



ANNALES

DE

L'INSTITUT FOURIER

Jesús ESCRIBANO

Nash triviality in families of Nash mappings

Tome 51, n° 5 (2001), p. 1209-1228.

http://aif.cedram.org/item?id=AIF_2001__51_5_1209_0

© Association des Annales de l'institut Fourier, 2001, tous droits réservés.

L'accès aux articles de la revue « Annales de l'institut Fourier » (<http://aif.cedram.org/>), implique l'accord avec les conditions générales d'utilisation (<http://aif.cedram.org/legal/>). Toute reproduction en tout ou partie cet article sous quelque forme que ce soit pour tout usage autre que l'utilisation à fin strictement personnelle du copiste est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

cedram

*Article mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques
<http://www.cedram.org/>*

NASH TRIVIALITY IN FAMILIES OF NASH MAPPINGS

by Jesús ESCRIBANO(*)

Introduction.

The aim of this paper is to study triviality in Nash families of proper Nash submersions, or, in a more general setting, triviality in pairs of proper Nash submersions.

We work over an arbitrary real closed field R . Let N and P be Nash manifolds over R and $g : N \rightarrow P$ a Nash mapping. We say that g is *Nash trivial* if there exist a point $p \in P$ and a Nash diffeomorphism $\gamma = (\gamma_0, g) : N \rightarrow g^{-1}(p) \times P$. Let M be a Nash manifold and $f : M \rightarrow N$ a Nash mapping. We say that (f, g) is *Nash trivial* if there exist p and γ as before and a Nash diffeomorphism $\theta = (\theta_0, g \circ f) : M \rightarrow (g \circ f)^{-1}(p) \times P$ such that $f \circ \theta_0 = \gamma_0 \circ f$. In other words, the following diagram is commutative:

$$\begin{array}{ccccc}
 M & \xrightarrow{f} & N & \xrightarrow{g} & P \\
 \downarrow \theta & & \downarrow \gamma & & \downarrow \text{id} \\
 (g \circ f)^{-1}(p) \times P & \xrightarrow{f \times \text{id}} & g^{-1}(p) \times P & \xrightarrow{\pi} & P.
 \end{array}$$

The main result of this paper is

THEOREM. — *Let R be a real closed field. Let M and N be Nash manifolds, and let $f : M \rightarrow N$, $g : N \rightarrow R^l$ be proper surjective Nash submersions. Then (f, g) is Nash trivial.*

(*) Partially supported by DGICYT, PB98-0756-C02-01.

Keywords: Nash manifold – Nash mapping – Nash triviality – Real spectrum.

Math. classification: 14P20 – 58A07.

(We recall that, as we work on arbitrary real closed fields, the usual notion of compactness is not well behaved. In this paper, *compact* just means closed and bounded. A mapping $f : A \rightarrow B$ between semialgebraic subset $A \subset R^n$, $B \subset R^m$, is said to be *proper* if $f^{-1}(K)$ is closed and bounded for any closed and bounded semialgebraic subset $K \subset B$.)

In [H], Hardt studies the local triviality of semialgebraic families of semialgebraic sets, but his results say nothing about smooth situations. In [CS1], M. Coste and M. Shiota study Nash families of Nash manifolds and they prove an important triviality result. This result is a Nash version of Thom's first isotopy lemma. In fact, these two authors prove a semialgebraic version of Thom's first isotopy lemma in [CS2]. Our main result can be seen as a Nash version of Thom's second isotopy lemma.

A useful tool for the study of the triviality of smooth families of smooth sets or mappings is integration of vector fields (see, for example, [GWPL]). We cannot use this tool in the semialgebraic context. In [CS1], the authors use the construction of *Nash models* of Nash manifolds over smaller real closed fields. That is, given a real closed field extension $R' \rightarrow R$ and a Nash manifold M defined over R , they construct a Nash manifold M' defined over R' such that the extension M'_R is Nash diffeomorphic to M . This construction, together with the use of *real spectrum theory*, allow us to substitute integration of vector fields.

We follow the same ideas in this paper, translating them to the relative situation. We extend the results on construction of Nash models of Nash manifolds to results on construction of Nash models of Nash proper submersions between Nash manifolds (Section 3). We also study with detail generic fibres at points of the real spectrum of Nash families of proper Nash submersions (Section 1). These results, together with some technical results in Section 2 allow us to prove our main theorem.

We remark here that in [Shi2], §II.6, there is a result of this type, although the proof is quite different and difficult to follow already over the reals.

Finally, in Section 5, we give some results on finiteness of topological types in families of Nash mappings. We also get some results on effectiveness of the theorem above, using that our results are true for any real closed field.

I thank professor Michel Coste for his help and support during the realization of this paper.

1. Nash families. Real spectrum.

In what follows, R will denote a real closed field.

We are interested in Nash families of Nash manifolds and mappings, and we want to study the generic fibres of these families at points of the real spectrum. In [BCR], Chapters 7 and 8, we can find a detailed exposition on real spectrum, semialgebraic families and the relationship between them. We adopt the notation from the above reference. In particular, we will usually call α to a point of the real spectrum $\widetilde{R^p}$. We can associate to each point α of the real spectrum an ultrafilter $\widehat{\alpha}$ of semialgebraic sets. If S is a semialgebraic subset of R^p , then \widetilde{S} will be identified with the subset of $\widetilde{R^p}$ consisting of those α such that S is in the ultrafilter $\widehat{\alpha}$. We will call $k(\alpha)$ to the real closed field associated to each point $\alpha \in \widetilde{R^p}$. For a semialgebraic family $X \subset R^n \times R^p$, we will write X_α for the generic fibre of X at a point $\alpha \in \widetilde{R^p}$. Finally, for a real closed field extension $R' \rightarrow R$ and for a semialgebraic subset $A \subset R^n$, we will write A_R for the extension of A to R .

In a few words, the philosophy about the “generalized fibre” is that something (expressible by a formula of the theory of real closed fields) holds in the fibre at α if and only if it holds over some S with $\alpha \in \widetilde{S}$.

An example on the philosophy about real spectrum is the following

PROPOSITION 1.1 ([BCR], 8.10.3). — Let $\alpha \in \widetilde{R^p}$, and let Ω be an open semialgebraic subset of $k(\alpha)^m$, and $\varphi : \Omega \rightarrow k(\alpha)$ a Nash function. There exist a Nash submanifold $M \subset R^p$, with $\alpha \in \widetilde{M}$, an open semialgebraic subset U of $R^m \times M$ and a semialgebraic family of functions $f : U \rightarrow R \times M$ parametrized by M , such that $U_\alpha = \Omega$, $f_\alpha = \varphi$ and f is a Nash mapping.

Following straightforward the proof of the proposition above, we can prove a useful extension of this result.

PROPOSITION 1.2. — Let $\alpha \in \widetilde{R^p}$, and let M and N be Nash manifolds over R . Let $\varphi : M_{k(\alpha)} \rightarrow N_{k(\alpha)}$ be a Nash mapping. There exist a Nash submanifold $S \subset R^p$, with $\alpha \in \widetilde{S}$, and a semialgebraic family of mappings $f : M \times S \rightarrow N \times S$ parametrized by S , such that $f_\alpha = \varphi$ and f is a Nash mapping.

