \cap Z$, and let Z' be the set $\{\zeta_t \mid t \in T - \mathcal{A}\}$. Then Z' is a holomorphic section of the \mathbb{C}^2 -bundle $\mathbb{P}(E) - D = (\mathbb{C}^*)^4$ over $T - \mathcal{A}$, and thus extends to a section of $\mathbb{P}(E) - D$ over T by Hartogs extension, a contradiction to Lemma 4.1. \square

Let us say that a meromorphic function f on M has fiberwise linear levels if for each $c \in \mathbb{P}_1$, the level sets $(f = c)$ intersect the fibers of π in hyperplanes.

LEMMA 4.3. — *If f is a nonconstant meromorphic function on X , then f has fiberwise linear levels.*

Proof. — Note first that the \mathbb{C}^2 -action (2) on $(\mathbb{C}^*)^4$ extends holomorphically to an action on X , which fixes D pointwise. Moreover, the action preserves fibers and is linear on them. If f is a meromorphic function on X with non-linear fiber levels, then by pulling back f with the \mathbb{C}^2 action, we could produce a second meromorphic function g with different level foliation on the fibers. Thus g and f would be algebraically independent, contradicting Lemma 4.2. \square

An easy consequence of the assumption that every line bundle on T is flat is

LEMMA 4.4. — *Let L be a line bundle on T . If there exists a non-zero map of line bundles $L \rightarrow \mathcal{O}$, then $L = \mathcal{O}$.*

Now we can complete the proof of Theorem 2 by

LEMMA 4.5. — *The manifold X has no meromorphic functions with fiberwise linear levels.*

Proof. — A level set of such a meromorphic function defines a rank-2 subbundle $F \subset E$ such that $\mathbb{P}F \neq D$. From the transition functions, we have the exact sequence

$$0 \longrightarrow \mathcal{O}^2 \longrightarrow E \longrightarrow \mathcal{O} \longrightarrow 0.$$

By Lemma 4.4, F must surject to \mathcal{O} and $\mathbb{P}F \cap (\mathbb{P}E - D)$ defines a rank-1 affine subbundle of $\pi : (\mathbb{C}^*)^4 \rightarrow T$, a contradiction to Lemma 4.1. \square

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