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Formal solutions of nonlinear first order totally characteristic type PDE with irregular singularity


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FORMAL SOLUTIONS OF NONLINEAR FIRST ORDER
TOTALLY CHARACTERISTIC TYPE PDE
WITH IRREGULAR SINGULARITY

by H. CHEN, Z. LUO and H. TAHARA

1. Introduction.

Let \((t, x) \in \mathbb{C}_t \times \mathbb{C}_x\), \(\mathbb{N} = \{1, 2, \ldots\}\), \(\mathbb{Z}_+ = \{0, 1, 2, \ldots\}\), and denote by \(\mathbb{C}[[t, x]]\) (resp. by \(\mathbb{C}[[x]]\)) the ring of formal power series in the variables \((t, x)\) (resp. in the variable \(x\)).

Let us consider the following nonlinear singular first order partial differential equation:

\[
\frac{\partial u}{\partial t} = F\left(t, x, u, \frac{\partial u}{\partial x}\right),
\]

where \(u = u(t, x)\) is an unknown function, and \(F(t, x, u, v)\) is a function defined in an open polydisc \(\Delta\) centered at the origin of \(\mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_u \times \mathbb{C}_v\).

Set \(\Delta_0 = \Delta \cap \{t = 0, u = 0\text{ and } v = 0\}\). We impose the following condition on \(F(t, x, u, v)\):

(F1) \(F(t, x, u, v)\) is a holomorphic function on \(\Delta\);
(F2) \(F(0, x, 0, 0) \equiv 0\) on \(\Delta_0\).

Then by the Taylor expansion in \((t, u, v)\) we can express \(F(t, x, u, v)\) in the form

\[
F(t, x, u, v) = a(x)t + b(x)u + \gamma(x)v + \sum_{i+j+\alpha \geq 2} a_{i,j,\alpha}(x)t^i u^j v^\alpha,
\]

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and $a(x), b(x), \gamma(x), a_{i,j}, a(x)$ are all holomorphic functions on $\Delta_0$.

If $\gamma(x) \equiv 0$ on $\Delta_0$, the equation (1.1) is called a non-linear Fuchsian type PDE (or is called a “Briot-Bouquet type PDE” in [4], [5]); this situation has been discussed by [4]–[7]. If $\gamma(0) \neq 0$, we can solve $\partial u/\partial x$ from the equation (1.1) and then we can apply the Cauchy-Kowalewski theorem. If $\gamma(x) \neq 0$ and $\gamma(0) = 0$, the indicial operator $C(\lambda, x, \partial/\partial x) = \lambda - b(x) - \gamma(x)\partial/\partial x$ is a singular differential operator; in this situation the equation (1.1) has been called a totally characteristic type PDE by [1], [2] and [3]. Thus, in this paper we assume:

(F3) $\gamma(x) = x^p c(x)$ for $p \in \mathbb{N}$ and $c(0) \neq 0$.

In the case $p = 1$ we already have the following result.

**Theorem 1.1** (Chen-Tahara [2]). — Assume $p = 1$ and $|i - nb(0) - jc(0)| \neq 0$ for any $(i, j) \in \mathbb{N} \times \mathbb{Z}^+$. Then we have

1. The equation (1.1) has a unique formal solution $u(t, x) \in \mathbb{C}[[t, x]]$ with $u(0, x) \equiv 0$.

2. Moreover, if $c(0) \in \mathbb{C} \setminus [0, \infty)$ holds the unique formal solution in (1) is convergent in a neighborhood of $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_x$.

In this paper we shall discuss the case $p \geq 2$. In this case the indicial operator $C(\lambda, x, \partial/\partial x) = \lambda - b(x) - x^p c(x)\partial/\partial x$ has an irregular singularity at $x = 0 \in \mathbb{C}$ and the formal power series solution of (1.1) is not convergent in general; but still it belongs to a formal Gevrey class.

**Definition.** — Let $s \geq 1$ and $\sigma \geq 1$. We say that a formal power series $f(t, x) = \sum_{i \geq 0, j \geq 0} f_{i,j} t^i x^j \in \mathbb{C}[[t, x]]$ belongs to the formal Gevrey class $G\{t, x\}_{(s, \sigma)}$ if the power series

\[
\sum_{i \geq 0, j \geq 0} \frac{f_{i,j}}{(i!)^{s-1}(j!)^{\sigma-1}} t^i x^j
\]

is convergent in a neighborhood of $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_x$.

The following result is a consequence of the main theorem (Theorem 2.1) of this paper.

**Theorem 1.2.** — Assume $p \geq 2$ and $b(0) \notin \mathbb{N}$. Then

1. The equation (1.1) has a unique formal solution $u(t, x) \in \mathbb{C}[[t, x]]$ with $u(0, x) \equiv 0$. 

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(2) Moreover, it belongs to the formal Gevrey class $G\{t, x\}_{(s, \sigma)}$ for any $s \geq p/(p - 1)$ and $\sigma \geq p/(p - 1)$.

The result of this type is often called a Maillet’s type theorem (see [6], [7], [9]).

In this paper, we have confined ourselves to the study of formal power series solutions $u(t, x) \in \mathbb{C}[[t, x]]$ of (1.1). The relation between true solutions of (1.1) and the formal solution obtained in this paper will be discussed in a forthcoming paper.

2. Main results.

We discuss the same equation (1.1) as in §1 under the conditions (F1), (F2), (F3), and $p \geq 2$.

Our equation is written as

$$
(2.1) \quad \left( t \frac{\partial}{\partial t} - b(x) - x^p c(x) \frac{\partial}{\partial x} \right) u = a(x) t + \sum_{i+j+\alpha \geq 2} a_{i,j,\alpha}(x) t^i u^j \left( \frac{\partial u}{\partial x} \right)^\alpha
$$

where $a(x), b(x), c(x), a_{i,j,\alpha}(x)$ are all holomorphic functions on $\Delta_0$, $c(0) \neq 0$, and the right hand side is a holomorphic function on $\Delta$ with $v = \partial u/\partial x$.

Set

$$
J = \{(i, j, \alpha); i + j + \alpha \geq 2, \alpha > 0, \text{ and } a_{i,j,\alpha}(0) \neq 0\}.
$$

We have

THEOREM 2.1. — Assume (F1), (F2), (F3), $p \geq 2$ and $b(0) \notin \mathbb{N}$. Then, the equation (2.1) has a unique formal solution $u(t, x) \in \mathbb{C}[[t, x]]$ with $u(0, x) \equiv 0$ and it belongs to the formal Gevrey class $G\{t, x\}_{(s, \sigma)}$ for any $(s, \sigma)$ satisfying

$$
(2.2) \quad s \geq 1 + \max \left[ 0, \sup_{(i,j,\alpha) \in J} \left( \frac{1}{(p-1)(i+j+\alpha-1)} \right) \right]
$$

and $\sigma \geq p/(p - 1)$.

The proof of this theorem will be given in §4. Note that

$$
1 + \frac{1}{(p-1)(i+j+\alpha-1)} \leq 1 + \frac{1}{(p-1)(2-1)} = \frac{p}{p-1}
$$
and therefore $s \geq p/(p - 1)$ implies the condition (2.2). Thus, Theorem 1.2 follows from Theorem 2.1.

As a particular case, we have

**Corollary 2.2.** — If $J = \emptyset$, the unique formal solution $u(t, x)$ belongs to the class $G\{t, x\}_{1, p/(p - 1)}$.

