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Semi-infinite cohomology and superconformal algebras


<http://aif.cedram.org/item?id=AIF_2001__51_3_745_0>
1. Introduction.

B. Feigin and E. Frenkel have introduced a semi-infinite analogue of the Weil complex based on the space

$$W^{\otimes +}* (\mathfrak{g}) = S^{\otimes +}* (\mathfrak{g}) \otimes \Lambda^{\otimes +}* (\mathfrak{g}).$$

In their construction $\mathfrak{g} = \oplus_{n \in \mathbb{Z}} \mathfrak{g}_n$ is a graded Lie algebra, $S^{\otimes +}* (\mathfrak{g})$ and $\Lambda^{\otimes +}* (\mathfrak{g})$ are some semi-infinite analogues of the symmetric and exterior power modules, [FF]. As in the classical case, two differentials, $d$ and $h$, are defined on $W^{\otimes +}* (\mathfrak{g})$. They are analogous to the differential in Lie algebra (co)homology and the Koszul differential, respectively. The semi-infinite Weil complex

$$\{W^{\otimes +}* (\mathfrak{g}), \, d + h\}$$

is acyclic similarly to the classical Weil complex. The cohomology of the complex

$$\{W^{\otimes +}* (\mathfrak{g}), \, d\}$$

Keywords: Weil complex – Semi-infinite cohomology – Superconformal algebra – Kähler geometry.
is called the semi-infinite cohomology of $g$ with coefficients in its “adjoint semi-infinite symmetric powers” $H^{\infty+\infty}(g, S^{\infty+\infty}(g))$. One can also define the relative semi-infinite Weil complex $W^{\infty+\infty}_{rel}(g)$ (relatively $g_0$), and the relative semi-infinite cohomology $H^{\infty+\infty}(g, g_0, S^{\infty+\infty}(g))$, [FF].

E. Getzler has shown that the semi-infinite Weil complex of the Virasoro algebra admits an action of the $N = 2$ superconformal algebra, [G].

Recall that a superconformal algebra (SCA) is a simple complex Lie superalgebra, such that it contains the centerless Virasoro algebra (i.e. the Witt algebra) $Witt = \oplus_{n \in \mathbb{Z}} \mathbb{C}L_n$ as a subalgebra, and has growth 1. The $\mathbb{Z}$-graded superconformal algebras are ones for which $adL_0$ is diagonalizable with finite-dimensional eigenspaces, [KL]:

\begin{equation}
\mathfrak{s} = \oplus_j \mathfrak{s}_j, \mathfrak{s}_j = \{ x \in \mathfrak{s} \mid [L_0, x] = jx \}.
\end{equation}

In this work we consider the semi-infinite Weil complex constructed for the next natural (after the Virasoro algebra) class of graded Lie algebras: the loop algebras of the complex finite-dimensional Lie algebras. The action of the Virasoro algebra on such complex is ensured by the fact that it has a structure of a vertex operator superalgebra (see [Ak]).

Let $g$ be a complex finite-dimensional Lie algebra, and $\widehat{g} = g \otimes \mathbb{C}[t, t^{-1}]$ be the corresponding loop algebra. We obtain a representation of the $N = 2$ SCA in the semi-infinite Weil complex $W^{\infty+\infty}(\widehat{g})$ and in the semi-infinite cohomology $H^{\infty+\infty}(\widehat{g}, S^{\infty+\infty}(\widehat{g}))$ with central charge $3\dim g$. We extend the representation of the $N = 2$ SCA in $W^{\infty+\infty}(\widehat{g})$ to a representation of the one-parameter family $\hat{S}'(2, \alpha)$ of deformations of the $N = 4$ SCA (see [Ad] and [KL]). In the case, when $g$ is endowed with a non-degenerate invariant symmetric bilinear form, we obtain a representation of $\hat{S}'(2, 0)$ in $H^{\infty+\infty}(\widehat{g}, S^{\infty+\infty}(\widehat{g}))$. Finally, there exists a representation of a central extension of the Lie superalgebra of all derivations of $S'(2, 0)$ in the relative semi-infinite cohomology $H^{\infty+\infty}(\widehat{g}, \widehat{g}_0, S^{\infty+\infty}(\widehat{g}))$.

It was shown in [FGZ] that the cohomology of the relative semi-infinite complex $C^*_\infty(l, l_0, V)$, where $l$ is a complex graded Lie algebra, and $V$ is a graded Hermitian $l$-module, has (under certain conditions) a structure analogous to that of the de Rham cohomology in Kähler geometry.

Recall that given a compact Kähler manifold $M$, there exists a number of classical operators on the space of differential forms on $M$, such as the differentials $\partial, \bar{\partial}, d, d_c$, their corresponding adjoint operators and the associated Laplacians (see [GH]). There also exists an action of $\mathfrak{sl}(2)$ on
$H^*(M)$ according to the Lefschetz theorem. All these operators satisfy a series of identities known as Hodge identities, [GH]. Naturally, the classical operators form a finite-dimensional Lie superalgebra.

We show that given a complex finite-dimensional Lie algebra $\mathfrak{g}$ endowed with a non-degenerate invariant symmetric bilinear form, there exist the analogues of the classical operators on the complex $W_{\text{rel}}^{\infty,\infty}(\mathfrak{g})$. We prove that the exterior derivations of $S'(2,0)$ form an $\mathfrak{sl}(2)$, and observe that they define an $\mathfrak{sl}(2)$-module structure on $H^{\infty,\infty}(\mathfrak{g}, \mathfrak{g}_0, S^{\infty,\infty}(\mathfrak{g}))$, which is the analogue of the $\mathfrak{sl}(2)$-module structure on the de Rham cohomology in Kähler geometry.

The action of $\tilde{S}'(2,0)$ provides $H^{\infty,\infty}(\mathfrak{g}, \mathfrak{g}_0, S^{\infty,\infty}(\mathfrak{g}))$ with eight series of quadratic operators. In particular, they include the semi-infinite Koszul differential $h$, and the semi-infinite analogue of the homotopy operator (cf. [Fu]). We prove that the degree zero part of the $\mathbb{Z}$-grading of $S'(2,0)$ defined by the element $L_0 \in \text{Witt}$, is isomorphic to the Lie superalgebra of classical operators in Kähler geometry.

It would be interesting to interpret the superconformal algebra $S'(2,0)$ as “affinization” of the classical operators in the case of an infinite-dimensional manifold.

This work is partly based on [P1]-[P3].

2. Semi-infinite Weil complex.

The semi-infinite Weil complex of a graded Lie algebra was introduced by B. Feigin and E. Frenkel in [FF]. Recall the necessary definitions. More generally, let $V = \bigoplus_{n \in \mathbb{Z}} V_n$ be a graded vector space over $\mathbb{C}$, such that $\dim V_n < \infty$. Let $V' = \bigoplus_{n \in \mathbb{Z}} V'_n$ be the restricted dual of $V$. The linear space $V \oplus V'$ carries non-degenerate skew-symmetric and symmetric bilinear forms: $(\cdot, \cdot)$ and $\{\cdot, \cdot\}$. Let $H(V)$ and $C(V)$ be the quotients of the tensor algebra $T^*(V \oplus V')$ by the ideals generated by the elements of the form $xy - yx - (x, y)$ and $xy + yx - \{x, y\}$, respectively, where $x, y \in V \oplus V'$. We fix $K \in \mathbb{Z}$. Let $V = V_+ \oplus V_-$ be the corresponding polarization of $V$: $V_+ = \bigoplus_{n \geq K} V_n$, $V_- = \bigoplus_{n \leq K} V_n$.