We will use these results in the proof of our main theorem. In particular, we will use the next result in the key step of the proof:

PROPOSITION 1.3. — *Let $M \subset R^m$ and $N \subset R^n$ be Nash manifolds, $f : M \times R^p \rightarrow N \times R^p$ and $g : M \times R^p \rightarrow N \times R^p$ be Nash families of Nash mappings. Let $\alpha \in \widetilde{R^p}$. Assume that we have diffeomorphisms $\gamma : M_{k(\alpha)} \rightarrow M_{k(\alpha)}$ and $\sigma : N_{k(\alpha)} \rightarrow N_{k(\alpha)}$ such that the following commutes:*

$$\begin{array}{ccc} M_{k(\alpha)} & \xrightarrow{\gamma} & M_{k(\alpha)} \\ \downarrow g_\alpha & & \downarrow f_\alpha \\ N_{k(\alpha)} & \xrightarrow{\sigma} & N_{k(\alpha)} \end{array}$$

(we recall that $(M \times R^p)_\alpha$ is just $M_{k(\alpha)}$). Then there exist a Nash submanifold $S \subset R^p$, $\alpha \in \widetilde{S}$, and two families of Nash diffeomorphisms $\zeta : M \times S \rightarrow M \times S$ and $\eta : N \times S \rightarrow N \times S$ such that the following diagram commutes:

$$\begin{array}{ccc} M \times S & \xrightarrow{\zeta} & M \times S \\ \downarrow g & & \downarrow f \\ N \times S & \xrightarrow{\eta} & N \times S. \end{array}$$

Proof. — We consider the Nash diffeomorphism

$$\begin{aligned} H : M_{k(\alpha)} \times N_{k(\alpha)} &\rightarrow M_{k(\alpha)} \times N_{k(\alpha)} \\ (x, y) &\mapsto (\gamma(x), \sigma(y)). \end{aligned}$$

This diffeomorphism maps the graph of g_α to the graph of f_α . Using Proposition 1.2 for H , we can assume that there exist a Nash submanifold $S \subset R^p$, $\alpha \in \widetilde{S}$, and a family of Nash mappings

$$\begin{aligned} \Xi : M \times N \times S &\rightarrow M \times N \times S \\ (x, y, t) &\mapsto (\zeta(x, t), \eta(x, t), t) \end{aligned}$$

such that $\Xi_\alpha = H$. Using again Proposition 1.2, and perhaps shrinking S , we can assume that there exists a family of Nash mappings $\Xi^* : M \times N \times S \rightarrow M \times N \times S$ such that $\Xi^*_\alpha = H^{-1}$. We have that $\text{Id}_{M_{k(\alpha)} \times N_{k(\alpha)}} = H \circ H^{-1} = \Xi_\alpha \circ \Xi^*_\alpha = (\Xi \circ \Xi^*)_\alpha$ so, by [BCR], Remark 7.4.5, shrinking again S , we can assume that $\Xi|_S \circ \Xi^*_S = \text{Id}_{M \times N \times S}$. Now, in the same way, we can assume also that $\Xi^*_S \circ \Xi|_S = \text{Id}_{M \times N \times S}$, so $\Xi^* = \Xi^{-1}$ over S , and, in fact, $\Xi|_S$ is a family of diffeomorphisms. Finally, perhaps shrinking again S , we can assume that the mapping Ξ sends the graph of g to the graph of f , so it induces the diagram above. \square

2. A preliminary lemma.

In this section we prove a lemma that is crucial for the proof of our main theorem. This lemma is the following:

LEMMA 2.1. — *Let M, N be Nash manifolds, $f : M \rightarrow N$ and $g : N \rightarrow R^l$ proper Nash submersions. Let $P \subset R^l$ be a Nash submanifold such that $(f|_{(g \circ f)^{-1}(P)}, g|_{g^{-1}(P)})$ is Nash trivial. Then, we can extend this Nash triviality to a neighbourhood of P , i.e., there exists an open neighbourhood U of P such that $(f|_{(g \circ f)^{-1}(U)}, g|_{g^{-1}(U)})$ is Nash trivial.*

Before proceeding with the proof of this result, we need three preliminary lemmas. We first state a very simple but useful topological fact:

LEMMA 2.2. — *Let N and T be semialgebraic sets and $h : N \rightarrow T$ a proper semialgebraic mapping. Let $P \subset T$ be a semialgebraic set and V an open semialgebraic subset of N such that $h^{-1}(P) \subset V$. Then, there exists an open semialgebraic subset $U \supset P$ such that $V \supset h^{-1}(U)$.*

Proof. — Take $U = T \setminus h(N \setminus V)$, which is open because h is closed. \square

Next lemma is a semialgebraic version of a known result on differential topology.

LEMMA 2.3. — *Let $f : M \rightarrow N$ be a proper surjective semialgebraic mapping between two semialgebraic sets M and N over R . Let $P \subset M$ be a closed semialgebraic subset and assume that f is a local homeomorphism for each $x \in P$ and $f|_P : P \rightarrow f(P)$ is injective. Then, there exists an open semialgebraic neighbourhood W of P in M such that $f|_W : W \rightarrow f(W)$ is a homeomorphism.*

Proof. — Let Ω be the set of points $x \in M$ such that f is a local homeomorphism at x . Since to be a semialgebraic local homeomorphism can be expressed with a first-order formula in the language of real closed fields, Ω is semialgebraic. Obviously, Ω is open and $P \subset \Omega$. Let us consider the semialgebraic set $M \times_N M := \{(x, y) \in M \times M : f(x) = f(y)\}$ and the following diagram:

$$\begin{array}{ccc}
 M \times_N M & \xrightarrow{\pi_2} & M \\
 \downarrow \pi_1 & & \downarrow f \\
 M & \xrightarrow{f} & N.
 \end{array}$$

Consider now the semialgebraic set $D = \{(w, w) : w \in \Omega\} \subset M \times_N M$. The set D is open, because f is a local homeomorphism in Ω , and $P \times_N P \subset D$. The restriction $f|_P : P \rightarrow N$ is proper since P is closed. The projection onto the first factor $\Omega \times_N P \rightarrow \Omega$ is also proper, because it is the pull-back of $f|_P$ by the diagram above. (See [DK] for properties on semialgebraic proper maps). By Lemma 2.2 there exists a semialgebraic neighbourhood $V_1 \supset P$ such that $P \times_N P \subset V_1 \times_N P \subset D$. We can assume that V_1 is closed and, applying the same argument for the proper projection $V_1 \times_N \Omega \rightarrow \Omega$, we obtain an open semialgebraic neighbourhood V_2 of P in Ω such that $V_1 \times_N V_2 \subset D$. Finally, we just need to take an open semialgebraic neighbourhood W of P such that $W \subset V_1 \cap V_2$. \square

Finally, using an standard argument (see [CS2], Lemma 4) and above Lemma 2.3, we can proof the following result on compatible tubular neighbourhoods.

LEMMA 2.4. — *Let $Y \subset R^n$, $Z \subset R^m$ be Nash manifolds and $F : Y \rightarrow Z$ a Nash mapping. Let us consider a Nash submanifold $X \subset Y$ such that $F|_X$ is a submersion. Then, there exists a Nash tubular neighbourhood U of X in Y , with a submersive Nash retraction $\tau : U \rightarrow X$ such that $F \circ \tau = F$ on U .*

Proof of Lemma 2.1. — Since the pair $(f|_{(g \circ f)^{-1}(P)}, g|_{g^{-1}(P)})$ is Nash trivial we have diffeomorphisms

$$\begin{aligned} \theta &= (\theta_0, g \circ f) : (g \circ f)^{-1}(P) \rightarrow (g \circ f)^{-1}(p) \times P \\ \gamma &= (\gamma_0, g) : g^{-1}(P) \rightarrow g^{-1}(p) \times P \end{aligned}$$

for a certain $p \in P$, such that the following diagram is commutative:

$$\begin{array}{ccccc} (g \circ f)^{-1}(P) & \xrightarrow{f} & g^{-1}(P) & \xrightarrow{g} & P \\ \downarrow \theta & & \downarrow \gamma & & \downarrow \text{id} \\ (g \circ f)^{-1}(p) \times P & \xrightarrow{f \times \text{id}} & g^{-1}(p) \times P & \xrightarrow{\pi} & P. \end{array}$$

Let U be a tubular neighborhood of P and $\tau : U \rightarrow P$ a Nash retraction. We want to extend the diffeomorphisms θ and γ to $\tilde{\theta} : (g \circ f)^{-1}(U) \rightarrow (g \circ f)^{-1}(p) \times U$ and $\tilde{\gamma} : g^{-1}(U) \rightarrow g^{-1}(p) \times U$ so that the corresponding diagram commutes.

