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UNIVERSAL FUNCTIONS ON NONSIMPLY CONNECTED DOMAINS

by **Antonios D. MELAS**

1. Introduction.

Let $\Omega \subseteq \mathbb{C}$ be a connected open set. For every function $f : \Omega \to \mathbb{C}$ holomorphic in Ω and $\zeta_0 \in \Omega$ we let

$$S_N(f,\zeta_0)(z) = \sum_{n=0}^{N} \frac{f^{(n)}(\zeta_0)}{n!}(z-\zeta_0)^n \tag{1.1}$$

denote the Nth partial sum of the Taylor development of f with center ζ_0. This sequence of polynomials converges to f uniformly on compact subsets of the open disc $D(\zeta_0, \rho)$ where $\rho = \operatorname{dist}(\zeta_0, \partial\Omega)$ and "generically" it will diverge on the complement of the closed disc $\overline{D(\zeta_0, \rho)}$. However certain subsequences of $(S_N(f,\zeta_0)(z))_{N=1}^{\infty}$ may have nice approximation properties outside Ω and this can be described by defining the following class.

Definition 1.— *A holomorphic function $f : \Omega \to \mathbb{C}$ is said to be universal with respect to Ω and ζ_0 if the following holds:*

For every compact set $K \subseteq \mathbb{C}$ such that $K \cap \Omega = \varnothing$ and its complement K^c is connected, and every function $h : K \to \mathbb{C}$ continuous on K and holomorphic in the interior K^o (if nonempty), there exists a sequence (λ_n)

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of nonnegative integers such that $S_{\lambda_n}(f)(z) \to h(z)$ uniformly on K. We denote by $\mathcal{U}(\Omega, \zeta_0)$ the class of all such functions.

Here the compact set K is allowed to contain pieces of the boundary $\partial\Omega$. The existence of such functions in the case where Ω is simply connected has been established in [7] where it actually has been shown that $\mathcal{U}(\Omega, \zeta_0)$ is a G_δ-dense subset of the Frechet space $H(\Omega)$ of all functions holomorphic in Ω. Similar definition but with K not allowed to meet the boundary $\partial\Omega$ has been given by Luh [4], where also existence has been proven when Ω is simply connected.

To explain this definition let us assume that $\zeta_0 = 0 \in \Omega$ and for $f \in \mathcal{U}(\Omega, 0)$ write $a_n = \frac{f^{(n)}(0)}{n!}$. Then the Taylor series $\sum_{n=0}^{\infty} a_n z^n$ defines a germ of analytic functions that can be analytically continued throughout Ω and its partial sums approximate everything one might hope in $\mathbb{C}\backslash\Omega$.

Several properties of these classes $\mathcal{U}(\Omega, \zeta_0)$ have been established in [5] in the case Ω is assumed to be contained in a half plane (actually in the complement of an angle suffices). Under this assumption it has been shown in [5]:

(a) That the class $\mathcal{U}(\Omega, \zeta_0)$ is empty if Ω is not simply connected.

(b) That $\mathcal{U}(\Omega, \zeta_1) = \mathcal{U}(\Omega, \zeta_2)$ for all $\zeta_1, \zeta_2 \in \Omega$.

(c) That for Ω simply connected any $f \in \mathcal{U}(\Omega, \zeta_0)$ has $\partial\Omega$ as its natural boundary, that is cannot be continued analytically across any portion of $\partial\Omega$, which answered a conjecture by J.-P. Kahane.

However nothing is known for the class $\mathcal{U}(\Omega, \zeta_0)$ in the case where Ω is not contained in the complement of an angle (except that it is G_δ-dense in $H(\Omega)$ if Ω is simply connected).

