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LOCAL REDUCTION THEOREMS AND INVARIANTS FOR SINGULAR CONTACT STRUCTURES

by B. JAKUBCZYK* and M. ZHITOMIRSKII

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1. Introduction.

A contact structure on a manifold M of dimension $n = 2k + 1$ is a smooth field D of hyperplanes $p \rightarrow D(p) \subset T_p M$ which satisfies the contact condition, also called the *Darboux condition*,

$$(DC) \quad (\omega \wedge (d\omega)^k)(p) = (\omega \wedge d\omega \wedge \cdots \wedge d\omega)(p) \neq 0, \quad p \in M,$$

where ω is a locally defined 1-form such that $D(q) = \ker \omega(q)$ for any point q near an arbitrarily chosen point $p \in M$. The Darboux theorem says that around any point p satisfying (DC) there exists a system $(x_1, \dots, x_k, y_1, \dots, y_k, z)$ of local coordinates such that

$$\omega = dz + x_1 dy_1 + \cdots + x_k dy_k.$$

In particular, this means that all 1-forms satisfying the Darboux condition are locally equivalent with respect to the natural action of the group of local diffeomorphisms.

In this paper we study local invariants of singular contact structures which are defined by nonvanishing 1-forms ω satisfying the Darboux condition only on a dense set of points.

To be precise, by a *local singular contact structure* we mean a module $P = (\omega)$ of germs at the origin of differential 1-forms on $M = \mathbb{K}^{2k+1}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) generated by the germ of a nonvanishing 1-form ω such that the set of noncontact points

$$S = \{p : (\omega \wedge (d\omega)^k)(p) = 0\}$$

contains the origin, but is nowhere dense in a neighbourhood of the origin. The module P is considered over the ring of function germs in a fixed category. The set S is called the *Martinet hypersurface*. It is the set of zeros of the *Martinet ideal* (H) generated by the function germ H defined by the relation

$$\omega \wedge (d\omega)^k = H dx_1 \wedge \cdots \wedge dx_n$$

in a given coordinate system.

Our aim is to identify sets of complete invariants (strictly speaking, covariants) of the C^∞ , real-analytic, and holomorphic local singular contact structures, with respect to the natural action of the group of local diffeomorphisms of the same category.

The simplest invariants are the Martinet hypersurface S and the restriction $P|_S$ of $P = (\omega)$ to S defined as follows. We say that S is *structurally smooth* at $p \in S$ if $dH(p) \neq 0$ (then it is smooth at p and it remains smooth in a neighbourhood of p when $P = (\omega)$ is replaced by $\tilde{P} = (\tilde{\omega})$ with $\tilde{\omega}$ close to ω in the C^2 topology). Let S^{reg} be the set of structurally smooth points of S . We prove (Proposition 3.1) that for any local singular contact structure this set is nonempty. The naturally defined pullback of P to S^{reg} will be called the *restriction of P to S* and denoted by $P|_S$ (Section 3.1).

The following question is natural.

Is the hypersurface S together with the restriction $P|_S$ a complete invariant ?

In restricted form, this question was asked in [Ma2] and [JP]. For first occurring singularities an affirmative answer follows from the Martinet normal form (see [Ma1] or Section 3.1). In [Zh1], [Zh2] normal forms for deeper singularities were obtained, which imply an affirmative answer to our question for singularities of codimension ≤ 3 in the C^∞ category. (For such singularities these normal forms imply an affirmative answer to the stronger conjecture, also mentioned in [Zh1], that the *characteristic line field* on S is a complete invariant.)

One of the main results of the present paper gives an affirmative answer to the above question for all singularities excluding degenerations of infinite codimension. The following statement follows from Theorem 3.1 and Proposition 3.4.

THEOREM 1.1. — (i) *In the space of germs at the origin of holomorphic nonvanishing 1-forms ω on \mathbb{K}^{2k+1} there exists a subset E of infinite codimension such that any two local singular contact structures $P = (\omega)$ and $\tilde{P} = (\tilde{\omega})$, with ω and $\tilde{\omega}$ not belonging to E , are equivalent if and only if the pairs $(S, P|_S)$ and $(\tilde{S}, \tilde{P}|_{\tilde{S}})$ are equivalent.*

(ii) *The same is true in the real-analytic category with the space of all germs replaced by the set of germs ω whose Martinet ideal has the property of zeros.*

(iii) *With the definition of the set E given below, the statement (ii) holds in the C^∞ category without the claim $\text{codim } E = \infty$ (we conjecture that this claim is also true).*

We recall that an ideal has the property of zeros if any function vanishing on the set of zeros of this ideal belongs to the ideal. The property of zeros of the Martinet ideal will be called *property (PZ)* (see Section 3.1).

The set E in our theorem can be described explicitly. Namely, we define the *characteristic form*

$$\eta = \omega \wedge (d\omega)^{k-1} \wedge dH$$

and the quotient ideal

$$I_{\text{ch}}(\omega) = I(H, \eta)/(H),$$

where $I(H, \eta)$ denotes the ideal generated by H and the coefficients of the $2k$ -form η . The ideal $I_{\text{ch}} = I_{\text{ch}}(\omega)$ will be called *characteristic ideal* of P (see Sections 3.2 and 3.3).

The set E consists of the germs ω for which $\text{depth}(I_{\text{ch}}(\omega)) < 2$.

In the holomorphic category the condition $\omega \notin E$ is equivalent to the condition $\text{codim}(S_1) = 2$ (in S) or $S_1 = \emptyset$, where the set $S_1 \subset S$ consists of the points of S at which the characteristic form η vanishes. This condition is natural since the set of all local singular contact structures such that $\eta(0) = 0$ is a stratified codimension 2 submanifold in the space of local singular contact structures (see Proposition 3.3 in Section 3.2).

We also introduce the following more subtle invariants:

- (i) an orientation \mathcal{O} (Sections 3.2 and 4.1),
- (ii) a line bundle L over the Martinet hypersurface S (Section 4.1),
- (iii) a canonical partial connection Δ_0 on L at $0 \in \mathbb{K}^n$ (Section 4.2),
- (iv) a 2-dimensional kernel $K(0) = \ker(\omega \wedge (d\omega)^{k-1})(0)$ (Section 4.2).

We establish independence of these invariants and obtain a few results stating that they (or part of them) form a complete set of local invariants. In particular, in the case where S is structurally smooth our results in Sections 2.1 and 4.2 include the following statements.

THEOREM 1.2. — *In the set of holomorphic local singular contact structures $P = (\omega)$ on \mathbb{C}^{2k+1} , with the Martinet hypersurface S structurally smooth, the following objects form complete sets of local invariants:*

- (a) S and $P|_S$, if $k = 1$,
- (b) S , $P|_S$ and $K(0)$, if $\ker \omega(0)$ is transversal to S ,
- (c) L and Δ_0 (under no a priori assumptions).

These statements also hold for C^∞ and real-analytic local singular contact structures on \mathbb{R}^{2k+1} if we add to the invariants the orientation \mathcal{O} , with the exception of statement (a) in the C^∞ category.

A version of this result was already announced in [JZh2].

Another result of this paper, Theorem 3.2, concerns the more general case; we allow S to have singularities. We introduce the condition $\text{gen}(I_{\text{ch}}) \geq 2$, where $\text{gen}(I_{\text{ch}})$ denotes the minimal number of generators of the ideal I_{ch} , which is slightly weaker than the condition $\omega \notin E$. We show that under this condition the restriction and the orientation are complete invariants.

Relations between the invariants are studied in Section 4.3. We prove that under the condition $\omega \notin E$ the restriction determines the orientation, and under the condition $\text{gen}(I_{\text{ch}}) \geq 2$ the line bundle determines the partial connection.

In Section 5 we present a realization theorem which describes the Pfaffian equations on the Martinet hypersurface S that can be obtained as restrictions of local singular contact structures to S (see also Section 2.1).

Our results can be used for obtaining a classification of certain classes of Pfaffian systems as we explain in Section 2.4. In particular, in dimension 3 our results allow reducing the classification problem of local singular contact structures in the smooth category to the known results on smooth orbital classification of singularities of vector fields [B], [Ro]. In the analytic categories the functional moduli appear in the local classification of Pfaffian equations in dimension 3, just as they appear in the orbital equivalence of holomorphic or real-analytic singularities of vector fields (Martinet-Ramis).

Other applications of our results in dimension 3 are presented in Section 2.3. In particular, our realization theorem allows us to introduce an interesting class of singular foliations orienting the plane. We observe that the characteristic vector field of a local singular contact structure introduced by us in Section 3.2 has a meaning in sub-Riemannian geometry (cf. e.g. [A], [Mon], and [LS]). Namely, the so-called abnormal curves appearing there are the integral curves of our characteristic vector field. Our results imply that in dimension 3 the abnormal curves determine the local Pfaffian equation up to equivalence.

Restrictions of contact structures to odd-dimensional generic submanifolds are singular contact structures. Thus our results and the relative Darboux theorem determine complete local invariants of generic submanifolds

of contact manifolds. Another reason for interest in the local structure of Pfaffian equations comes from the geometric theory of differential equations [BC3G], [VKL].

2. Dimension 3.

2.1. Main results.

We begin formulating our results with the somewhat easier case of dimension 3 and structurally smooth Martinet surface S . Let us recall that by equivalence of germs of two objects (or tuples of objects) we mean the fact that one of them (or one tuple) can be transformed to the other by the germ of a diffeomorphism.

THEOREM 2.1. — *Let $P = (\omega)$ and $\tilde{P} = (\tilde{\omega})$ be local singular contact structures on \mathbb{C}^3 with structurally smooth S and \tilde{S} . Then, in the holomorphic category, equivalence of the pairs $(S, P|_S)$ and $(\tilde{S}, \tilde{P}|_{\tilde{S}})$ implies equivalence of P and \tilde{P} .*

THEOREM 2.2. — *The same is true in \mathbb{R}^3 in the C^∞ and real-analytic categories provided that the 1-form $\alpha = \omega|_S$ on S either does not vanish or has an algebraically isolated singularity at the origin.*

The condition that α has an algebraically isolated singularity means that factorizing the ring of all function germs over the ideal generated by the coefficients α_1, α_2 of $\alpha = \alpha_1(x, y)dx + \alpha_2(x, y)dy$ we obtain a linear space of finite dimension.

The difference between the holomorphic and the other two categories is due to the existence of a canonical orientation \mathcal{O}_S on the Martinet surface S which is, in general, an independent invariant. The canonical orientation is defined as follows.

We use the following convention on inducing orientation from an n -manifold M to its boundary ∂M . Let $M \subset \mathbb{R}^n$ and, locally, $\partial M = \{h = 0\}$, $M = \{p \in \mathbb{R}^n : h(p) \geq 0\}$, where $h : \mathbb{R}^n \rightarrow \mathbb{R}$ is a function regular on ∂M . Then any orientation on M given by an n -form Ω induces an orientation on ∂M which, by our convention, is the orientation defined by the unique volume form $\Omega_{\partial M}$ on ∂M satisfying $\Omega_{\partial M}(p) \wedge dh(p) = \Omega(p)$, $p \in \partial M$.

When $S = \{p \in \mathbb{R}^3 : (\omega \wedge d\omega)(p) = 0\}$ is structurally smooth at $0 \in \mathbb{R}^3$, then S divides a neighbourhood $U \subset \mathbb{R}^3$ of the origin into two

disjoint connected open subsets U^+ and U^- such that $U \setminus S = U^+ \cup U^-$. The differential 3-form $\omega \wedge d\omega$ (where ω is a generator of P) does not vanish on U^+ and on U^- , and so it defines orientations \mathcal{O}^+ and \mathcal{O}^- on U^+ and U^- , respectively. These orientations depend on P but are independent of the choice of the generator ω (since $\phi\omega \wedge d(\phi\omega) = \phi^2\omega \wedge d\omega$). As S is part of the boundary of U^+ and U^- , the orientations \mathcal{O}^+ and \mathcal{O}^- induce orientations \mathcal{O}_S^+ and \mathcal{O}_S^- on S . It follows from the structural smoothness of S that the 3-form $\omega \wedge d\omega$ “changes sign” when passing from U^+ to U^- . Therefore \mathcal{O}_S^+ and \mathcal{O}_S^- coincide. We define* the orientation \mathcal{O}_S as $\mathcal{O}_S = \mathcal{O}_S^+ = \mathcal{O}_S^-$.

Note that a volume form Ω_S on S defining \mathcal{O}_S can be uniquely determined from the equation $\Omega_S(p) \wedge dH(p) = (H^{-1}\omega \wedge d\omega)(p)$, for $p \in S$, where H is any generator of the Martinet ideal (see also Section 3.5).

THEOREM 2.3. — *Theorem 2.1 holds in \mathbb{R}^3 , in the real-analytic category, with the pairs $(S, P|_S)$ and $(\tilde{S}, \tilde{P}|_{\tilde{S}})$ replaced by the triples $(S, P|_S, \mathcal{O}_S)$ and $(\tilde{S}, \tilde{P}|_{\tilde{S}}, \tilde{\mathcal{O}}_S)$. It also holds in \mathbb{R}^3 in the smooth category, with the same replacement, provided that the pullback $\omega|_S$ is either not flat at 0 or $\omega|_S \equiv 0$.*

Above and thereafter by a flat form we mean a form with coefficients being flat functions (i.e., having vanishing Taylor series).

Theorems 2.1, 2.2, and 2.3 are explained in the next subsection. There we also show that the orientation \mathcal{O}_S is, in general, an independent invariant, i.e., Theorem 2.1 does not hold in the C^∞ and real-analytic categories. The additional “nonflatness” requirement in Theorem 2.3 in the smooth case can not be removed. This will be shown in Section 4 where we introduce (for any dimension $n \geq 3$) one more invariant - a canonical partial connection.

Theorems 2.1 and 2.2 imply many corollaries provided that we know what Pfaffian equations on \mathbb{R}^2 are realizable, i.e., can be obtained by restricting a singular contact structure to the Martinet surface. The realization theorem in \mathbb{R}^3 is as follows.

THEOREM 2.4. — *Let P be a local singular contact structure on \mathbb{R}^3 with structurally smooth Martinet surface S , and let (α) be the restriction*

* This definition was suggested to us by V.I. Arnold.

of P to S . Then there exists a 1-form β on S such that

$$(2.1) \quad d\alpha = \alpha \wedge \beta, \quad d\beta(0) \neq 0$$

and the canonical orientation \mathcal{O}_S is defined by $d\beta$. Vice versa, if α and β are local 1-forms on a smooth surface $S \subset \mathbb{R}^3$ satisfying (2.1) then there exists a local singular contact structure P on \mathbb{R}^3 with structurally smooth Martinet surface S , such that $P|_S = (\alpha)$, and the canonical orientation \mathcal{O}_S is defined by $d\beta$. Both statements hold in the categories C^∞ and C^ω . They also hold on \mathbb{C}^3 in the holomorphic category, with the conditions involving the orientation removed.

In the following subsection we use the realization theorem to show that if the restriction (α) has an algebraically isolated singularity then it uniquely determines \mathcal{O}_S .

2.2. Determinacy of the orientation by the restriction.

In this subsection we explain the relation between the restriction $P|_S$ and the orientation \mathcal{O}_S , in the real categories C^ω and C^∞ .