This implies that the formal solution is holomorphic in the variable $t$.

For $f(x) = \sum_{j \geq 0} f_j x^j \in \mathbb{C}[[x]]$ we write $f(x) \gg 0$ if $f_j \gg 0$ holds for all $j \geq 0$. The following proposition asserts that our condition (2.2) is the best possible result in a generic case.

**Proposition 2.3.** — Assume $(F1), (F2), (F3), p \geq 2$ and $b(0) \not\in \mathbb{N}$. Moreover, assume the following additional conditions:

- $c1) a(0) > 0$, $(\partial a/\partial x)(0) > 0$ and $a(x) \gg 0$;
- $c2) b(0) < 1$ and $(b(x) - b(0)) \gg 0$;
- $c3) c(0) > 0$ and $c(x) \gg 0$;
- $c4) a_{i,j,\alpha}(x) \gg 0$ (for $i + j + \alpha \geq 2$).

Then, the unique formal solution $u(t, x)$ in Theorem 2.1 belongs to the class $G\{t, x\}_{(s, \sigma)}$ if and only if $(s, \sigma)$ satisfies (2.2) and $\sigma \geq p/(p - 1)$.

The proof of this proposition will be given in §5.

Thus, we may say that the index $(s_0, \sigma_0)$ defined by

$$s_0 = 1 + \max \left[0, \sup_{(i,j,\alpha) \in J} \left( \frac{1}{(p - 1)(i + j + \alpha - 1)} \right) \right], \quad \sigma_0 = \frac{p}{p - 1}$$

is the formal Gevrey index of the equation (2.1).

For other types of partial differential equations, the formal Gevrey index is calculated by [6], [7], [8], [9].

**Example 2.4.** — Let $p, q, l, m, n \in \mathbb{Z}_+$ satisfying $p \geq 2$, $n \geq 1$ and $l + m + n \geq 2$. Let us consider

$$t \frac{\partial u}{\partial t} = (1 + x)t + x^p \frac{\partial u}{\partial x} + x^q t^l u^m \left( \frac{\partial u}{\partial x} \right)^n.$$  

We have

1) (2.4) has a unique formal solution $u(t, x) \in \mathbb{C}[[t, x]]$ with $u(0, x) \equiv 0$.  

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2) When \( q > 1 \), \( u(t, x) \) belongs to the class \( G\{t, x\}_{(s, \sigma)} \) if and only if
\[
s \geq 1 \quad \text{and} \quad \sigma \geq \frac{p}{p-1}.
\]

3) When \( q = 0 \), \( u(t, x) \) belongs to the class \( G\{t, x\}_{(s, \sigma)} \) if and only if
\[
s \geq 1 + \frac{1}{(p-1)(l + m + n - 1)} \quad \text{and} \quad \sigma \geq \frac{p}{p-1}.
\]

3. Preparatory discussions.

Before the proof of Theorem 2.1 we shall present some preparatory lemmas.

For \( f(x) = \sum_{j \geq 0} f_j x^j \in \mathbb{C}[x] \), we write
\[
|f|(x) = \sum_{j \geq 0} |f_j| x^j,
\]
\[
S(f)(x) = \sum_{j \geq 0} f_{j+1} x^j = \frac{f(x) - f(0)}{x},
\]
\[
B_\sigma(f)(x) = \sum_{j \geq 0} \frac{f_j}{(j!)^{\sigma-1}} x^j, \quad \sigma > 1.
\]
\( B_\sigma(f)(x) \) is a variation of the Borel transform of \( f(x) \). For \( f(x) = \sum_{j \geq 0} f_j x^j \), \( g(x) = \sum_{j \geq 0} g_j x^j \) we write \( f(x) \ll g(x) \) if \( |f_j| \leq g_j \) holds for all \( j \geq 0 \).

It is easy to show (see also [7]):

**Lemma 3.1.** — For \( \sigma > 1 \), \( a(x), \phi(x) \), \( f(x) \in \mathbb{C}[x] \) we have

1) \( |a\phi|(x) \ll |a|(x) |\phi|(x) \);

2) \( B_\sigma(a\phi)(x) \ll B_\sigma(|a|(x))B_\sigma(|\phi|(x)) \);

3) if \( c \neq 0 \) and \( \phi(0) = 0 \) then
\[
B_\sigma\left(\frac{1}{c + \phi}\right)(x) \ll \frac{1}{|c| - B_\sigma(|\phi|(x))};
\]

4) \( B_\sigma\left(x \frac{\partial f}{\partial x}\right)(x) = x \frac{\partial}{\partial x} B_\sigma(f)(x) \ll x \frac{\partial}{\partial x} B_\sigma(|f|)(x) \);

5) if \( p \geq 2 \) and \( \sigma \geq p/(p-1) \) then
\[
B_\sigma\left(x^p \frac{\partial f}{\partial x}\right)(x) \ll x^{p-1} B_\sigma(|f|)(x);
\]
6) \( S(f)(x) \ll \frac{\partial}{\partial x} |f|(x) \) and \( B_\sigma(S(f))(x) \ll B_\sigma \left( \frac{\partial}{\partial x} |f| \right)(x) \).

We say that \( f(x) \in \mathbb{C}[[x]] \) belongs to the formal Gevrey class \( G\{x\}_\sigma \) if \( B_\sigma(f)(x) \) is convergent in a neighborhood of \( x = 0 \). The following lemma is used to construct a formal solution of (2.1).

**Lemma 3.2.** Let \( b(x), c(x) \in \mathbb{C}[[x]] \), \( p \geq 2 \), \( k \in \mathbb{N} \) and assume that \( b(0) \neq k \). We have

1) For any \( g(x) \in \mathbb{C}[[x]] \), the equation

\[
\left( k - b(x) - x^p c(x) \frac{\partial}{\partial x} \right) w = g(x)
\]

has a unique solution \( w(x) \in \mathbb{C}[[x]] \).

2) If \( b(x), c(x), g(x) \in G\{x\}_\sigma \) for some \( \sigma \geq p/(p - 1) \) we have \( w(x) \in G\{x\}_\sigma \) and moreover if \( |k - b(0)| \geq \rho k \) with \( \rho > 0 \) we have

\[
B_\sigma(|w|)(x) \ll \frac{1}{k \rho - \Phi(x)} B_\sigma(|g|)(x)
\]

where \( \Phi(x) = xB_\sigma(|S(b)|)(x) + x^{p-1}B_\sigma(|c|)(x) \gg 0 \). Note that \( \Phi(0) = 0 \) holds.

**Proof.** 1) is verified by a calculation. Since (3.1) is written as

\[
(k - b(0)) w = xS(b)(x)w + x^p c(x) \frac{\partial w}{\partial x} + g(x),
\]

by using the \( B_\sigma \)-transformation and 5) of Lemma 3.1 we have

\[
pk B_\sigma(|w|)(x)
\ll xB_\sigma(|S(b)|)(x)B_\sigma(|w|)(x) + x^{p-1}B_\sigma(|c|)(x)B_\sigma(|w|)(x) + B_\sigma(|g|)(x)
\]

\[
= \Phi(x)B_\sigma(|w|)(x) + B_\sigma(|g|)(x)
\ll k\Phi(x)B_\sigma(|w|)(x) + B_\sigma(|g|)(x)
\]

which leads us to the conclusion of 2). Lemma 3.2 is proved.

In order to estimate the term \( B_\sigma(\partial u/\partial x) \) we need the following lemma.