The only results known in this direction are that the corresponding class where the compact set is not allowed to meet the boundary $\partial\Omega$ is G_δ-dense in $H(\Omega)$ if Ω is connected and equal to the complement of a connected compact set (see [1]) and that for the special case where $\Omega = \mathbb{C}\backslash\{1\}$ the class $\mathcal{U}(\mathbb{C}\backslash\{1\}, 0)$ is nonempty (see [8]). It is not known whether the class $\mathcal{U}(\mathbb{C}\backslash A, 0)$ is nonempty if A is a finite set or if the result in [1] extends for the class $\mathcal{U}(\Omega, \zeta_0)$ and also whether some of the above mentioned properties (a)-(c) hold in these cases.

Here we will give some answers to the above questions. First we will prove the following.

THEOREM 1.— *Let $K \subset \mathbb{C}$ be a connected compact set such that $\Omega = \mathbb{C}\backslash K$ is also connected. Then for every $\zeta_0 \in \Omega$ the class $\mathcal{U}(\Omega, \zeta_0)$ is a G_δ-dense subset of $H(\Omega)$, hence nonempty.*

Thus the result in [1] actually does extend for the class $\mathcal{U}(\Omega, \zeta_0)$. We mention here that it is not known whether $\mathcal{U}(\Omega, \zeta_0)$ is nonempty when the compact set K in the above theorem is not assumed to be connected for example if Ω is equal to the complement of two disjoint closed discs and ζ_0 is any point of Ω.

To proceed with the other questions let E be any countable subset of $\mathbb{C}\backslash D$ (where D is the unit disc), let B be any discrete subset of $\mathbb{C}\backslash D$ such that $1 \in B$ and let $W = \mathbb{C}\backslash B$. We will denote by $\mathcal{V}(W, E)$ the class of all functions $f \in H(W)$ (if any) having the following property:

"For every finite subset $E' \subset E$ and any function $h : E' \to \mathbb{C}$ there exists a strictly increasing sequence of positive integers (λ_k) such that

$$(1.2) \qquad S_{\lambda_k}(f, 0)(z) \to h(z) \text{ for every } z \in E'."$$

Then we have the following.

THEOREM 2.— *For every E and B as above the class $\mathcal{V}(W, E)$ is a G_δ-dense subset of $H(W)$, hence nonempty.*

By taking $E = (\mathbb{C}\backslash D) \cap (\mathbb{Q} + i\mathbb{Q})$ and $B = \{1\}$ then Theorem 2 provides us with an $F \in \mathcal{V}(\mathbb{C}\backslash\{1\}, E)$. Consider also any infinite discrete subset L of E such that $1 \in L$ and let $\Omega = \mathbb{C}\backslash L$. Then on the one hand clearly $f = F\,|_\Omega \in \mathcal{U}(\Omega, 0)$ (compact subsets of $\mathbb{C}\backslash\Omega$ are finite). However on the other hand f extends analytically to the larger domain $\mathbb{C}\backslash\{1\}$. Also $f \notin \mathcal{U}(\Omega, \zeta_0)$ for every $\zeta_0 \in \Omega$ such that $D(\zeta_0, |\zeta_0 - 1|) \cap L \neq \varnothing$ since then $\{(S_N(f, \zeta_0)(z)\}_{N=1}^{\infty}$ will converge to $F(z)$ for any $z \in D(\zeta_0, |\zeta_0 - 1|) \cap L$, F being holomorphic in $\mathbb{C}\backslash\{1\}$ and so $\mathcal{U}(\Omega, 0) \neq \mathcal{U}(\Omega, \zeta_0)$ for every such ζ_0. It is easy to see that we can arrange L so that every $\zeta_0 \in \Omega\backslash D$ has this property. Moreover the sequence $\{(S_N(f, 0)(z)\}_{N=1}^{\infty}$ will be dense for every $z \in E\backslash L$ which is a nonempty subset of Ω. Hence we have proved the following.