Let us fix a surface S and a Pfaffian equation (α) on $S \subset \mathbb{R}^3$, which is realizable in the sense of Theorem 2.4. Consider arbitrary local singular contact structures P and \tilde{P} on \mathbb{R}^3 with the structurally smooth Martinet surface S and the same restriction (α) to S . A priori there are 3 possibilities:

(a) Any such P and \tilde{P} define the same orientation $\mathcal{O}_S = \tilde{\mathcal{O}}_S$ on S (then we say that $P|_S$ determines \mathcal{O}_S).

(b) There exist such P and \tilde{P} which define different canonical orientations $\mathcal{O}_S \neq \tilde{\mathcal{O}}_S$. There is a diffeomorphism $\Phi : S \rightarrow S$ that preserves (α) and reverses the orientation of S .

(c) There exist such P and \tilde{P} which define different canonical orientations $\mathcal{O}_S \neq \tilde{\mathcal{O}}_S$. Any diffeomorphism $\Phi : S \rightarrow S$ that preserves (α) also preserves the orientation \mathcal{O}_S .

Note that in the cases (a) and (b) the triples $(S, P|_S, \mathcal{O}_S)$ and $(\tilde{S}, \tilde{P}|_{\tilde{S}}, \tilde{\mathcal{O}}_S)$ are equivalent if and only if so are the pairs $(S, P|_S)$ and $(\tilde{S}, \tilde{P}|_{\tilde{S}})$.

We will show below that all the cases (a), (b), and (c) are realizable.

PROPOSITION 2.1. — *If $\alpha(0) = 0$ and 0 is an algebraically isolated zero of α then the restriction $P|_S$ determines the orientation \mathcal{O}_S , i.e., the statement (a) holds.*

Proof. — Let P and \tilde{P} be local singular contact structures with the same S and the same restriction (α) to S . Let \mathcal{O}_S and $\tilde{\mathcal{O}}_S$ be the canonical orientations of S defined by P and \tilde{P} . By Theorem 2.4 we have $d\alpha = \alpha \wedge \beta$ and $d\alpha = \alpha \wedge \tilde{\beta}$ for some 1-forms β and $\tilde{\beta}$ on S with nonvanishing differentials and defining the orientations \mathcal{O}_S and $\tilde{\mathcal{O}}_S$, respectively. It follows that $\alpha \wedge (\beta - \tilde{\beta}) = 0$. Since α has an algebraically isolated singularity at the origin we can use a result in [Mou] to conclude that $\beta - \tilde{\beta} = f\alpha$ for some function f . Since $\alpha(0) = 0$ then, by Theorem 2.4, $d\alpha(0) = 0$ and consequently $d\beta(0) = d\tilde{\beta}(0)$. Therefore $\mathcal{O}_S = \tilde{\mathcal{O}}_S$. Q.E.D.

Note that, by Proposition 2.1, Theorem 2.2 follows from Theorem 2.3 and the following remark.

We have the case (b) if the restriction is not singular, i.e., $\alpha(0) \neq 0$. The classical Martinet theorem says that in this case P is equivalent to the fixed singular contact structure $(dy - x^2dz)$, therefore P (and so \mathcal{O}_S) is determined up to a diffeomorphism by $P|_S$. For example let us take $(P, \tilde{P}) = (P_+, P_-) = (dy \pm x^2dz)$. Then $S = S_+ = S_- = \{x = 0\}$ and we may take $\alpha = \alpha_+ = \alpha_- = dy$. The canonical orientations on S are defined by 2-forms $\pm dz \wedge dy$. The existence of an orientation reversing symmetry of (α) is obvious.

The following example shows that the case (c) is realizable, i.e., the orientation \mathcal{O}_S is an independent invariant. We have to take α with singularity of infinite codimension.

Example 2.1. — Define $\alpha = xdy + (3y + x^3y^2)dx$. The Pfaffian equation $(x^2\alpha)$ does not admit symmetries reversing the orientation of the plane (x, y) (the proof of this fact is given below) and is realizable. Namely, the Pfaffian equations P_\pm generated by 1-forms $\omega_\pm = dz + x^2\alpha + z(2x^2ydx \pm \alpha)$ have the same structurally smooth Martinet surface $S = \{z = 0\}$ and the same restriction $(x^2\alpha)$ to S . They define different canonical orientations on S . Therefore P_+ is not equivalent to P_- .

The proof that the Pfaffian equation $(x^2\alpha)$ on \mathbb{R}^2 does not admit orientation-reversing symmetries is as follows. Let Ψ be a symmetry of (α) . Then Ψ is an orbital symmetry of the vector field

$$(2.2) \quad V = x^2V_1, \quad \text{where} \quad V_1 = x \frac{\partial}{\partial x} - (3y + x^3y^2) \frac{\partial}{\partial y},$$

which means that $\Psi_*V = QV$, $Q(0) \neq 0$ for some function $Q(x, y)$. The y -axis is the set of singular points of V , therefore Ψ brings the function

x to a function $A(x, y)x$, $A(0) \neq 0$. Then $x^2A^2(x, y)\Psi_*V_1 = x^2V_1$ and so $V_1 = A^2\Psi_*V_1$. Let $u = x^3y$ and $w = u \circ \Psi$. Note that $V_1(u) = -u^2$. This relation and $V_1 = A^2\Psi_*V_1$ imply

$$(2.3) \quad V_1(w) = -A^2(x, y)w^2.$$

The coordinate axis are invariant manifolds of the vector field V_1 , therefore Ψ has the form $x \rightarrow A(x, y)x$, $y \rightarrow B(x, y)y$. Let $T = A^3(x, y)B(x, y)$. Then it follows from (2.3) that

$$(2.4) \quad -V_1(T) + Tu = A^2T^2u.$$

The vector field V_1 has the property: for any function T the formal series of the function $V_1(T)$ does not contain the resonant term x^3y . Therefore, taking the formal series of the relation (2.4) we obtain $T(0) = A^2(0)T^2(0)$ and, consequently, $T(0) = A^3(0)B(0) > 0$. Thus $\det \Psi'(0) = A(0)B(0) > 0$, i.e., Ψ does not change the orientation of the plane.

2.3. Corollaries.

This and the next subsections contain corollaries of Theorems 2.1 - 2.4.

At first we note that Proposition 2.1 and Theorem 2.4 allow us to define an interesting class of foliations of a plane orienting the plane. Observe that any Pfaffian equation (α) on the plane defines a foliation F_α (with singularities) of the plane by curves. These curves are the phase curves of the vector field defined by $\dot{x} = \alpha_2(x, y)$, $\dot{y} = -\alpha_1(x, y)$, where $\alpha = \alpha_1(x, y)dx + \alpha_2(x, y)dy$.

COROLLARY 2.1. — *Any foliation of the plane defined by a C^∞ Pfaffian equation (α) such that α has an algebraically isolated singularity at the origin and satisfies the condition (2.1), for some 1-form β , defines a unique orientation of the plane described by the 2-form $d\beta$.*

To prove this statement we show that the foliation F_α can be identified with the Pfaffian equation (α) . More precisely, if α and $\tilde{\alpha}$ have algebraically isolated singularity at the origin and $F_\alpha = F_{\tilde{\alpha}}$ then $(\alpha) = (\tilde{\alpha})$. Namely, the equality of the foliations implies that $\alpha \wedge \tilde{\alpha} = 0$. Using a result in [Mou] and the condition of algebraic isolation we can conclude that $\alpha = f\tilde{\alpha}$ and $\tilde{\alpha} = \tilde{f}\alpha$ for some functions f and \tilde{f} . Therefore the Pfaffian equations (α)

and $(\tilde{\alpha})$ are the same and the corollary follows from Proposition 2.1 and Theorem 2.4.

These arguments allow us to reformulate Theorem 2.2 in the following geometric way.

COROLLARY 2.2. — *Let D be a 2-distribution on \mathbb{R}^3 described by a 1-form ω , with the Martinet surface S corresponding to (ω) structurally smooth. Let F be the foliation of S cut by D . If the restriction $\alpha = \omega|_S$ is nonsingular, or it has an algebraically isolated singularity at the origin, then the foliation F is a complete invariant of D in the categories C^∞ and C^ω .*

The foliation F plays a special role in the sub-Riemannian geometry and control theory. It consists of so-called abnormal (or singular) curves that are the critical points of the endpoint map, see [A], [Mon], [LS]. The significance of abnormal curves as local minimizers in sub-Riemannian geometry was showed by R. Montgomery, see [Mon]. The last corollary can be reformulated as follows.

Under the given mild condition on α the set of all abnormal curves is a complete invariant of the distribution D .

The case (b) of the previous subsection, when the restriction $P|_S$ does not define the orientation O_S , but the restriction $P|_S$ admits an orientation reversing symmetry, is not exhausted by the Martinet case $\alpha(0) \neq 0$. Another simple example is the case $\alpha \equiv 0$ which means that the field of planes defined by P is everywhere tangent to the Martinet hypersurface S . This case is realizable, by Theorem 2.4, and an example of P satisfying this requirement is as follows: $P_\pm = (dz \pm xzdy)$ (the canonical orientations on $S = \{z = 0\}$ corresponding to the signs $+$ and $-$ are different). Obviously, (α) admits symmetries reversing the orientation of S and, by Theorems 2.2 and 2.3, we obtain the following corollary.

COROLLARY 2.3. — *Any field of planes on \mathbb{K}^3 described by $\ker \omega$, which has nonempty and structurally smooth Martinet surface S , and which is tangent to S at any point $p \in S$, is locally equivalent to the field of planes described by the 1-form $dz + xzdy$ (this holds in holomorphic, real analytic, and C^∞ categories).*

Our construction of the orientation \mathcal{O}_S and Proposition 2.1 imply the following result, which might be used in the global study of singular contact structures.

COROLLARY 2.4. — *Let $P = (\omega)$ be a global singular contact structure on a 3-manifold M^3 with structurally smooth Martinet surface S and ω nonvanishing. Then S is an orientable surface, with the orientation O_S invariantly related to P . If there exists a point $p \in S$ at which the pullback $\alpha = \omega|_S$ has an algebraically isolated singularity, then the orientation O_S is uniquely determined by the germ of α at p .*

2.4. Local classification of singularities of plane fields on \mathbb{R}^3 .

Below we use the reduction Theorem 2.2 and the realization Theorem 2.4 in order to reduce the local classification of singular contact structures to the well known problem of the orbital classification of vector fields on the plane.

Let P be a local singular contact structure on \mathbb{R}^3 given by a C^∞ nonvanishing 1-form ω with structurally smooth Martinet surface S . Let (α) be the restriction of P to S . By Theorem 2.2, P is determined up to equivalence by (α) , provided that $\alpha(0) \neq 0$ or α has an algebraically isolated singularity at the origin. This reduces the classification of P 's (satisfying the given conditions) to classification of Pfaffian equations $(\alpha) = P|_S$. The latter problem coincides with the problem of orbital classification of vector fields on the plane (where equivalence is defined up to diffeomorphisms and multiplication by nonvanishing functions).

Assume that $\alpha(0) = 0$. Then the realizability condition (2.1) implies that $d\alpha(0) = 0$. Let X be a vector field obtained from α via a nondegenerate volume form Ω on S : $X \lrcorner \Omega = \alpha$. The vector field X defined up to multiplication by a nonvanishing function is invariantly related to P . It is called *characteristic vector field on S* . The condition $d\alpha(0) = 0$ is equivalent to the fact that the sum of the eigenvalues of the linear part of X at the origin is equal to zero. Therefore we can distinguish *hyperbolic, elliptic, and parabolic singularities* of local singular contact structures on \mathbb{R}^3 with S structurally smooth. They correspond, respectively, to the eigenvalues $\pm 1, \pm i$ and $0, 0$ (up to multiplication by a common factor).

In the hyperbolic case the characteristic vector field X is reducible, by a formal change of coordinates, to the resonant normal form

$$(2.5) \quad \dot{x}_1 = x_1(1 + f(w)), \quad \dot{x}_2 = x_2(-1 + g(w)), \quad w = x_1x_2,$$

where $f(0) = g(0) = 0$, see [AI]. One can show that the realizability condition (2.1), considered for the Pfaffian equation (α) corresponding to

a vector field having the formal normal form (2.5), is equivalent to the condition

$$(2.6) \quad f'(0) + g'(0) \neq 0.$$

It is known that a smooth vector field X whose resonant normal form (2.5) satisfies (2.6) is smoothly orbitally equivalent to the normal form

$$\dot{x}_1 = x_1(1 + x_1x_2 + b(x_1x_2)^2), \quad \dot{x}_2 = -x_2,$$

where b is a modulus, see [AI], [Bo]. It follows that the restriction to the Martinet surface of any hyperbolic singular contact structure is generated, in suitable coordinates, by the 1-form

$$(2.7) \quad \alpha = x_2dx_1 + x_1(1 + x_1x_2 + b(x_1x_2)^2)dx_2.$$

Note that $d\alpha = \alpha \wedge \beta$ where,

$$(2.8) \quad \beta = (2x_1 + 3bx_1^2x_2)dx_2.$$

The singular contact structure $(dz + \alpha + z\beta)$ has the Martinet surface $S = \{z = 0\}$ structurally smooth and its restriction to S is generated by α . Thus, from Theorem 2.2 we obtain the following normal form for hyperbolic singularities of singular contact structures.

COROLLARY 2.5. — *Any C^∞ singular contact structure on \mathbb{R}^3 having a hyperbolic singularity at the origin is locally C^∞ equivalent to the singular contact structure generated by the 1-form $dz + \alpha + z\beta$, where α and β have the form (2.7) and (2.8).*

In the elliptic case X is reducible by a formal diffeomorphism to the resonant normal form

$$(2.9) \quad \dot{x}_1 = -x_2 + x_1f(w), \quad \dot{x}_2 = x_1 + x_2g(w), \quad w = x_1^2 + x_2^2,$$

where $f(0) = g(0) = 0$. Like in the hyperbolic case, the realizability condition (2.1), expressed in terms of this normal form, is equivalent to the condition (2.6). In the elliptic case this condition means that the first focus number of X is different from zero, see [AI]. A smooth vector field X whose resonant normal form (2.9) satisfies (2.6) is smoothly orbitally equivalent to the normal form

$$\dot{x}_1 = -x_2 + x_1(x_1^2 + x_2^2 + b(x_1^2 + x_2^2)^2), \quad \dot{x}_2 = x_1 + x_2(x_1^2 + x_2^2 + b(x_1^2 + x_2^2)^2),$$

where b is a modulus, see [AI], [Bo]. It follows that the restriction to the Martinet surface of any elliptic singular contact structure is generated, in suitable coordinates, by the 1-form

$$(2.10) \quad \alpha = -(x_1 + x_2(x_1^2 + x_2^2 + b(x_1^2 + x_2^2)^2))dx_1 + (-x_2 + x_1(x_1^2 + x_2^2 + b(x_1^2 + x_2^2)^2))dx_2.$$

Note that $d\alpha = \alpha \wedge \beta$, where

$$(2.11) \quad \beta = (4 + 6b(x_1^2 + x_2^2))(x_2dx_1 - x_1dx_2).$$

By Theorem 2.2 we obtain the following normal form for elliptic singularities of singular contact structures.

COROLLARY 2.6. — *Any C^∞ singular contact structure on \mathbb{R}^3 having an elliptic singularity at the origin is locally C^∞ equivalent to the singular contact structure generated by the 1-form $dz + \alpha + z\beta$, where α and β have the form (2.10) and (2.11).*

The results of Corollaries 2.5 and 2.6 are not new; equivalent normal forms were obtained in [Zh1] by rather complicated techniques. The new classification results of the present subsection concern analytic hyperbolic and elliptic singularities and smooth parabolic singularities.