**Lemma 3.3.** Let \( \sigma > 1 \) and \( 0 < R < 1 \). If \( f(x) \in G\{x\}_\sigma \) satisfies

\[
B_\sigma(f)(x) \ll \frac{C}{(R - x)^\alpha}
\]

where \( \alpha \in \mathbb{N} \) and \( C > 0 \) is a constant.
for some $C > 0$ and $a \geq 1$, we have

$$B_\sigma \left( x \frac{\partial f}{\partial x} \right) (x) \leq \frac{aC}{(R - x)^{a+1}} \leq \frac{aC}{(R - x)^{a+\sigma}},$$

and

$$B_\sigma \left( \frac{\partial f}{\partial x} \right) (x) \leq \frac{e^\sigma (a + \sigma)^\sigma C}{(R - x)^{a+\sigma}}.$$

Proof. — Assume that $f(x) \in G\{x\}_\sigma$ satisfies (3.3). Then

$$B_\sigma \left( x \frac{\partial f}{\partial x} \right) (x) = x \frac{\partial}{\partial x} B_\sigma (f)(x) \leq x \frac{\partial}{\partial x} \frac{C}{(R - x)^a} = \frac{xaC}{(R - x)^{a+1}}.$$

Combining this with

$$\frac{x}{R - x} \leq \frac{R}{R - x} \leq \frac{1}{R - x} \leq \frac{R^{\sigma-1}}{(R - x)^\sigma} \leq \frac{1}{(R - x)^\sigma}$$

(since $0 < R < 1$) we obtain (3.4). Note that the function $1/(R - x)^a$ is expressed as

$$\frac{1}{(R - x)^a} = \sum_{j \geq 0} \frac{1}{R^{a+j}} \frac{\Gamma(a + j)}{\Gamma(a) \Gamma(j + 1)} x^j.$$

Therefore, if we prove the inequality

$$\sup_{a \geq 1, j \geq 1} \left( \frac{j^{\sigma-1}}{(a + \sigma)^\sigma} \frac{\Gamma(a + j)}{\Gamma(a) \Gamma(a + j + \sigma - 1)} \right) \leq e^\sigma,$$

a simple calculation shows that (3.5) follows easily from (3.3).

Since a sharp form of the Stirling’s formula for the $\Gamma$-function guarantees

$$1 < \frac{\Gamma(x)}{\sqrt{2\pi} x^{x-1/2} e^{-x}} < \exp \left( \frac{1}{12x} \right) < \sqrt{e} \quad \text{for } x \geq 1$$

(see [10]), the inequality (3.6) is verified as follows:

$$\frac{j^{\sigma-1}}{(a + \sigma)^\sigma} \frac{\Gamma(a + j)}{\Gamma(a) \Gamma(a + j + \sigma - 1)} \leq \left( \frac{a}{a + \sigma} \right)^{\sigma/2} \left( 1 + \frac{\sigma}{a} \right)^{a} \frac{j^{\sigma-1}(a + j)^{a+j-1/2}}{(a + j + \sigma - 1)^{(\sigma-1)+(a+j-1/2)}}$$

$$\leq \left( 1 + \frac{\sigma}{a} \right)^{a} \leq e^\sigma.$$

**Lemma 3.4.** — Let $k \geq 2$, $i, j, \alpha \in \mathbb{Z}_+$, $m_1, \ldots, m_j \in \mathbb{N}$, and $n_1, \ldots, n_\alpha \in \mathbb{N}$. Assume $2 \leq i + j + \alpha \leq k$ and $i + |m| + |n| = k$,
where $|m| = m_1 + \cdots + m_j$ and $|n| = n_1 + \cdots + n_\alpha$. Then we have 1) 

$$(m_1 - 1)! \cdots (m_j - 1)! (n_1 - 1)! \cdots (n_\alpha - 1)! \leq (k - 2)! \leq (k - 1)!;$$

2) $(m_1 - 1)! \cdots (m_j - 1)! (n_1 - 1)! \cdots (n_\alpha - 1)! \leq \frac{(i + j + \alpha)}{k^{i + j + \alpha - 1}} (k - 1)!$;

3) $\frac{1}{m_1 \cdots m_j n_1 \cdots n_\alpha} \leq \frac{i + j + \alpha}{k}$.

**Proof.** — 1) is verified by

$$(m_1 - 1)! \cdots (m_j - 1)! (n_1 - 1)! \cdots (n_\alpha - 1)!$$

$$\leq (|m| + |n| - j - \alpha)! = (i + |m| + |n| - i - j - \alpha)!$$

$$= (k - i - j - \alpha)!$$

$$\leq (k - 2)! \leq (k - 1)!.$$ 

By using the Stirling’s formula (3.7) we have

$$\frac{(m_1 - 1)! \cdots (m_j - 1)! (n_1 - 1)! \cdots (n_\alpha - 1)!}{(k - 1)!} \leq \frac{(k - i - j - \alpha)!}{(k - 1)!} = \frac{\Gamma(k - i - j - \alpha + 1)}{\Gamma(k)}$$

$$\leq \frac{\sqrt{2\pi}(k - i - j - \alpha + 1)^{k - i - j - \alpha + 1 - 1/2} e^{-k + i + j + \alpha - 1} e}{\sqrt{2\pi} k^{k - 1/2} e^{-k}}$$

$$= \left( \frac{k - i - j - \alpha + 1}{k} \right)^{k - i - j - \alpha + 1 - 1/2} \frac{e^{i + j + \alpha}}{k^{i + j + \alpha - 1}}$$

$$\leq \frac{e^{i + j + \alpha}}{k^{i + j + \alpha - 1}}$$

which proves 2). Since $m_p \geq 1$ and $n_q \geq 1$, we have

$$(m_1 + \cdots + m_j + n_1 + \cdots + n_\alpha) \leq (j + \alpha) (m_1 \cdots m_j n_1 \cdots n_\alpha)$$

and therefore

$$k = i + |m| + |n| \leq i + (j + \alpha) (m_1 \cdots m_j n_1 \cdots n_\alpha)$$

$$\leq (i + j + \alpha) (m_1 \cdots m_j n_1 \cdots n_\alpha)$$

which proves 3). Thus Lemma 3.4 is proved.

**4. Proof of Theorem 2.1.**

Now, by using Lemmas 3.1 ~ 3.4 we shall give here a proof of Theorem 2.1.
In this section we set \( \sigma = p/(p-1) \); then the condition (2.2) is written as
\[
(4.1) \quad s \geq 1 + \max \left[ 0, \sup_{(i,j,\alpha) \in J} \left( \frac{\sigma - 1}{i + j + \alpha - 1} \right) \right].
\]
Since \( b(0) \not\in \mathbb{N} \) is assumed, we can find a \( \rho > 0 \) such that \( |k - b(0)| \geq \rho k \) holds for all \( k \in \mathbb{N} \).