COROLLARY 1.— *There exists a nonsimply connected domain $\Omega \subseteq \mathbb{C}$ with $0 \in D \subseteq \Omega$ and $\mathbb{C}\backslash\Omega$ is infinite and discrete such that*

(a) *The class $\mathcal{U}(\Omega, 0)$ is G_δ-dense in $H(\Omega)$.*

(b) $\mathcal{U}(\Omega,0)$ *contains functions that can be analytically continued on the larger domain* $\mathbb{C}\backslash\{1\}$*, hence Kahane's conjecture cannot be generalized in this case.*

(c) $\mathcal{U}(\Omega,0)\neq\mathcal{U}(\Omega,\zeta)$ *for every* $\zeta\in\Omega\backslash D\neq\varnothing$.

(d) *There exists* $f\in\mathcal{U}(\Omega,0)$ *such that sequence of complex numbers* $\{(S_N(f,0)(z)\}_{N=1}^{\infty}$ *is dense for some points* z *in* Ω.

Thus the situation is completely different if the complement of Ω is small. Property (d) appears to be completely new.

It is not known and natural to ask here how small must $\mathbb{C}\backslash\Omega$ be for Ω to share the above properties. For example if $\Omega=\mathbb{C}\backslash[1,+\infty)$, is there a function $f\in\mathcal{U}(\Omega,0)$ that can be analytically continued across some subinterval of $(1,+\infty)$? Also in this case one may ask whether we can find $\zeta\in\Omega$ such that $\mathcal{U}(\Omega,0)\neq\mathcal{U}(\Omega,\zeta)$.

In Section 2 we will prove Theorem 1. Then in Section 3 we will establish a technical proposition that will be used in Section 4 to prove Theorem 2.

2. Proof of Theorem 1.

Applying a translation, a rotation and a dilation we may assume that $\zeta_0=0$, $D\subset\Omega$, $1\in K$ and that there exists a real number $\alpha\geqslant 1$ such that $\alpha\in K$ and $\alpha=\max\{|z|:z\in K\}$.

By Lemma 3.3 in [1], for every open neighborhood V of K there is a connected and simply connected open set U such that $K\subset U\subset V$. Hence it is easy to construct an exhausting sequence $(L_m)_{m=1}^{\infty}$ of compact subsets of $\Omega=\mathbb{C}\backslash K$ such that for every m, $\mathbb{C}\backslash L_m$ has exactly two connected components, one of which is bounded and contains K and the other unbounded.

Now for every polynomial p and every $\varepsilon>0$ we define

$$\Gamma(p,\varepsilon,\Omega)$$
$$=\{f\in H(\Omega):\ \text{there exists } n \text{ such that } \max_{z\in K}|S_n(f,0)(z)-p(z)|<\varepsilon\}.$$

Then letting $(p_j)_{j=1}^{\infty}$ denote an enumeration of all polynomials with coefficients in $\mathbb{Q}+i\mathbb{Q}$ we have the following.

LEMMA 1.— $\mathcal{U}(\Omega,0)=\bigcap_{j,s=1}^{\infty}\Gamma(p_j,\frac{1}{s},\Omega)$.

The proof of the above lemma is well known (see for example [5]), noticing that every compact subset of Ω is of course contained in K.

It is also easy to show that each $\Gamma(p,\varepsilon,\Omega)$ is an open subset of $H(\Omega)$ (see [7], [5]). Hence, in view of Baire's Theorem, to complete the proof of Theorem 1 it suffices to prove the following.

LEMMA 2.— *For every polynomial p and every $\varepsilon>0$ the set $\Gamma(p,\varepsilon,\Omega)$ is dense in $H(\Omega)$.*

Proof.— It suffices to show that for every $g\in H(\Omega)$, $\delta>0$ and $m\geqslant 1$ there exists $f\in\Gamma(p,\varepsilon,\Omega)$ such that $\max\limits_{L_m}|f-g|<\delta$.