In the real-analytic category our reduction and realization theorems also reduce (in the same way) the classification of plane fields on \mathbb{R}^3 to the orbital classification of vector fields having the resonant formal normal form (2.5) or (2.9) and satisfying the condition (2.6). The classification of such vector fields in the real-analytic category is much more complicated than in the C^∞ category. It was obtained by Martinet and Ramis [MR]. The orbitally nonequivalent germs of such vector fields are distinguished by the modulus b of the formal normal forms (2.7) and (2.10) and certain functional moduli of the kind of Ecalle-Voronin. We obtain

COROLLARY 2.7. — *These moduli are the only ones in the classification of hyperbolic and elliptic singularities of analytic local singular contact structures.*

Now we consider the case where a singular contact structure P has a parabolic singularity at the origin. In this case the characteristic vector field X has zero eigenvalues. We will say that a parabolic singularity is *nilpotent* if $j_0^1 X \neq 0$, i.e., the differential equation corresponding to the linear approximation of X is equivalent to $\dot{x}_1 = x_2$, $\dot{x}_2 = 0$. A nilpotent singularity will be called algebraically isolated if the origin is

an algebraically isolated singular point of X . In this case X is formally orbitally equivalent to the normal form $\dot{x}_1 = x_2$, $\dot{x}_2 = x_1^m + x_2g(x_1)$, where $m \geq 2$, $g(0) = 0$, see [AI]. Therefore the restriction of P to the Martinet surface is formally equivalent to the Pfaffian equation generated by a 1-form $\alpha = (x_1^m + x_2g(x_1))dx_1 - x_2dx_2$. It is easy to see that the realizability condition (2.1) is equivalent to the condition that $g(x_1) = x_1^m h(x_1)$, where $h'(0) \neq 0$. Under this condition $d\alpha = \alpha \wedge \beta$, where $\beta = x_1^m h^2(x_1)dx_1 - h(x_1)dx_2$. Using Borel's theorem on existence of a smooth function with a prescribed Taylor series and Theorem 2.1 in the C^∞ category, we obtain the following classification result.

COROLLARY 2.8. — *Any local singular contact structure having an algebraically isolated nilpotent singularity at the origin is formally equivalent to the local singular contact structure generated by the 1-form $dz + \alpha + z\beta$,*

$$\alpha = -x_1^m(1 + x_2h(x_1))dx_1 + x_2dx_2, \quad \beta = x_1^m h^2(x_1)dx_1 - h(x_1)dx_2,$$

where $m \geq 2$ and h is a C^∞ -function with $h'(0) \neq 0$.

3. Reduction theorems for general S .

In the present section we extend our results in Section 2.1 to arbitrary dimensions, without assuming S to be regular. We introduce the characteristic ideal (Section 3.2) and state a theorem which says that under a natural condition on the depth of the characteristic ideal the pair $(S, P|_S)$ is a complete set of invariants (Section 3.3). Generalizing the notion of orientation \mathcal{O} to arbitrary dimension and using a weaker condition on the minimal number of generators of the characteristic ideal, we state a theorem on sufficiency of the triple of invariants $(S, P|_S, \mathcal{O})$ (Section 3.4).

In our considerations we fix a category \mathcal{C} which is any of the three categories: holomorphic, $\mathcal{C} = C^h$, real-analytic, $\mathcal{C} = C^\omega$, or smooth, $\mathcal{C} = C^\infty$.

We denote by R the ring of germs at $0 \in \mathbb{K}^n$ of functions of category \mathcal{C} .

3.1. Basic notions and notation.

Let $P = (\omega)$ be a *local Pfaffian equation*, i.e., a module over R of germs at the origin of differential 1-forms of category \mathcal{C} on \mathbb{K}^n , $n = 2k + 1$.

Given a generator ω of P and the germ at the origin of a nondegenerate n -form Ω on \mathbb{K}^n of category \mathcal{C} , we define the function germ

$$H = \frac{\omega \wedge (d\omega)^k}{\Omega}.$$

The ideal (H) of R generated by H is called the *Martinet ideal* of P and it is independent of the choice of ω and Ω . The germ of the set of noncontact points $S = \{p : H(p) = 0\}$, called the *Martinet hypersurface*, is the set of zeros of the Martinet ideal (H) . By definition, P is a *local singular contact structure* if a representative of S is a nowhere dense subset defined in a neighbourhood of $0 \in \mathbb{K}^n$ (equivalently, H is a noninvertible nonzerodivisor in R).

We call S *smooth* at $0 \in S$ if S is a smooth submanifold of $M = \mathbb{K}^n$ of codimension 1 in a neighbourhood of 0. A slightly stronger property says that S is *structurally smooth* at $0 \in S$ if $dH(0) \neq 0$. Note that structural smoothness of S is the property of the ideal (H) but this abuse of language should not cause confusion. Given a representative H , we define the *regular part* of S as $S^{\text{reg}} = \{p \in S : dH(p) \neq 0\}$, and the *singular part* $S^{\text{sing}} = \{p \in S : dH(p) = 0\}$. Again, S^{reg} and S^{sing} depend on the ideal (H) .

(PZ) We say that a local Pfaffian equation P has the property of zeros, called briefly property (PZ), if the ideal (H) of R has the property of zeros, i.e., any function germ $\psi \in R$ which vanishes on $S = \{H = 0\}$ belongs to the ideal (H) .

If P is a local Pfaffian equation with S structurally smooth, then P is a local singular contact structure and the property of zeros (PZ) holds automatically. Additionally, when our ideal has the property of zeros, it is defined uniquely in a given category by the hypersurface S (as the set of function germs vanishing on S). Then the structural smoothness and the regular and singular parts S^{reg} , S^{sing} are defined by S itself (and not only by the ideal (H)). The following fact will be proved in Section 3.6.

PROPOSITION 3.1. — If $P = (\omega)$ is a local singular contact structure which has the property (PZ), then the regular part S^{reg} of S is nonempty. In the holomorphic category S^{reg} is dense in S if and only if P has the property (PZ).

Example 3.1. — In the real categories the regular part of S may be not dense in S . Let $\omega = dz + (\frac{1}{3}x^3 - xzy^2)dy$. Then the Martinet ideal is

generated by the function $H = x^2 - zy^2$. Its zero set (called Whitney umbrella) contains the halfline $x = 0, y = 0, z < 0$ which is the set $S \setminus cl(S^{reg})$.

Remark (Realization of S). — Any function H which does not vanish at generic points near $0 \in \mathbb{K}^{2k+1}$ (i.e., such that H is a nonzerodivisor in R) can be realized as a generator of the Martinet ideal of a singular contact structure $P = (\omega)$ of category \mathcal{C} . In particular, in the C^∞ category S can be the germ of any closed, nowhere dense set. This follows from the fact that, given an arbitrary function germ H of category \mathcal{C} , the Pfaffian equation generated by the 1-form $\omega = dz + h(x, y, z)dy_1 + x_2dy_2 + \cdots + x_kdy_k$, where $\partial h / \partial x_1 = H$, $x = (x_1, \dots, x_k)$, and $y = (y_1, \dots, y_k)$, has the Martinet ideal equal to (H) .

Let $P = (\omega)$ be a local singular contact structure of class \mathcal{C} and assume that $0 \in \mathbb{K}^n$ is a structurally smooth point of S . Then S is the germ of a regular hypersurface in M . We define the *restriction $P|_S$ of P to S* as the module (over the ring of function germs on S) of differential 1-forms, generated by $\omega|_S$, where $\omega|_S$ denotes the pullback of ω to S .

The notion of the restriction $P|_S$ can be generalized to include the case where the Martinet hypersurface is not structurally smooth. Since the regular part S^{reg} is a submanifold of \mathbb{K}^n , then the pullback $\omega|_{S^{reg}}$ is well defined. We introduce the ring $R|_S$ as the ring of (the germs of) functions on S which have extensions of category \mathcal{C} to a neighbourhood of 0 in \mathbb{K}^n . By definition the *restriction of P to S* , denoted by $P|_S$, is the module over $R|_S$ of the germs of differential 1-forms on S^{reg} , generated by $\omega|_{S^{reg}}$. Note that if P has the property (PZ), then the ring $R|_S$ is isomorphic to the quotient ring $R/(H)$.

Remark. — Note that the ring $R|_S$ determines S and so, with our definition of the restriction $P|_S$, the restriction determines S . In our theorems we will not use this implicit encoding of S in $P|_S$ and we will list S as a separate, though dependent, invariant.

It was proved by Martinet [Ma1] that all local singular contact structures $P = (\omega)$ which satisfy the condition (called below *Martinet condition*)

$$(MC) \quad (\omega \wedge (d\omega)^{k-1} \wedge dH)(0) \neq 0$$

are equivalent, and are equivalent to P generated by the 1-form

$$\omega = dz + x_1^2 dy_1 + x_2 dy_2 \cdots + x_k dy_k.$$

For this normal form the Martinet ideal is (x_1) and the Martinet hypersurface is given by the equation $x_1 = 0$. The restriction $P|_S$ is generated by $\alpha = dz + x_2 dy_2 \cdots + x_k dy_k$. This 1-form satisfies the genericity condition $(\alpha \wedge (d\alpha)^{k-1})(0) \neq 0$, therefore, it defines a quasicontract structure on S .

The Martinet singularity characterized by the condition (MC) is the least degenerated singularity of a singular contact structure and its codimension is 1 (which corresponds to the condition $H(0) = 0$). It is easy to see that condition (MC) is violated if either $dH(0) = 0$, i.e., if S is not structurally smooth, or $dH(0) \neq 0$, but $P|_S$ is not a quasicontract structure. In order to study the singularities which violate condition (MC) we introduce the following notion.

3.2. Characteristic ideal.

We define the *characteristic differential form* of a local singular contact structure $P = (\omega)$ on \mathbb{K}^n , $n = 2k + 1$, as the germ at the origin of the $2k$ -form

$$\eta = \omega \wedge (d\omega)^{k-1} \wedge dH.$$

This form depends on the choice of the generators ω and H of the Pfaffian equation P and the Martinet ideal (H) , respectively. However, the module of germs of $2k$ -forms generated by η depends only on P if the $2k$ -forms and the function germs are taken modulo (H) , i.e., if the ring R is replaced by the quotient ring $R/(H)$.

We will denote by $I(H, \eta)$ the ideal of the ring of germs R generated by H and the coefficients of the characteristic $2k$ -form η , in some coordinate system. (Equivalently, $I(H, \eta)$ is the ideal generated by H and all function germs of the form $\eta(X_1, \dots, X_{n-1})$, where X_1, \dots, X_{n-1} are germs of vector fields.) We introduce the *characteristic ideal* of P as the following ideal of the quotient ring $R/(H)$

$$I_{\text{ch}} = I(H, \eta)/(H).$$

The ideals $I(H, \eta)$ and I_{ch} do not depend on the choice of the generators H and ω , neither they depend on the choice of coordinates used to take coefficients of η .

Note that the Martinet condition (MC) means that the characteristic form does not vanish at 0 and it is equivalent to the condition $I_{\text{ch}} = R/(H)$.

Sometimes, instead of the characteristic form, it is more convenient to use a dual object called characteristic vector field. It can be defined as follows (see Section 2.4 for a special case).

Let Ω be the germ at $0 \in \mathbb{K}^n$ of a nondegenerate n -form of category \mathcal{C} on \mathbb{K}^n . At first we introduce the *characteristic vector field* Z on \mathbb{K}^n to be the germ of the vector field defined by

$$Z \lrcorner \Omega = \omega \wedge (d\omega)^{k-1} \wedge dH.$$

The characteristic ideal can be equivalently defined as $I_{\text{ch}} = I(H, Z)/(H)$, where $I(H, Z)$ denotes the ideal generated by H and the coefficients of the characteristic vector field Z .

Note that the vector field Z is tangent to S . Namely, if $p \in S^{\text{sing}}$, then $dH(p) = 0$ and $Z(p) = 0$. We also have $(Z \lrcorner dH)\Omega = -dH \wedge (Z \lrcorner \Omega) = -dH \wedge \omega \wedge (d\omega)^{k-1} \wedge dH = 0$ which means that $Z \lrcorner dH = 0$ and so Z is tangent to S .

This allows us to define the *characteristic vector field on S* as the restriction of Z to S ,

$$X = Z|_S.$$

The characteristic vector field X depends on the choice of the generator ω and on the choice of Ω . However, the module (X) (over the ring $R|_S$ of function germs on S) generated by X is uniquely defined by P . If the Martinet ideal has the property of zeros (PZ) then the rings $R|_S$ and $R/(H)$ are isomorphic. This implies the isomorphism $I_{\text{ch}} \simeq I(X)$, where $I(X)$ denotes the ideal in $R|_S$ generated by the coefficients of X (i.e., the coefficients of Z expressed in a coordinate system in \mathbb{K}^n and restricted to S).

Now we will show that the ideal I_{ch} is 2-generated if S is structurally smooth or if the weaker condition $\omega \wedge (d\omega)^{k-1}(0) \neq 0$ holds. We will also construct its generators. Define

$$K(p) = \ker(\omega \wedge (d\omega)^{k-1})(p), \quad p \in S,$$

where by the kernel of an exterior form γ we mean the space of vectors v such that $v \lrcorner \gamma = \gamma(v, \cdot, \dots, \cdot) = 0$.

LEMMA 3.1. — (a) *If S is structurally smooth then $\omega \wedge (d\omega)^{k-1}(0) \neq 0$.*

(b) The latter condition implies that $\dim K(p) = 2$, for any $p \in S$, and that there exist two germs of vector fields X_1 and X_2 on \mathbb{K}^n of category \mathcal{C} such that $X_1(p)$ and $X_2(p)$ span $K(p)$ for any $p \in S$.

Proof. — (a) If $\omega \wedge (d\omega)^{k-1}(0) = 0$ then the rank of $d\omega(0)$ restricted to $\ker \omega(0)$ is less than $k - 1$. In this case $H \in \mathcal{M}^2$, with \mathcal{M} the maximal ideal in R , which contradicts to the structural smoothness of S .

(b) We have $\omega \wedge (d\omega)^{k-1}(0) \neq 0$ and $\omega \wedge (d\omega)^k(0) = 0$, thus P is equivalent to a Pfaffian equation (ω), with ω of the form $dz + fdy_1 + x_2dy_2 + \dots + x_kdy_k$ and $f \in \mathcal{M}^2$ (cf. [Zh1], Section 16). For this Pfaffian equation we have $S = \{\partial f/\partial x_1 = 0\}$ and $K(p) = \text{span}\{\partial/\partial x_1, \partial/\partial y_1 - f\partial/\partial z\}$, for $p \in S$. Q.E.D.

PROPOSITION 3.2. — If $P = (\omega)$ has the property (PZ) and $\omega \wedge (d\omega)^{k-1}(0) \neq 0$ then the characteristic ideal I_{ch} is isomorphic to the ideal $I(f_1, f_2)$ generated by two function germs f_1, f_2 on S , where

$$f_i(p) = (X_i(H))(p), \quad p \in S, \quad i = 1, 2,$$

and X_1, X_2 are germs of linearly independent vector fields X_1 and X_2 on \mathbb{K}^n of category \mathcal{C} such that $K(p) = \text{span}\{X_1(p), X_2(p)\}$, for $p \in S$.