We consider the Nash submanifold $g^{-1}(P) \subset N$. Using Lemma 2.4 with $Y = g^{-1}(U)$, $Z = P$, $X = g^{-1}(P)$ and $F = \tau \circ g : g^{-1}(U) \rightarrow P$, and perhaps shrinking U (we can do it by Lemma 2.2), we obtain a submersive retraction $\tilde{\tau} : g^{-1}(U) \rightarrow g^{-1}(P)$ such that $g \circ \tilde{\tau} = \tau \circ g$. We can define now

the Nash mapping

$$\begin{aligned} \tilde{\gamma} : g^{-1}(U) &\rightarrow g^{-1}(p) \times U \\ x &\mapsto (\gamma_0 \tilde{\tau}(x), g(x)). \end{aligned}$$

By Lemma 2.3 applied to $\tilde{\gamma}$, there exists an open neighbourhood W of $g^{-1}(P)$ in $g^{-1}(U)$ such that $\tilde{\gamma} : W \rightarrow \tilde{\gamma}(W)$ is a diffeomorphism. The map $g : g^{-1}(U) \rightarrow U$ is proper hence, by Lemma 2.2, there exists an open neighbourhood U^* of P in U such that $g^{-1}(U^*) \subset W$. So we have the diffeomorphism $\tilde{\gamma} : g^{-1}(U^*) \rightarrow \tilde{\gamma}(g^{-1}(U^*))$. But $\tilde{\gamma}(g^{-1}(U^*)) \subset g^{-1}(p) \times U^*$, hence applying again Lemma 2.2 to the proper map $\pi : g^{-1}(p) \times U^* \rightarrow U^*$ ($g^{-1}(p)$ is compact), we obtain an open neighbourhood U^{**} of P in U^* such that $g^{-1}(p) \times U^{**} \subset \tilde{\gamma}(g^{-1}(U^*))$. So $\tilde{\gamma} : g^{-1}(U^{**}) \rightarrow g^{-1}(p) \times U^{**}$ is a diffeomorphism. Now, renaming U^{**} as U we have that $\tilde{\gamma}$ is a diffeomorphism and that for all $x \in g^{-1}(U)$, $\pi \circ \tilde{\gamma}(x) = g(x)$, hence the corresponding diagram is commutative.

In the same way, using Lemma 2.4 with $Y = (g \circ f)^{-1}(U)$, $X = (g \circ f)^{-1}(P)$, $Z = g^{-1}(P)$, $F = \tilde{\tau} \circ f$ and noting that $F|_X = f|_{(g \circ f)^{-1}(P)}$ is Nash trivial, we can construct a submersive retraction $\tilde{\sigma} : (g \circ f)^{-1}(U) \rightarrow (g \circ f)^{-1}(P)$ such that $f \circ \tilde{\sigma} = \tilde{\tau} \circ f$, and, similarly, we can define

$$\begin{aligned} \tilde{\theta} : (g \circ f)^{-1}(U) &\rightarrow (g \circ f)^{-1}(p) \times U \\ x &\mapsto (\theta_0 \tilde{\sigma}(x), (g \circ f)(x)). \end{aligned}$$

We obtain then that $\tilde{\theta}$ is a diffeomorphism that extends our trivialization to U . □

To finish this section, we state a useful corollary of Lemma 2.4. We can set in a natural way ([Shi1]) the notions of C^r Nash function and C^r Nash manifold. We can also define a topology in the space of C^r Nash mappings between C^r Nash manifolds. We will call this topology simply C^r topology. See [Shi1] for the definition and properties of this topology.

COROLLARY 1. — *Let $f : M \rightarrow N$ be a Nash (resp. C^r Nash, $r > 0$) submersion between Nash (resp. C^r Nash) submanifolds M and N . Let $\tilde{f} : M \rightarrow N$ be a Nash (resp. C^r Nash) mapping sufficiently close to f (for the C^1 topology). Then, there exists a Nash (resp. C^r Nash) diffeomorphism $\tau : M \rightarrow M$ such that $\tilde{f} = f \circ \tau$.*

Proof. — For the Nash case, let us consider $\text{graph}(f) \subset M \times N$. The natural projection $\pi : \text{graph}(f) \rightarrow N$ is a submersion because f is a submersion. Hence, by Lemma 2.4, there exists an open neighbourhood U

of $\text{graph}(f)$ in $M \times N$ and a submersive retraction $\sigma : U \rightarrow \text{graph}(f)$ such that $\pi \circ \sigma = \pi$ on U . If we define $\tau(x)$ as $\sigma_1(x, \tilde{f}(x))$, by definition we have that $\sigma_2(x, \tilde{f}(x)) = f(\tau(x))$. And, if \tilde{f} is close enough to f , then τ is close enough to the identity, so we can assume that τ is a diffeomorphism. We can use this proof of the C^r Nash case, as it is easy to prove a C^r Nash version of Lemma 2.4. (See [CS2], Lemma 4). □

3. A model for a Nash proper submersion.

In this section, given a Nash proper submersion $f : M \rightarrow N$, we find a Nash model for f over a real closed subfield of R . We construct this model over the elements of a semialgebraic open covering of N . The first lemma will allow us to glue local models in order to obtain a global one.

LEMMA 3.1. — *Let Λ, V and N be C^r Nash manifolds over R ($0 < r < \infty$) and Ω and U open sets in N such that $\Omega \cup U = N$. Let $f : \Lambda \rightarrow \Omega, g : V \rightarrow U$ be proper C^r Nash submersions, and let us assume that there exists a C^r Nash diffeomorphism $\psi : f^{-1}(\Omega \cap U) \rightarrow g^{-1}(\Omega \cap U)$ such that $f|_{f^{-1}(\Omega \cap U)} = g|_{g^{-1}(\Omega \cap U)} \circ \psi$. Then there exist a C^r Nash manifold M , embeddings $i : \Lambda \rightarrow M, j : V \rightarrow M$ and a proper C^r Nash submersion $F : M \rightarrow N$ such that $j \circ \psi = i$ and the following diagram is commutative:*

$$\begin{array}{ccccc}
 \Lambda & \xrightarrow{i} & M & \longleftarrow & V \\
 & f \searrow & F \downarrow & \swarrow & g \\
 & & N & &
 \end{array}$$

Proof. — We may assume that $\Lambda \subset R^n$ and $V \subset R^m$. Let $\{h, 1-h\}$ be a C^r Nash partition of unity subordinated to $\{\Omega, U\}$, so that $\{h \neq 0\} \cap \{h \neq 1\} \subset \Omega \cap U$.

We consider

$$M_1 = \{(h(f(x))x, (1 - h(f(x)))\psi(x), f(x)) : x \in \Lambda\},$$

$$M_2 = \{(h(g(y))\psi^{-1}(y), (1 - h(g(y)))y, g(y)) : y \in V\},$$

$M = M_1 \cup M_2 \subset R^n \times R^m \times N$ and the projection $F = (\pi_3)|_M : M \rightarrow N$.

It is clear that M is a semialgebraic set and F is a semialgebraic map. We also have that $F^{-1}(\Omega) = M_1$ is diffeomorphic to Λ . In fact the mapping $i : \Lambda \rightarrow M_1 : x \mapsto (h(f(x))x, (1 - h(f(x)))\psi(x), f(x))$ is a diffeomorphism.