The symmetric algebra $S^*(V_+ \oplus V_-)$ is a subalgebra of $H(V)$ and the exterior algebra $\Lambda^*(V_+ \oplus V_-)$ is a subalgebra of $C(V)$. Let $S^{\infty,\infty}(V)$, $\Lambda^{\infty,\infty}(V)$ be the representations of $H(V)$ and $C(V)$ induced from the trivial representations $<1_S>$ and $<1_A>$ of $S^*(V_+ \oplus V_-)$ and of $\Lambda^*(V_+ \oplus V_-)$.
respectively. Thus we obtain some semi-infinite analogues of symmetric and exterior power modules. Denote the actions of $H(V)$ and $C(V)$ on these modules by $\beta(x)$, $\gamma(x')$ and $\tau(x)$, $\varepsilon(x')$, respectively, for $x \in V$, $x' \in V'$. Notice that each element of $S_{\infty}^{\infty}+(V)$ and of $\Lambda_{\infty}^{\infty}+(V)$ is a finite linear combination of the monomials of the type $\gamma(x'_1) \ldots \gamma(x'_k)\beta(y_1) \ldots \beta(y_m)1_S$ and of the type $\varepsilon(x'_1) \ldots \varepsilon(x'_k)\tau(y_1) \ldots \tau(y_m)1_A$, respectively, where $x'_1, \ldots, x'_k \in V'_+$, $y_1, \ldots, y_m \in V_-$. Let $\text{Deg}_\varepsilon(x') = \text{Deg}_\gamma(x') = 1$, and $\text{Deg}_\tau(x) = \text{Deg}_\beta(x) = -1$. Correspondingly, we obtain $\mathbb{Z}$-gradings on the spaces of semi-infinite power modules: $S_{\infty}^{\infty}+(V) = \oplus_{i \in \mathbb{Z}} S_{\infty}^{\infty+i}(V)$, $\Lambda_{\infty}^{\infty}+(V) = \oplus_{i \in \mathbb{Z}} \Lambda_{\infty}^{\infty+i}(V)$.

Let $\{e_i\}_{i \in \mathbb{Z}}$ be a homogeneous basis of $V$ so that if $i \in \mathbb{Z}$, then $e_i \in V_n$ for some $n \in \mathbb{Z}$, and if $e_i \in V_n$, then $e_{i+1} \in V_n$ or $e_{i+1} \in V_{n+1}$. Let $\{e'_i\}_{i \in \mathbb{Z}}$ be the dual basis. Let $i_0 \in \mathbb{Z}$ be such that $e_{i_0} \in V_K$ and $e_{i_0+1} \in V_{K+1}$.

Notice that one can think of $\Lambda_{\infty}^{\infty}+(V)$ as the vector space spanned by the elements $w = e'_1 \wedge e'_2 \wedge \ldots$ such that there exists $N(w) \in \mathbb{Z}$ such that $i_{n+1} = i_n - 1$ for $n > N(w)$. Then $1_A = e'_{i_0} \wedge e'_{i_0-1} \wedge \ldots$ is a vacuum vector in this space. The actions of $\varepsilon(x'), \tau(x)$ are, respectively, the exterior multiplication and contraction in the space of semi-infinite exterior products.

Let $g = \oplus_{n \in \mathbb{Z}} g_n$ be a graded Lie algebra over $\mathbb{C}$, such that $\dim g_n < \infty$. Let $\phi$ be a representation of $g$ in $V$ so that

$$\phi(g_n)V_k \subset V_{k+n}. \quad (2.1)$$

One can define the projective representations $\rho$ and $\pi$ of $g$ in $\Lambda_{\infty}^{\infty}+(V)$ and $S_{\infty}^{\infty}+(V)$, respectively

$$\rho(x) = \sum_{i \in \mathbb{Z}} : \tau(\phi(x)e_i)\varepsilon(e'_i) :; \quad (2.2)$$

$$\pi(x) = \sum_{i \in \mathbb{Z}} : \beta(\phi(x)e_i)\gamma(e'_i) :; \quad (2.3)$$

where $x \in g$, and where the double colons $: :$ denote a normal ordering operation:

$$: \tau(e_j)\varepsilon(e'_i) : = \begin{cases} \tau(e_j)\varepsilon(e'_i) \text{ if } i \leq i_0 \\ -\varepsilon(e'_i)\tau(e_j) \text{ if } i > i_0 \end{cases},$$

$$: \beta(e_j)\gamma(e'_i) : = \begin{cases} \beta(e_j)\gamma(e'_i) \text{ if } i \leq i_0 \\ \gamma(e'_i)\beta(e_j) \text{ if } i > i_0 \end{cases}. \quad (2.4)$$
Thus
\begin{equation}
\rho(x)1_A = \pi(x)1_S = 0 \text{ for } x \in \mathfrak{g}_0
\end{equation}
and
\begin{align}
\left[ \rho(x), \rho(y) \right] &= \rho([x, y]) + c_A(x, y), \\
\left[ \pi(x), \pi(y) \right] &= \pi([x, y]) + c_S(x, y),
\end{align}
where \( x, y \in \mathfrak{g} \) and \( c_A, c_S \) are 2-cocycles. Notice that \( -c_S \).

Let since the cocycles corresponding to the projective representations cancel, the representation \( \theta(x) = \rho(x) + \pi(x) \) of \( \mathfrak{g} \) in \( W^{[2]} +\ast (V) \) is well-defined. We define a \( \mathbb{Z} \)-grading on \( W^{[2]} +\ast (V) \) setting
\begin{equation}
W^{[2]} +i(V) = \bigoplus_{2l+j=i} S^{[2]} + l(V) \otimes \Lambda^{[2]} + j(V).
\end{equation}

Let \( V = \mathfrak{g} = \oplus_{n \in \mathbb{Z}} \mathfrak{g}_n \) and \( \phi \) be the adjoint representation of \( \mathfrak{g} \). We define two differentials on the space \( W^{[2]} +\ast (\mathfrak{g}) \):
\begin{align}
d &= \sum_{i < j} : \tau([e_i, e_j]) \varepsilon(e'_j) \varepsilon(e'_i) : + \sum_{i,j} \beta([e_j, e_i]) \gamma(e'_j) \varepsilon(e'_i), \\
h &= \sum_i \gamma(e'_i) \tau(e_i).
\end{align}

We obtain the \textit{semi-infinite Weil complex}
\begin{equation}
\{ W^{[2]} +\ast (\mathfrak{g}), \ d + h \}.
\end{equation}
The differential \( d \) is the analogue of the classical differential for the Lie algebra (co)homology, and \( h \) is the analogue of the Koszul differential. Notice that
\begin{equation}
d^2 = 0, h^2 = 0, [d, h] = 0, (d + h)^2 = 0.
\end{equation}
Notice also that if \( \mathfrak{g} \) is a finite-dimensional Lie algebra, then applying the definitions given above to the polarization \( \mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_- \), where \( \mathfrak{g}_+ = \mathfrak{g}, \mathfrak{g}_- = 0 \), we obtain the classical Weil complex.

As in the case of the classical Weil complex, one can construct two filtrations, \( F_1^p \) and \( F_2^p \), on \( W^{[2]} +\ast (\mathfrak{g}) \):
\begin{align}
F_1^p &= \bigoplus_{l+j \geq p} S^{[2]} + l(\mathfrak{g}) \otimes \Lambda^{[2]} + j(\mathfrak{g}), \\
F_2^p &= \bigoplus_{2l \geq p} S^{[2]} + l(\mathfrak{g}) \otimes \Lambda^{[2]} +\ast (\mathfrak{g}).
\end{align}
For filtration $F^*_i$ the complex is acyclic, the second term of the spectral sequence associated to filtration $F^*_2$ is the semi-infinite cohomology of Lie algebra $\mathfrak{g}$ with coefficients in its “adjoint semi-infinite symmetric powers” $H^{\infty,+*}_2(\mathfrak{g}, S^{\infty,+*}_2(\mathfrak{g}))$ (see [FF]). Let

\begin{equation}
W^{\infty,+*}_{\text{rel}}(V) = \{ w \in W^{\infty,+*}_2(V) \mid \tau(x)w = 0 \text{ for all } x \in V_0, \theta(x)w = 0 \text{ for all } x \in \mathfrak{g}_0 \}. \end{equation}

The differential $d$ preserves the space $W^{\infty,+*}_{\text{rel}}(\mathfrak{g})$ since

\begin{equation}
[d, \tau(x)] = d\tau(x) + \tau(x)d = \theta(x), \end{equation}

and

\begin{equation}
[d, \theta(x)] = 0, \end{equation}

for any $x \in \mathfrak{g}$. The complex $\{W^{\infty,+*}_{\text{rel}}(\mathfrak{g}), d\}$ is called the relative semi-infinite Weil complex. Its cohomology is called the relative semi-infinite cohomology $H^{\infty,+*}_2(\mathfrak{g}, \mathfrak{g}_0, S^{\infty,+*}_2(\mathfrak{g}))$.