Proof. — Let X_1, X_2 be germs of vector fields on \mathbb{K}^n which span $\ker \omega \wedge (d\omega)^{k-1}(p)$ for $p \in S$. Let X_3, \dots, X_n be the germs of other vector fields such that X_1, \dots, X_n are linearly independent. The ideal $I(\eta)$ generated by the coefficients of the characteristic form $\eta = \omega \wedge (d\omega)^{k-1} \wedge dH$ is generated by the functions $\phi_i = \eta(X_1, \dots, \check{X}_i, \dots, X_n)$, $i = 1, \dots, n$ (\check{X}_i indicates absence of X_i). Since $X_1(p), X_2(p) \in \ker(\omega \wedge (d\omega)^{k-1})(p)$, $p \in S$, it follows that only two such functions are nonzero on S , namely $\phi_i = \psi dH(X_i)$, $i = 1, 2$, where $\psi = (\omega \wedge (d\omega)^{k-1})(X_3, \dots, X_n)$ is nonvanishing. We conclude that the ideal $I(\eta)$ restricted to S is generated by the functions $f_i(p) = dH(p)(X_i(p)) = (X_i(H))(p)$, $i = 1, 2$. By the property (PZ) the characteristic ideal $I_{ch} = I(H, \eta)/(H)$ is isomorphic to the ideal $I(f_1, f_2)$. Q.E.D.

An important invariant is the set of zeros on S of the characteristic ideal I_{ch} , given by

$$S_1 = \{p : H(p) = 0, (\omega \wedge (d\omega)^{k-1} \wedge dH)(p) = 0\}.$$

This is exactly the set of points where the Martinet condition (MC) is violated. If S is structurally smooth then it follows from the above

proposition that, independently of the dimension, the set S_1 is defined by the three equations $H = 0$, $f_1 = 0$ and $f_2 = 0$. The following fact holds without assuming structural smoothness of S .

PROPOSITION 3.3 [Ma1]. — *The set Z_1 of 2-jets at 0 of 1-forms ω such that*

$$H(0) = 0 \quad \text{and} \quad (\omega \wedge (d\omega)^{k-1} \wedge dH)(0) = 0$$

is a stratified algebraic subset of codimension 3 in the space of all 2-jets at 0 of 1-forms.

The stratification of Z_1 into submanifolds of the space of 2-jets at 0 is

$$Z_1 = Z^0 \cup Z^1 \cup Z^2,$$

where Z^0 is a codimension n stratum defined by the equality $\omega(0) = 0$, the stratum Z^1 has the smallest codimension, equal to 3, and is defined by the conditions

$$(\omega \wedge (d\omega)^{k-1})(0) \neq 0, \quad H(0) = 0, \quad f_1(0) = 0, \quad f_2(0) = 0$$

(we use notation from Proposition 3.2), and Z^2 is a stratified submanifold of codimension 6 distinguished by the condition $(\omega \wedge (d\omega)^{k-1})(0) = 0$, see [Ma1].

The set S_1 is equal to the pullback of Z_1 under the 2-jet extension of the map defined by ω . This implies, in the holomorphic category, that $\text{codim } S_1 \leq \text{codim } Z_1 = 3$ if S_1 is nonempty. We will show that $\text{codim } S_1 = \text{codim } Z_1 = 3$ except of degenerations of infinite codimension. In the following section we introduce an algebraic version of the condition $\text{codim } S_1 = 3$ which works in all categories.

3.3. Determination by the restriction.

We will use certain notions from commutative algebra (for an account of basic facts concerning these notions see Appendix 1). We recall that a sequence of elements a_1, \dots, a_r of a proper ideal $I \subset R$ is called *regular* if it satisfies the following condition:

$$a_i \text{ is a nonzerodivisor on the quotient ring } R/(a_1, \dots, a_{i-1}),$$

for $i = 1, \dots, r$

(in particular, a_1 is a nonzerodivisor in R). Here (a_1, \dots, a_i) denotes the ideal generated by a_1, \dots, a_i .

By the *depth* of a proper ideal $I \subset R$, denoted by $\text{depth}(I)$, we mean the supremum of lengths of regular sequences in I . We also define $\text{depth}(I) = \infty$, if $I = R$.

Given a local singular contact structure P , we introduce the following condition:

$$(A) \quad \text{depth}(I_{\text{ch}}) \geq 2$$

on the characteristic ideal $I_{\text{ch}} = I(H, \eta)/(H)$. It follows from the properties of regular sequences in Noetherian rings (cf. Appendix 1) that in the holomorphic and real-analytic categories condition (A) is equivalent to the condition

$$(\tilde{A}) \quad \text{depth}(I(H, \eta)) \geq 3.$$

Clearly, (A) implies (\tilde{A}) in any category. Note that the set S_1 is the set of zeros of the ideal $I(H, \eta)$. The following proposition explains why condition (A) is natural.

PROPOSITION 3.4. — (a) *In the holomorphic category \mathcal{C}^h condition (A) is equivalent to the condition that the set S_1 is of codimension 3 (as the germ of an analytic subset of \mathbb{C}^n), if it is nonempty.*

(b) *In the real-analytic and holomorphic categories the inequality $\text{depth}(I_{\text{ch}}) \leq 2$ holds for any local singular contact structure $P = (\omega)$ such that $\eta(0) = 0$.*

(c) *Condition (A) excludes a subset of infinite codimension in the space of germs of 1-forms ω of category $\mathcal{C} = \mathcal{C}^h$ or $\mathcal{C} = \mathcal{C}^\omega$. More precisely, for any fixed 2-jet ξ of a 1-form $\hat{\omega} \in \mathcal{C}$, there exists an algebraic provariety E_ξ^f of infinite codimension, in the space of infinite jets of 1-forms with the initial 2-jet equal to ξ , such that if the infinite jet of a 1-form $\omega \in \mathcal{C}$ does not belong to E_ξ^f , then either $P = (\omega)$ is a local contact structure, or $P = (\omega)$ is a local singular contact structure and (A) holds.*

Statement (b) means that inequality in condition (A) may be replaced by equality for P satisfying $\eta(0) = 0$. Note that $\text{depth}(I_{\text{ch}}) = \infty$, if $\eta(0) \neq 0$. We refer the reader to Appendix 2 for the proof of the proposition and for the definition of algebraic provariety of infinite codimension.

We are ready for stating our first main result.

THEOREM 3.1. — *Let $P = (\omega)$ and $\tilde{P} = (\tilde{\omega})$ be local singular contact structures of category \mathcal{C} which have the property (PZ), and let P satisfy condition (A). Then equivalence of the pairs $(S, P|_S)$ and $(\tilde{S}, \tilde{P}|_{\tilde{S}})$ implies equivalence of P and \tilde{P} .*

In dimension 3 condition (A) can be changed for a simpler condition.

COROLLARY 3.1. — *For $n = 3$ condition (A) in Theorem 3.1 can be replaced by the following condition implying (A) : the 2-form $\eta = \omega \wedge dH$ does not vanish at the origin, or the origin is an algebraically isolated zero for the ideal $I(H, \eta)$.*

Remark. — If S is structurally smooth then the condition that the origin is an algebraically isolated zero holds for the ideal $I(H, \eta)$ if and only if it holds for the ideal of function germs on S generated by the coefficients of the 1-form $\alpha = \omega|_S$. Therefore Corollary 3.1 generalizes Theorem 2.2.

Our result may be stated more explicitly in the case of structurally smooth S . In this case the condition (PZ) is automatically satisfied and the characteristic ideal is isomorphic to the 2-generated ideal $I(f_1, f_2)$ defined in Proposition 3.2. In the holomorphic and real-analytic categories the condition $\text{depth}(I(f_1, f_2)) = 2$ is equivalent to any of the following conditions:

- (A1) *The sequence f_1, f_2 is a regular sequence.*
- (A2) *The function germs f_1 and f_2 do not have a common noninvertible factor.*
- (A3) *The origin is an algebraically isolated zero for the ideal $I(f_1, f_2)$ on S .*

Therefore, we obtain the following version of Theorem 3.1 concerning the case where S is structurally smooth and the Martinet condition (MC) does not hold.

COROLLARY 3.2. — *Theorem 3.1 remains valid, in the holomorphic and real-analytic categories, if the condition (PZ) is replaced by structural smoothness of the Martinet hypersurface and the condition (A) is replaced by any of the conditions (A1), (A2), or (A3).*

Equivalence of the conditions $\text{depth}(I(f_1, f_2)) = 2$ and (A1) follows from the following general fact in local Noetherian rings: if $\text{depth}(a_1, \dots, a_r) = r$, then a_1, \dots, a_r is a regular sequence. Proving equivalence of (A1) and (A2) is an easy exercise which uses the fact that the ring of function germs R is factorial in the analytic categories. Equivalence of the conditions $\text{depth}(f_1, f_2) = 2$ and condition (A3) is a well known fact.

Remark. — In the holomorphic category condition (A) implies condition (PZ). This will follow from Proposition 3.6 in the following subsection.

3.4. Determination by the restriction and orientations.

In this subsection we introduce a canonical orientation \mathcal{O} that is invariantly related to P and state a theorem on the determinacy of P by the pair $P|_S, \mathcal{O}$ under a condition that is weaker than (A). Namely, we introduce the condition

$$(B) \quad \text{gen}(I_{\text{ch}}) \geq 2.$$

where $\text{gen}(I)$ denotes the minimal number of generators of an ideal I . Since $\text{depth}(I) \leq \text{gen}(I)$ for any ideal I , it follows that (A) implies (B).

For the case where S is not structurally smooth we need one more condition that is an algebraic analogon of the condition (in the holomorphic category) that the singular part of S has codimension at least two in S . The singular part of S is defined by the equalities $H = 0, dH = 0$, therefore it is natural to consider the ideal, in the ring R of function germs, generated by H and the coefficients of the 1-form dH . We denote this ideal by $I(H, dH)$, and define the ideal $I_H(dH)$ of the quotient ring $R/(H)$ by

$$I_H(dH) = I(H, dH)/(H).$$

The algebraic analogon of the condition $\text{codim } S^{\text{sing}} \geq 2$ is the condition

$$(C) \quad \text{depth}(I_H(dH)) \geq 2.$$

The condition (C) always holds if S is structurally smooth. In general case, it is also implied by (A). Namely, it follows from the definition of the characteristic form η that $I(H, \eta) \subset I(H, dH)$ and so $I_{\text{ch}} \subset I_H(dH)$, consequently, $\text{depth}(I_{\text{ch}}) \leq \text{depth}(I_H(dH))$.

Now we introduce a canonical orientation. For $n = 2k + 1$ with k odd we use the same method as in dimension 3 to introduce an orientation on S (more precisely, on the regular part of S). Namely, assume that $\mathbb{K} = \mathbb{R}$ and consider a 1-form ω defined in a neighborhood of $0 \in \mathbb{R}^n$ which generates a local singular contact structure P . Consider a point $p \in S^{\text{reg}}$ together with a small ball $U \subset \mathbb{R}^n$ containing p such that $S \cap U \subset S^{\text{reg}}$ and the n -form $\omega \wedge (d\omega)^k$ is nondegenerate on $U \setminus S$. Then $U \setminus S$ has two connected components U^+ and U^- with well defined orientations \mathcal{O}^+ and \mathcal{O}^- given by the n -form $\omega \wedge (d\omega)^k$ (note that multiplying ω by an invertible function ϕ changes the n -form by the factor $(\phi)^{k+1}$ which is positive since we assume k to be odd). The orientations \mathcal{O}^+ and \mathcal{O}^- have “different signs” when compared to a fixed orientation of the whole U , which follows from the condition $dH(p) \neq 0$. Therefore, they induce a unique orientation of $U \cap S^{\text{reg}}$. In this way we obtain a *canonical orientation* \mathcal{O}_S of S^{reg} which depends on P only.

In order to extend our definition to the case of k even we introduce the distribution E by intersecting $D = \ker \omega$ with the tangent bundle of S^{reg} . More precisely, E is given by

$$E(p) = \ker \omega(p) \cap T_p S = \ker(\omega \wedge dH)(p)$$

and it is defined on the open subset S_E of S^{reg} , where

$$S_E = \{ p \in S : (\omega \wedge dH)(p) \neq 0 \}.$$

The following proposition is proved at the end of this subsection.

PROPOSITION 3.5. — *If $k \geq 2$ and (P) satisfies conditions (PZ) and (C) then S_E is nonempty.*

The above construction of the orientation \mathcal{O}_S , repeated for k even, gives an orientation $\mathcal{O}_S(\omega)$ of S^{reg} which depends on the choice of the generator ω . Consider this orientation on the set $(T_p S)^+$ consisting of tangent vectors $\xi \in T_p S$ such that $\omega(\xi) \geq 0$. By the condition $(\omega \wedge dH)(p) \neq 0$ the set $(T_p S)^+$ is a half-space in $T_p S$. The orientation $\mathcal{O}_S(\omega)$ on $T_p S$ induces the orientation of the boundary of $(T_p S)^+$, i.e., on the space $E(p)$. This orientation will be denoted $\mathcal{O}_E(p)$. Multiplying ω by a negative function ϕ reverses the orientation $\mathcal{O}_S(\omega)$ and it reverses the direction of the covector $\omega(p)|_{T_p S}$, thus changing the half-space $(T_p S)^+$ for the opposite one. This means that the induced orientation on $E(p)$ is the same as earlier and so defined orientation $\mathcal{O}_E(p)$ of $E(p)$ is independent of the choice of ω .

The orientation on the distribution E obtained in this way is invariantly related to P , for k even, and is called *canonical orientation* \mathcal{O}_E .

THEOREM 3.2. — (a) *Let P and \tilde{P} be local singular contact structures which have the property (PZ). If P satisfies conditions (B) and (C) then equivalence of the pairs $(S, P|_S)$ and $(\tilde{S}, \tilde{P}|_{\tilde{S}})$ implies equivalence of P and \tilde{P} , in the holomorphic category.*

(b) *The same holds in the smooth and real-analytic categories with the pairs $(S, P|_S)$ and $(\tilde{S}, \tilde{P}|_{\tilde{S}})$ replaced by the triples $(S, P|_S, \mathcal{O})$ and $(\tilde{S}, \tilde{P}|_{\tilde{S}}, \tilde{\mathcal{O}})$, where \mathcal{O} denotes the canonical orientation \mathcal{O}_S , if k is odd, and the canonical orientation \mathcal{O}_E , if k is even.*

Remarks. — (a) It follows from Proposition 3.2 that if $(\omega \wedge (d\omega)^{k-1})(0) \neq 0$, then the characteristic ideal is generated by two functions, $I_{ch} \simeq I(f_1, f_2)$, where $f_i = X_i(H)$ and X_1, X_2 are vector fields such that $K(p) = \text{span}\{X_1(p), X_2(p)\}$, for $p \in S$. Then condition (B) means that neither of these two functions is divisible over the other one.

(b) If $(\omega \wedge (d\omega)^{k-1})(0) = 0$ then the characteristic ideal might be complicated (in particular, not 2-generated), but in this case our proof of Theorem 3.2 shows that the condition (B) is not needed, i.e., the determinacy by the restriction and the orientation holds under condition (C) only.

(c) In the holomorphic category the condition (PZ) in Proposition 3.5 is not needed, and it is not needed in Theorem 3.2 if we require that both P and \tilde{P} satisfy condition (C). This is because, in this category, condition (PZ) is implied by (C), and even by the weaker condition $\text{depth}(I_H(dH)) \geq 1$ as the following proposition says.