Similarly, $F^{-1}(U) = M_2$, and M_2 is C^r Nash diffeomorphic to V . Then, M is a C^r Nash manifold and F is a proper C^r Nash submersion. \square

Now we prove an easy but useful lemma:

LEMMA 3.2. — *Let N be a Nash manifold over a real closed field R . Then we can find a finite covering of N by open semialgebraic subsets which are Nash diffeomorphic to affine space.*

Proof. — By [BCR], Th. 9.1.4, there exists a Nash stratification $\{S_i\}$ of N such that S_i is Nash diffeomorphic to $R^{\dim S_i}$ for each i . Consider a stratum S_i and assume that $\dim S_i < s = \dim N$. Let T_i be an open tubular neighbourhood of S_i in N and $\tau_i : T_i \rightarrow S_i$ a Nash retraction. By the construction of the tubular neighbourhood ([BCR], 8.9.3) using the normal bundle to S_i , and by the Nash triviality of this bundle (see [BCR], 12.7.7 and 12.7.15), we can assume that $\tau_i : T_i \rightarrow S_i$ is Nash trivial over S_i . That is, we can assume that there exists an open ball B_i of radius ϵ_i in $R^{s-\dim S_i}$ such that T_i is Nash diffeomorphic to $S_i \times B_i$. But then it is clear that T_i is Nash diffeomorphic to $R^{\dim S_i} \times R^{s-\dim S_i} = R^s$. So, if we define U_i to be T_i if $\dim S_i < s$ and S_i otherwise, we can assume that $N = \bigcup_{i=1}^k U_i$, where U_i is open and diffeomorphic to R^s for all i . \square

LEMMA 3.3. — *Let $R' \rightarrow R$ be a real closed field extension. Let A and B be Nash manifolds over R' , and $\delta : A_R \rightarrow B_R$ be a Nash mapping satisfying certain (finitely many) conditions which can be formulated as first-order formulas of the language of ordered fields with parameters in R' . Then there exists a Nash mapping $\delta' : A \rightarrow B$ (over R') satisfying the same conditions. Moreover, we can choose δ' such that there exists a Nash mapping $\Delta : A_R \times [0, 1]_R \rightarrow B_R$ such that $\Delta_0 = \delta'_R$, $\Delta_1 = \delta$ and, for every $t \in [0, 1]_R$, Δ_t satisfies the same first-order conditions.*

Proof. — We consider the graph of δ . We can write

$$\text{graph}(\delta) = \bigcup_{i=1}^d \{x \in R^N : f_i(a, x) = 0, g_{i1}(a, x) > 0, \dots, g_{iu_i}(a, x) > 0\}$$

where $f_i, g_{ij} \in \mathbb{Z}[Y, X]$, for certain $a \in R^q$. Let a' be an element in R'^q . We can consider the semialgebraic set

$$X(a') = \bigcup_{i=1}^d \{x \in R'^N : f_i(a', x) = 0, g_{i1}(a', x) > 0, \dots, g_{iu_i}(a', x) > 0\}$$

and let B' be the set of points $a' \in R'^q$ such that $X(a')$ is the graph of a Nash mapping between A and B satisfying the above first-order conditions. The condition for a semialgebraic set in R'^N to be the graph of a semialgebraic mapping between A and B can be expressed with a first order formula in the language of real closed fields. From [R], Prop. 3.5, we obtain the following result:

Given a semialgebraic set S and a semialgebraic family $\{f_t : A \rightarrow B\}_{t \in S}$, the set $\{t \in S : f_t \text{ is Nash}\}$ is semialgebraic.

Hence, the subset B' is semialgebraic and $a \in B'_R$. Now, we can choose a Nash stratification $\{S_i\}_{i=1, \dots, \nu}$ of B' , where each S_i is a connected Nash manifold. Then $\{S_{iR}\}_{i=1, \dots, \nu}$ is a Nash stratification of B'_R and we can assume that a is a point in the connected Nash manifold S_{1R} . We observe that $S_1 \neq \emptyset$ because $S_{1R} \neq \emptyset$ ([BCR], 4.1.1). Then we can consider the semialgebraic family $\Theta : A \times S_1 \rightarrow B \times S_1$ such that for each $t \in S_1$, Θ_t is the Nash mapping corresponding to $X(t)$. Now, by Proposition 8.10.1 in [BCR] and Proposition 1.1 we can assume that S_1 is a finite union $\bigcup_{i=1}^k M_i$ of Nash submanifolds such that $\Theta : A \times M_i \rightarrow B \times M_i$ is Nash (also with respect to the second parameter) for $i = 1, \dots, k$ (see [BCR], 8.10.4). Moreover, substratifying if necessary, we can assume that M_i is Nash diffeomorphic to $R'^{\dim M_i}$ for each $i = 1, \dots, k$. Again we can assume that $a \in M_{1R}$ and $M_{1R} \neq \emptyset$. We can choose $a' \in M_1$ so that $X(a')$ is the graph of a Nash mapping $\delta' : A \rightarrow B$ defined over R' , satisfying the first-order conditions. As M_{1R} is a Nash submanifold diffeomorphic to $R^{\dim M_1}$ we can choose a Nash path between a and a' in M_{1R} . This means that there exists a “Nash path” between δ'_R and δ , i.e., there exists a Nash family

$$\Delta : A_R \times [0, 1]_R \rightarrow B_R \times [0, 1]_R$$

such that $\Delta_0 = \delta'_R$, $\Delta_1 = \delta$ and Δ_t is a Nash mapping satisfying the above first-order conditions with parameters in R' , for each $t \in [0, 1]_R$. □

After this preliminary lemmas, we state the main result of this section.

PROPOSITION 3.4 (Semialgebraic models). — *Let $R' \rightarrow R$ be a real closed field extension. Let $f : M \rightarrow N$ be a proper Nash submersion between the Nash submanifolds M and N of R^n . Then, there exist two Nash submanifolds M' and N' of R'^n , a proper Nash submersion $f' : M' \rightarrow N'$ and diffeomorphisms $\alpha : N'_R \rightarrow N$, $\beta : M'_R \rightarrow M$, such that $f \circ \beta = \alpha \circ f'_R$.*

Proof. — The idea of the proof is to build “local” models for f over “big” open semialgebraic subsets of N , that is, over a finite open

semialgebraic covering of N . Then, by a rather technical argument, we glue all these local models to obtain a global one.

By [CS1], there exist a Nash submanifold N' , defined over R' , and a Nash diffeomorphism $\alpha : N'_R \rightarrow N$. By Lemma 3.2, we can assume that $N' = \bigcup_{i=1}^k U'_i$, where U'_i is semialgebraic open and Nash diffeomorphic to R'^s for all i . Let us consider now the open subsets $U_i = \alpha((U'_i)_R)$. Then, $N = \bigcup_{i=1}^k U_i$, and U_i is diffeomorphic to R^s for all i .

We have that $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \rightarrow U_i$ is a Nash proper submersion. So, by the semialgebraic version of the first isotopy lemma [CS2], there exist a compact Nash manifold F_i and a Nash diffeomorphism $\beta_i : f^{-1}(U_i) \rightarrow F_i \times U_i$ such that $f = \pi_i \circ \beta_i$, where π_i is the projection $F_i \times U_i \rightarrow U_i$.

For each i , there exists a compact Nash manifold F'_i over R' whose extension to R is diffeomorphic to F_i , that is, we have a diffeomorphism $e_i : F'_{iR} \rightarrow F_i$. Now we are going to “glue” the sets $\{F'_i \times U'_i\}_{i=1, \dots, k}$ to build a manifold M' and a proper Nash submersion $f' : M' \rightarrow N'$, defined over R' . We proceed by induction on k .