We fix $K = 0$ from this point on. Correspondingly, $V = V_+ \oplus V_-$, where $V_+ = \oplus_{n > 0} V_n$, $V_- = \oplus_{n \leq 0} V_n$.

3. The $N = 2$ superconformal algebra.

Recall that the $N = 2$ SCA is spanned by the Virasoro generators $\mathfrak{L}_n$, the Heisenberg generators $H_n$, two fermionic fields $G^+_r$, and a central element $C$, where $n \in \mathbb{Z}, r \in \mathbb{Z} + 1/2$, and where the non-vanishing commutation relations are as follows, [FST]:

\begin{equation}
[\mathfrak{L}_n, \mathfrak{L}_m] = (n - m)\mathfrak{L}_{n+m} + \frac{C}{12} (n^3 - n) \delta_{n,-m},
\end{equation}

\begin{equation}
[\mathfrak{L}_n, H_m] = -mH_{n+m}, [\mathfrak{L}_n, G^+_r] = \left(\frac{n}{2} - r\right) G^+_{n+r},
\end{equation}

\begin{equation}
[G^+_r, G^-_s] = 2\mathfrak{L}_{r+s} + (r - s)H_{r+s} + \frac{C}{3} \left(r^2 - \frac{1}{4}\right) \delta_{r,-s},
\end{equation}

\begin{equation}
[H_n, H_m] = \frac{C}{3} \eta \delta_{n,-m}, [H_n, G^+_r] = \pm G^+_{n+r}.
\end{equation}

Let $\text{Witt} = \oplus_{i \in \mathbb{Z}} \mathbb{C} L_i$ be the Witt algebra:

\begin{equation}
[L_i, L_j] = (i - j)L_{i+j}.
\end{equation}
Let $\lambda, \mu \in \mathbb{C}$. Let $\mathcal{F}_{\lambda, \mu} = \oplus_{m \in \mathbb{Z}} \mathbb{C}u_m$ be a module over Witt defined as follows:

\begin{equation}
\phi(L_n)u_m = (-m + \mu - (n - 1)\lambda)u_{n+m}.
\end{equation}

**Remark 3.1.** — The module $\mathcal{F}_{\lambda, \mu} = \oplus_{m \in \mathbb{Z}} \mathbb{C}u_m$ is isomorphic to the module $\mathcal{F}_{-\lambda, \mu+1} = \oplus_{j \in \mathbb{Z}} \mathbb{C}f_j$ over the Witt algebra defined in [Fu]. The isomorphism is given by the correspondence $u_m \leftrightarrow f_{-m-1}$.

**Theorem 3.1.** — The space $W^{\mathbb{Z}}_{\mathbb{Z}}(\mathcal{F}_{\lambda, \mu})$ is a module over the $N = 2$ SCA with central charge $3 - 6\lambda$.

**Proof.** — Set

\begin{equation}
h_n = \frac{1}{\sqrt{2}} G^+_n, p_n = \frac{1}{\sqrt{2}} G^-_n.
\end{equation}

We define a representation of Witt in $W^{\mathbb{Z}}_{\mathbb{Z}}(\mathcal{F}_{\lambda, \mu})$ as follows:

\begin{equation}
\theta(L_n) = \sum_{m \in \mathbb{Z}} (-m + \mu - n\lambda + \lambda) \{ \tau(u_{m+n})\varepsilon(u'_m) : + : \beta(u_{m+n})\gamma(u'_m) : \}.
\end{equation}

Let us extend $\theta$ to a representation of the $N = 2$ SCA in $W^{\mathbb{Z}}_{\mathbb{Z}}(\mathcal{F}_{\lambda, \mu})$:

\begin{equation}
\theta(H_n) = \lambda \sum_{m \in \mathbb{Z}} \tau(u_m)\varepsilon(u'_{m+n}) : + (\lambda - 1) \sum_{m \in \mathbb{Z}} \beta(u_m)\gamma(u'_{m+n}) : + \mu \delta_{n,0},
\end{equation}

\begin{equation}
\theta(h_n) = \sum_{m \in \mathbb{Z}} \gamma(u'_{m+n})\tau(u_m),
\end{equation}

\begin{equation}
\theta(p_n) = \sum_{m \in \mathbb{Z}} (m - \mu - (n + 1)\lambda) \beta(u_{m-n})\varepsilon(u'_m),
\end{equation}

\begin{equation}
\theta(l_n) = -\theta(L_n) + \frac{n+1}{2} \theta(H_n).
\end{equation}

We calculate the central charge by checking the commutation relations on the vacuum vector $1 = 1_S \otimes 1_A$. Let $n > 0$. Then

\begin{equation}
\theta([H_n, H_{-n}])1 = -\theta(H_{-n})\theta(H_n)1.
\end{equation}

\begin{equation}
= -\theta(H_{-n}) \left( \lambda \sum_{m=1-n}^{0} \tau(u_m)\varepsilon(u'_{m+n}) \right)
\end{equation}

TOME 51 (2001), FASCICULE 3
Thus the central charge is $3 - 6\lambda$. The other commutation relations on the vacuum vector $1$ are calculated in the same way.

$\square$

Remark 3.2. — In the case $\lambda = 1$, the module $\mathcal{F}_{\lambda, \mu}$ is the adjoint representation of Witt. Thus we obtain a representation of the $N = 2$ SCA in the semi-infinite Weil complex of the Witt algebra (cf. [G]).

Theorem 3.2. — Let $V$ be a complex finite-dimensional vector space, $\tilde{V} = V \otimes \mathbb{C}[t, t^{-1}]$. There exists a representation of the $N = 2$ SCA in $W^{\infty}_{\mathbb{Z}}(\tilde{V})$ with central charge $3\dim V$.

Proof. — There is the natural $\mathbb{Z}$-grading $\tilde{V} = \oplus_{n \in \mathbb{Z}} \tilde{V}_n$, where $\tilde{V}_n = V \otimes t^n$. Let $u$ run through a fixed basis of $V$, $u_n$ stand for $u \otimes t^n$, and let $\{u'_n\}$ be the dual basis of $\tilde{V}'$. Define the following quadratic expansions by analogy with (3.5) and (3.6), where $\lambda = 0, \mu = 0$:

$$L_n = - \sum_u \sum_{m \in \mathbb{Z}} (m : \tau(u_{m+n})\varepsilon(u'_m) : + m : \beta(u_{m+n})\gamma(u'_m) : )$$

$$H_n = - \sum_u \sum_{m \in \mathbb{Z}} : \gamma(u'_{m+n})\beta(u_m) :$$

$$h_n = \sum_u \sum_{m \in \mathbb{Z}} \gamma(u'_{m+n})\tau(u_m),$$

$$p_n = \sum_u \sum_{m \in \mathbb{Z}} m\beta(u_{m-n})\varepsilon(u'_m).$$

Set

$$\mathfrak{g}_n = -L_{-n} + \frac{n + 1}{2}H_n.$$
Then $\mathfrak{L}_n, H_n, h_n$, and $p_n$ span the centerless $N = 2$ SCA.

Let $n > 0$. Then $H_{-n}1 = 0$. Hence

\begin{equation}
[H_n, H_{-n}] 1 = -H_{-n} \left( -\sum_{u} \sum_{m=1-n}^{0} \gamma(u_{m+n}')\beta(u_m) \right) 1
\end{equation}

\begin{align*}
&= \left( -\sum_{u} \sum_{m=1-n}^{0} \gamma(u_{m-n}')\beta(u_m) \right) \left( \sum_{u} \sum_{m=1-n}^{0} \gamma(u_{m+n}')\beta(u_m) \right) 1 \\
&= -\sum_{u} \sum_{m=1-n}^{0} \gamma(u_{m}')(\gamma(u_{m+n})\beta(u_m)) 1 \\
&= -\text{dim}V(-n)1,
\end{align*}

since $\gamma(u_{i}')\beta(u_i) - \beta(u_i)\gamma(u_{i}') = 1$. Notice that

\begin{equation}
[H_{n}, H_{m}] 1 = 0, \text{ if } m \neq -n.
\end{equation}

Hence

\begin{equation}
[H_n, H_{m}] 1 = n\text{dim}V\delta_{n,-m}1.
\end{equation}

Thus the central charge is $3\text{dim}V$. \hfill \square

**Corollary 3.1.** — *Let $\mathfrak{g}$ be a complex finite-dimensional Lie algebra, let $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. There exists a representation of the $N = 2$ SCA in $H^{\infty, \ast}(\tilde{\mathfrak{g}}, S^{\infty, \ast}(\tilde{\mathfrak{g}}))$ with central charge $3\text{dim}\mathfrak{g}$.*