PROPOSITION 3.6. — *For an arbitrary local singular contact structure P the condition $\text{depth}(I_H(dH)) \geq 1$ implies that H does not have a nontrivial double factor, i.e., there is no function germ ϕ , $\phi(0) = 0$, such that $H = \phi^2\psi$ for some function germ ψ . In particular, the condition (C) (and consequently condition (A)) implies the property (PZ) in the holomorphic category.*

Proof. — Assume that the contrary holds, i.e., $H = \phi^2\psi$, $\phi(0) = 0$. Then $dH = \phi d\tilde{H}$, where $d\tilde{H} = 2\psi d\phi + \phi d\psi$. We have that $\phi\psi dH = \phi^2\psi d\tilde{H} = 0 \pmod{(H)}$. The condition $\text{depth}(I_H(dH)) \geq 1$ implies that the ideal generated by the coefficients of dH in the quotient ring $R/(H)$ contains a nonzerodivisor. This fact together with the equality $\phi\psi dH = 0$

mod (H) implies that $\phi\psi = 0 \text{ mod } (H)$, i.e., there exists a function germ h such that $\phi\psi = hH$. It follows that $H = \phi^2\psi = \phi hH = \tilde{\phi}H$, where $\tilde{\phi}(0) = 0$. We conclude that $H \equiv 0$ which contradicts our assumption. The second statement follows from the first one and the fact that the ring of holomorphic function germs is factorial. Q.E.D.

Proof of Proposition 3.5. — Assume that S_E is empty. Then $(\omega \wedge dH)(p) = 0$ for $p \in S$ and from the property of zeros of (H) we get $\omega \wedge dH = 0$ modulo (H) . It follows from (C) and Corollary 1 in Appendix 1 that dH has 1-division property modulo (H) (i.e., in the module \hat{R}^n , where \hat{R} is the quotient ring $\hat{R} = R/(H)$). Therefore, $\omega = fdH + H\gamma$ and so $d\omega = (df - \gamma) \wedge dH + Hd\gamma$, for some f and 1-form γ . Computing now $\omega \wedge (d\omega)^k$ and taking into account that $k \geq 2$ we see that the Martinet ideal (H) of $P = (\omega)$ is contained in (H^2) . We conclude that $H \equiv 0$ which contradicts condition (C). Q.E.D.

3.5. Division of exterior forms and orientations.

The orientations \mathcal{O}_S and \mathcal{O}_E can be defined by volume forms. The definitions below, based on the division of exterior forms, are convenient for their evaluations. For further use we introduce the following notation.

PROPOSITION 3.7. — *If η_1, η_2 , and η_3 are exterior forms on a vector space V such that*

$$\eta_1 \wedge \eta_2 = \eta_3, \quad \eta_2 \neq 0,$$

then the form η_1 restricted to the kernel $\ker \eta_2 = \{v \in V : v \lrcorner \eta_2 = 0\}$ is uniquely determined by this equation and will be denoted

$$\frac{\eta_3}{\eta_2} = \eta_1|_{\ker \eta_2}.$$

Proof. — Let $\tilde{\eta}_1$ be another exterior form satisfying the equation $\tilde{\eta}_1 \wedge \eta_2 = \eta_3$. Subtracting one equation from the other we obtain that $\gamma \wedge \eta_2 = 0$, where $\gamma = \eta_1 - \tilde{\eta}_1$. Given that η_1 is a p -form and η_2 is a q -form, we take collections of vectors $X_1, \dots, X_p \in \ker \eta_2$ and Y_1, \dots, Y_q such that $\eta_2(Y_1, \dots, Y_q) \neq 0$. Then we have

$$0 = (\gamma \wedge \eta_2)(X_1, \dots, X_p, Y_1, \dots, Y_q) = \gamma(X_1, \dots, X_p) \eta_2(Y_1, \dots, Y_q).$$

Since the second factor does not vanish, we deduce that $\gamma(X_1, \dots, X_p) = 0$ for any vectors $X_1, \dots, X_p \in \ker \eta_2$. It follows that $\gamma = \eta_1 - \tilde{\eta}_2$ vanishes on the kernel of η_2 . Q.E.D.

Note that the division of η_3 over η_2 makes sense only when $\eta_2 \neq 0$ and there exists a form η_1 such that $\eta_3 = \eta_1 \wedge \eta_2$. We distinguish between two equal forms defined on different kernels, for example

$$\frac{dx_1 \wedge dx_2}{dx_2} \neq \frac{dx_1 \wedge dx_3}{dx_3}.$$

Using this notation we can easily construct the volume forms defining the orientations \mathcal{O}_S and \mathcal{O}_E . The orientation \mathcal{O}_S is defined by the following nondegenerate differential (n-1)-form on S^{reg} :

$$\frac{(H^{-1}\omega \wedge (d\omega)^k)(p)}{dH(p)}, \quad p \in S^{\text{reg}}.$$

The orientation \mathcal{O}_E is defined by the following nondegenerate exterior (n-2)-form on E :

$$\frac{(H^{-1}\omega \wedge (d\omega)^k)(p)}{(\omega \wedge dH)(p)}, \quad p \in S_E.$$

Here ω is any generator of P , and H is any generator of the Martinet ideal. It is easy to check that the choice of ω and H is irrelevant for \mathcal{O}_S and \mathcal{O}_E .

3.6. Proof of Proposition 3.1.

Suppose that S^{reg} is empty. We will prove that this assumption implies that $H \equiv 0$, which will contradict our assumption that P is a local singular contact structure.

The equality $S^{\text{reg}} = \emptyset$ means that $dH(p) = 0$ for any $p \in S$. From the property (PZ) it follows that any coefficient of the 1-form dH belongs to the ideal (H) , in particular

$$\frac{\partial H}{\partial x_n} = f(x_1, \dots, x_n)H,$$

where f is a function germ. Solving this linear differential equation of order 1 we get

$$H(x_1, \dots, x_n) = \exp\left(\int_0^{x_n} f(x_1, \dots, x_{n-1}, s) ds\right) H_1(x_1, \dots, x_{n-1}),$$

where $H_1(x_1, \dots, x_{n-1}) = H(x_1, \dots, x_{n-1}, 0)$ represents the initial condition on the hypersurface $x_n = 0$. As the first function in the above product

is nonvanishing, we obtain equality of ideals $(H) = (H_1)$, where H_1 depends on $n - 1$ variables only. Repeating this procedure $n - 1$ times (eliminating one variable each time) we will get that the ideal (H) is generated by a function of zero variables, i.e., a constant. As $H(0) = 0$, this constant must be zero and we obtain that $H \equiv 0$.

In the holomorphic category the ring R^h is a unique factorization domain, therefore H can be written as a product of irreducible factors $H = h_1 \cdots h_r$. From the property (PZ) it follows that the same factor can not be repeated twice. Then $S = S_1 \cup \cdots \cup S_r$, where $S_i = \{h_i = 0\}$, and the ideals (h_i) generated by h_i also have the property of zeros (as h_i are irreducible). Applying the same argument (as in the first part of the proof) to the ideals (h_i) we see that the regular parts $S_i^{\text{reg}} = \{p \in S : h_i(p) = 0, dh_i(p) \neq 0\}$ are nonempty. As the smooth part of an irreducible germ of a complex analytic set is connected and dense in this set (cf. [Lo]), it follows that S_i^{reg} is dense in S_i (its complement is a proper analytic subset of S_i and so it is nowhere dense). In order to conclude that S^{reg} is dense in S it is enough to take into account the equality

$$dH(p) = (h_1 \cdots \check{h}_i \cdots h_r dh_i)(p), \quad \text{for } p \in S_i,$$

(\check{h}_i means that this factor is absent in the product) and the property that the set $S_i \cap S_j$ is nowhere dense in S_i for $i \neq j$. This last property follows from the fact that h_i and h_j are mutually prime. We obtain that the set where $dH(p) \neq 0$ is dense in each S_i , and so it is dense in S .

If (H) does not have the property of zeros then, in the holomorphic category, H has a multiple factor in its prime decomposition. Writing

$$H = h_1^{i_1} \cdots h_r^{i_r}, \quad S = S_1 \cup \cdots \cup S_r, \quad S_i = \{h_i = 0\}$$

with $i_1 > 1$ we have that $dH = h_1 \alpha$, with α a holomorphic 1-form, and so dH vanishes on $S_1 = \{h_1 = 0\}$. It follows that $S^{\text{reg}} \subset S_2 \cup \cdots \cup S_r$ and so it is not dense in S . Q.E.D.

4. Complete invariants for structurally smooth S .

In this section we assume that the Martinet hypersurface S is structurally smooth. We introduce three new invariants: a line bundle L over S (which is an invariant stronger than the restriction $P|_S$), a canonical orientation \mathcal{O}_L of L , and a partial connection on L . Under no a priori assumptions this triple of invariants forms a complete set of invariants for

local singular contact structures (Section 4.2). We show relations between the invariants (Section 4.3).

4.1. The line bundle L and orientation \mathcal{O}_L .

Let $P = (\omega)$ be a local singular contact structure of category \mathcal{C} on $M = \mathbb{K}^n$, $n = 2k + 1$. Consider the field of lines $p \in S \rightarrow L_p$, where L_p is the punctured line in the cotangent space T_p^*M spanned by $\omega(p)$, i.e., $L_p = \mathbb{K}_* \omega(p)$, where $\mathbb{K}_* = \mathbb{K} \setminus \{0\}$. We denote by L the subset of the cotangent bundle defined by

$$L = \bigcup_{p \in S} L_p$$

which will be called the *canonical line bundle* or simply the *line bundle* associated to P . The name comes from the fact that, with the canonical projection $L \rightarrow S$, the subset $L \subset T^*M$ is a (multiplicative) line bundle over the set germ S .

Note that while the line bundle L is determined by the covectors $\omega(p) \in T^*M$, $p \in S$, the restriction $P|_S = (\omega|_S)$ used in the earlier sections is determined by the restrictions of $\omega(p)$ to $T_p S$. Therefore, the line bundle L determines the pair of invariants $(S, P|_S)$. We will show in Section 4.3 that the pair $(S, P|_S)$ determines L , up to equivalence, and so it is not a stronger invariant when considered separately. Its role becomes more transparent when we define two other invariants which live on L .

When k is even and $\mathbb{K} = \mathbb{R}$ we can define a canonical orientation on L using a construction analogous to Section 3.4. Namely, given a point $p \in S$ and its small neighbourhood U in M , the differential n -form $\omega \wedge (d\omega)^k$ defines two orientations \mathcal{O}^+ and \mathcal{O}^- of the half-neighbourhoods U^+ and U^- obtained from U by removing $S \cap U$. Both orientations induce the same orientation \mathcal{O} on $S \cap U$ (\mathcal{O}^+ and \mathcal{O}^- are reverse when compared to a fixed orientation of U which follows from $dH(p) \neq 0$, $p \in S$). This orientation depends on the choice of the generator ω . Consider the line bundle L which is, locally, the product of S and a real line. Having the orientation \mathcal{O} on S and the natural orientation on the fiber L_p defined by the covector $\omega(p)$, we have the product orientation \mathcal{O}_L defined locally on L . This orientation does not depend on the choice of the generator ω since reversing the sign of ω changes the signs of the orientation \mathcal{O} on S (since k is even) and of the orientation of the fiber. The orientation \mathcal{O}_L on L defined in this way, for k even, is called the *canonical orientation on L* associated to P .

Note that \mathcal{O}_L is well defined even if the set of points $p \in S$ such that $(\omega \wedge dH)(p) \neq 0$ is empty, contrary to the orientation \mathcal{O}_E defined in Section 3.4. However, in order to compare two orientations \mathcal{O}_L corresponding to two different local singular contact structures we have to make their line bundles equal, while for comparing the orientations \mathcal{O}_E it is enough to have their restrictions to S equal.

4.2. The partial connection and complete invariants.

In order to introduce our last invariant, the connection, let us consider a generator ω of a local singular contact structure P . Since we assume that $dH(p) \neq 0$, for $p \in S$, we have $(\omega \wedge (d\omega)^{k-1})(p) \neq 0$ (Lemma 3.1). We define the *field of kernels* K by

$$K(p) = \ker(\omega \wedge (d\omega)^{k-1})(p), \quad p \in S,$$

which is invariantly related to P . Changing the generator ω for $\tilde{\omega} = \phi\omega$ does not change K since $\tilde{\omega} \wedge (d\tilde{\omega})^{k-1} = \phi^k \omega \wedge (d\omega)^{k-1}$. We also have $(d\tilde{\omega})^k = d(\tilde{\omega} \wedge (d\tilde{\omega})^{k-1}) = d(\phi^k \omega \wedge (d\omega)^{k-1})$ and so

$$(4.1) \quad (d(\phi\omega))^k = k\phi^{k-1}d\phi \wedge \omega \wedge (d\omega)^{k-1} + \phi^k(d\omega)^k.$$

This transformation rule implies the following property.

LEMMA 4.1. — *If S is structurally smooth, then there exists a generator ω of P such that $(d\omega)^k(0) = 0$. If $\tilde{\omega} = \phi\omega$ is another generator with this property, then $d\phi(0)|_{K(0)} = 0$.*

Proof. — From Lemma 3.1 we have $(\omega \wedge (d\omega)^{k-1})(0) \neq 0$, which together with the equality $(\omega \wedge (d\omega)^k)(0) = 0$, implies, by the algebraic Darboux lemma, that P is equivalent to $\tilde{P} = (\tilde{\omega})$, where the 1-jet of $\tilde{\omega}$ coincides with the 1-jet of $dz + x_2dy_2 + \cdots + x_kdy_k$. Clearly, this implies the first statement. The second statement follows from the transformation rule for $(d\tilde{\omega})^k$. Q.E.D.

Using Lemma 4.1 we introduce the following invariant. Take a generator ω of P which satisfies the condition $(d\omega)^k(0) = 0$. We will call such generators *good generators*. Any generator ω defines a section of the cotangent bundle T^*M . Two good generators differ by a function ϕ such that $d\phi(0)|_{K(0)} = 0$. This means that the sections passing through a given point

$q = \omega(0) \in T_0^*M$ which correspond to the good generators are all tangent to a 2-dimensional subspace

$$\Delta(q) \subset T_q(T^*M)$$

such that $\pi_*(\Delta(q)) = K(0)$, where $\pi : T^*M \rightarrow M$ is the natural projection and π_* is the tangent map to π . Multiplying q by a constant $a \in \mathbb{K}_* = \mathbb{K} \setminus \{0\}$ and a good generator ω by the same constant we see that $a\omega$ is also good. This shows that the subspace $\Delta(q)$ defines a unique subspace $\Delta(q')$ at the point $q' = aq \in T_0^*M$, and vice versa (multiplication by $a \in \mathbb{K}_*$ can be treated as a map $\Phi_a : T^*M \rightarrow T^*M$ and $\Delta(q')$ is the image of $\Delta(q)$ by the tangent map $d\Phi_a(q)$). We get the family of subspaces

$$\Delta_0 = \{\Delta(q)\}_{q \in L_0}.$$

This family is determined by any of the subspaces $\Delta(q)$ and it is invariantly defined by our Pfaffian equation P . We will call it the *partial connection* at 0 of P or the *canonical partial connection* at 0. Note that our definition implies that Δ_0 determines the kernel $K(0)$. Further properties of this invariant will be discussed later.

Our main result in this section says that the line bundle L , the orientations \mathcal{O}_S of S or \mathcal{O}_L of L , and the partial connection Δ_0 at the origin form a complete set of invariants of local singular contact structures.