First, let us look for a formal solution \( u(t, x) \) of the form
\[
(4.2) \quad u(t, x) = \sum_{k \geq 1} u_k(x) t^k, \quad u_k(x) \in G\{x\}_\sigma \quad (\text{for } k \geq 1).
\]
Under (4.2) the equation (2.1) is decomposed into the following recurrent family:
\[
(4.3) \quad \left( 1 - b(x) - x^p c(x) \frac{\partial}{\partial x} \right) u_1 = a(x),
\]
and for \( k \geq 2 \)
\[
(4.4) \quad \left( k - b(x) - x^p c(x) \frac{\partial}{\partial x} \right) u_k = \sum_{2 \leq i+j+\alpha \leq k} a_{i,j,\alpha}(x) \left[ \sum_{i+|m|+|n|=k} u_{m_1} \cdots u_{m_j} \times \frac{\partial u_{n_1}}{\partial x} \cdots \frac{\partial u_{n_\alpha}}{\partial x} \right],
\]
where \( |m| = m_1 + \cdots + m_j \) and \( |n| = n_1 + \cdots + n_\alpha \). Therefore, if \( b(0) \not\in \mathbb{N} \) by Lemma 3.2 we can determine \( u_k(x) \in G\{x\}_\sigma \) \( (k = 1, 2, \ldots) \) inductively on \( k \). Thus, we have obtained a unique formal solution \( u(t, x) \) in (4.2).

Next, let us prove that this formal solution \( u(t, x) \) belongs to the formal Gevrey class \( G\{t, x\}_{s,\sigma} \) if \( s \) satisfies the condition (4.1). To do so, we set
\[
w_k(x) = S(u_k)(x) \in G\{x\}_\sigma, \quad k = 1, 2, \ldots.
\]
Then we have \( u_k(x) = u_k(0) + x w_k(x) \) and by (4.3),(4.4) we have
\[
(4.5) \quad \left( 1 - b(0) \right) u_1(0) = a(0),
\]
\[
(4.6) \quad \left( 1 - b(x) - x^{p-1} c(x) - x^p c(x) \frac{\partial}{\partial x} \right) w_1 = S(b)(x) u_1(0) + S(a)(x),
\]
and for \( k \geq 2 \)
\[
(4.7) \quad (k - b(0)) u_k(0) = \sum_{2 \leq i+j+\alpha \leq k} a_{i,j,\alpha}(0) \left[ \sum_{i+|m|+|n|=k} u_{m_1}(0) \cdots \cdots u_{m_j}(0) w_{n_1}(0) \cdots w_{n_\alpha}(0) \right],
\]
Choose $0 < R < 1$ and $A > 0$ so that $|u_1(0)| \leq A$ and

\[
(4.8) \quad \left( k - b(x) - x^{p-1}c(x) - x^pc(x) \frac{\partial}{\partial x} \right) w_k
= S(b)(x)u_k(0)
+ \sum_{2 \leq i+j+\alpha \leq k} S(a_{i,j,\alpha})(x) \left[ \sum_{i+|m|+|n|=k} \left( u_{m_1}(0) + x w_{m_1} \right) \times \cdots \times \left( u_{m_j}(0) + x w_{m_j} \right) \times \left( w_{n_1} + x \frac{\partial w_{n_1}}{\partial x} \right) \cdots \left( w_{n_\alpha} + x \frac{\partial w_{n_\alpha}}{\partial x} \right) \right]
\]
\[+ \sum_{2 \leq i+j+\alpha \leq k} a_{i,j,\alpha}(0) \left[ \sum_{i+|m|+|n|=k} \left( \frac{1}{x} \left( u_{m_1}(0) + x w_{m_1} \right) \times \cdots \times \left( u_{m_j}(0) + x w_{m_j} \right) \times \left( w_{n_1} + x \frac{\partial w_{n_1}}{\partial x} \right) \cdots \left( w_{n_\alpha} + x \frac{\partial w_{n_\alpha}}{\partial x} \right) \right) \right] - u_{m_1}(0) \cdots u_{m_j}(0)w_{n_1} \cdots w_{n_\alpha} \right] + u_{m_1}(0) \cdots u_{m_j}(0)S(w_{n_1} \cdots w_{n_\alpha}) \right].
\]

Using these constants, let us consider the following functional equation with respect to $Y$:

\[
(4.10) \quad B(\sigma)(w_1)(x) \ll \frac{A}{(R-x)^{\sigma}}.
\]

Put $\Phi(x) = xB(\sigma)(|S(b)|)(x) + 2x^{p-1}B(\sigma)(|c|)(x)$ and take $B > 0$ such that

\[
\frac{B(\sigma)(|S(b)|)(x)}{\rho - \Phi(x)} \ll \frac{B}{(R-x)^{\sigma}}.
\]

Similarly, choose $A^{(0)}_{i,j,\alpha} \geq 0$ and $A_{i,j,\alpha} \geq 0$ so that $|a_{i,j,\alpha}(0)| \leq A^{(0)}_{i,j,\alpha}$,

\[
\frac{|a_{i,j,\alpha}(0)|}{\rho - \Phi(x)} \ll \frac{A^{(0)}_{i,j,\alpha}}{(R-x)^{\sigma}}, \quad \frac{B(\sigma)(|S(a_{i,j,\alpha})|)(x)}{\rho - \Phi(x)} \ll \frac{A_{i,j,\alpha}}{(R-x)^{\sigma}}
\]

and that

\[
\sum_{i+j+\alpha \geq 2} A^{(0)}_{i,j,\alpha} t^i u^j v^\alpha \quad \text{and} \quad \sum_{i+j+\alpha \geq 2} A_{i,j,\alpha} t^i u^j v^\alpha
\]

are convergent in a neighborhood of the origin of $\mathbb{C}_t \times \mathbb{C}_u \times \mathbb{C}_v$. We may assume that $A^{(0)}_{i,j,\alpha} = 0$ if $a_{i,j,\alpha}(0) = 0$.

Using these constants, let us consider the following functional equation with respect to $Y$:

\[
(4.10) \quad Y = \frac{A}{(R-x)^{2\sigma}} t + \frac{1}{(R-x)^{\sigma}} \sum_{i+j+\alpha \geq 2} \frac{C_{i,j,\alpha}}{(R-x)^{\sigma(2i+2j+2\alpha-3)}} t^i (2Y)^j (2\beta Y)^\alpha
\]

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where $\beta = (4e\sigma)^\sigma$ and

$$C_{i,j,\alpha} = \left( (1 + B / \rho) A_{i,j,\alpha}^{(0)} + A_{i,j,\alpha} \right) (i+j+\alpha)^{\sigma-1} + A_{i,j,\alpha} (e^{i+j+\alpha})^{s-1}. $$

Note that by $i + j + \alpha \geq 2$ we have $4i + 2j + 2\alpha - 3 \geq 1$.

Since (4.10) is an analytic functional equation with respect to $Y$, by the implicit function theorem we see that (4.10) has a unique holomorphic solution $Y = Y(t, x)$ in a neighborhood of the origin of $\mathbb{C}_t \times \mathbb{C}_x$ with $Y(0, x) \equiv 0$. If we expand $Y$ into the form

$$Y(t, x) = \sum_{k \geq 1} Y_k(x) t^k$$

we see that the coefficients $Y_k(x)$ ($k \geq 1$) are determined by the following recurrent formula:

$$Y_1 = \frac{A}{(R - x)^{2\sigma}}$$

and for $k \geq 2$

$$Y_k = \frac{1}{(R - x)^{\sigma}} \sum_{2 \leq i+j+\alpha \leq k} \frac{C_{i,j,\alpha}}{(R - x)^{\sigma(4i+2j+2\alpha-3)}} \left[ \sum_{i+|m|+|n|=k} \left( 2Y_{m_1} \right) \cdots \left( 2Y_{m_l} \right) \left( 2\beta Y_{n_1} \right) \cdots \left( 2\beta Y_{n_\alpha} \right) \right].$$

Moreover we can prove by induction on $k$ that $Y_k(x)$ has the form

$$Y_k(x) = \frac{M_k}{(R - x)^{\sigma(4k-2)}}$$

for $k \geq 1$ with constants $M_1 = A$ and $M_k > 0$ for $k \geq 2$.