**Proof.** — We will show that the expansions (3.9) commute with the differential $d$. Recall that

\begin{equation}
d = d^{(1)} + d^{(2)},
\end{equation}

where

\begin{align*}
d^{(1)} &= (1/2) \sum_{u, v, i, j} :\tau([u_i, v_j])\varepsilon(v_j')\varepsilon(u_i') :, \\
d^{(2)} &= \sum_{u, v, i, j} :\beta([u_i, v_j])\gamma(v_j')\varepsilon(u_i') :,
\end{align*}

$u, v$ run through a fixed basis of $\mathfrak{g}$, and $i, j \in \mathbb{Z}$. Then

\begin{align*}
[L_n, d^{(1)}] &= (1/2) \sum_{u, v, i, j} :-(i + j)\tau([u_i, v_{i+j+n}])\varepsilon(v_j')\varepsilon(u_i') : \\
&\quad + :\tau([u_i, v_j])(j-n)\varepsilon(v_{j-n}')\varepsilon(u_i') : \\
&\quad + :\tau([u_i, v_j])\varepsilon(v_j')(i-n)\varepsilon(u_{i-n}') := 0
\end{align*}
and

\[ [L_n, d^{(2)}] = \sum_{u,v,i,j} : -(i + j)\beta([u,v]_{i+j+n})\gamma(v'_j)\varepsilon(u'_i) : + : \beta([u_i,v_j])(j - n)\gamma(v'_{j-n})\varepsilon(u'_i) : + : \beta([u_i,v_j])\gamma(v'_j)(i - n)\varepsilon(u'_{i-n}) := 0. \]

Clearly,

\[ [H_n, d^{(1)}] = 0, \]

and

\[ [H_n, d^{(2)}] = \sum_{u,v,i,j} : -\beta([u,v]_{i+j-n})\gamma(v'_j)\varepsilon(u'_i) : + \beta([u,v]_{i+j})\gamma(v'_{j+n})\varepsilon(u'_i) : = 0. \]

Next,

\[ [h_n, d^{(1)}] = (1/2) \sum_{u,v,i,j} : -\tau([u,v]_{i+j})\gamma(v'_{j+n})\varepsilon(u'_i) : + : \tau([u,v]_{i+j})\varepsilon(v'_j)\gamma(u'_{i+n}) : = - \sum_{u,v,i,j} : \tau([u,v]_{i+j})\gamma(v'_{j+n})\varepsilon(u'_i) :, \]

\[ [h_n, d^{(2)}] = \sum_{u,v,i,j} : \tau([u,v]_{i+j-n})\gamma(v'_j)\varepsilon(u'_i) : + : \beta([u,v]_{i+j})\gamma(v'_j)\gamma(u'_{i+n}) : = \sum_{u,v,i,j} : \tau([u,v]_{i+j})\gamma(v'_{j+n})\varepsilon(u'_i) :, \]

since \( \sum_{u,v,i,j} : \beta([u,v]_{i+j})\gamma(v'_j)\gamma(u'_{i+n}) := 0. \) Hence

\[ [h_n, d^{(2)}] = -[h_n, d^{(1)}]. \]

Finally,

\[ [p_n, d^{(1)}] = (1/2) \sum_{u,v,i,j} : (i + j)\beta([u,v]_{i+j-n})\varepsilon(v'_j)\varepsilon(u'_i) :, \]

\[ [p_n, d^{(2)}] = \sum_{u,v,i,j} : -\beta([u,v]_{i+j})(j + n)\varepsilon(v'_{j+n})\varepsilon(u'_i) :. \]
Hence

\[
\sum_{u,v,i,j} \beta([u,v]_{i+j-n})j \varepsilon(v'_j) \varepsilon(u'_i) = -(1/2) \sum_{u,v,i,j} (j + i) \beta([u,v]_{i+j-n}) \varepsilon(v'_j) \varepsilon(u'_i).
\]

Recall the necessary definitions, [KL]. Let \( W(N) \) be the superalgebra of all derivations of \( \mathbb{C}[t, t^{-1}] \otimes \Lambda(N) \), where \( \Lambda(N) \) is the Grassmann algebra in \( N \) variables \( \theta_1, \ldots, \theta_N \), and \( p(t) = 0 \), \( p(\theta_i) = 1 \) for \( i = 1, \ldots, N \). Let \( \partial_i \) stand for \( \partial/\partial \theta_i \), and \( \partial_t \) stand for \( \partial/\partial t \). Let

\[
S(N, \alpha) = \{ D \in W(N) \mid \text{Div}(t^\alpha D) = 0 \} \quad \text{for} \quad \alpha \in \mathbb{C}.
\]

Recall that

\[
\text{Div}(f \partial_t + \sum_{i=1}^N f_i \partial_i) = \partial_t f + \sum_{i=1}^N (-1)^{p(f_i)} \partial_i f_i
\]

where \( f, f_i \in \mathbb{C}[t, t^{-1}] \otimes \Lambda(N) \), and

\[
\text{Div}(f D) = Df + f \text{Div} D,
\]

where \( f \) is an even function. Let \( S'(N, \alpha) = [S(N, \alpha), S(N, \alpha)] \) be the derived superalgebra. Assume that \( N > 1 \). If \( \alpha \notin \mathbb{Z} \), then \( S(N, \alpha) \) is simple, and if \( \alpha \in \mathbb{Z} \), then \( S'(N, \alpha) \) is a simple ideal of \( S(N, \alpha) \) of codimension 1:

\[
0 \rightarrow S'(N, \alpha) \rightarrow S(N, \alpha) \rightarrow \mathbb{C} t^{-\alpha} \theta_1 \cdots \theta_N \partial_t \rightarrow 0.
\]

Notice that

\[
S(N, \alpha) \cong S(N, \alpha + n) \quad \text{for} \quad n \in \mathbb{Z}.
\]

The superalgebra \( S'(N, \alpha) \) has, up to equivalence, only one non-trivial 2-cocycle if and only if \( N = 2 \), which is important for our task. Let

\[
\{ E_n, H_n, F_n, h_n, p_n, x_n, y_n \}_{n \in \mathbb{Z}}
\]
be the basis of $S'(2, \alpha)$ defined as follows:

\begin{equation}
\mathcal{L}_n^\alpha = -t^n(t \partial_t + \frac{1}{2}(n + \alpha + 1)(\theta_1 \partial_1 + \theta_2 \partial_2)),
\end{equation}

\begin{align*}
E_n &= t^n \theta_2 \partial_1, \\
H_n &= t^n(\theta_2 \partial_2 - \theta_1 \partial_1), \\
F_n &= t^n \theta_1 \partial_2, \\
h_n^\alpha &= t^n \theta_2 \partial_t - (n + \alpha)t^{n-1}\theta_1 \theta_2 \partial_1, \\
p_n &= -t^{n+1} \partial_2, \\
x_n &= t^{n+1} \partial_1, \\
y_n^\alpha &= t^n \theta_1 \partial_t + (n + \alpha)t^{n-1}\theta_1 \theta_2 \partial_2.
\end{align*}

The non-vanishing commutation relations between these elements are

\begin{align}
[E_n, F_k] &= H_{n+k}, [H_n, E_k] = 2E_{n+k}, [H_n, F_k] = -2F_{n+k}, \\
[L_n^\alpha, E_k] &= -kE_{n+k}, [L_n^\alpha, H_k] = -kH_{n+k}, [L_n^\alpha, F_k] = -kF_{n+k}, \\
[L_n^\alpha, h_n^\alpha] &= \frac{1}{2}(n - 2k + 1 - \alpha)h_{n+k}^\alpha, [L_n^\alpha, p_k] = \frac{1}{2}(n - 2k - 1 + \alpha)p_{n+k}, \\
[L_n^\alpha, x_k] &= \frac{1}{2}(n - 2k - 1 + \alpha)x_{n+k}, [L_n^\alpha, y_k^\alpha] = \frac{1}{2}(n - 2k + 1 - \alpha)y_{n+k}^\alpha, \\
[E_n, y_k^\alpha] &= h_{n+k}^\alpha, [F_n, h_k^\alpha] = y_{n+k}^\alpha, [E_n, p_k] = x_{n+k}, [F_n, x_k] = p_{n+k}, \\
[H_n, h_k^\alpha] &= h_{n+k}^\alpha, [H_n, y_k^\alpha] = -y_{n+k}^\alpha, [H_n, x_k] = x_{n+k}, [H_n, p_k] = -p_{n+k}, \\
h_{n+k}^\alpha, x_k = (k + 1 - n - \alpha)E_{n+k}, [p_k, y_k^\alpha] = (k - n - 1 + \alpha)F_{n+k}, \\
h_{n+k}^\alpha, p_k = L_{n+k}^\alpha - \frac{1}{2}(k - n + 1 - \alpha)H_{n+k}, \\
x_n, y_k^\alpha = -L_{n+k}^\alpha + \frac{1}{2}(k - n - 1 + \alpha)H_{n+k}.
\end{align}