THEOREM 4.1. — *Let P and \tilde{P} be local singular contact structures with structurally smooth Martinet hypersurfaces. Then equivalence of the pairs (L, Δ_0) and $(\tilde{L}, \tilde{\Delta}_0)$ in the holomorphic category (respectively, equivalence of the triples $(L, \Delta_0, \mathcal{O})$ and $(\tilde{L}, \tilde{\Delta}_0, \tilde{\mathcal{O}})$ in the real-analytic and smooth categories) implies equivalence of P and \tilde{P} . Here \mathcal{O} denotes the orientation \mathcal{O}_S , when k is odd, and the orientation \mathcal{O}_L when k is even.*

By definition Δ_0 determines $K(0)$. If the kernel $\ker \omega(0)$ is not tangent to S , then also $K(0)$ determines Δ_0 . Moreover, the following proposition is proved in Section 4.3.

PROPOSITION 4.1. — *Let P and \tilde{P} have the same structurally smooth S , the same line bundle L , and the kernels $K(0), \tilde{K}(0)$ tangent to S . Let ω and $\tilde{\omega}$ be generators of P and \tilde{P} which are equal at any point of S . Then the following holds:*

- (i) *If $\ker \omega(0), \ker \tilde{\omega}(0)$ are not tangent to S , then $K(0) = \tilde{K}(0)$ if and only if $\Delta_0 = \tilde{\Delta}_0$.*

(ii) If $\ker \omega(0)$, $\ker \tilde{\omega}(0)$ are tangent to $S = \tilde{S}$, then the equality $K(0) = \tilde{K}(0)$ holds and the equality $\Delta_0 = \tilde{\Delta}_0$ is equivalent to the equality $(d\omega)^k(0) = (d\tilde{\omega})^k(0)$.

The first statement of the proposition implies that in the case of nontangency of $\ker \omega(0)$ to S the kernel $K(0)$ and the partial connection are equivalent invariants. This is not so in the tangency case as the following example shows.

Example 4.1. — We present nonequivalent C^∞ local singular contact structures on \mathbb{R}^3 , with different connections Δ_0 , but with the same structurally smooth Martinet surface S , the same restriction to S , the same line bundle L , and the same orientation \mathcal{O}_S .

Consider the local Pfaffian equation P_a on \mathbb{R}^3 generated by the germ at 0 of the 1-form

$$\omega_a = dz + f(x)dx + z(a + y)dx,$$

where $a \in \mathbb{R}$ is a parameter. Let f be a C^∞ function on \mathbb{R} , flat at 0, with the set of zeros $Z = \{0\} \cup \{1/n : n \in \mathbb{N}\}$, where \mathbb{N} is the set of positive integers. One can easily check that all P_a , $a \in \mathbb{R}$, have the same Martinet surface $S = \{z = 0\}$ which is structurally smooth, they have the same restriction $\omega_a|_S = \alpha = f(x)dx$, the same line bundle L , the same orientation \mathcal{O}_S of S , and the same kernel $K(0) = \ker \omega_a(0) = \ker dz$. We have $d\omega_a(0) = adz \wedge dx$ and $d(\phi\omega_a)(0) = 0$, where $\phi = 1 + ax$. Thus $(1 + ax)\omega_a$ is a good generator of P_a and its partial connection Δ_0^a is determined by the tangent space to the section $(1 + ax)(dz + f(x)dx)$ of L . It is clear that $\Delta_0^a = \Delta_0^{\tilde{a}}$ if and only if $a = \tilde{a}$.

Now we will prove that P_a is not equivalent to $P_{\tilde{a}}$ if $a \neq \tilde{a}$; by Theorem 4.1 this means that Δ_0^a can not be transformed to $\Delta_0^{\tilde{a}}$ preserving the other invariants. Consider two constants $a \neq \tilde{a}$ and let $P = P_a = (\omega_a)$ and $\tilde{P} = P_{\tilde{a}} = (\omega_{\tilde{a}})$. Assume that P_a can be transformed to $P_{\tilde{a}}$ by the germ of a diffeomorphism Ψ . Then $\hat{\omega}_a = \Psi^*(\omega_a) = \phi\omega_{\tilde{a}}$, where ϕ is a nonvanishing function. This diffeomorphism preserves the line bundle L and its restriction Φ to S is a symmetry of the restriction $P_a|_S = (\alpha) = P_{\tilde{a}}|_S$, where $\alpha = f(x)dx$. Below we will show that the symmetries of (α) are of the form $(x, y) \rightarrow (x + g(x), \phi_2(x, y))$, with g a smooth function which is flat at zero. Thus the diffeomorphism Ψ is of the form $(x, y, z) \rightarrow (x + g(x) + zh_1, \phi_2(x, y) + zh_2, zh_3)$, where h_1 , h_2 and h_3 are functions of (x, y, z) . At the points where $z = 0$ we have

$\Psi^*(\omega_a) = \Psi^*(dz + f dx) = h_3 dz + f(1 + g') dx + f h_1 dz = \phi \omega_{\tilde{a}} = \phi(dz + f dx)$. This implies that $\phi(0) = h_3(0)$. Comparing the terms at dx we get $1 + g' = \phi$. Therefore, $\phi|_S - 1$ is a flat function on S and so $d\phi|_S(0) = 0$ and $d\phi(0) \wedge dz = 0$. From our assumption $\hat{\omega}_a = \Psi^*(\omega_a) = \phi \omega_{\tilde{a}}$ it follows that $d\hat{\omega}_a = d(\phi \omega_{\tilde{a}})$. Computing both sides at the origin we get $d\hat{\omega}_a(0) = \Psi^*(d\omega_a(0)) = \Psi^*(adz \wedge dx) = ah_3(0) dz \wedge dx$ and $d(\phi \omega_{\tilde{a}})(0) = \phi(0) d\omega_{\tilde{a}}(0) + d\phi(0) \wedge \omega_{\tilde{a}}(0) = \tilde{a}\phi(0) dz \wedge dx + d\phi(0) \wedge dz = \tilde{a}h_3(0) dz \wedge dx$. Since $h_3(0) \neq 0$ and $a \neq \tilde{a}$, these 2-forms are not equal. This contradicts our assumption that $\hat{\omega}_a = \Psi^*(\omega_a) = \phi \omega_{\tilde{a}}$ and implies that P_a and $P_{\tilde{a}}$ are not equivalent.

Let us prove that any symmetry Φ of the Pfaffian equation (α) on \mathbb{R}^2 has the form

$$\Phi : (x, y) \rightarrow (\phi_1(x), \phi_2(x, y)), \quad \text{where } \phi_1(x) = x + g(x)$$

and g is the germ of a flat at 0 function. We first observe that any symmetry Φ preserves the germ of the foliation of the (x, y) -plane by the lines $x = \text{const}$ (since this is the integral foliation of α), therefore the first component ϕ_1 of Φ depends on the variable x , only. Additionally, Φ preserves the set Z of zeros of α and so we get $\phi_1 = \text{id}$ on Z . This implies that $g(x) = \phi_1(x) - x$ is a flat function at 0.

Remark. — Example 4.1 shows that, in dimension 3, the connection Δ_0 is independent of the other invariants L and \mathcal{O} and Theorem 4.1 does not hold if Δ_0 is omitted. The following example shows that the same holds in higher dimensions and, moreover, the kernel $K(0)$ is independent of L and \mathcal{O} . These examples also show that condition (A) in Theorem 3.1 and condition (B) in Theorem 3.2 (violated in the examples) are essential.

Example 4.2. — Let $k \geq 2$ and $P = (\omega)$, $\tilde{P} = (\tilde{\omega})$, where

$$\omega = dy_1 + x_2 dy_2 + \dots + x_{k-1} dy_{k-1} + x_1^2 dx_k + z(dx_1 + y_k dx_k),$$

$$\tilde{\omega} = dy_1 + x_2 dy_2 + \dots + x_{k-1} dy_{k-1} + x_1^2 dx_k + z(dx_k - y_k dx_1).$$

Then P and \tilde{P} have the same structurally smooth Martinet hypersurface $S = \{z = 0\}$, the same restriction to S , the same line bundle L , and the same orientations \mathcal{O}_S , or \mathcal{O}_L , \mathcal{O}_E , depending on parity of k .

Note that $S_1 = \{z = x_1 = 0\}$ for both P and \tilde{P} . On the other hand, computing the kernels

$$K(0) = \ker(\omega \wedge (d\omega)^{k-1})(0) = \text{span}\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_k}\right)$$

$$\tilde{K}(0) = \ker(\tilde{\omega} \wedge (d\tilde{\omega})^{k-1})(0) = \text{span}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_k}\right)$$

we see that $K(0)$ is transversal to S_1 whereas $\tilde{K}(0)$ is not. Therefore P and \tilde{P} are not equivalent. The invariant distinguishing the equivalence classes of P and \tilde{P} is the space $K(0)$.

Remark. — The canonical orientation is also independent of the other invariants and can not be omitted in Theorems 3.2 and 4.1, in the smooth and real-analytic categories. Namely, in Example 2.1 we have two Pfaffian equations P_+ and P_- , with the generators $\omega_{\pm} = dz + x^2\alpha + z(2x^2ydx + \alpha)$, where $\alpha = xdy + (3y + x^3y^2)dx$. They have the same Martinet hypersurface $S = \{z = 0\}$ and the same line bundle L with the fibers L_p generated by $(dz + x^2\alpha)(p)$. They also have the same kernel $K(0) = \ker dz$ tangent to S . Both ω_+ and ω_- are good generators and they are tangent to each other at the origin, when considered as sections of the bundle L . Therefore, P_+ and P_- define the same partial connection Δ_0 . We have shown in Example 2.1 that P_+ and P_- define different orientations \mathcal{O}_S and they are nonequivalent. This means that \mathcal{O}_S is independent of the pair of invariants L and Δ_0 .

4.3. Relations between the invariants.

If condition (A) introduced in Section 3.3 holds then the partial connection and the orientation are determined by L or even by the pair $(S, P|_S)$. Recall that when S is structurally smooth and $\eta(0) = 0$, then condition (A) is equivalent to the condition

$$(A') \quad \text{depth}(I(f_1, f_2)) = 2,$$

where $I(f_1, f_2)$ is the ideal generated by the functions $f_i = X_i(H)$, $i = 1, 2$, where X_1 and X_2 are germs of vector fields on \mathbb{K}^n which span $K(p)$, for $p \in S$ (cf. Proposition 3.2).

THEOREM 4.2. — *Let P and \tilde{P} be local singular contact structures with structurally smooth Martinet hypersurfaces and the characteristic forms vanishing at $0 \in \mathbb{K}^n$. If P and \tilde{P} satisfy condition (A') or (A) then we have the following implications:*

$$\begin{aligned} (S, P|_S) = (\tilde{S}, \tilde{P}|_{\tilde{S}}) &\implies \mathcal{O}_S = \tilde{\mathcal{O}}_{\tilde{S}}, & \text{when } k \text{ is odd,} \\ (S, P|_S) = (\tilde{S}, \tilde{P}|_{\tilde{S}}) &\implies \mathcal{O}_E = \tilde{\mathcal{O}}_{\tilde{E}}, & \text{when } k \text{ is even,} \\ L = \tilde{L} &\implies \mathcal{O}_L = \tilde{\mathcal{O}}_{\tilde{L}}, & \text{when } k \text{ is even.} \end{aligned}$$

When S is structurally smooth and $\eta(0) = 0$, then condition (B) in Section 3.4, $\text{gen}(I_{\text{ch}}) \geq 2$, is equivalent to the condition

$$(B') \quad \text{gen}(I(f_1, f_2)) = 2,$$

where $\text{gen}(I)$ denotes the minimal number of generators of I .

THEOREM 4.3. — *Let P and \tilde{P} be local singular contact structures with structurally smooth Martinet hypersurface and the characteristic forms vanishing at $0 \in \mathbb{K}^n$. If P and \tilde{P} satisfy condition (B') (or equivalent condition (B)) then the following implication holds:*

$$L = \tilde{L} \quad \implies \quad \Delta_0 = \tilde{\Delta}_0.$$

The following proposition (proved in Section 6.2) explains the relation between the restriction and the line bundle.

PROPOSITION 4.2. — *If P and \tilde{P} are local singular contact structures with the same Martinet ideal (H) having the property of zeros and satisfying condition (C) from Section 3.4, then equality of the restrictions $P|_S$ and $\tilde{P}|_{\tilde{S}}$ implies existence of a local diffeomorphism, equal identity on S , which transforms the line bundle L into \tilde{L} .*

The assumptions in this proposition are satisfied when S is structurally smooth which means that in this case the line bundle L is not a stronger invariant than the pair $(S, P|_S)$.

Dealing with the canonical partial connection defined as in Section 4.2 may be troublesome. Therefore we will also use an equivalent invariant, a *connection 1-form* defined on the kernel $K(0) = \ker(\omega \wedge (d\omega)^{k-1})(0)$. The definition will use the following property.

LEMMA 4.2. — *Let ω be a generator of a Pfaffian equation P with S structurally smooth. There exists an exterior 1-form $\beta(0)$ on T_0M such that*

$$(d\omega)^k(0) = k\beta(0) \wedge (\omega \wedge (d\omega)^{k-1})(0).$$

This form is unique when restricted to $K(0)$. Changing the generator ω for $\tilde{\omega} = \phi\omega$ transforms $\beta(0)$ into

$$\tilde{\beta}(0) = \beta(0) + (\phi(p))^{-1}d\phi(0).$$

Proof. — By Lemma 4.1 we can choose a generator $\tilde{\omega} = \phi\omega$ of P so that $(d\tilde{\omega})^k(0) = 0$. Then it follows from the transformation rule (4.1) for $(d\tilde{\omega})^k$ that the first property holds with $\beta = -k\phi^{-1}d\phi$. Uniqueness of $\beta(0)|_{K(0)}$ is obvious (cf. Proposition 3.7). The transformation rule for $\beta(0)$ follows from

$$(d\tilde{\omega})^k(0) = k((\phi^{-1}d\phi + \beta) \wedge \tilde{\omega} \wedge (d\tilde{\omega})^{k-1})(0), \quad \text{where } \tilde{\omega} = \phi\omega,$$

which is a consequence of the transformation rule for $(d\tilde{\omega})^k$. Q.E.D.

Lemma 4.2 says that the exterior form $(d\omega)^k(0)$ is divisible over $(\omega \wedge (d\omega)^{k-1})(0)$ in the sense of Proposition 3.7. This allows us to define the unique 1-form at 0

$$c(0) = \beta(0)|_{K(0)} = \frac{1}{k} \frac{(d\omega)^k(0)}{(\omega \wedge (d\omega)^{k-1})(0)},$$

called the *connection 1-form* at 0.

Let us show that the connection 1-form $c(0)$ and the partial connection Δ_0 are equivalent invariants (see [JZh2] for interpretation of this invariant as covariant derivative).

PROPOSITION 4.3. — *Let P and \tilde{P} have the same structurally smooth S , the same line bundle L , and the kernels $K(0)$, $\tilde{K}(0)$ tangent to S . Let ω and $\tilde{\omega}$ be generators of P and \tilde{P} such that $\omega(p) = \tilde{\omega}(p)$ for all $p \in S$. Then $\Delta_0 = \tilde{\Delta}_0$ if and only if $c(0) = \tilde{c}(0)$.*

Proof. — Our generators define the same section in L . Assume that $\Delta_0 = \tilde{\Delta}_0$. Then $K(0) = \tilde{K}(0)$. Let ω be good, i.e., defining a section tangent to Δ_0 . Then $\tilde{\omega}$ defines the same section which is tangent to $\tilde{\Delta}_0$. This means that $\tilde{\omega}$ is good, too. Thus we have $(d\omega)^k(0) = 0 = (d\tilde{\omega})^k(0)$ and so $c(0) = 0|_{K(0)} = \tilde{c}(0)$. If ω is not good, then we can find a function ϕ such that $\phi\omega$ is good (Lemma 4.1). Then equality of the connection 1-forms corresponding to the good generators $\phi\omega$ and $\phi\tilde{\omega}$ implies equality of the connection 1-forms corresponding to ω and $\tilde{\omega}$ (they are changed by the same term $(\phi(0))^{-1}d\phi(0)$, Lemma 4.2).