In addition, we have the following lemma.

**Lemma 4.1.** Let $\beta = (4e\sigma)^\sigma$, and let $u(t, x)$ be the unique formal solution in (4.2). If $s$ satisfies the condition (4.1) we have the following estimates for all $k \in \mathbb{N}$:

$$|u_k(0)| \ll \frac{(k-1)!^{s-1}}{k^\sigma} Y_k(x),$$

$$B_\sigma (|w_k|) (x) \ll \frac{(k-1)!^{s-1}}{k^\sigma} Y_k(x),$$

$$B_\sigma \left( x \frac{\partial}{\partial x} |w_k| \right) (x) \ll \frac{(k-1)!^{s-1}}{k^{\sigma-1}} \beta Y_k(x),$$
This implies that our formal solution \( u(t, x) \) in (4.2) belongs to the class 

Thus, to complete the proof of Theorem 2.1 it is sufficient to give a proof of Lemma 4.1.

**Proof of Lemma 4.1.** Assume that \( s \) satisfies the condition (4.1). We have

\[
(i + j + \alpha - 1)(s - 1) \geq \sigma - 1 \quad \text{for any} \quad (i, j, \alpha) \in J.
\]

First let us prove the case \( k = 1 \). Since \( |u_1(0)| \leq A \) is assumed, we have

\[
|u_1(0)| \leq A \ll \frac{A}{(R - x)^{2\sigma}} = Y_1(x)
\]

which is (4.15)_1. Using (4.9) and Lemma 3.3 we can verify (4.16)_1, (4.17)_1, (4.18)_1 as follows:

\[
B_\sigma \left( |w_1| \right)(x) \ll \frac{A}{(R - x)^{\sigma}} \ll \frac{A}{(R - x)^{2\sigma}} = Y_1(x),
\]

\[
B_\sigma \left( x \frac{\partial}{\partial x} |w_1| \right)(x) \ll \frac{\sigma A}{(R - x)^{2\sigma}} = \sigma Y_1(x) \ll \beta Y_1(x),
\]

\[
B_\sigma \left( \frac{\partial}{\partial x} |w_1| \right)(x) \ll \frac{e^\sigma (\sigma + \sigma)^{\sigma} A}{(R - x)^{2\sigma}} = (2e\sigma)^{\sigma} Y_1(x) \ll \beta Y_1(x).
\]

Here we used the conditions \( 1 \ll 1/(R - x)^{\sigma} \) (since \( 0 < R < 1 \)) and \( \beta = (4e\sigma)^{\sigma} \).

Next, let us show the general case \( k \geq 2 \) by induction on \( k \).
Let $k \geq 2$ and suppose that $(4.15)_i \sim (4.18)_i$ are already proved for all $i \leq k - 1$. Then by (4.7) and the induction hypotheses we have

$$|u_k(0)| \ll \frac{1}{k \rho} \sum_{2 \leq i+j+\alpha \leq k} |a_{i,j,\alpha}(0)|$$

$$\times \left[ \sum_{i+|m|+|n|=k} \left( \frac{(m_1-1)!^{s-1}}{m_1^{\sigma}Y_{m_1}} \right) \cdots \left( \frac{(m_j-1)!^{s-1}}{m_j^{\sigma}Y_{m_j}} \right) \right.$$

$$\left. \times \left( \frac{(n_1-1)!^{s-1}}{n_1^{\sigma}Y_{n_1}} \right) \cdots \left( \frac{(n_\alpha-1)!^{s-1}}{n_\alpha^{\sigma}Y_{n_\alpha}} \right) \right].$$

Therefore, by 1), 3) of Lemma 3.4 and by using the inequality $(i+j+\alpha)/k \leq 1$ we have

$$|u_k(0)| \ll \frac{1}{k \rho} \sum_{2 \leq i+j+\alpha \leq k} |a_{i,j,\alpha}(0)| \left[ \sum_{i+|m|+|n|=k} (k-1)!^{s-1} \right.$$

$$\times \left( \frac{i+j+\alpha}{k} \right)^{s-1} \left( \frac{i+j+\alpha}{k} \right)^{\sigma-1} Y_{m_1} \cdots Y_{m_j} \times Y_{n_1} \cdots Y_{n_\alpha} \right]$$

$$\ll \frac{(k-1)!^{s-1}}{k^\sigma} \frac{1}{\rho} \sum_{2 \leq i+j+\alpha \leq k} |a_{i,j,\alpha}(0)| (i+j+\alpha)^{\sigma-1}$$

$$\times \left[ \sum_{i+|m|+|n|=k} Y_{m_1} \cdots Y_{m_j} \times Y_{n_1} \cdots Y_{n_\alpha} \right].$$

Hence, if we note that

$$\frac{1}{\rho} |a_{i,j,\alpha}(0)| \ll \frac{|a_{i,j,\alpha}(0)|}{\rho - \Phi(x)} \ll \frac{A_{i,j,\alpha}^{(0)}}{(R-x)^{\sigma}},$$

we have

$$|u_k(0)| \ll \frac{(k-1)!^{s-1}}{k^\sigma} \sum_{2 \leq i+j+\alpha \leq k} A_{i,j,\alpha}^{(0)} (i+j+\alpha)^{\sigma-1}$$

$$\times \left[ \sum_{i+|m|+|n|=k} Y_{m_1} \cdots Y_{m_j} \times Y_{n_1} \cdots Y_{n_\alpha} \right].$$

By comparing this with (4.13) and by using $A_{i,j,\alpha}^{(0)} (i+j+\alpha)^{\sigma-1} \leq C_{i,j,\alpha}$, $4i+2j+2\alpha-3 \geq 1$ and $1 \ll 1/(R-x)^{\sigma}$ we can easily obtain (4.15)$_k$. 
Let us show (4.16)_k, (4.17)_k and (4.18)_k. To do so, it is sufficient to prove

\begin{equation}
B_\sigma(|w_k|)(x) \ll \frac{(k-1)!^{s-1}}{k^\sigma} \frac{(R-x)^\sigma Y_k(x)}{M_k} \frac{1}{(R-x)^{(4k-3)\sigma}}
\end{equation}

(see (4.14)). In fact, if we know this, by using \(1 \ll 1/(R-x)^\sigma\) and (4.14) we have (4.16)_k, and by applying Lemma 3.4 we can obtain (4.17)_k and (4.18)_k.