By taking m sufficiently large we may assume that $\overline{D(0,\frac{1}{2})}\subset L_m$. Fix such g, δ, m. Then there exists $\beta>\alpha$ such that β belongs to the bounded component of $\mathbb{C}\backslash L_m$ and hence by Runge's Theorem there exists a rational function Q with poles only at β such that

$$\max_{L_m}|g-Q|<\frac{\delta}{2}\ \text{and}\ \max_{K}|p-Q|<\frac{\varepsilon}{2}. \tag{2.1}$$

Since the Taylor development of Q with center 0 converges to Q uniformly on compact subsets of $D(0,\beta)$ and by the choice of α, β the compact set K is contained in $D(0,\beta)$ we conclude that

$$S_N(Q,0)(z)\to Q(z)\ \text{uniformly for}\ z\in K\ \text{as}\ N\to\infty. \tag{2.2}$$

Hence in view of (2.1) there is N_0 sufficiently large such that

$$\max_{z\in K}|S_{N_0}(Q,0)(z)-p(z)|<\frac{\varepsilon}{2}. \tag{2.3}$$

Fix such a N_0. The only problem is that Q is not holomorphic in Ω as β does not belong to K. However by applying again Runge's Theorem we conclude that there exists a sequence $(R_M)_{M=1}^{\infty}$ of rational functions with poles only at $\alpha\in K$ such that $R_M\to Q$ uniformly in L_m.

Since $\overline{D(0,\frac{1}{2})}\subset L_m$ the Cauchy's estimates imply that $S_{N_0}(R_M,0)(z)\to S_{N_0}(Q,0)(z)$ uniformly for $z\in K$ as $M\to\infty$, since N_0 is fixed and K is compact.

Hence there is M_0 sufficiently large such that $f = R_{M_0} \in H(\Omega)$ satisfies

$$\max_{L_m} |f - Q| < \frac{\delta}{2} \text{ and } \max_{z \in K} |S_{N_0}(f,0)(z) - S_{N_0}(Q,0)(z)| < \frac{\varepsilon}{2}. \tag{2.4}$$

Therefore by (2.1), (2.3) and (2.4) we have $\max_{L_m} |f - g| < \delta$ and $f \in \Gamma(p, \varepsilon, \Omega)$. This completes the proof of the lemma and hence of Theorem 1. □

3. A property of certain linear systems.

In this section we will prove a technical result that will be essential for the proof of Theorem 2.

For any positive integers m, N and any $z \in \mathbb{C}$ we define

$$A_N(m, z) = \sum_{n=1}^{N-1} n^m z^n. \tag{3.1}$$

If s is another positive integer and $z_1, ..., z_s \in \mathbb{C}$ we also define

$$\Delta_N(m, z_1, ..., z_s) = \det[A_N(m + r, z_j)]_{j,r=1}^{s} \tag{3.2}$$

the determinant of the corresponding $s \times s$ matrix.

Consider also the meromorphic function

$$\psi(\zeta) = \frac{1}{e^\zeta - 1}. \tag{3.3}$$

Then we have:

Lemma 3. — *If $z = e^\zeta \in \mathbb{C}$ is such that $|z| \geqslant 1$, $z \neq 1$ then for every $m > s > 1$*

$$N^{-m} z^{-N} A_N(m, z) = \sum_{\lambda=0}^{s-1} \binom{m}{\lambda} N^{-\lambda} \psi^{(\lambda)}(\zeta) + \rho_N(m, s, z) \tag{3.4}$$

where $|\rho_N(m, s, z)| \leqslant CN^{-s}$ the constant C depending only on m, s and z.

Proof. — Since $e^\zeta \neq 1$ we have

$$\begin{aligned} A_N(m,z) &= \sum_{n=1}^{N-1} n^m e^{n\zeta} = \left(\frac{d}{d\zeta}\right)^m \left(\sum_{n=1}^{N-1} e^{n\zeta}\right) = \left(\frac{d}{d\zeta}\right)^m \left(\frac{e^{N\zeta}-1}{e^\zeta - 1}\right) \\ &= \sum_{\lambda=0}^{m} \binom{m}{\lambda} \left(\frac{d}{d\zeta}\right)^{m-\lambda} (e^{N\zeta}-1)\psi^{(\lambda)}(\zeta) \\ &= \sum_{\lambda=0}^{s-1} \binom{m}{\lambda} N^{m-\lambda} z^N \psi^{(\lambda)}(\zeta) + \sum_{\lambda=s}^{m-1} \binom{m}{\lambda} N^{m-\lambda} z^N \psi^{(\lambda)}(\zeta) \\ &\quad + (z^N - 1)\psi^{(m)}(\zeta) \end{aligned}$$