For $k = 1$, there is nothing to prove. We consider the case $k > 1$. By induction hypothesis, we can assume that there exist two open subsets $\Omega, U \subset N$ such that $N = \Omega \cup U$, and that there exist models Λ' and V' of $\Lambda = f^{-1}(\Omega)$ and $V = f^{-1}(U)$, respectively, and Nash diffeomorphisms $\psi_1 : \Lambda'_R \rightarrow \Lambda$ and $\psi_2 : V'_R \rightarrow V$. We can also assume that there exist open semialgebraic subsets $\Omega', U' \subset N'$ such that $\alpha(\Omega'_R) = \Omega$, $\alpha(U'_R) = U$, and proper Nash submersions $\phi^1 : \Lambda' \rightarrow \Omega'$ and $\phi^2 : V' \rightarrow U'$ such that the following commutative diagrams hold:

$$\begin{array}{ccc}
 \Lambda'_R & \xrightarrow{\psi_1} & \Lambda \\
 \downarrow \phi^1_R & & \downarrow f \\
 \Omega'_R & \xrightarrow{\alpha} & \Omega,
 \end{array}
 \qquad
 \begin{array}{ccc}
 V'_R & \xrightarrow{\psi_2} & V \\
 \downarrow \phi^2_R & & \downarrow f \\
 U'_R & \xrightarrow{\alpha} & U.
 \end{array}$$

Note that $\psi_1((\phi^1)^{-1}(\Omega' \cap U')_R) = \Lambda \cap V = \psi_2((\phi^2)^{-1}(\Omega' \cap U')_R)$.

We consider the diffeomorphism

$$\delta = (\psi_2)^{-1} \circ \psi_1 : (\phi_1)^{-1}(U' \cap \Omega')_R \rightarrow (\phi_2)^{-1}(U' \cap \Omega')_R$$

over $(U' \cap \Omega')_R$ compatible with ϕ^1_R and ϕ^2_R , that is, $\phi^2_R \circ \delta = \phi^1_R$ over $(U' \cap \Omega')_R$. By Lemma 3.3 there exist a Nash diffeomorphism

$$\delta' : (\phi_1)^{-1}(U' \cap \Omega')_R \rightarrow (\phi_2)^{-1}(U' \cap \Omega')_R$$

defined over R' and compatible with ϕ^1 and ϕ^2 , and a Nash family of Nash diffeomorphisms

$$\Delta : (\phi_1)^{-1}(U' \cap \Omega')_R \times [0, 1]_R \rightarrow (\phi_2)^{-1}(U' \cap \Omega')_R \times [0, 1]_R$$

such that $\Delta_0 = \delta'_R$, $\Delta_1 = \delta$ and Δ_t is compatible with ϕ_R^1 and ϕ_R^2 for each $t \in [0, 1]_R$.

So, using Lemma 3.1 for δ' , we have a C^r Nash manifold M' over R' and a proper C^r Nash submersion $f' : M' \rightarrow N'$ ($0 < r < \infty$). We claim that M'_R is C^r Nash diffeomorphic to M . By the proof of Lemma 3.1, M'_R is obtained by “glueing” Λ'_R and V'_R via the diffeomorphism δ'_R . This means that $M'_R = (M'_1)_R \cup (M'_2)_R$ where $(M'_1)_R$ and $(M'_2)_R$ are defined as in Lemma 3.1 for ϕ_R^1 , ϕ_R^2 , δ'_R and h_R , and $\{h, 1 - h\}$ is a C^r Nash partition of unity subordinated to $\{\Omega', U'\}$. Now, let M^* be the manifold obtained by glueing Λ'_R and V'_R via the diffeomorphism $\delta : (\phi_1)^{-1}(U' \cap \Omega')_R \rightarrow (\phi_2)^{-1}(U' \cap \Omega')_R$. It is not difficult to show that M'_R is diffeomorphic to M^* .

Let us see now that $M^* \simeq M$. We have that $M = \Lambda \cup V$, and $M^* = M^*_1 \cup M^*_2$, where M^*_1 and M^*_2 are defined in the same way as above. We define the mapping

$$\begin{aligned} \Lambda &\xrightarrow{\overline{\gamma}_1} M^*_1 \\ x &\longmapsto (h_R(\phi_R^1(p))p, (1 - h_R(\phi_R^1(p))\delta(p), \phi_R^1(p)) \end{aligned}$$

where $p = \psi_1^{-1}(x)$. This is clearly a Nash diffeomorphism. On the other hand, we have

$$\begin{aligned} V &\xrightarrow{\overline{\gamma}_2} M^*_2 \\ x &\longmapsto (h_R(\phi_R^2(q))\delta^{-1}(q), (1 - h_R(\phi_R^2(q))q, \phi_R^2(q)) \end{aligned}$$

where $q = \psi_2^{-1}(x)$. This is also a Nash diffeomorphism. It is easy to see that $\overline{\gamma}_1 = \overline{\gamma}_2$ over $\Lambda \cap V$.

Hence we can define a diffeomorphism $\overline{\gamma} : M \rightarrow M^*$.

Finally, composing both diffeomorphisms, we obtain a C^r Nash diffeomorphism $\beta : M'_R \rightarrow M$. Due to the compatibility conditions that are verified by the above diffeomorphisms, we obtain our result in the C^r Nash category.

To finish the proof, we just need to use the approximation theorem in the C^r Nash category ([Shi1]), to obtain the result in the Nash category. We have that M' is a C^r Nash manifold, so there exist a Nash manifold M'' defined over R' and a C^r Nash diffeomorphism $\zeta : M'' \rightarrow M'$. Then, replacing β by $\beta \circ \zeta_R$ and f' by $f' \circ \zeta$, we can assume that M' is a Nash manifold. Let $\tilde{f}' : M' \rightarrow N'$ a Nash approximation of f' . If the approximation is close enough, we can assume that \tilde{f}' is a proper Nash submersion and, by Corollary 2.5, we can assume that there exists a C^r

Nash diffeomorphism $\tau' : M' \rightarrow M'$ such that $\tilde{f}' = f' \circ \tau'$. Hence, replacing f' with $\tilde{f}' = f' \circ \tau'$ and β with $\beta \circ \tau'_R$, we can assume that f' is a Nash mapping. Finally, we can approximate the C^r Nash diffeomorphism $\beta : M'_R \rightarrow M$ by a Nash mapping $\tilde{\beta}$. If the approximation is close enough, we can assume that $\tilde{\beta}$ is in fact a Nash diffeomorphism. Since $\tilde{\beta}$ is a close enough Nash approximation of β , then $\alpha \circ f'_R \circ \tilde{\beta}^{-1}$ is a close enough Nash approximation of f . Hence, by Corollary 2.5, there exists a Nash diffeomorphism $\tau : M \rightarrow M$ such that $\alpha \circ f'_R \circ \tilde{\beta}^{-1} = f \circ \tau$. Replacing then β with $\tau \circ \tilde{\beta}$ we have the result in the Nash category. \square

Finally, the above proposition allows us to prove a “Hardt’s type” theorem.

THEOREM 3.5. — *Let $B \subset R^p$, $X \subset R^n \times B$ and $Y \subset R^m \times B$ be semialgebraic sets, and let $f : X \rightarrow Y$ be a semialgebraic family of semialgebraic mappings. We assume that $f_t : X_t \rightarrow Y_t$ is a proper Nash submersion between Nash manifolds for each $t \in B$. Then, we can stratify B in a disjoint union of Nash manifolds, say $B = S_1 \sqcup \dots \sqcup S_r$, such that, for each S_i , we can find a Nash trivialization of f over S_i of the form*

$$\begin{array}{ccc} X_{t_i} \times S_i & \xrightarrow{\sim} & X|_{S_i} \\ \downarrow f_{t_i} \times \text{id} & & \downarrow f|_{S_i} \\ Y_{t_i} \times S_i & \xrightarrow{\sim} & Y|_{S_i}, \end{array}$$

for certain $t_i \in S_i$.