Let us prove (4.21) from now. By applying 2) of Lemma 3.2 to (4.8) we have

\[ B_\sigma(|w_k|)(x) \ll I_1 + I_2 + I_3 \]

with

\begin{align*}
I_1 &= \frac{1}{k} B_\sigma(|S(b)||)(x) \frac{u_k(0)}{\rho - \Phi(x)} \\
I_2 &= \frac{1}{k} \sum_{2 \leq i+j+\alpha \leq k} \frac{B_\sigma(|S(a_{i,j,\alpha})||)(x)}{\rho - \Phi(x)} \left[ \sum_{i+|m|+|n|=k} \left( |u_{m_1}(0)| + xB_\sigma(|w_{m_1}|) \right) \right. \\
& \quad \times \left( |u_{m_j}(0)| + xB_\sigma(|w_{m_j}|) \right) \\
& \quad \times \left( B_\sigma(|w_{n_1}|) + B_\sigma \left( x \frac{\partial}{\partial x} |w_{n_1}| \right) \right) \cdots \left( B_\sigma(|w_{n_\alpha}|) + B_\sigma \left( x \frac{\partial}{\partial x} |w_{n_\alpha}| \right) \right),
\end{align*}

\begin{align*}
I_3 &= \frac{1}{k} \sum_{2 \leq i+j+\alpha \leq k} \frac{|a_{i,j,\alpha}(0)|}{\rho - \Phi(x)} \left[ \sum_{i+|m|+|n|=k} \left( \frac{1}{x} \left( |u_{m_1}(0)| + xB_\sigma(|w_{m_1}|) \right) \right. \\
& \quad \times \left. \left( |u_{m_j}(0)| + xB_\sigma(|w_{m_j}|) \right) \right. \\
& \quad \times \left( B_\sigma(|w_{n_1}|) + xB_\sigma \left( x \frac{\partial}{\partial x} |w_{n_1}| \right) \right) \cdots \left( B_\sigma(|w_{n_\alpha}|) + xB_\sigma \left( x \frac{\partial}{\partial x} |w_{n_\alpha}| \right) \right) \\
& \quad - |u_{m_1}(0)| \cdots |u_{m_j}(0)| B_\sigma(|w_{n_1}|) \cdots B_\sigma(|w_{n_\alpha}|) \\
& \quad + |u_{m_1}(0)| \cdots |u_{m_j}(0)| B_\sigma \left( |S(w_{n_1} \cdots w_{n_\alpha})| \right). \]
\end{align*}

\(I_1\) is estimated by (4.20):
Since \( Y(x) \) has the form (4.14) and \( 0 < R < 1 \) is assumed, we have

\[
xY_l(x) \ll RY_l(x) \ll Y_l(x).
\]

By using this and the induction hypotheses, we see

\[
I_2 \ll \frac{1}{k} \sum_{2 \leq i+j+\alpha \leq k} \frac{A_{i,j,\alpha}}{(R-x)^{\sigma}} \left[ \sum_{i+|m|+|n|=k} \frac{(m_1-1)!^{s-1}}{m_1^{\sigma}} (2Y_{m_1}) \right] \\
\times \cdots \times \left( \frac{(m_j-1)!^{s-1}}{m_j^{\sigma}} (2Y_{m_j}) \right) \left( \frac{(n_1-1)!^{s-1}}{n_1^{\sigma-1}} \left( \frac{1}{n_1} + \beta \right) Y_{n_1} \right) \\
\times \cdots \times \left( \frac{(n_{\alpha}-1)!^{s-1}}{n_{\alpha}^{\sigma-1}} \left( \frac{1}{n_{\alpha}} + \beta \right) Y_{n_{\alpha}} \right).
\]

Therefore, by 1), 3) of Lemma 3.4 and by the same argument as in (4.20) we obtain

\[
(4.23) \quad I_2 \ll \frac{(k-1)!^{s-1}}{k^{\sigma}} \sum_{2 \leq i+j+\alpha \leq k} \frac{A_{i,j,\alpha}(i+j+\alpha)^{\sigma-1}}{(R-x)^{\sigma}} \left[ \sum_{i+|m|+|n|=k} \left( 2Y_{m_1} \right) \right] \\
\times \cdots \times \left( 2Y_{m_j} \right) \times \left( \left( \frac{1}{n_1} + \beta \right) Y_{n_1} \right) \cdots \left( \left( \frac{1}{n_{\alpha}} + \beta \right) Y_{n_{\alpha}} \right).
\]

In order to estimate \( I_3 \) we note
\[ B_\sigma \left( |S(w_{n_1} \cdots w_{n_\alpha})| \right)(x) \leq B_\sigma \left( \frac{\partial}{\partial x} |w_{n_1} \cdots w_{n_\alpha}| \right)(x) \]
\[ \leq \sum_{i=1}^{\alpha} B_\sigma \left( |w_{n_1}| \right) \cdots B_\sigma \left( \frac{\partial}{\partial x} |w_{n_i}| \right) \cdots B_\sigma \left( |w_{n_\alpha}| \right) \]
\[ \leq \alpha \beta \left( \frac{(n_1 - 1)!^{s-1}}{1} Y_{n_1} \right) \cdots \left( \frac{(n_\alpha - 1)!^{s-1}}{1} Y_{n_\alpha} \right) \]
\[ \leq \left( \frac{(n_1 - 1)!^{s-1}}{1} \beta Y_{n_1} \right) \cdots \left( \frac{(n_\alpha - 1)!^{s-1}}{1} \beta Y_{n_\alpha} \right); \]
here we used 6) of Lemma 3.1, the induction hypotheses, the inequality \( \alpha \beta \leq \beta^\alpha \), and \( B_\sigma \left( |w_n| \right) \leq \frac{(n - 1)!^{s-1}}{n^\sigma} Y_n \leq \frac{(n_1 - 1)!^{s-1}}{1} Y_n. \)
Therefore, using this and \( xY_i(x) \leq Y_i(x) \) we can estimate \( I_3 \) in the following way:

\[
I_3 \leq \frac{1}{k} \sum_{2 \leq i+j+\alpha \leq k} \frac{A_{i,j,\alpha}^{(0)}}{(R - x)^\sigma} \left[ \sum_{x+|m|+|n|=k} \left( \frac{(m_1 - 1)!^{s-1}}{m_1^\sigma} 2Y_{m_1} \right) \right. \\
\left. \times \cdots \times \left( \frac{(m_j - 1)!^{s-1}}{m_j^\sigma} 2Y_{m_j} \right) \right. \\
\left. \times \left( \frac{(n_1 - 1)!^{s-1}}{1} \beta Y_{n_1} \right) \cdots \left( \frac{(n_\alpha - 1)!^{s-1}}{1} \beta Y_{n_\alpha} \right). \right]
\]

If \( \alpha = 0 \), then by 1), 3) of Lemma 3.4 we have

\[
\frac{(m_1 - 1)!^{s-1}}{m_1^\sigma} \cdots \frac{(m_j - 1)!^{s-1}}{m_j^\sigma} \leq \frac{(k - 1)!^{s-1}}{k^\sigma - 1} (i + j + \alpha)^{\sigma - 1}
\]

as in the proof of (4.20). If \( \alpha > 0 \) and \( a_{i,j,\alpha}(0) = 0 \), we have \( A_{i,j,\alpha}^{(0)} = 0 \) and nothing to do. If \( \alpha > 0 \) and \( a_{i,j,\alpha}(0) \neq 0 \), we know that \( s \) satisfies the condition (4.19); in this case by 2) of Lemma 3.4 we have

\[
\frac{(m_1 - 1)!^{s-1}}{m_1^\sigma} \cdots \frac{(m_j - 1)!^{s-1}}{m_j^\sigma} \frac{(n_1 - 1)!^{s-1}}{1} \cdots \frac{(n_\alpha - 1)!^{s-1}}{1}
\leq (m_1 - 1)!^{s-1} \cdots (m_j - 1)!^{s-1} (n_1 - 1)!^{s-1} \cdots (n_\alpha - 1)!^{s-1}
\leq \left( \frac{e^{i+j+\alpha}}{k^{i+j+\alpha-1}} \right)^{s-1}(k - 1)!^{s-1} = \frac{(e^{i+j+\alpha})^{s-1}}{k^{(i+j+\alpha-1)(s-1)}}(k - 1)!^{s-1}
\leq \frac{(e^{i+j+\alpha})^{s-1}}{k^{\sigma - 1}}(k - 1)!^{s-1}.
\]
Hence, applying (4.25) and (4.26) to (4.24) we obtain

\[ I_3 \ll \frac{(k-1)!^{s-1}}{k^\sigma} \sum_{2 \leq i+j+\alpha \leq k} \frac{A_{i,j,\alpha}^{(0)}}{(R-x)^\sigma} \left( (i+j+\alpha)^{\sigma-1} + (e^{i+j+\alpha})^{s-1} \right) \]
\[ \times \left[ \sum_{i+|m|+|n|=k} (2Y_{m_1}) \cdots (2Y_{m_j}) \times (2\beta Y_{n_1}) \cdots (2\beta Y_{n_\alpha}) \right]. \]