from which the lemma follows easily since $|z| \geqslant 1$. □

LEMMA 4. — *Let $z_1 = e^{\zeta_1}, \ldots, z_s = e^{\zeta_s} \in \mathbb{C}\setminus(D \cup \{1\})$ and $m \geqslant 1$. Then for all sufficiently large N we have*

$$|\Delta_N(m, z_1, \ldots, z_s)| \geqslant c\,|z_1 \ldots z_s|^N N^{(m+1)s} \tag{3.5}$$

where $c > 0$ doesn't depend on N.

Proof. — By Lemma 2 we have

$$\begin{aligned} &A_N(m+r, z_j) \\ &\quad = N^{m+r} z_j^N \left(\sum_{\lambda=0}^{s-1} \binom{m+r}{\lambda} N^{-\lambda} \psi^{(\lambda)}(\zeta_j) + \rho_N(m+r, s, z_j)\right) \end{aligned} \tag{3.6}$$

hence

$$\begin{aligned} (z_1 \ldots z_s)^{-N} N^{-sm-1-2-\ldots-s} \Delta_N(m, z_1, \ldots, z_s) = \det[\psi(\zeta_j) \\ + (m+r)\frac{\psi'(\zeta_j)}{N} + \ldots \\ + \binom{m+r}{s-1} \frac{\psi^{(s-1)}(\zeta_j)}{N^{s-1}} \\ + \rho_N(m+r, s, z_j)]_{j,r=1}^s. \end{aligned}$$

Now it is easy to see that each column of the above determinant is the sum of $s+1$ columns. The rth column is thus the sum of $[\psi(\zeta_j)]_{j=1}^s, \ldots, \left[\binom{m+r}{s-1} \frac{\psi^{(s-1)}(\zeta_j)}{N^{s-1}}\right]_{j=1}^s$ and the column of "errors" $[\rho_N(m+r, s, z_j)]_{j=1}^s$.

We can thus expand this determinant into a sum of $(s+1)^s$ simple determinants. This sum can be further partitioned into two sums. The first consists of all determinants that don't contain any column of errors $[\rho_N(m+r,s,z_j)]_{j=1}^s$ and thus this sum is equal to the following determinant:

$$\text{(3.7)} \qquad \det\left[\sum_{\lambda=0}^{s-1}\binom{m+r}{\lambda}N^{-\lambda}\psi^{(\lambda)}(\zeta_j)\right]_{j,r=1}^s = \det[\psi^{(k-1)}(\zeta_j)]_{j,k=1}^s \cdot \det\left[N^{-k+1}\binom{m+r}{k-1}\right]_{k,r=1}^s$$

according to the determinant of the product formula.

Now $\det\left[N^{-k+1}\binom{m+r}{k-1}\right]_{k,r=1}^s = N^{-(0+1+\ldots+(s-1))}\det\left[\binom{m+r}{k-1}\right]_{k,r=1}^s$ and it is easy to see by induction (using the formula $\binom{m+r+1}{k-2}-\binom{m+r}{k-2}=\binom{m+r}{k-2}$) that $\det\left[\binom{m+r}{k-1}\right]_{k,r=1}^s = 1$.

The second sum consists of those determinants that contain at least one column of "errors" $[\rho_N(m+r,s,z_j)]_{j=1}^s$. Observing that such a determinant will be zero if it contains at least two columns of the same type i.e. $\left[\binom{m+r_1}{\lambda}\frac{\psi^{(\lambda)}(\zeta_j)}{N^\lambda}\right]_{j=1}^s$ and $\left[\binom{m+r_2}{\lambda}\frac{\psi^{(\lambda)}(\zeta_j)}{N^\lambda}\right]_{j=1}^s$ for $r_1 \neq r_2$ and for the same λ we conclude that each nonzero determinant in the second sum has absolute value at most

$$C'N^{-(0+1+\ldots+(s-2))}.N^{-s} = C'N^{-\frac{s(s-1)}{2}-1}$$

where C' dosn't depend on N. Hence the second sum has absolute value at most $CN^{-\frac{s(s-1)}{2}-1}$ where C depends only on m, s and the z_j's.