Vice versa, let $c(0) = \tilde{c}(0)$. Then $K(0) = \tilde{K}(0)$. Let ω be good. Then $(d\omega)^k(0) = 0$ and $c(0) = 0|_{K(0)} = \tilde{c}(0)$. The latter equality and the displayed formula in Lemma 4.2 imply $(d\tilde{\omega})^k(0) = 0$, and so $\tilde{\omega}$ is good. Since both good generators ω and $\tilde{\omega}$ define the same section in L , they define the same tangent spaces $\Delta(q) = \tilde{\Delta}(q)$, where $q = \omega(0) = \tilde{\omega}(0)$, and so the same connections. Q.E.D.

Consider two local singular contact structures P and \tilde{P} with the same structurally smooth S and Martinet ideal (H) , and the same line bundle L . Then there exist generators ω of P and $\tilde{\omega}$ of \tilde{P} such that

$$(4.2) \quad \tilde{\omega} = \omega + H\gamma,$$

where γ is a 1-form. In the main proofs we will need the equality

$$(4.3) \quad (\omega \wedge (d\omega)^{k-1} \wedge \gamma)(0) = 0.$$

PROPOSITION 4.4. — For generators ω and $\tilde{\omega}$ of P and \tilde{P} satisfying (4.2), with $S = \tilde{S}$ structurally smooth and $0 \in S_1$, the following statements hold:

- (i) The equality $\Delta_0 = \tilde{\Delta}_0$ holds if and only if (4.3) holds.
- (ii) Let $(\omega \wedge dH)(0) \neq 0$. Then $K(0) = \tilde{K}(0)$ if and only if (4.3) holds.

Proof. — From (4.2) it follows that the line bundles L and \tilde{L} coincide, and both forms ω and $\tilde{\omega}$ define the same section of $L = \tilde{L}$. The conditions $dH(0) \neq 0$ and $d\tilde{H}(0) \neq 0$ imply that $\omega \wedge (d\omega)^{k-1}(0) \neq 0$ and $\tilde{\omega} \wedge (d\tilde{\omega})^{k-1}(0) \neq 0$. Consequently, these forms have 2-dimensional kernels $K(0)$ and $\tilde{K}(0)$. Proposition 4.3 says that $\Delta_0 = \tilde{\Delta}_0$ if and only if $c(0) = \tilde{c}(0)$, i.e.,

$$(4.4) \quad \frac{(d\omega)^k}{\omega \wedge (d\omega)^{k-1}}(0) = \frac{(d\tilde{\omega})^k}{\tilde{\omega} \wedge (d\tilde{\omega})^{k-1}}(0).$$

With no loss of generality (Lemma 4.1) we assume that $(d\omega)^k(0) = 0$. Let $\tilde{\omega} = \omega + H\gamma$, then $d\tilde{\omega}(0) = d\omega(0) + (dH \wedge \gamma)(0)$. Computing the denominator in the right hand side of (4.4) gives

$$\tilde{\omega} \wedge (d\tilde{\omega})^{k-1}(0) = \omega \wedge (d\omega)^{k-1}(0) + (k-1)(\omega \wedge (d\omega)^{k-2} \wedge dH \wedge \gamma)(0)$$

and equality of kernels $K(0)$ and $\tilde{K}(0)$ is equivalent to the condition (empty, when $k = 1$)

$$(4.5) \quad K(0) \subset \ker(\omega \wedge (d\omega)^{k-2} \wedge dH \wedge \gamma)(0).$$

Similarly, computing the nominator we obtain

$$(d\tilde{\omega})^k(0) = (d\omega)^k(0) + k((d\omega)^{k-1} \wedge dH \wedge \gamma)(0)$$

and the canonical connections are equal if and only if (4.5) holds along with the condition

$$(4.6) \quad K(0) \subset \ker((d\omega)^{k-1} \wedge dH \wedge \gamma)(0).$$

Assume that $(\omega \wedge dH)(0) \neq 0$. Then either $k = 1$, or $k > 1$ and the (2k-2)-form

$$(\omega \wedge (d\omega)^{k-2} \wedge dH)(0)$$

is nonzero, decomposable, and has 3-dimensional kernel (this follows from the conditions $(\omega \wedge (d\omega)^{k-2} \wedge dH)(0) = 0$, $(\omega \wedge (d\omega)^{k-1})(0) \neq 0$). In both cases the subspace $K(0)$ is contained in the 3-dimensional vector space V which is $T_0\mathbb{K}^3$, if $k = 1$, or otherwise V is the kernel of the differential form $(\omega \wedge (d\omega)^{k-2} \wedge dH)(0)$. Therefore, the condition (4.5) implies that $\gamma|_{K(0)} = 0$ and, consequently, (4.5) implies (4.3). Vice versa, if (4.3) holds then $\gamma|_{K(0)} = 0$ and the condition (4.5) is valid. Since (4.5) is equivalent to the equality of the kernels $K(0)$ and $\tilde{K}(0)$, we obtain the

proof of the statement (ii). In order to prove the statement (i) in the case $(\omega \wedge dH)(0) \neq 0$ it remains to show that the condition (4.3) implies (4.6). Note that $K(0)$ is contained in $\ker dH(0)$ (since $0 \in S_1$) and in the kernel of the form $(d\omega)^{k-1}(0)$ (due to the condition $(d\omega)^k(0) = 0$). Since the condition (4.3) implies that $\gamma|_{K(0)} = 0$, we conclude that (4.3) implies (4.6). It remains to consider the case $(\omega \wedge dH)(0) = 0$, $\omega(0) \neq 0$, $dH(0) \neq 0$. In this case (4.5) holds automatically, whereas (4.6) is still equivalent to the condition (4.3) since the forms $((d\omega)^{k-1} \wedge dH)(0)$ and $\omega \wedge (d\omega)^{k-1}(0)$ are the same up to a nonzero numerical factor. It follows that statement (i) holds in this case as well. Q.E.D.

Proof of Proposition 4.1. — (i) From the definition of Δ_0 it follows that $\Delta_0 = \tilde{\Delta}_0$ implies $K(0) = \tilde{K}(0)$. If $(\omega \wedge dH)(0) \neq 0$, then the converse implication follows from Proposition 4.4. Namely, $K(0) = \tilde{K}(0)$ implies (4.3) and this condition implies that $\Delta_0 = \tilde{\Delta}_0$.

To prove the statement (ii) note that if $(\omega \wedge dH)(0) = 0$, then we have the equality $(\omega \wedge (d\omega)^{k-1})(0) = (\tilde{\omega} \wedge (d\tilde{\omega})^{k-1})(0)$ which follows from $\omega(0) = \tilde{\omega}(0)$, $d\omega(0)|_{T_0S} = d\tilde{\omega}(0)|_{T_0S}$ (since $\omega(p) = \tilde{\omega}(p)$ on S), and $\ker \omega(0) = \ker \tilde{\omega}(0) = T_0S$. Thus $K(0) = \tilde{K}(0)$. Using the above equality and the formula $(d\omega)^k(0) = k c(0) \wedge (\omega \wedge (d\omega)^{k-1})(0)$ (which follows from the definition of the connection 1-form) we see that (ii) follows from Proposition 4.3. Q.E.D.

5. A realization theorem.

We complete our analysis of local singular contact structures with a theorem which describes all local Pfaffian equations on \mathbb{K}^{2k} which can be obtained as the restrictions $P|_S$, or the restriction to a smooth component of S , when S is not smooth.

We call a local Pfaffian equation (α) on \mathbb{K}^{2k} *realizable* (respectively, *realizable with structurally smooth S*) if there exists a nonvanishing 1-form ω on \mathbb{K}^{2k+1} such that $\mathbb{K}^{2k} \subset S = S(\omega)$ and $(\alpha) = (\omega)|_{\mathbb{K}^{2k}}$, (respectively, $\mathbb{K}^{2k} = S$, S is structurally smooth, and $(\alpha) = (\omega)|_S$). Here we treat \mathbb{K}^{2k} as a subspace of \mathbb{K}^{2k+1} .

THEOREM 5.1. — (a) *A local Pfaffian equation (α) on \mathbb{K}^{2k} is realizable if and only if the ideal in the exterior algebra of germs of differential forms*

on \mathbb{K}^{2k} generated by the form $\alpha \wedge (d\alpha)^{k-1}$ is a differential ideal (i.e., it is closed under differentiation).

(b) A local Pfaffian equation (α) on \mathbb{K}^{2k} is realizable with structurally smooth S if and only if there exists a local differential 1-form β such that the following equalities hold:

$$(i) \quad (d\alpha)^k = k\alpha \wedge (d\alpha)^{k-1} \wedge \beta,$$

$$(ii) \quad ((d\alpha)^{k-1} \wedge d\beta - (k-1)\alpha \wedge (d\alpha)^{k-2} \wedge \beta \wedge d\beta)(0) \neq 0,$$

(when $k = 1$, these conditions take the form $d\alpha = \alpha \wedge \beta$, and $d\beta(0) \neq 0$).

Proof. — (a) Assume that α is a realizable 1-form on $\mathbb{K}^{2k} = \hat{S}$, i.e., there is a Pfaffian equation P on \mathbb{K}^{2k+1} , generated by a nonvanishing 1-form, such that $\hat{S} \subseteq S = S(P)$, and $P|_{\hat{S}} = (\alpha)$. Let ω be a generator of P such that $\omega|_{\hat{S}} = \alpha$. Since $\omega(0) \neq 0$ there exists a nonvanishing vector field V which is transversal to \hat{S} and such that $V|_{\omega} \equiv 1$. We can choose coordinates (x_1, \dots, x_{2k}, z) such that \hat{S} is given by the equation $z = 0$ and $V = \partial/\partial z$. In these coordinates $\alpha = a_1(x)dx_1 + \dots + a_{2k}(x)dx_{2k}$ and $\omega = dz + \alpha + z\beta$ for some 1-form $\beta = b_1(x, z)dx_1 + \dots + b_{2k}(x, z)dx_{2k}$ on \mathbb{K}^{2k+1} . We compute

$$\begin{aligned} \omega \wedge (d\omega)^k &= dz \wedge ((d\alpha)^k - k\alpha \wedge (d\alpha)^{k-1} \wedge \beta) + kz\alpha \wedge (d\alpha)^{k-1} \wedge d\beta \\ &\quad + zdz \wedge ((d\alpha)^{k-1} \wedge d\beta - k(k-1)\alpha \wedge (d\alpha)^{k-2} \wedge \beta \wedge d\beta) \pmod{(z^2)}, \end{aligned}$$

(we omit the term $z\beta \wedge (d\alpha)^k$ which vanishes as product of $2k+1$ terms of the form dx_1, \dots, dx_{2k}). Since $\omega \wedge (d\omega)^k = 0$ modulo (z) , comparing both sides on the hypersurface $\{z = 0\}$ we obtain that

$$(d\alpha)^k = \alpha \wedge (d\alpha)^{k-1} \wedge (k\beta|_S).$$

The latter relation means that the ideal generated by $\alpha \wedge (d\alpha)^{k-1}$ is a differential ideal which completes the proof of the "only if" part of statement (a).

In order to prove the "if" part of statement (a) consider $\alpha = a_1(x)dx_1 + \dots + a_{2k}(x)dx_{2k}$ and assume that the ideal generated by $\alpha \wedge (d\alpha)^{k-1}$ is a differential ideal, i.e., $(d\alpha)^k = \alpha \wedge (d\alpha)^{k-1} \wedge \beta$ for some 1-form $\beta = b_1(x)dx_1 + \dots + b_{2k}(x)dx_{2k}$. Put $\omega = dz + \alpha + k^{-1}z\beta$. Computing $\omega \wedge (d\omega)^k$ we obtain $\{z = 0\} \subset S(\omega)$, $\omega|_{\{z=0\}} = \alpha$, and it follows that (α) is realizable.

(b) The proof of realizability criteria with S structurally smooth goes along the same lines. Let us assume that $(\alpha \wedge (d\alpha)^{k-1})(0) = 0$ (otherwise we can bring α to the normal form $dx_2 + x_3dx_4 + \dots + x_{2k-1}dx_{2k}$ and the

proof is trivial). If (α) is realizable on $\hat{S} = \mathbb{K}^{2k}$, with structurally smooth Martinet hypersurface S , then $\hat{S} = S$. We choose coordinates as before and compute $\omega \wedge (d\omega)^k$. The first term vanishes by the first realizability criterion. The second and the third term give the expression

$$kz dz \wedge (\sigma + \delta), \quad \text{where } \sigma = -\alpha \wedge (d\alpha)^{k-1} \wedge \left(\sum \frac{\partial b_i}{\partial z} dx_i \right)$$

and δ is the $2k$ -form which appears in condition (ii). Since $\sigma(0) = 0$, it follows that condition (ii) is necessary for the Martinet hypersurface to be structurally smooth.

Vice versa, if conditions (i) and (ii) hold then the construction of $P = (\omega)$ proposed in the proof of statement (a) leads to a local Pfaffian equation (ω) , $\omega = dz + \alpha + k^{-1}z\beta$, which has the Martinet hypersurface $S = \{z = 0\}$ structurally smooth. Q.E.D.

Remark. — Condition (i) in statement (b) coincides with the condition that the ideal generated by $\alpha \wedge (d\alpha)^{k-1}$ is a differential ideal. This condition is equivalent to the condition $\operatorname{div}(Z) \in I(\alpha \wedge (d\alpha)^{k-1})$, where $I(\delta)$ denotes the ideal of function germs generated by the coefficients of the differential form $\delta = \alpha \wedge (d\alpha)^{k-1}$, Z denotes the characteristic vector field Z on S (see Section 3.2), and $\operatorname{div}(Z)$ is the divergence of Z defined by the equality $\operatorname{div}(Z)\nu = d(Z \lrcorner \nu)$, where ν is a nondegenerate $2k$ -form on \mathbb{K}^{2k} . Writing $\alpha \wedge (d\alpha)^{k-1} = \sum_1^{2k} a_i d\check{x}_i$, where $d\check{x}_i = dx_1 \wedge \cdots \wedge dx_i \wedge \cdots \wedge dx_{2k}$ with dx_i omitted, we see that our realizability condition (i) is equivalent to

$$\sum_1^{2k} (-1)^i \frac{\partial a_i}{\partial x_i} \in (a_1, \dots, a_{2k}).$$

6. Main proofs.

In Section 6.1 we prove the results of Sections 3 and 4 (Theorems 3.1, 3.2, 4.1, 4.2, and 4.3) using the same scheme. Proposition 6.1 plays the central role in the proofs. In Section 6.2 we prove lemmas used in the proof. The results of Section 2 are proved in Section 6.3. Throughout the proofs all objects are germs at the origin of category \mathcal{C} and all paths (of 1-forms, functions, etc.) are smooth with respect to the parameter t .