Thus, by (4.22), (4.23) and (4.27) we have

\[ B_{\sigma} (|w_k|) (x) \ll I_1 + I_2 + I_3 \]
\[ \ll \frac{(k-1)!^{s-1}}{k^\sigma} \sum_{2 \leq i+j+\alpha \leq k} \frac{C_{i,j,\alpha}}{(R-x)^\sigma} \left[ \sum_{i+|m|+|n|=k} (2Y_{m_1}) \right. \]
\[ \left. \times \cdots \times (2Y_{m_j}) (2\beta Y_{n_1}) \cdots (2\beta Y_{n_\alpha}) \right] \]

and by comparing this with (4.13) we obtain

\[ B_{\sigma} (|w_k|) (x) \ll \frac{(k-1)!^{s-1}}{k^\sigma} (R-x)^\sigma Y_k(x) \]

which proves (4.21).

Thus, the proof of Lemma 4.1 is completed.

The proof of Theorem 2.1 is also completed.

By the above proof, we can say more. Let \( p \geq 2 \) and \( \sigma \geq p/(p-1) \). Assume the conditions: (i) \( \hat{a}(x), \hat{b}(x), \hat{c}(x) \) and \( \hat{a}_{i,j,\alpha}(x) \) are all formal power series in \( x \) belonging to the class \( G\{x\}_\sigma \); and (ii) the series

\[ \sum_{i+j+\alpha \geq 2} B_{\sigma} (\hat{a}_{i,j,\alpha}) (x) t^i u^j v^\alpha \]

is convergent in a neighborhood of the origin of \( \mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_u \times \mathbb{C}_v \).

Let us consider the following formal equation:

\[ (4.28) \left( t \frac{\partial}{\partial t} - \hat{b}(x) - x^p \hat{c}(x) \frac{\partial}{\partial x} \right) \hat{u} = \hat{a}(x) t^j \hat{u}^j \left( \frac{\partial \hat{u}}{\partial x} \right)^\alpha. \]

Then we have

**Theorem 4.2.** — Let \( p \geq 2 \) and \( \sigma \geq p/(p-1) \). Assume the above conditions (i) and (ii). Then, if \( \hat{b}(0) \notin \mathbb{N} \), the formal equation (4.28) has a
unique formal power series solution \( \hat{u}(t,x) \in \mathbb{C}[t,x] \) with \( \hat{u}(0,x) \equiv 0 \) and it belongs to the formal Gevrey class \( G\{t,x\}_{(s,\sigma)} \) for any \( s \) satisfying

\[
s \geq 1 + \max \left[ 0, \sup_{(i,j,\alpha) \in J} \left( \frac{\sigma - 1}{i + j + \alpha - 1} \right) \right],
\]

where \( J = \{(i,j,\alpha); i + j + \alpha \geq 2, \alpha > 0, \text{ and } \hat{a}_{i,j,\alpha}(0) \neq 0\} \).

5. Proof of Proposition 2.3.

Before the proof of Proposition 2.3 we shall show the following lemma.

LEMMA 5.1. — Let \( p \geq 2 \) and \( q \geq 1 \) be integers, let \( A > 0, C > 0, K > 0 \), and let us consider

\[
t \frac{\partial u}{\partial t} = Axt + Cx^p \frac{\partial u}{\partial x} + Kt^q \frac{\partial u}{\partial x}.
\]

We have

1) (5.1) has a unique formal solution \( u(t,x) \in \mathbb{C}[[t,x]] \) with \( u(0,x) \equiv 0 \).

2) \( u(t,x) \) belongs to the class \( G\{t,x\}_{(s,\sigma)} \) if and only if

\[
s \geq 1 + \frac{1}{(p - 1)q} \quad \text{and} \quad \sigma \geq \frac{p}{p - 1}.
\]

Proof. — Let \( u(t,x) \) be the formal solution of (5.1) in 1). Since Theorem 2.1 is already proved, we have only to show that \( u(t,x) \in G\{t,x\}_{(s,\sigma)} \) implies the condition (5.2). Note that in case \( p = 2 \) the condition (5.2) is given in [11].

Suppose that \( u(t,x) \in G\{t,x\}_{(s,\sigma)} \) holds. Without loss of generality we may assume \( A \geq 1, C \geq 1 \) and \( K \geq 1 \); if otherwise, we apply the change of variables \( t \rightarrow h_1 t, x \rightarrow h_2 x \) for sufficiently large \( h_1, h_2 \) and we can reduce the equation to the case where \( A \geq 1, C \geq 1 \) and \( K \geq 1 \) hold. Then, the formal solution \( w(t,x) \in \mathbb{C}[[t,x]] \) of

\[
t \frac{\partial w}{\partial t} = xt + x^p \frac{\partial w}{\partial x} + t^q \frac{\partial w}{\partial x}
\]

with \( w(0,x) \equiv 0 \) satisfies \( 0 \ll w(t,x) \ll u(t,x) \) and therefore we have \( w(t,x) \in G\{t,x\}_{(s,\sigma)} \); in particular, we have \( w(t,0) \in G\{t\}_s \) and \( (\partial w/\partial t)(0,x) \in G\{x\}_\sigma \).
It is easy to see that \( w(t, x) \) has the form
\[
    w(t, x) = \sum_{k \geq 0} w_{1+kq}(x)t^{1+kq}, \quad w_{1+kq}(x) \in \mathbb{C}[[x]] \text{ (for } k \geq 0) 
\]
and the coefficients are determined by the following recurrent formula:
\[
    (1 + kq)w_{1+kq} = x^p \frac{\partial w_{1+kq}}{\partial x} + \frac{\partial w_{1+(k-1)q}}{\partial x}. \quad (5.4)
\]
and for \( k \geq 1 \)
\[
    (1 + kq)w_{1+kq} = x^p \frac{\partial w_{1+kq}}{\partial x} + \frac{\partial w_{1+(k-1)q}}{\partial x}. \quad (5.5)
\]
By solving the equation (5.4) we have
\[
    w_1(x) = x^p + \sum_{l \geq 1} \left( (1+(p-1))(1+2(p-1)) \cdots (1+l(p-1)) \right) x^{p+l(p-1)} 
    \geq x^p \sum_{l \geq 1} (p-1)^l! x^{l(p-1)}. 
\]
Since \( w_1(x) = (\partial w/\partial t)(0, x) \in G\{x\}_\sigma \) is known, we have
\[
    \sum_{l \geq 1} (p-1)^l! x^{l(p-1)} \in G\{x\}_\sigma, 
\]
which immediately leads us to the condition \( \sigma \geq p/(p-1) \).