Therefore

$$\text{(3.8)} \qquad |\Delta_N(m,z_1,\ldots,z_s)| \geqslant |z_1\ldots z_s|^N N^{s(m+1)}\left[\left|\det[\psi^{(k-1)}(\zeta_j)]_{j,k=1}^s\right| - \frac{C}{N}\right].$$

Hence to complete the proof of the lemma it suffices to prove that $\det[\psi^{(k-1)}(\zeta_j)]_{j,k=1}^s \neq 0$. But it is easy to show by induction that for every $k \geqslant 1$ we have

$$\text{(3.9)} \qquad \psi^{(k-1)}(\zeta) = a_{1,k}\psi(\zeta)+\ldots+a_{k,k}\psi(\zeta)^k$$

for certain constants $a_{1,k}, ..., a_{k,k} \in \mathbb{C}$ where $a_{k,k} = (-1)^k(k-1)! \neq 0$. Hence simplifying the corresponding determinant we get

$$\det[\psi^{(k-1)}(\zeta_j)]_{j,k=1}^s = a_{1,1}...a_{k,k} \det[\psi(\zeta_j)^k]_{j,k=1}^s \neq 0 \tag{3.10}$$

since the last determinant is a Vandermonde one and the corresponding complex numbers $\psi(\zeta_1), ..., \psi(\zeta_s)$ are distinct. This completes the proof of the lemma. □

Remark. — We can get a similar asymptotic behavior in the case that one of the z_j's is equal to 1. However since we won't be needing this in the proof of Theorem 2 we don't include it.

Now we can prove the following.

PROPOSITION 1. — *Let $z_1, ..., z_s \in \mathbb{C}\backslash(D \cup \{1\})$ be distinct and let $(\gamma_N^1), ..., (\gamma_N^s)$ be s sequences of complex numbers such that*

$$\left|\gamma_N^j\right| \leqslant C\left|z_j\right|^N N^d \tag{3.11}$$

for all $N \geqslant 1$ and $1 \leqslant j \leqslant s$ where C, d are constants that do not depend on N. Then there exists an integer $m > d$ such that for all sufficiently large N the linear system

$$\sum_{r=1}^{s} A_N(m+r, z_j)t_{r,N} = \gamma_N^j \quad \text{for } j = 1, 2, ..., s \tag{3.12}$$

has a unique solution $t_{1,N}, ..., t_{s,N} \in \mathbb{C}$ and moreover

$$|t_{1,N}| + ... + |t_{s,N}| \to 0 \text{ as } N \to \infty. \tag{3.13}$$

Proof. — We choose $m = d + s(s+3)$. Then for every sufficiently large N Lemma 3 implies that the determinant of the system (3.12) which is $\Delta_N(m, z_1, ..., z_s)$ satisfies the estimate (3.5) in particular it is nonzero and hence the system has a unique solution $t_{1,N}, ..., t_{s,N} \in \mathbb{C}$. Now each $t_{r,N}$ can be computed according to Cramer's rule and is equal to the ratio of two determinants, the one in the denominator being $\Delta_N(m, z_1, ..., z_s)$ and the other in the numerator is produced by $\Delta_N(m, z_1, ..., z_s)$ by replacing the rth column of $\Delta_N(m, z_1, ..., z_s)$ by $[\gamma_N^j]_{j=1}^s$. Expanding such a determinant we get a sum of $s!$ terms of the form $\pm A_N(m+r_1, z_{j_1})...A_N(m+r_{s-1}, z_{j_{s-1}})\gamma_N^{j_s}$