A non-trivial 2-cocycle on $S'(2, \alpha)$ is

\begin{equation}
c(L_n^\alpha, L_k^\alpha) = \frac{C}{12}n(n^2 - 1)\delta_{n,-k},
\end{equation}
c(E_n, F_k) = \frac{C}{6} n \delta_{n,-k}, \quad c(H_n, H_k) = \frac{C}{3} n \delta_{n,-k},

c(h_n^\alpha, p_k) = \frac{C}{6} \left( (n - 1 + \frac{\alpha + 1}{2})^2 - \frac{1}{4} \right) \delta_{n,-k},

c(x_n, y_k^\alpha) = -\frac{C}{6} \left( (-n - 1 + \frac{\alpha + 1}{2})^2 - \frac{1}{4} \right) \delta_{n,-k};

see [KL]. Let \( \hat{S}'(2, \alpha) \) be the corresponding central extension of \( S'(2, \alpha) \). In particular, \( \hat{S}'(2, 0) \) is isomorphic to the \( N = 4 \) SCA (see [Ad]).

**Remark 4.1** — Notice that

\[
S'(2, \alpha)_0 = \text{Witt} \ltimes \mathfrak{sl}(2), \quad \text{where}
\]

\[
\text{Witt} = \langle \mathfrak{L}_n^\alpha \rangle_{n \in \mathbb{Z}}, \mathfrak{sl}(2) = \langle E_n, H_n, F_n \rangle_{n \in \mathbb{Z}},
\]

and

\[
S'(2, \alpha)_1 = \langle h_n^\alpha, y_n^\alpha \rangle_{n \in \mathbb{Z}} \oplus \langle p_n, x_n \rangle_{n \in \mathbb{Z}}
\]

is a direct sum of two standard (odd) \( \mathfrak{sl}(2) \)-modules.

**Remark 4.2** — For any \( \alpha \in \mathbb{C} \) one can consider the subalgebra of \( \hat{S}'(2, \alpha) \), spanned by \( \mathfrak{L}_n^\alpha, H_n, h_n^\alpha, p_n \), and \( \mathbb{C} \). Thus we obtain a one-parameter family of superalgebras, which are isomorphic to the \( N = 2 \) SCA. The isomorphism

\[
\varphi : \langle \mathfrak{L}_n^\alpha, H_n, h_n^\alpha, p_n, \mathbb{C} \rangle \rightarrow \langle \mathfrak{L}_n, H_n, h_n, p_n, \mathbb{C} \rangle
\]

is given as follows:

\[
\varphi(\mathfrak{L}_n^\alpha) = \mathfrak{L}_n - \frac{\alpha}{2} H_n + \frac{\alpha^2}{24} \delta_{n,0} \mathbb{C},
\]

\[
\varphi(H_n) = H_n - \frac{\alpha}{6} \delta_{n,0} \mathbb{C},
\]

\[
\varphi(h_n^\alpha) = h_n, \quad \varphi(p_n) = p_n, \quad \varphi(\mathbb{C}) = \mathbb{C}.
\]

Notice that formulae (4.13) correspond to the spectral flow transformation for the \( N = 2 \) SCA (cf. [FST]).

Let \( \text{Der}S'(2, \alpha) \) be the Lie superalgebra of all derivations of \( S'(2, \alpha) \), and \( \text{Der}_{ext}S'(2, \alpha) \) be the exterior derivations of \( S'(2, \alpha) \) (see [Fu]).

**TOME 51 (2001), FASCICULE 3**
THEOREM 4.1.
1) If $\alpha \in \mathbb{Z}$, then $\text{Der}_e S'(2, \alpha) \cong \mathfrak{sl}(2) = \langle \mathcal{E}, \mathcal{H}, \mathcal{F} \rangle$, where

\begin{equation}
[\mathcal{H}, \mathcal{E}] = 2\mathcal{E}, [\mathcal{H}, \mathcal{F}] = -2\mathcal{F}, [\mathcal{E}, \mathcal{F}] = \mathcal{H}.
\end{equation}

The action of $\mathfrak{sl}(2)$ is given as follows:

\begin{align}
[\mathcal{E}, h^\alpha_k] &= x_{k-1+\alpha}, [\mathcal{E}, y^\alpha_k] = p_{k-1+\alpha}; \\
[\mathcal{F}, x_k] &= h^{\alpha}_{k+1-\alpha}, [\mathcal{F}, p_k] = y^\alpha_{k+1-\alpha}; \\
[\mathcal{H}, x_k] &= x_k, [\mathcal{H}, h^\alpha_k] = -h^\alpha_k, \\
[\mathcal{H}, p_k] &= p_k, [\mathcal{H}, y^\alpha_k] = -y^\alpha_k.
\end{align}

2) If $\alpha \in \mathbb{C} \setminus \mathbb{Z}$, then $\text{Der}_e S'(2, \alpha) = \langle \mathcal{H} \rangle$.

Proof. — Recall that the exterior derivations of a Lie (super) algebra can be identified with its first cohomology with coefficients in the adjoint representation (see [Fu]). Thus

\begin{equation}
\text{Der}_e S'(2, \alpha) \cong H^1(S'(2, \alpha), S'(2, \alpha)).
\end{equation}

The superalgebra $S'(2, \alpha)$ has the following $\mathbb{Z} \pm \alpha$-grading deg:

\begin{align}
\deg \mathcal{L}^\alpha_n &= n, \deg E_n = n + 1 - \alpha, \deg F_n = n - 1 + \alpha, \\
\deg H_n = n, \deg h^\alpha_n = n, \deg p_n = n, \deg x_n = n + 1 - \alpha, \\
\deg y^\alpha_n &= n - 1 + \alpha.
\end{align}

Let

\begin{equation}
L_0 = -\mathcal{L}^\alpha_0 + \frac{1}{2} (1 - \alpha) H_0.
\end{equation}

Then

\begin{equation}
[L_0, s] = (\deg s) s
\end{equation}

for a homogeneous $s \in S'(2, \alpha)$. Accordingly,

\begin{equation}
[L_0, D] = (\deg D) D
\end{equation}

for a homogeneous $D \in \text{Der}_e S'(2, \alpha)$. On the other hand, since the action of a Lie superalgebra on its cohomology is trivial (see [Fu]), then one must have

\begin{equation}
[L_0, D] = 0.
\end{equation}
Hence the non-zero elements of $\text{Der}_{\text{ext}}S'(2,\alpha)$ have deg = 0, and they preserve the superalgebra $S'(2,\alpha)_{\text{deg}=0}$. Let $\alpha \in \mathbb{Z}$. Then one can check that the exterior derivations of $S'(2,\alpha)_{\text{deg}=0}$ form an $\mathfrak{sl}(2)$, and extend them to the exterior derivations of $S'(2,\alpha)$ as in (4.15). One should also note that if the restriction of a derivation of $S'(2,\alpha)$ to $S'(2,\alpha)_{\text{deg}=0}$ is zero, then this derivation is inner.

Finally, notice that the exterior derivations $\mathcal{E}$ and $\mathcal{F}$ interchange $\{h^a_k\}$ with $\{x_k\}$. If $\alpha \notin \mathbb{Z}$, then deg $h^a_k - \text{deg } x_n \notin \mathbb{Z}$ for any $k, n \in \mathbb{Z}$. Hence $\mathcal{E}$ and $\mathcal{F}$ cannot have deg = 0. By this reason, $\text{Der}_{\text{ext}}S'(2,\alpha) = \langle \mathcal{H} \rangle$ for $\alpha \in \mathbb{C} \setminus \mathbb{Z}$.