Proof. — Let us take a point α in the real spectrum \tilde{B} of B . Then the fibre $f_\alpha : X_\alpha \rightarrow Y_\alpha$ is a proper Nash submersion over $k(\alpha)$. By Proposition 3.4, we have a Nash submersion $f' : X' \rightarrow Y'$ between Nash manifolds X', Y' , defined over R , and diffeomorphisms γ and σ such that the following diagram commutes:

$$\begin{array}{ccc} X'_{k(\alpha)} & \xrightarrow{\gamma} & X_\alpha \\ \downarrow f'_{k(\alpha)} & & \downarrow f_\alpha \\ Y'_{k(\alpha)} & \xrightarrow{\sigma} & Y_\alpha. \end{array}$$

But, since $f'_{k(\alpha)}$ is the fibre $(f' \times \text{id})_\alpha$ of the constant family $f' \times \text{id} : X' \times B \rightarrow Y' \times B$, by Proposition 1.3, there exists a Nash manifold $S \subset R$, $\alpha \in \tilde{S}$, and diffeomorphisms ζ and η , compatible with the projections, such that the following diagram commutes:

$$\begin{array}{ccc}
 X' \times S & \xrightarrow{\zeta} & X|_S \\
 \downarrow f' \times \text{id} & & \downarrow f \\
 Y' \times S & \xrightarrow{\eta} & Y|_S.
 \end{array}$$

Then, by the compactness property of \tilde{B} , and perhaps substratifying, we can assume that there exists a finite stratification $\{S_i\}$ of B such that a diagram like the one above holds for each S_i . □

4. Proof of the main theorem.

The map $p : N \rightarrow R^l$ is a proper submersion, so, by the semialgebraic version of Thom’s first isotopy lemma [CS2], there exist a compact Nash manifold F and a Nash diffeomorphism $\psi = (\psi_0, g) : N \rightarrow F \times R^l$. Similarly, $g \circ f : M \rightarrow R^l$ is a proper Nash submersion, so there exist a compact Nash manifold G and a Nash diffeomorphism $\varphi = (\varphi_0, g \circ f) : M \rightarrow G \times R^l$. We can consider then the following diagram:

$$\begin{array}{ccccccc}
 M & \xrightarrow{f} & N & \xrightarrow{g} & R^l \\
 \downarrow \varphi & & \downarrow \psi & & \downarrow \text{id} \\
 G \times R^l & \xrightarrow{f'} & F \times R^l & \xrightarrow{\pi_1} & R^l,
 \end{array}$$

where $f' = \psi f \varphi^{-1}$ ($\pi_1 : F \times R^l \rightarrow R^l$ is the projection). We have that $\pi_1 f'(x, t) = t$. Hence, we can assume that M, N are compact and $f : M \times R^l \rightarrow N \times R^l$ is a proper submersion of the form $f(x, t) = (f_t(x), t)$. In other words, we have a Nash family of Nash submersions $\{f_t : M \rightarrow N\}_{t \in R^l}$ and we can forget about g . We argue by induction on l .

First, we consider the case $l = 1$. By Theorem 3.5 there exists a finite Nash stratification $\{S_i\}$ of R such that $f|_{M \times S_i}$ is Nash trivial for each i . We can assume that the S_i are open intervals or singletons, and by Lemma 2.1 we can also replace the latter by open intervals. Thus we have trivializations of the family f over a covering of R by open intervals, and we must glue all of them.

For example, let us glue a trivialization over the interval $(0, 2)$ with a trivialization over $(-1, 1)$. Namely

$$\begin{array}{ccccc}
 M \times (-1, 1) & \xrightarrow{\bar{\zeta}_1} & M \times (-1, 1) & & M \times (0, 2) & \xrightarrow{\bar{\zeta}_2} & M \times (0, 2) \\
 \downarrow f'_1 \times \text{id} & & \downarrow f & & \downarrow f'_2 \times \text{id} & & \downarrow f \\
 N \times (-1, 1) & \xrightarrow{\bar{\eta}_1} & N \times (-1, 1), & & N \times (0, 2) & \xrightarrow{\bar{\eta}_2} & N \times (0, 2),
 \end{array}$$

where $f'_1, f'_2 : M \rightarrow N$ are proper Nash submersions, $\bar{\zeta}_i$ is a Nash diffeomorphism of the form $\bar{\zeta}_i(x, t) = (\zeta_i(x, t), t)$ and $\bar{\eta}_i$ is a Nash diffeomorphism of the form $\bar{\eta}_i(x, t) = (\eta_i(x, t), t)$ for $i = 1, 2$.

Let us consider $t_0 = 1/2$. Then

$$\eta_{1,t_0} \circ f'_1 \circ (\zeta_{1,t_0})^{-1} = f_{t_0} = \eta_{2,t_0} \circ f'_2 \circ (\zeta_{2,t_0})^{-1},$$

hence

$$f'_2 = (\eta_{2,t_0})^{-1} \circ \eta_{1,t_0} \circ f'_1 \circ (\zeta_{1,t_0})^{-1} \circ \zeta_{2,t_0}.$$

Then, if we replace $\bar{\zeta}_2$ by $\bar{\zeta}_2 \circ ((\zeta_{2,t_0})^{-1} \circ \zeta_{1,t_0} \times \text{id})$ and $\bar{\eta}_2$ by $\bar{\eta}_2 \circ ((\eta_{2,t_0})^{-1} \circ \eta_{1,t_0} \times \text{id})$, we can assume that $f'_1 = f'_2$. We will write f' instead of f_i , $i = 1, 2$.

Over the interval $(0, 1)$ we have the following commutative diagram:

$$\begin{array}{ccccc} M \times (0, 1) & \xrightarrow{\bar{\zeta}_1} & M \times (0, 1) & \xleftarrow{\bar{\zeta}_2} & M \times (0, 1) \\ \downarrow f' \times \text{id} & & \downarrow f & & \downarrow f' \times \text{id} \\ N \times (0, 1) & \xrightarrow{\bar{\eta}_1} & N \times (0, 1) & \xleftarrow{\bar{\eta}_2} & N \times (0, 1), \end{array}$$

so $(\eta_1^{-1} \eta_2)_t f' = f'(\zeta_1^{-1} \zeta_2)_t$.

We consider a C^1 Nash function $u : R \rightarrow R$, such that $u(t) = \frac{1}{2}$ if $t \leq 1/4$ and $u(t) = t$ if $t \geq 3/4$. We define then the C^1 Nash diffeomorphisms $\bar{\phi} = (\phi_t(x), t)$ as $\phi_t(x) = \zeta_{1,u(t)}^{-1} \circ \zeta_{2,u(t)}$ and $\bar{\psi} = (\psi_t(x), t)$ as $\psi_t(x) = \eta_{1,u(t)}^{-1} \circ \eta_{2,u(t)}$. We observe that $\psi_t \circ f' = f' \circ \phi_t$ for each $t \in (0, 1)$. Then we can consider the Nash diffeomorphism over $(0, 1)$ $\bar{\zeta}^*_2 = \bar{\zeta}_1 \circ \bar{\phi}$ and $\bar{\eta}^*_2 = \bar{\eta}_1 \circ \bar{\psi}$. This new diffeomorphism makes the corresponding diagrams commutative. Over $(3/4, 1)$ we have that $\bar{\zeta}^*_2 = \bar{\zeta}_2$ and $\bar{\eta}^*_2 = \bar{\eta}_2$. Over $(0, 1/4)$, $\bar{\zeta}^*_2 = \bar{\zeta}_1 \circ (\mu \times \text{id})$ and $\bar{\eta}^*_2 = \bar{\eta}_1 \circ (\nu \times \text{id})$ where $\mu = \zeta_{1,1/2}^{-1} \circ \zeta_{2,1/2} : M \rightarrow M$ and $\nu = \eta_{1,1/2}^{-1} \circ \eta_{2,1/2} : N \rightarrow N$ are Nash diffeomorphisms such that $f' \circ \mu = \nu \circ f'$. This means that we can glue ζ_1 and ζ_2 to obtain a Nash diffeomorphism $\bar{\zeta}^*(x, t) = (\zeta^*_t(x), t)$ defined by

$$\zeta^*_t = \begin{cases} \zeta_{1,t} \circ \mu, & -1 < t \leq 1/4 \\ \zeta_{2,t}, & 1/4 \leq t \leq 1 \\ \zeta_{2,t}, & 3/4 \leq t < 2, \end{cases}$$

In a similar way, we glue η_1 and η_2 to obtain a Nash diffeomorphism $\bar{\eta}^*(x, t) = (\eta^*_t(x), t)$ such that the diagram

$$\begin{array}{ccc} M \times (-1, 2) & \xrightarrow{\bar{\zeta}^*} & M \times (-1, 2) \\ \downarrow f' \times \text{Id} & & \downarrow f \\ N \times (-1, 2) & \xrightarrow{\bar{\eta}^*} & N \times (-1, 2) \end{array}$$

is commutative. Hence, we obtain a global C^1 Nash trivialization over $(-1, 2)$.