where the $j_1, ..., j_s$ are distinct and $r_1 < ... < r_{s-1}$ belong to $\{1, 2, ..., s\}$. Since

$$(3.14) \qquad |A_N(m+r,z)| \leqslant \sum_{n=1}^{N-1} n^{m+r} |z|^N < N^{m+r+1} |z|^N$$

whenever $|z| \geqslant 1$ we conclude that any such term has absolute value at most

$$N^{m+r_1+1} |z_{j_1}|^N ... N^{m+r_{s-1}+1} |z_{j_{s-1}}|^N C N^d |z_{j_s}|^N \leqslant C N^{m(s-1)+3+...+(s+2)+d} |z_1...z_s|^N .$$

Therefore using Lemma 4 we conclude that for every sufficiently large N we have

$$(3.15) \qquad |t_{1,N}| + ... + |t_{s,N}| \leqslant s \frac{C s! N^{m(s-1)+3+...+(s+2)+d} |z_1...z_s|^N}{c N^{(m+1)s} |z_1...z_s|^N} = s \frac{Cs!}{c} N^{-m+\frac{s(s+3)}{2}+d}$$

which tends to 0 as $N \to \infty$ by the choise of m. This completes the proof of the proposition. □

4. Proof of Theorem 2.

In this section we will prove Theorem 2. For this purpose let $(h_\ell)_{\ell=1}^{\infty}$ be an enumeration of all functions $h : E' \to \mathbb{Q} + i\mathbb{Q}$ where $E' \subseteq E$ is finite. Then we define

$$\Gamma(\ell, k, W) = \Big\{ f \in H(W) : \text{there exists } n \geqslant 1 \text{ such that } |S_n(f,0)(z) - h_\ell(z)| < \frac{1}{k} \text{ for every } z \in \text{dom } h_\ell \Big\}.$$

It is easy to see that $\bigcap_{\ell,k=1}^{\infty} \Gamma(j,k,W) = \mathcal{V}(W,E)$ and it is also easy to see that each $\Gamma(\ell,k,W)$ is an open subset of $H(W)$. Hence to prove Theorem 2 it suffices to prove the following.

PROPOSITION 2. — *Let $\{z_1, ..., z_s\}$ be a finite subset of $\mathbb{C} \setminus D$, $a_1, ..., a_s \in \mathbb{C}$ and $\varepsilon > 0$. Then the set*

$$\Gamma = \{ f \in H(W) : \textit{there exists } n \geqslant 1 \textit{ such that } |S_n(f,0)(z_j) - a_j| < \varepsilon \textit{ for every} j = 1, 2, ..., s \}$$

is a dense subset of $H(W)$.

Proof. — It suffices to prove that given any $\delta > 0$, $g \in H(W)$ and compact set $L \subseteq W$ there exists $f \in \Gamma$ such that $\max_L |g - f| < \delta$.

Since $B = \mathbb{C} \backslash W$ is discrete we may assume that L is of the form $\{z \in \mathbb{C} : |z| \leqslant R \text{ and } \operatorname{dist}(z, B) \geqslant \eta\}$ for certain $R > 2$ and $\eta > 0$.

Then $\mathbb{C} \backslash L$ has finitely many bounded components each containing at least one point of B. Therefore by Runge's Theorem there exists a rational function Q with poles in $B \subseteq \mathbb{C} \backslash D$ such that $\max_L |g - Q| < \frac{\delta}{2}$. Expanding Q in simple fractions we may write

$$Q(z) = \sum_{\mu=1}^{M} \sum_{k=1}^{K} \frac{b_{\mu,k}}{(1 - z/w_\mu)^k} \tag{4.1}$$

where $w_1, ..., w_M \in B$ and so each $|w_\mu| \geqslant 1$.