Remark 4.3. — If $\alpha \in \mathbb{Z}$, then one can identify $\mathcal{F}$ with $-t^{-\alpha} \theta_1 \theta_2 \partial_t$ (see (4.4)).

5. An action of $\hat{S}'(2,\alpha)$ on the semi-infinite Weil complex of a loop algebra.

We will consider a more general case, i.e. when $V$ is a complex finite-dimensional vector space, and $\tilde{V} = V \otimes \mathbb{C}[t, t^{-1}]$. Let $\hat{\text{Der}}S'(2,\alpha)$ be a non-trivial central extension of $\text{Der}S'(2,\alpha)$.

Theorem 5.1.

1) The space $W_{\mathbb{Z}}^{\infty} \oplus \ast(\tilde{V})$, where $\alpha \in \mathbb{C}$, is a module over $\hat{S}'(2,\alpha)$ with central charge $3 \dim V$;  

2) if $\alpha \in \mathbb{C} \setminus \mathbb{Z}$, then $W_{\mathbb{Z}}^{\infty} \oplus \ast(\tilde{V})$ is a module over $\hat{\text{Der}}S'(2,\alpha)$.

Proof. — Let $u$ run through a fixed basis of $V$, $u_n$ stand for $u \otimes t^n$, and $\{u'_n\}$ be the dual basis of $\tilde{V}'$. One can define a representation of Witt in $W_{\mathbb{Z}}^{\infty} \oplus \ast(\tilde{V})$ by analogy with (3.5), where $\lambda = 0, \mu = \alpha/2$:

\begin{equation}
\theta(L_n) = -\sum_u \sum_m \left(m - \frac{\alpha}{2}\right) : \tau(u_{m+n}) \in (u'_m) : + : \beta(u_{m+n}) \gamma(u'_m) : ,
\end{equation}

then extend it to a representation of the $N = 2$ SCA, and apply (4.13). We obtain the following representation of $\hat{S}'(2,\alpha)$:

\begin{equation}
\theta(H_n) = -\sum_u \sum_m : \beta(u_m) \gamma(u'_{m+n}) ,
\end{equation}
Theorem 5.2. Let $g$ be a complex finite-dimensional Lie algebra endowed with a non-degenerate invariant symmetric bilinear form. Then $H^{\otimes+\otimes}(\tilde{g}, S^{\otimes+\otimes}(\tilde{g}))$ is a module over $\breve{S}'(2, 0)$ with central charge $3\dim g$.

Proof. — Let $\{v_i\}$ be a basis of $g$ so that with respect to the given form $\langle v_i, v_j \rangle = \delta_{i,j}$. Let $u$ run through this basis. Then by Theorem 5.1, there is a representation of $\breve{S}'(2, 0)$ in $W^{\otimes+\otimes}(\tilde{g})$. Notice that we can identify the elements of $S'(2, 0)$ with the quadratic expansions obtained by putting $\alpha = 0$ in the equations (5.2). One can check that the commutation relations (4.8) (where $\alpha = 0$) are fulfilled. One can notice that

\[
\theta(L^\alpha_n) = -\theta(L_{-n}) + \frac{n+1-\alpha}{2} \theta(H_n) + \left(\frac{\alpha}{4} - \frac{\alpha^2}{8}\right) \dim V \delta_{n,0},
\]

\[
\theta(h^\alpha_n) = \sum_u \sum_m \gamma(u^\prime_{m+n}) \tau(u_m),
\]

\[
\theta(p_n) = \sum_u \sum_m \left(m - \frac{\alpha}{2}\right) \beta(u_{m-n}) \varepsilon(u'_m),
\]

\[
\theta(E_n) = -(1/2)i \sum_u \sum_m \gamma(u'_m) \gamma(u'_{1-m+n}),
\]

\[
\theta(F_n) = -(1/2)i \sum_u \sum_m \beta(u_m) \beta(u'_{1-m-n}),
\]

\[
\theta(y^\alpha_n) = i \sum_u \sum_m \beta(u_m) \tau(u_{1-m-n}),
\]

\[
\theta(x_n) = -i \sum_u \sum_m \left(m - \frac{\alpha}{2}\right) \gamma(u'_{1-m+n}) \varepsilon(u'_m),
\]

\[
\theta(h) = - \sum_u \sum_m \tau(u_m) \varepsilon(u'_m).
\]

One can check that the central charge is $3\dim V$ in the same way as in Theorem 3.2. \qed

Recall that

\[
[S'(2, 0), d] = 0.
\]

In fact, since $\langle \cdot, \cdot \rangle$ is an invariant symmetric bilinear form on $g$, then the elements $E_n, H_n,$ and $F_n$ commute with $\pi(g)$ for any $g \in \tilde{g}$. Hence they commute with $d$. According to Corollary 3.1,

\[
[h^0_n, d] = [p_n, d] = 0.
\]

Recall that

\[
S'(2, 0)_1 = \langle h^0_n, y^0_n, p_n, x_n \rangle_{n \in \mathbb{Z}}.
\]
Since

\[ [E_n, p_k] = x_{n+k}, [F_n, h_k^0] = y_{n+k}^0, \]

then

\[ [S'(2, 0)_{\bar{1}}, d] = 0. \]

Since

\[ S'(2, 0)_{\bar{0}} = [S'(2, 0)_{\bar{1}}, S'(2, 0)_{\bar{1}}], \]

then (5.3) follows. \( \Box \)

To define an action of \( \hat{\text{Der}}S'(2, 0) \), one should consider a relative semi-infinite Weil complex.

Let \( g \) be a complex finite-dimensional Lie algebra, \( \phi \) be a representation of \( g \) in \( V \), \( \langle \cdot, \cdot \rangle \) be a non-degenerate \( g \)-invariant symmetric bilinear form on \( V \). One can naturally extend \( \phi \) to a representation of \( \hat{g} \) in \( \hat{V} \):

\[ \phi(g \otimes t^n)(v \otimes t^k) = (\phi(g)v) \otimes t^{n+k}, \quad \text{for } g \in g, v \in V. \]

**Theorem 5.3.** — The space \( W_{\text{rel}}^{\infty+} (\hat{V}) \) is a module over \( \hat{\text{Der}}S'(2, 0) \) with central charge \( 3\dim V \).

**Proof.** — Let \( \{v_i\} \) be a basis of \( V \) so that \( \langle v_i, v_j \rangle = \delta_{i,j} \). Let \( u \) run through this basis. Then by Theorem 5.1, there is a representation of \( \hat{S}'(2, 0) \) in \( W_{\text{rel}}^{\infty+} (\hat{V}) \). We can identify the elements of \( S'(2, 0) \) with the expansions (5.2) where \( \alpha = 0 \).

Since the form \( \langle \cdot, \cdot \rangle \) is \( g \)-invariant, then there is an action of \( \hat{S}'(2, 0) \) on \( W_{\text{rel}}^{\infty+} (\hat{V}) \). To extend this representation to \( \hat{\text{Der}}S'(2, 0) \), we have to define it on \( SL(2) = (\mathcal{F}, \mathcal{H}, \mathcal{E}) \). Let

\[ \mathcal{E} = i \sum_u \sum_{m > 0} m \epsilon(u'_m) \epsilon(u'_m), \]

\[ \mathcal{H} = - \sum_u \sum_{m \neq 0} : \tau(u_m) \epsilon(u'_m) :, \]

\[ \mathcal{F} = -i \sum_u \sum_{m > 0} (1/m) \tau(u_m) \tau(u_{-m}). \]

Notice that \( SL(2) \) acts on \( W_{\text{rel}}^{\infty+} (\hat{V}) \). The commutation relations between \( \mathcal{E}, \mathcal{H}, \mathcal{F} \) and the elements of \( S'(2, 0) \) coincide with the relations (4.15),
where $\alpha = 0$, up to some terms which contain elements $\tau(u_0)$. Since the action of $\tau(u_0)$ on $W_{rel}^{\infty+*}(\tilde{V})$ is trivial, then a representation of $\check{\text{Der}}S'(2,0)$ in $W_{rel}^{\infty+*}(\tilde{V})$ is well-defined. 

**Corollary 5.1.** — $H^{\frac{\infty}{2}+*}(\tilde{g}, \tilde{g}_0, S^{\frac{\infty}{2}+*}(\tilde{g}))$ is a module over $\check{S}'(2,0)$ with central charge $3\dim g$.