Now we can approximate ζ^* and η^* by Nash mappings $\zeta : M \times (-1, 2) \rightarrow M$ and $\eta : N \times (-1, 2) \rightarrow N$ respectively. If the approximations are close enough, we can assume that the Nash mapping $\bar{\zeta} : M \times (-1, 2) \rightarrow M \times (-1, 2)$ (resp. $\bar{\eta} : N \times (-1, 2) \rightarrow N \times (-1, 2)$) defined as $\bar{\zeta}(x, t) = (\zeta(x, t), t)$ (resp. $\bar{\eta}(x, t) = (\eta(x, t), t)$) is a Nash diffeomorphism. Moreover, we can assume that $\bar{\eta} \circ (f' \times \text{Id}) \circ \bar{\zeta}^{-1}$ is a close Nash approximation to f . Then, by Corollary 2.5, there exists a Nash diffeomorphism $\tau : M \times (-1, 2) \rightarrow M \times (-1, 2)$ such that $\bar{\eta} \circ (f' \times \text{Id}) \circ \bar{\zeta}^{-1} = f \circ \tau$. This last condition in fact implies that τ is a diffeomorphism over $(-1, 2)$. Hence, replacing $\bar{\eta}^*$ with $\bar{\eta}$ and $\bar{\zeta}^*$ with $\tau \circ \bar{\zeta}$ we get the result in the Nash category. (Observe that ζ and η are just θ^{-1} and γ^{-1} , respectively, in the statement of the theorem.)

If $l > 1$, we see f as a parametrized family over R :

$$f : (M \times R^{l-1}) \times R \rightarrow (N \times R^{l-1}) \times R$$

$$(x, s, t) \mapsto (f_t(x, s), t) = (f_{(s,t)}(x), s, t).$$

Let $\alpha \in \tilde{R}$ and let us consider the fibre $f_\alpha : M_{k(\alpha)} \times k(\alpha)^{l-1} \rightarrow N_{k(\alpha)} \times k(\alpha)^{l-1}$. By the induction hypothesis, there exists a Nash trivialization

$$\begin{array}{ccc} M_{k(\alpha)} \times k(\alpha)^{l-1} & \xrightarrow{\zeta^\alpha} & M_{k(\alpha)} \times k(\alpha)^{l-1} \\ \downarrow \tilde{f}_\alpha \times \text{id} & & \downarrow f_\alpha \\ N_{k(\alpha)} \times k(\alpha)^{l-1} & \xrightarrow{\eta^\alpha} & N_{k(\alpha)} \times k(\alpha)^{l-1} \end{array}$$

for certain Nash submersion $\tilde{f}_\alpha : M_{k(\alpha)} \rightarrow N_{k(\alpha)}$. Applying Proposition 3.4 to the proper Nash submersion \tilde{f}_α we get two Nash manifolds M' and N' defined over R , two Nash diffeomorphisms $\zeta' : M'_{k(\alpha)} \rightarrow M_{k(\alpha)}$ and $\eta' : N'_{k(\alpha)} \rightarrow N_{k(\alpha)}$ and a proper Nash submersion $f' : M' \rightarrow N'$ such that the following diagram commutes:

$$\begin{array}{ccc} M'_{k(\alpha)} & \xrightarrow{\zeta'} & M_{k(\alpha)} \\ \downarrow f'_{k(\alpha)} & & \downarrow \tilde{f}_\alpha \\ N'_{k(\alpha)} & \xrightarrow{\eta'} & N_{k(\alpha)}. \end{array}$$

The manifolds $M'_{k(\alpha)}$ and $M_{k(\alpha)}$ are Nash diffeomorphic, so, by Lemma 3.3, there exists a Nash diffeomorphism $\zeta : M' \rightarrow M$ defined over R ; similarly, we have a Nash diffeomorphism $\eta : N' \rightarrow N$ defined over R . So, replacing ζ^α by $\zeta^\alpha \circ (\zeta' \times \text{id}) \circ (\zeta^{-1} \times \text{id})_{k(\alpha)}$, η^α by $\eta^\alpha \circ (\eta' \times \text{id}) \circ (\eta^{-1} \times \text{id})_{k(\alpha)}$,

f' by $f' = \eta \circ f' \circ \zeta^{-1} : M \rightarrow N$ and $\widetilde{f_\alpha}$ by $f'_{k(\alpha)}$, we can assume that in fact we have a trivialization of the form

$$\begin{array}{ccc} M_{k(\alpha)} \times k(\alpha)^{l-1} & \xrightarrow{\zeta^\alpha} & M_{k(\alpha)} \times k(\alpha)^{l-1} \\ \downarrow f'_{k(\alpha)} \times \text{id} & & \downarrow f_\alpha \\ N_{k(\alpha)} \times k(\alpha)^{l-1} & \xrightarrow{\eta^\alpha} & N_{k(\alpha)} \times k(\alpha)^{l-1}. \end{array}$$

This implies (by Proposition 1.3) that there exists a Nash trivialization

$$\begin{array}{ccc} M \times R^{l-1} \times S & \xrightarrow{\zeta^S} & M \times R^{l-1} \times S \\ \downarrow f' \times \text{id} \times \text{id} & & \downarrow f \\ N \times R^{l-1} \times S & \xrightarrow{\eta^S} & N \times R^{l-1} \times S \end{array}$$

where S is a semialgebraic subset of R with $\alpha \in \widetilde{S}$. Now, by the compactness of the real spectrum, we have a partition $R = \bigcup S_i$ such that over each $R^{l-1} \times S_i$ we have a Nash trivialization as above. We can assume that the S_i are open intervals or singletons. In the case $S_i = \{a\}$, for $a \in R$, by the induction hypothesis, f is Nash trivial over $R^{l-1} \times \{a\}$ so, by Lemma 2.1, we can find a Nash trivialization over an open neighbourhood of $R^{l-1} \times \{a\}$. In other words, we can find a positive Nash function $\delta : R^{l-1} \rightarrow R$ such that f is Nash trivial over $R^{l-1} \times (a_i - \delta, a_i + \delta) := \{(x, t) \in R^{l-1} \times R : |t - a_i| < \delta(x)\}$. As before, we have to glue these trivializations. The argument is the same as above. For example, we are going to glue the trivialization over $R^{l-1} \times (0, 1)$,

$$\begin{array}{ccc} M \times R^{l-1} \times (0, 1) & \xrightarrow{\zeta_1} & M \times R^{l-1} \times (0, 1) \\ \downarrow f_1 \times \text{id} \times \text{id} & & \downarrow f \\ N \times R^{l-1} \times (0, 1) & \xrightarrow{\eta_1} & N \times R^{l-1} \times (0, 1) \end{array}$$

and the trivialization over $R^{l-1} \times (-\delta, \delta)$,

$$\begin{array}{ccc} M \times R^{l-1} \times (-\delta, \delta) & \xrightarrow{\zeta_2} & M \times R^{l-1} \times (-\delta, \delta) \\ \downarrow f_2 \times \text{id} \times \text{id} & & \downarrow f \\ N \times R^{l-1} \times (-\delta, \delta) & \xrightarrow{\eta_2} & N \times R^{l-1} \times (-\delta, \delta) \end{array}$$

for a certain C^1 Nash function $\delta : R^{l-1} \rightarrow R, 0 < \delta < 1$.