Since each $\frac{1}{(1-z/w_\mu)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} \left(\frac{z}{w_\mu}\right)^n$ has partial sums with absolute value

$$\left| \sum_{n=0}^{N} \binom{n+k-1}{k-1} \left(\frac{z}{w_\mu}\right)^n \right| \leqslant C_k N^{k-1} |z|^N \tag{4.2}$$

for every z with $|z| \geqslant 1$ we conclude that

$$|S_N(Q, 0)(z)| \leqslant C |z|^N N^{K-1} \tag{4.3}$$

for every $z \in \mathbb{C} \backslash D$ where C depends only on Q.

Now we define for any integer $p \geqslant 0$ the power series $\sum_{n=0}^{\infty} n^p z^n$. It is easy to see that there exist rational functions R_p with pole at 1 only such that

$$R_p(z) = \sum_{n=0}^{\infty} n^p z^n \text{ on } D. \tag{4.4}$$

In fact $R_0(z) = \frac{1}{1-z} - 1$ and $R_{p+1}(z) = \frac{d}{dz}(z R_p(z))$.

We fix a large integer m to be defined, choose a complex number of absolute value 1, $e^{i\theta} \notin \{z_1, ..., z_s\}$ belonging to the same component of $\mathbb{C} \backslash L$ that contains 1 (using $e^{i\theta}$ will not be necessary unless $1 \in \{z_1, ..., z_s\}$) and consider the function

$$G(z) = \sum_{r=1}^{s} \lambda_r R_{m+r}(z e^{-i\theta}) \tag{4.5}$$

where $\lambda_1, ..., \lambda_s$ are to be determined.

Clearly G is a rational function with pole at $e^{\imath\theta}$ only.

Then

$$S_N(G,0)(z) = \sum_{r=1}^{s} A_N(m+r, ze^{-\imath\theta})\lambda_r \tag{4.6}$$

and so using (4.3) and chosing m as in Proposition 1 applied to the numbers $z_1e^{-i\theta}, ..., z_se^{-i\theta} \in \mathbb{C}\backslash(D \cup \{1\})$ we conclude that given $\varepsilon_1 > 0$ there exist N sufficiently large and $\lambda_1, ..., \lambda_s \in \mathbb{C}$ such that

$$S_N(G,0)(z_j) + S_N(Q,0)(z_j) = a_j \text{ for } j = 1, 2, ..., s \tag{4.7}$$

and

$$|\lambda_1| + ... + |\lambda_s| < \varepsilon_1. \tag{4.8}$$

We choose $\varepsilon_1 > 0$ such that (4.8) implies that

$$\max_{z\in L} |G(z)| < \frac{\delta}{2} \tag{4.9}$$

and fix the corresponding $N = N_0$ for which (4.7) and (4.8) hold. This is possible since L is compact and each $R_{m+r}(ze^{-\imath\theta})$ has no poles in L.

Then by Runge's Theorem there is a sequence $(F_q)_{q=1}^{\infty}$ of rational functions having poles only at 1 such that $F_q \to G$ uniformly on $T = \{z \in \mathbb{C} : |z| \leqslant R,\ |z-1| \geqslant \eta\}$. Since T contains a disc around 0 the Cauchy's estimates impy that

$$S_{N_0}(F_q,0)(z_j) \to S_{N_0}(G,0)(z_j) \text{ for } j = 1, 2, ..., s \tag{4.10}$$

as $q \to \infty$.

Therefore we can choose q_0 sufficiently large such that the rational function $F = F_{q_0}$ that has pole only at 1 satisfies

$$\max_{z\in L} |F(z)| < \frac{\delta}{2} \quad \text{and } |S_{N_0}(F+Q,0)(z_\jmath) - a_j| < \varepsilon \text{ for } j = 1, 2, ..., s. \tag{4.11}$$

Then by taking $f = F + Q \in H(W)$ we have

$$\max_L |g - f| \leqslant \max_L |g - Q| + \max_L |F| < \delta \tag{4.12}$$

and

$$|S_{N_0}(f,0)(z_j) - a_j| < \varepsilon \text{ for } j = 1, 2, ..., s \tag{4.13}$$

hence $f \in \Gamma$. This completes the proof of the proposition and hence of Theorem 2. □

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