**Proof.** — Follows from Theorem 5.2. 

### 6. Relative semi-infinite cohomology and Kähler geometry.

Let $M$ be a compact Kähler manifold with associated $(1,1)$-form $\omega$, let $\dim_{\mathbb{C}}M = n$. There exists a number of operators on the space $\mathcal{A}^*(M)$ of differential forms on $M$ such as $\partial, \bar{\partial}, d, d_c$, their corresponding adjoint operators and the associated Laplacians (see [GH]). Recall that

\begin{align}
\partial : A^{p,q}(M) &\rightarrow A^{p+1,q}(M), \\
\bar{\partial} : A^{p,q}(M) &\rightarrow A^{p,q+1}(M), \\
d &= \partial + \bar{\partial}, \\
d_c &= i(\partial - \bar{\partial}), \\
\Delta &= dd^* + d^*d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}.
\end{align}

The Hodge $*$-operator maps

\begin{equation}
* : A^{p,q}(M) \rightarrow A^{n-q,n-p}(M),
\end{equation}

so that $*^2 = (-1)^{p+q}$ on $A^{p,q}(M)$. Correspondingly, the Hodge inner product is defined on each of $A^{p,q}(M)$:

\begin{equation}
(\varphi, \psi) = \int_M \varphi \wedge *\psi.
\end{equation}

In addition, $\mathcal{A}^*(M)$ admits an $\mathfrak{sl}(2)$-module structure. Namely, $\mathfrak{sl}(2) = \langle L, H, \Lambda \rangle$, where

\begin{equation}
\end{equation}

The operator

\begin{equation}
L : A^{p,q}(M) \rightarrow A^{p+1,q+1}(M),
\end{equation}
is defined by

\[ L(\varphi) = \varphi \wedge \omega. \]

Let \( \Lambda = L^* \) be its adjoint operator:

\[ \Lambda : A^{p,q}(M) \to A^{p-1,q-1}(M), \]

and

\[ H \mid_{A^{p,q}(M)} = p + q - n. \]

According to the Lefschetz theorem, there exists the corresponding action of \( \mathfrak{sl}(2) \) on \( H^*(M) \). These operators satisfy a series of identities, known as the Hodge identities (see [GH]). Consider the Lie superalgebra spanned by the classical operators:

\[ S := \langle \triangle, L, H, \Lambda, d, d^*, d_c, d_c^* \rangle. \]

The non-vanishing commutation relations in \( S \) are as follows:

\[ [H, L] = 2L, [H, \Lambda] = -2\Lambda, \]
\[ [L, d] = H, [H, d] = 2L, [H, \Lambda] = -2\Lambda, \]
\[ [d, d^*] = dd^* + d^*d = \triangle, \]
\[ [d_c, d_c^*] = d_c d_c^* + d_c^*d_c = \triangle, \]
\[ [H, d] = d, [H, d^*] = -d^*, \]
\[ [H, d_c] = d_c, [H, d_c^*] = -d_c^*, \]
\[ [L, d^*] = -d_c, [L, d_c^*] = d, \]
\[ [\Lambda, d] = d_c^*, [\Lambda, d_c] = -d^*. \]

**Theorem 6.1.** — Let \( \mathfrak{g} \) be a complex finite-dimensional Lie algebra with a non-degenerate invariant symmetric bilinear form. Then there exist operators on \( W_{rel}^{\infty, +*}(\mathfrak{g}) \), which are analogous to the classical operators in Kähler geometry.

**Proof.** — It was shown in [FGZ] that a relative semi-infinite complex \( C_\infty^*(l, l_0, V) \), where \( l = \oplus_{n \in \mathbb{Z}} l_n \) is a complex \( \mathbb{Z} \)-graded Lie algebra, and \( V \) is a graded Hermitian \( l \)-module, has a structure, which is similar to that of the de Rham complex in Kähler geometry. It is assumed that there exists a 2-cocycle \( \gamma \) on \( l \) such that \( \gamma \mid_{l_n \times l_{-n}} \) is non-degenerate if \( n \in \mathbb{Z} \setminus 0 \) and it is zero otherwise. Then there exist operators on \( C_\infty^*(l, l_0, V) \) analogous to the classical ones.
We will define analogues of the classical operators on $W_{\text{rel}}^\infty (+,\tilde{\mathfrak{g}})$. Using the form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ we obtain the 2-cocycle $\gamma$ on $\tilde{\mathfrak{g}}$:

\begin{equation}
\gamma(g_1 \otimes t^n, g_2 \otimes t^m) = n \langle g_1, g_2 \rangle \delta_{n,-m}, \text{ for } g_1, g_2 \in \mathfrak{g}.
\end{equation}

Notice that $\gamma|_{\mathfrak{g}^n \times \tilde{\mathfrak{g}}^{-n}}$ is non-degenerate if $n \in \mathbb{Z} \setminus 0$ and zero otherwise. Let

\begin{equation}
\Lambda_{\text{rel}}^\infty (+,\tilde{\mathfrak{g}}) = \bigoplus_{a,b \geq 0} \Lambda^a(n'_+) \wedge \Lambda^b(n'_-).
\end{equation}

For a homogeneous element in $\Lambda^a(n'_+) \wedge \Lambda^b(n'_-)$, $a$ is the number of added elements, and $b$ is the number of missing elements with respect to the vacuum vector $\mathbf{1}_{\text{rel}}$. Let

\begin{equation}
C^{a,b}(\tilde{\mathfrak{g}}) = [S_{\text{rel}}^\infty (+,\tilde{\mathfrak{g}}) \otimes \Lambda^a(n'_+) \wedge \Lambda^b(n'_-)]\tilde{\mathfrak{g}}_0.
\end{equation}

We obtain a bigrading on the relative semi-infinite Weil complex, such that

\begin{equation}
W_{\text{rel}}^{\infty,+} (\tilde{\mathfrak{g}}) = \bigoplus_{a-b=i} C^{a,b}(\tilde{\mathfrak{g}}).
\end{equation}

Let $d$ be the restriction of the differential to the relative subcomplex. Notice that

\begin{equation}
d : C^{a,b}(\tilde{\mathfrak{g}}) \longrightarrow C^{a+1,b}(\tilde{\mathfrak{g}}) \oplus C^{a,b-1}(\tilde{\mathfrak{g}}).
\end{equation}

Define $d_1$ and $d_2$ such that

\begin{equation}
d = d_1 + d_2,
\end{equation}

\begin{equation}
d_1 : C^{a,b}(\tilde{\mathfrak{g}}) \longrightarrow C^{a+1,b}(\tilde{\mathfrak{g}}),
\end{equation}

\begin{equation}
d_2 : C^{a,b}(\tilde{\mathfrak{g}}) \longrightarrow C^{a,b-1}(\tilde{\mathfrak{g}}).
\end{equation}

Let

\begin{equation}
d_c = i(d_1 - d_2).
\end{equation}

To define the adjoint operators, we have to introduce a Hermitian form on $W_{\text{rel}}^{\infty,+}(\tilde{\mathfrak{g}})$.

It was shown in [FGZ] that if a $\mathbb{Z}$-graded Lie algebra $\mathfrak{l}$ admits an antilinear automorphism $\sigma$ of order 2 such that $\sigma(l_n) = l_{-n}$, then there exists a Hermitian form on $\Lambda_{\text{rel}}^\infty (+,\mathfrak{l})$ such that

\begin{equation}
\varepsilon(x')^* = -\varepsilon(\sigma(x')), \quad \tau(x)^* = -\tau(\sigma(x)),
\end{equation}

where $x \in \mathfrak{l}, x' \in \mathfrak{l}'$.}

\textit{Annales de L'Institut Fourier}
To define a Hermitian form \{\cdot, \cdot\} on $\Lambda_{\text{rel}}^{\infty+*}(\tilde{\mathfrak{g}})$, we set \{1_{\text{rel}}, 1_{\text{rel}}\} = 1.

We fix a basis \{v_i\} of $\mathfrak{g}$ so that $\langle v_i, v_j \rangle = \delta_{i,j}$. Let $u$ run through this basis.