Since \( w_{1+kq}(x) \gg 0 \) is known, by (5.5) we have
\[
    w_{1+kq}(x) = \frac{1}{1+kq} \left( x^p \frac{\partial}{\partial x} w_{1+kq}(x) + \frac{\partial}{\partial x} w_{1+(k-1)q}(x) \right) 
    \geq \frac{1}{1+kq} \frac{\partial}{\partial x} w_{1+(k-1)q}(x) 
\]
and by repeating this \( k \)-times we have
\[
    w_{1+kq}(x) \geq \frac{1}{(1+q)(1+2q) \cdots (1+kq)} \left( \frac{\partial}{\partial x} \right)^k w_1(x). 
\]
Since \( w_1(x) \) is given explicitly in the equality (5.6), by putting \( k = p + l(p-1) \) and \( x = 0 \) we have
\[
    w_{1+(p+l(p-1))q}(0) 
    \geq \frac{(p + l(p-1))!}{(1+q)(1+2q) \cdots (1+(p+l(p-1))q)} \frac{\Gamma(1/q)}{\Gamma(1/(p-1))} q^{p-l(p-1)}(p-1)^l! 
\]

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and therefore
\[ u(t, 0) \gg \sum_{l \geq 1} w_{1+(p+l(p-1))q}(0) t^{1+(p+l(p-1))q} \]
\[ \gg t^{1+pq} \sum_{l \geq 1} \frac{\Gamma(1/q)}{\Gamma(1/(p-1))} q^{-p-l(p-1)}(p-1)^l l! t^{l(p-1)q}. \]

Thus, by the condition \( u(t, 0) \in G\{t\}_s \) we obtain
\[ \sum_{l \geq 1} (p-1)^l l! t^{l(p-1)q} \in G\{t\}_s \]
which immediately leads us to the condition \( s > 1 + (1/(p-1)q) \).

Thus, we have proved that \( u(t, x) \in G\{t, x\}_{(s, \sigma)} \) implies the condition (5.2).

**Proof of Proposition 2.3.** — Let \( u(t, x) \) be the unique formal power series solution of (2.1) with \( u(0, x) \equiv 0 \). Since Theorem 2.1 is already proved, to complete the proof of Proposition 2.3 it is sufficient to show that \( u(t, x) \in G\{t, x\}_{(s, \sigma)} \) implies the condition (2.2) and \( \sigma > p/(p-1) \). If \( J = \emptyset \) we have nothing to do; hence from now we assume that \( J \neq \emptyset \) holds.

By the conditions c1) \sim c4) we see that \( u(t, x) \gg 0 \) and we can choose \( M > 0 \) so that \( 0 < k - b(0) \leq Mk \) for all \( k \in \mathbb{N} \). Put \( a_0 = a(0) > 0 \) and \( a_1 = (\partial a/\partial x)(0) > 0 \). Take any \( (i, j, \alpha) \in J \). Then,
\[
Mt \frac{\partial u}{\partial t} \gg \left( t \frac{\partial}{\partial t} - b(0) \right) u
= xS(b)(x)u + x^pc(x)\frac{\partial u}{\partial x} + a(x)t + \sum_{k+l+m \geq 2} a_{k,l,m}(x)t^ku^l \left( \frac{\partial u}{\partial x} \right)^m
\gg x^pc(0)\frac{\partial u}{\partial x} + (a_0 + a_1)x + a_{i,j,\alpha}(0)t^i u^j \left( \frac{\partial u}{\partial x} \right)^\alpha.
\]

Therefore, we can see that the unique formal solution \( w(t, x) \in \mathbb{C}[t, x] \) of
\[
(5.7) \quad t \frac{\partial w}{\partial t} = \frac{1}{M} \left[ (a_0 + a_1)x + x^pc(0)\frac{\partial w}{\partial x} + a_{i,j,\alpha}(0)t^i u^j \left( \frac{\partial w}{\partial x} \right)^\alpha \right]
\]
with \( w(0, x) \equiv 0 \) satisfies \( 0 \ll w(t, x) \ll u(t, x) \) and therefore we have \( w(t, x) \in G\{t, x\}_{(s, \sigma)} \).

Moreover, \( w(t, x) \) has the form
\[
w(t, x) = \left( \frac{a_0}{M} + \frac{a_1}{M^2} \right) t + O(t^2)
\]
and by (5.7) we have
\[
\frac{t}{\partial t} \frac{\partial w}{\partial t} = \frac{1}{M} \left[ (a_0 + a_1 x) t + x^p c(0) \frac{\partial w}{\partial x} + a_{i,j,a}(0)^{i} \left( \left( \frac{a_0}{M} + \frac{a_1}{M} x \right) t + O(t^2) \right)^j \left( \left( \frac{a_0}{M} \right)^{\alpha-1} + O(t^2) \right)^{\alpha-1} \frac{\partial w}{\partial x} \right] 
\]
\[
\gg \frac{1}{M} \left[ a_1 x t + x^p c(0) \frac{\partial w}{\partial x} + a_{i,j,a}(0)^{i} \left( \frac{a_0}{M} \right)^{\alpha-1} t^{i+j+\alpha-1} \frac{\partial w}{\partial x} \right].
\]
Thus we can see also that the unique formal solution \( W(t, x) \in C[[t, x]] \) of (5.8)
\[
\frac{t}{\partial t} W = \frac{1}{M} \left[ a_1 x t + x^p c(0) \frac{\partial W}{\partial x} + a_{i,j,a}(0)^{i} \left( \frac{a_0}{M} \right)^{\alpha-1} \frac{\partial W}{\partial x} \right]
\]
with \( W(0, x) \equiv 0 \) satisfies \( 0 < W(t, x) < w(t, x) \) and \( W(t, x) \in G\{t, x\}_{(s, \sigma)} \).

Now, let us apply Lemma 5.1 to (5.8). Since \( W(t, x) \in G\{t, x\}_{(s, \sigma)} \) is known, we can conclude that \((s, \sigma)\) satisfies
\[
s \geq 1 + \frac{1}{(p-1)(i+j+\alpha-1)} \quad \text{and} \quad \sigma \geq \frac{p}{(p-1)}.
\]
Since \((i, j, \alpha) \in J\) is taken arbitrarily, we obtain
\[
s \geq 1 + \sup_{(i,j,\alpha) \in J} \left( \frac{1}{(p-1)(i+j+\alpha-1)} \right)
\]
which implies the condition (2.2).

Thus, the proof of Proposition 2.3 is completed.

**Remark.** — By the above proof we can see the following: if the equation (2.1) satisfies
\[
(i, j, \alpha) \in J \implies j = 0,
\]
we can remove the assumption \( a(0) > 0 \) from the condition c1) in Proposition 2.3.

**Example 5.2.** — Let \( p, l, n \in \mathbb{Z}_+ \) satisfying \( p \geq 2, n \geq 1 \) and \( l + n \geq 2 \). Let us consider
\[
\frac{t}{\partial t} u = xt + x^p \frac{\partial u}{\partial x} + t^l \left( \frac{\partial u}{\partial x} \right)^n.
\]
Then, the unique formal solution \( u(t, x) \in C[[t, x]] \) with \( u(0, x) \equiv 0 \) belongs to the class \( G\{t, x\}_{(s, \sigma)} \) if and only if
\[
s \geq 1 + \frac{1}{(p-1)(l+n-1)} \quad \text{and} \quad \sigma \geq \frac{p}{p-1}.
\]
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BIBLIOGRAPHY