We can assume as in the case $l = 1$ that $f_1 = f_2 = f' : M \rightarrow N$. We start considering the semialgebraic open subsets $U_1 = R^{l-1} \times (0, \frac{3}{4}\delta)$ and $U_2 = R^{l-1} \times (\frac{1}{4}\delta, \delta)$ of $R^{l-1} \times (0, \delta)$. We consider then a C^1 partition of unity of $R^{l-1} \times (0, \delta)$, $\{h, 1 - h\}$ subordinated to the covering $\{U_1, U_2\}$. We define then the semialgebraic mapping $u^* : R^{l-1} \times (0, \delta) \rightarrow R^{l-1} \times (0, \delta)$ given by $u^*(s, t) = ((1 - h(s, t))s, \frac{3}{4}\delta((1 - h(s, t))s))$ for $t \leq \frac{3}{4}\delta(s)$ and $u^*(s, t) = (s, t)$ for $t \geq \frac{3}{4}\delta(s)$. We observe that for $0 < t < \frac{1}{4}\delta(s)$,

$h(s, t) = 1$ and hence $u^*(s, t) = (0, \frac{3}{4}\delta(0))$ is constant. Moreover, for $t = \frac{3}{4}\delta(s)$ we have that $h(s, t) = 0$ and hence $u^*(s, t) = (s, \frac{3}{4}\delta(s)) = (s, t)$, so in fact u^* is continuous. So we can approximate u^* by a C^1 Nash mapping $u : R^{l-1} \times (0, \delta) \rightarrow R^{l-1} \times (0, \delta)$ such that $u(s, t) = (0, \frac{3}{4}\delta(0))$ for $t \leq \frac{1}{4}\delta(s)$ and $u(s, t) = (s, t)$ for $t \geq \frac{3}{4}\delta(s)$.

As in the case $l = 1$, taking the above function u , we can “glue” the above trivializations in order to obtain a global trivialization over $(-1, \delta)$. This trivialization will be of class C^1 Nash, but repeating the same argument than in the case $l = 1$, we obtain the result in the Nash category. \square

5. Finiteness and effectiveness results.

In this section we apply the main theorem to obtain some finiteness and effectiveness results.

Let us consider two Nash mappings $f_1 : M_1 \rightarrow N_1, f_2 : M_2 \rightarrow N_2$ between Nash manifolds defined over R . We say that f_1 and f_2 are Nash equivalent if there exist Nash diffeomorphisms $\gamma : M_1 \rightarrow M_2$ and $\delta : N_1 \rightarrow N_2$ such that $f_2 \circ \gamma = \delta \circ f_1$.

THEOREM 5.1. — *Let $B \subset R^p, X \subset R^n \times B$ and $Y \subset R^m \times B$ be semialgebraic sets, and let $f : X \rightarrow Y$ be a semialgebraic family of semialgebraic mappings. We assume that $f_t : X_t \rightarrow Y_t$ is a proper Nash submersion between Nash manifolds for each $t \in B$. Then, there exists a finite number of elements of B , say t_1, \dots, t_r , such that, for each $t \in B$, the mapping f_t is Nash equivalent to f_{t_i} for some $i = 1, \dots, r$.*

Proof. — We are in the situation of Theorem 3.5. So, taking S_i as in the theorem and choosing $t_i \in S_i$ we have the result. \square

Before stating the following theorem, we have to define the degree of a semialgebraic set. Given a semialgebraic set $S \subset R^n$, we can write S as

$$S = \bigcup_{k=1}^p \bigcap_{l=1}^q \{x \in R^n : f_{k,l} \Upsilon_{k,l} 0\}$$

where the $f_{k,l}$ are polynomials and $\Upsilon_{k,l} \in \{<, >, =\}$. We define the degree of S to be the sum of the degrees of the polynomials which appear in the above description (so, to be accurate, we should speak of the degree of a

description of a semialgebraic set). The degree of a semialgebraic map will be the degree of its graph.

THEOREM 5.2. — *Given positive integers n , m and c , there are integers s and d , Nash submanifolds X_1, \dots, X_s of R^n , Nash submanifolds Y_1, \dots, Y_s of R^m and proper Nash submersions $f_i : X_i \rightarrow Y_i$, for $i = 1, \dots, s$ of degree $\leq c$ such that, for any pair of Nash submanifolds $X \subset R^n$, $Y \subset R^m$ and any proper Nash submersion $f : X \rightarrow Y$ of degree $\leq c$, the mapping f is equivalent to some f_i and the Nash diffeomorphisms involved in the equivalence have degrees $\leq d$. Moreover, s and d are bounded by recursive functions of n , m and c .*

Proof. — For the proof of this result we just have to follow the proof of Theorem B in [CS1]. We just need to apply Theorem 5.1 to the family of proper Nash submersions between Nash manifolds $X \subset R^n$ and $Y \subset R^m$ of given degree (this is in fact a semialgebraic family, see [R]). Following the same argument, with the appropriate modifications, we have the result. \square

Finally, following again the proof of [CS1], Proposition 5.2, together with our Proposition 3.4, we obtain the following.

PROPOSITION 5.3. — *Given integers n , m , l and d , there is an integer e such that for any pair of Nash manifolds $M \subset R^n$ and $N \subset R^m$ and any pair of proper Nash submersions $f : M \rightarrow N$ and $p : N \rightarrow R^l$ of degree $\leq d$, there are Nash diffeomorphisms h and g as in the main theorem of degree $\leq e$. Moreover, e may be bounded by a recursive function of n , m , l and d .*

BIBLIOGRAPHY

- [BCR] J. BOCHNAK, M. COSTE, M-F. ROY, Real Algebraic Geometry, *Ergeb. Math. Grenzgeb.*, (3) 36, Springer-Verlag, Berlin - Heidelberg - New York 1998.
- [CS1] M. COSTE, M. SHIOTA, Nash triviality in families of Nash manifolds, *Invent. Math.*, 108 (1992), 349–368.
- [CS2] M. COSTE, M. SHIOTA, Thom’s first isotopy lemma: a semialgebraic version, with uniform bound, in *Real Analytic and Algebraic Geometry* (Ed. F. Broglia, M. Galbiati, A. Tognoli), Walter de Gruyter, Berlin (1995), 83–101.
- [DK] H. DELFS, M. KNEBUSH, Locally Semialgebraic Spaces, *Lecture Notes in Mathematics*, 1173, Berlin, Springer-Verlag (1985).

- [GWPL] C.G. GIBSON, K. WIRTHMÜLLER, A.A. du PLESSIS, E.J.N. LOOIJENGA, Topological stability of smooth mappings, Lecture Notes in Mathematics, 552, Berlin, Springer-Verlag (1976).
- [H] R. HARDT, Semi-algebraic local triviality in semi-algebraic mappings, Amer. J. Math., 102 (1980), 291–302.
- [R] R. RAMANAKORAISINA, Complexité des fonctions de Nash, Commun. Algebra, 17 -6 (1989), 1395–1406.
- [Shi1] M. SHIOTA, Nash Manifolds, Lecture Notes in Mathematics 1269 Springer-Verlag, Berlin (1987).
- [Shi2] M. SHIOTA, Geometry of Subanalytic and Semialgebraic Sets, Progress in Mathematics, vol. 150, Boston, Birkhäuser (1997).

Manuscrit reçu le 21 novembre 2000,
accepté le 9 mars 2001.

Jesús ESCRIBANO,
Universidad Complutense
Facultad de CC. Matemáticas
Departamento de Sistemas Informáticos
y Programación
E-28040 Madrid (Spain).
jesusesc@eucmax.sim.ucm.es