We define an antilinear automorphism $\sigma$ of $\tilde{\mathfrak{g}}$ as follows:

\begin{equation}
\sigma(u_n) = iu_{-n}.
\end{equation}

Correspondingly,

\begin{equation}
\sigma(u'_n) = -iu'_{-n}.
\end{equation}

We introduce a Hermitian form on $\Lambda_{\text{rel}}^{\infty+*}(\tilde{\mathfrak{g}})$ so that the relations (6.18), where

\begin{equation}
x \in \tilde{\mathfrak{g}}_n, x' \in \tilde{\mathfrak{g}}'_n \text{ for } n \neq 0
\end{equation}

hold. In the similar way we introduce a Hermitian form on $S_{\text{rel}}^{\infty+*}(\tilde{\mathfrak{g}})$, such that

\begin{equation}
\gamma(x')^* = \gamma(\sigma(x')), \quad \beta(x)^* = -\beta(\sigma(x)).
\end{equation}

Then we obtain a Hermitian on $W_{\text{rel}}^{\infty+*}(\tilde{\mathfrak{g}})$ by tensoring these two forms. It gives a pairing: $C^{a,b}(\tilde{\mathfrak{g}}) \rightarrow C^{b,a}(\tilde{\mathfrak{g}})$.

To define a Hermitian form on $C^{a,b}(\tilde{\mathfrak{g}})$, we use the linear map

\begin{equation}
*: C^{a,b}(\tilde{\mathfrak{g}}) \rightarrow C^{b,a}(\tilde{\mathfrak{g}}),
\end{equation}

defined as follows:

\begin{equation}
*v \left( v \otimes (\varepsilon(u'_{n_1}) \cdots \varepsilon(u'_{n_a}) \tau(u_{m_1}) \cdots \tau(u_{m_b}) 1_{\text{rel}}) \right) = v \otimes (\varepsilon(u'_{-m_1}) \cdots \varepsilon(u'_{-m_b}) \tau(u_{-n_1}) \cdots \tau(u_{-n_a}) 1_{\text{rel}}),
\end{equation}

where $v \in S_{\text{rel}}^{\infty+*}(\tilde{\mathfrak{g}}), \{n_i\}_{i=1}^a > 0$ and $\{m_i\}_{i=1}^b < 0$. Finally, the Hermitian form on $C^{a,b}(\tilde{\mathfrak{g}})$ is defined by $(w_1, w_2) = \{i^{a+b} * w_1, w_2\}$ (cf. [FGZ]).

We introduce the adjoint operators $d^*, d^*_c$ and the Laplace operator $\Delta = dd^* + d^*d$.

It was pointed out in [FGZ] that as in the classical theory (see [GH]), there exists an action of $\mathfrak{sl}(2)$ on $H^{\infty}_\lambda(l, l_0, V)$. One can identify $l'_n$ with $L_{-n}$ by means of the cocycle $\gamma$. If $\{e_i\}$ is a homogeneous basis in $l$, then $\mathfrak{sl}(2) = \langle L, H, \Lambda \rangle$ is defined as follows:

\begin{equation}
L = (i/2) \sum_{m \in \mathbb{Z} \setminus 0} \varepsilon(e_m)\varepsilon(e'_m),
\end{equation}

TOME 51 (2001), FASCICULE 3
We identify $g_n$ with $\mathfrak{g}_{-n}$ by means of the cocycle $\gamma$ (see (6.11)), and set

\begin{align*}
H = - \sum_{m \in \mathbb{Z} \setminus 0} \tau(e_m)e(e_m), \\
\Lambda = (i/2) \sum_{m \in \mathbb{Z} \setminus 0} \tau(e_m)e(e_m).
\end{align*}

We identify $\mathfrak{g}_n'$ with $\mathfrak{g}_{-n}$ by means of the cocycle $\gamma$ (see (6.11)), and set

\begin{equation}
(6.26) \quad \mathcal{E} = L, \mathcal{H} = H, \mathcal{F} = \Lambda.
\end{equation}

Then we obtain the $\mathfrak{s}\mathfrak{l}(2) = \langle \mathcal{E}, \mathcal{H}, \mathcal{F} \rangle$ defined in (5.10). The operators

\begin{equation}
(6.27) \quad \{\triangle, \mathcal{E}, \mathcal{H}, \mathcal{F}, d, d^*, d_c, d_e^*\}
\end{equation}

are the analogues of the classical operators (6.9).

\[ \square \]

**Theorem 6.2.** — Let $\mathfrak{g}$ be a complex finite-dimensional Lie algebra with a non-degenerate invariant symmetric bilinear form. Then $H^{\infty}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}_0, S_{\mathfrak{g}}^{\infty}(\mathfrak{g}))$ is a module over $\mathfrak{d}er S'(2,0)$ with central charge $3dimg$.

**Proof.** — By Theorem 5.3, $W_{\mathfrak{rel}}^{\infty}(\mathfrak{g})$ is a module over $\mathfrak{d}er S'(2,0)$ with central charge $3dimg$. By Corollary 5.1, there is an action of $\mathfrak{h}'(2,0)$ on $H^{\infty}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}_0, S_{\mathfrak{g}}^{\infty}(\mathfrak{g}))$. We have proved that

\begin{equation}
(6.28) \quad \mathfrak{d}er_{\mathfrak{ext}} S'(2,0) = \mathfrak{s}\mathfrak{l}(2) = \langle \mathcal{E}, \mathcal{H}, \mathcal{F} \rangle,
\end{equation}

see (5.10). Notice that as in the classical case, the element $\mathcal{F}$ and the differential $d$ do not commute. Nevertheless, there exists an action of $\mathfrak{s}\mathfrak{l}(2)$ on the relative semi-infinite cohomology according to [FGZ].

\[ \square \]

**Theorem 6.3.** — The degree zero part of the $\mathbb{Z}$-grading deg of $S'(2,0)$ is isomorphic to the Lie superalgebra of classical operators in Kähler geometry.

**Proof.** — Recall that the $\mathbb{Z}$-grading deg of $S'(2,0)$ is defined by the element $L_0 \in \text{Witt}$, see (4.17)-(4.19). One can easily check that

\begin{equation}
(6.29) \quad S'(2,0)_{\text{deg}=0} = \langle L_0, E_{-1}, H_0, F_1, h^0_0, p_0, x_{-1}, y^0_{1/2} \rangle.
\end{equation}

The isomorphism of Lie superalgebras

\begin{equation}
(6.30) \quad \psi : \mathcal{S} \longrightarrow S'(2,0)_{\text{deg}=0}
\end{equation}
is given as follows:

\begin{equation}
\psi(\Delta) = L_0, \psi(L) = E_{-1}, \psi(H) = H_0, \psi(\Lambda) = F_1,
\psi(d) = h_0^0, \psi(d^*) = -p_0, \psi(d_c) = x_{-1}, \psi(d_c^*) = y_1^0.
\end{equation}

\[\square\]

**Corollary 6.1.** — The action of \(S'(2, 0)_{\text{deg}=0}\) defines a set of quadratic operators on \(W^{\infty}_{\text{rel}}+(\vec{\mathfrak{g}})\) (correspondingly, on \(H^{\infty}_{\text{rel}}+(\vec{\mathfrak{g}}, \vec{\mathfrak{g}}_0, S^{\infty}_{\text{rel}}+(\vec{\mathfrak{g}}))\)), which are analogues of the classical ones, and include the semi-infinite Koszul differential \(h = h_0^0\) and the semi-infinite homotopy operator \(p_0\).

**Remark 6.1.** — In this work we have realized superconformal algebras by means of quadratic expansions on the generators of the Heisenberg and Clifford algebras related to \(\vec{\mathfrak{g}}\). Note that the differentials on a semi-infinite Weil complex are represented by cubic expansions. One can possibly define an additional (to the already known) action of the \(N = 2\) SCA on \(W^{\infty}_{\text{rel}}+(\vec{\mathfrak{g}})\), considering Fourier components of the differentials \(d\) and \(d^*\), [Fe].

**Acknowledgements.** — This work has been partly done at the Max-Planck-Institut für Mathematik in Bonn, L’Institut des Hautes Études Scientifiques in Bures-sur-Yvette, and the Institute for Advanced Study in Princeton. I wish to thank MPI, IHES, and IAS for their hospitality and support. I am grateful to B. Feigin, A. Givental, M. Kontsevich, V. Serganova, and V. Schechtman for very useful discussions.

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**TOME 51 (2001), FASCICULE 3**


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