6.1. Proofs of results in Sections 3 and 4.

Theorems 3.1, 3.2, and 4.1 will be proved by the reduction to Proposition 6.1 below. Theorems 4.2 and 4.3 will be proved using lemmas

realizing this reduction. To formulate Proposition 6.1 we need the following simple statement.

LEMMA 6.1. — *Assume that nonvanishing 1-forms ω and $\tilde{\omega}$ define the same Martinet ideal (H) and $\omega = \tilde{\omega} \bmod (H)$. Then, for the path of 1-forms $\omega_t = \omega + t(\tilde{\omega} - \omega)$, $t \in [0, 1]$, we have*

$$(6.1) \quad \omega_t \wedge (d\omega_t)^k = R_t H \Omega,$$

where Ω is a nondegenerated volume form and R_t is a family of functions such that $R_t(0)$ depends quadratically on t and $R_0(0) \neq 0$, $R_1(0) \neq 0$.

Proof. — Since $\omega = \tilde{\omega} \bmod (H)$, there exists a 1-form γ such that

$$(6.2) \quad \tilde{\omega} = \omega + H\gamma$$

and then $\omega_t = \omega + tH\gamma$ and $d\omega_t = d\omega + t dH \wedge \gamma + tH d\gamma$. Expressing $\omega_t \wedge (d\omega_t)^k$ in terms of ω and γ we see that $\omega_t \wedge (d\omega_t)^k = \Omega_1 + t\Omega_2 + t^2 H\Omega_3 \bmod (H^2)$, where Ω_i 's are n -forms. Since (H) is the Martinet ideal of $(\omega_0) = (\omega)$ and $(\omega_1) = (\tilde{\omega})$, the forms $\omega_0 \wedge (d\omega_0)^k = \Omega_1$ and $\omega_1 \wedge (d\omega_1)^k = \Omega_1 + \Omega_2$ are divisible over H . Therefore, Ω_2 is also divisible over H . It follows that (6.1) holds with the family R_t depending quadratically on $t \bmod (H)$ and a nondegenerated volume form Ω . Using again the condition that (H) is the Martinet ideal of ω_0 and ω_1 we conclude that $R_0(0) \neq 0$ and $R_1(0) \neq 0$.

PROPOSITION 6.1. — *Assume that two local singular contact structures P and \tilde{P} define the same Martinet ideal (H) and are generated by 1-forms ω and $\tilde{\omega}$ that are equal modulo (H) . Assume also that one of the following two conditions holds:*

(a) *the Martinet hypersurface S is structurally smooth and P and \tilde{P} define the same canonical partial connection Δ_0 ;*

(b) *P satisfies the condition (B) in Section 3, (H) has the property (PZ), and $0 \in S_1$.*

Then in the holomorphic case P and \tilde{P} are equivalent. The same is true in real cases (C^∞ and real-analytic categories) provided that the function $t \rightarrow R_t(0)$ has no zeros on the segment $[0, 1]$. Here R_t is the path of functions in (6.1).

Recall that the condition $0 \in S_1$ means that the Martinet condition (MC) is violated. If (MC) is valid then Proposition 6.1 is a trivial corollary of the Martinet theorem.

Proof of Proposition 6.1. — Define a function $f(t)$ on the interval $[0, 1]$ as follows: in the real cases $f(t) = t$, and in the holomorphic category

$f(t)$ is a complex-valued function such that $f(0) = 0, f(1) = 1$ and $R_{f(t)}(0) \neq 0$ as $t \in [0, 1]$, where R_t is the path of functions in (6.1). Such a function exists since $R_0(0) \neq 0, R_1(0) \neq 0$, and $t \rightarrow R_t(0)$ is quadratic. Let γ be the 1-form satisfying the relation (6.2). Consider the path $\omega_t = \omega + f(t)H\gamma, t \in [0, 1]$ joining ω to $\tilde{\omega}$. Then by Lemma 6.1

$$(6.3) \quad \omega_t \wedge (d\omega_t)^k = Q_t H\Omega, \quad Q_t = R_{f(t)}.$$

The construction of the function $f(t)$ and the assumption $R_t(0) \neq 0$ for $t \in [0, 1]$, in the real cases, imply that $Q_t(0) \neq 0, t \in [0, 1]$. This means that the Martinet ideal of ω_t is constant along the path and is equal to (H) . We will use this property of the path to prove that there exists a family of diffeomorphisms $\Phi_t, \Phi_t(0) = 0$, such that $\Phi_t^*(\omega_t) = (\omega_0)$ and in particular $\Phi_1^*(\tilde{\omega}) = (\omega)$.

We use the homotopy method (cf. [Ro] or [Zh1]): the existence of the required family Φ_t is obtained by solving the differential equation $d\Phi_t/dt = X_t(\Phi_t), \Phi_0 = id$. It suffices to prove that the ‘‘homological’’ equation

$$L_{X_t}\omega_t + f'(t)H\gamma = 0 \text{ mod } \omega_t,$$

obtained by differentiating with respect to t the equation $\Phi_t^*\omega_t = \phi_t\omega_0$, is solvable with respect to a path X_t of vector fields satisfying the condition $X_t(0) = 0$. Here L_{X_t} is the Lie derivative along X_t . An explicit solution of the homological equation is the family X_t defined by the relation

$$(6.4) \quad X_t \lrcorner \Omega = -kQ_t^{-1}f'(t)\gamma \wedge \omega_t \wedge (d\omega_t)^{k-1}.$$

To check that X_t is a solution of the homological equation note that (6.4) implies that $X_t \lrcorner \omega_t = 0$ and consequently $L_{X_t}\omega_t = X_t \lrcorner d\omega_t$. Then (6.4) and (6.3) imply that

$$(L_{X_t}\omega_t + f'(t)H\gamma) \wedge \omega_t \wedge (d\omega_t)^{k-1} = 0.$$

This relation implies that X_t is a solution of the homological equation by the following observations: (i) if a 1-form ω is contact at a point p and α is another 1-form such that $(\alpha \wedge \omega \wedge (d\omega)^{k-1})(p) = 0$ then $\alpha(p) = 0 \text{ mod } \omega(p)$. Therefore if the set of contact points of ω is dense and ω does not vanish then the relation $\alpha \wedge \omega \wedge (d\omega)^{k-1} = 0$ implies that $\alpha = 0 \text{ mod } \omega$; (ii) the set of contact points of ω_t coincides with that of $\omega_0 = \omega$ since these forms have the same Martinet ideal (H) . This set is dense since H is a nonzero divisor. The forms ω_t do not vanish since $\omega_t(0) = \omega(0) \neq 0$.

To complete the proof it remains to show that $\Phi_t(0) = 0$, i.e., the vector fields X_t defined by (6.4) satisfy $X_t(0) = 0$. Note that

$(\omega_t \wedge (d\omega_t)^{k-1} \wedge \gamma)(0) = (\omega \wedge (d\omega)^{k-1} \wedge \gamma)(0)$ by (6.2). Therefore to prove that $X_t(0) = 0$ it suffices to prove that

$$(6.5) \quad (\omega \wedge (d\omega)^{k-1} \wedge \gamma)(0) = 0.$$

We will prove that (6.5) follows from any of the assumptions (a) or (b) in Proposition 6.1. Actually, the fact that the assumption (a) implies (6.5) is already proved - it is one of the statements of Proposition 4.4 in Section 4. Therefore to complete the proof we need the following statement.

LEMMA 6.2. — *Assume that ω satisfies the condition (B) and $0 \in S_1$. If $\tilde{\omega}$ and ω have the same Martinet ideal (H) satisfying (PZ) and $\tilde{\omega} = \omega + H\gamma$, then (6.5) holds.*

Proposition 6.1 is proved modulo Lemma 6.2. Note that Lemma 6.2 immediately implies Theorem 4.3.

Proof of Theorem 4.3. — The equality of the line bundles allows to choose generators ω and $\tilde{\omega}$ that are equal modulo (H), where (H) is the Martinet ideal. Then we have (6.2), and by Lemma 6.2 the condition (B) implies (6.5). The relations (6.2) and (6.5) imply the equality of the partial connections by Proposition 4.4 in Section 4.

Now we will prove Theorems 3.1, 3.2, and 4.1 by the reduction to Proposition 6.1. In each of these theorems we have to prove the equivalence of P and \tilde{P} satisfying certain assumptions. The assumption on the equivalence of invariants will be replaced (without loss of generality) by the assumption of their equality. Therefore in each of the theorems we assume that the Martinet hypersurfaces are equal: $S = \tilde{S}$. Then, by the condition (PZ), which is an assumption in each of our theorems, we also have the equality of the Martinet ideals: $(H) = (\tilde{H})$. If $0 \notin S_1$ then Theorems 3.1, 3.2, and 4.1 are trivial corollaries of the Martinet theorem. Therefore we will assume that $0 \in S_1$.

We begin with the proof of Theorem 4.1 in the holomorphic category since in this case the reduction to Proposition 6.1 is straightforward.

Proof of Theorem 4.1 – Holomorphic case. — The equality of the line bundles defined by P and \tilde{P} allows to choose generators ω and $\tilde{\omega}$ that are equal modulo (H). The equivalence of P and \tilde{P} follows from Proposition 6.1 since the condition (a) in this proposition is an assumption in Theorem 4.1.

Now we pass to the holomorphic case in Theorems 3.1 and 3.2. The reduction to Proposition 6.1 requires the following statement, slightly stronger than Proposition 4.2.

LEMMA 6.3. — *Let P and \tilde{P} be local singular contact structures, with the same Martinet ideal (H) satisfying (PZ) and the condition (C), and the same restriction to the regular part S^{reg} of the Martinet hypersurface S . Then there exists a diffeomorphism Φ , equal identity on S , and a generator $\tilde{\omega}$ of Φ^*P such that $\tilde{\omega} = \omega \bmod (H)$.*

Proof of Theorems 3.1 and 3.2 – Holomorphic case. — The condition (C) is implied by the condition (A) (assumption in Theorem 3.1) and it is assumed in Theorem 3.2. Therefore we can apply Lemma 6.3. A diffeomorphism equal to identity on S does not change the Martinet ideal (H) , therefore by Lemma 6.3 we can assume that P and \tilde{P} have the same Martinet ideal (H) and are generated by 1-forms ω and $\tilde{\omega}$ equal modulo (H) . Since the condition (A) implies (B) then the assumption (b) of Proposition 6.1 is valid, and by this proposition P is equivalent to \tilde{P} .

Now we reduce the real cases in Theorems 3.1, 3.2, and 4.1 to Proposition 6.1. The reduction is more difficult than in the holomorphic case since it requires analyzing the quadratic function $t \rightarrow R_t(0)$ in (6.1),

$$R_t(0) = a + bt + ct^2.$$

LEMMA 6.4. — *Assume that ω and $\tilde{\omega}$ have the same Martinet ideal (H) satisfying (PZ) and $\tilde{\omega} = \omega \bmod (H)$. Assume also that $0 \in S_1$ and ω satisfies the condition (A). Let R_t be the path of functions in (6.1). Then $b = c = 0$, i.e., $R_t(0) \equiv R_0(0)$, for $t \in [0, 1]$.*

Proof of Theorem 3.1 – Real cases. — Repeating the arguments in the proof of Theorem 3.1 in the holomorphic case, we see that there is no loss of generality to assume that P and \tilde{P} have the same Martinet ideal (H) and are generated by 1-forms ω and $\tilde{\omega}$ equal modulo (H) . Using Lemma 6.4 and the fact that $(A) \Rightarrow (B)$ we see the validity of the assumptions of Proposition 6.1, and by this proposition P is equivalent to \tilde{P} .

To prove Theorems 3.2 and 4.1 in real cases, we first show that the function $t \rightarrow R_t(0)$ can be made affine, i.e., $c = 0$.

LEMMA 6.5. — *Let ω and $\tilde{\omega}$ be 1-forms which are equal modulo (H) , they have the same Martinet ideal (H) satisfying (PZ) and $\Delta_0 = \tilde{\Delta}_0$.*

(i) If $k = 1$ or if $dH(0) = 0$, then the path R_t in (6.1) is affine at the origin: $R_t(0) = a + bt$.

(ii) Assume that $k \geq 2$, S is structurally smooth, and $0 \in S_1$. Then there exists a diffeomorphism Φ preserving the invariants (H) , $\tilde{\mathcal{O}} = \tilde{\mathcal{O}}_S, \tilde{\mathcal{O}}_E, \tilde{\mathcal{O}}_L$, and $\tilde{\Delta}_0$ and bringing $\tilde{\omega}$ to a 1-form $\hat{\omega} = \Phi^*\tilde{\omega}$ such that $\hat{\omega} = \omega \bmod (H)$ and, after the replacement of $\tilde{\omega}$ by $\hat{\omega}$, the path R_t in (6.1) is affine at the origin, $R_t(0) = a + bt$.

Proof of Theorems 3.2 and 4.1 – Real cases. — Repeating the arguments used in the proofs of the holomorphic case we see that there is no loss of generality to assume that P and \tilde{P} have the same Martinet ideal (H) , they are generated by 1-forms ω and $\tilde{\omega}$ equal modulo (H) (we use Lemma 6.3 in the case of Theorem 3.2), and they satisfy all the assumptions of Proposition 6.1 except the last one: the assumption that $R_t(0) \neq 0$ as $t \in [0, 1]$. Using Lemma 6.5 we see that, without losing generality, we can additionally assume that the function $t \rightarrow R_t(0)$ is affine. This assumption implies that, in order to prove Theorems 3.2 and 4.1, it is enough to show that the numbers $R_0(0)$ and $R_1(0)$ have the same sign (we know, from Lemma 6.1, that these numbers are not equal to 0), then $R_t(0) \neq 0$ as $t \in [0, 1]$. The proof is completed by the fact that the requirement that $R_0(0)$ and $R_1(0)$ have the same sign follows from equality of canonical orientations (assumed in Theorems 3.2 and 4.1). Namely, we have the following lemma.

LEMMA 6.6. — *Let ω and $\tilde{\omega}$ be 1-forms with the same Martinet ideal (H) , i.e., $\omega \wedge (d\omega)^k = RH\Omega$ and $\tilde{\omega} \wedge (d\tilde{\omega})^k = \tilde{R}H\Omega$, where Ω is a nondegenerated volume form, and R and \tilde{R} are nonvanishing functions. Then any of the canonical orientations $\mathcal{O}_S, \mathcal{O}_E$, or \mathcal{O}_L defined by (ω) coincides with the same orientation defined by $(\tilde{\omega})$ if and only if the numbers $R(0)$ and $\tilde{R}(0)$ have the same sign.*

Proof of Lemma 6.6. — This lemma easily follows from our formulas for the volume forms defining the canonical orientations $\mathcal{O}_S, \mathcal{O}_E$ (Section 3.5) and \mathcal{O}_L (Section 4.1), and Propositions 3.1 and 3.5 stating that the sets S^{reg} and S_E (on which the orientations $\mathcal{O}_S, \mathcal{O}_L$ and, respectively, \mathcal{O}_E are defined) are not empty.

We have completed the proofs of Theorems 3.1, 3.2, 4.1 and 4.3 modulo the lemmas used in the proofs. Now we will prove Theorem 4.2 as a consequence of some of these results.

Proof of Theorem 4.2. — The condition $0 \in S_1$, meaning that the origin is not a Martinet point, will be essentially used. At first let us prove the second statement, that under condition (A) the equality of the line bundles implies the equality of the orientations $\mathcal{O}_L = \tilde{\mathcal{O}}_L$. Since the line bundles are equal, we can choose generators ω and $\tilde{\omega}$ to be equal modulo (H) , where (H) is the Martinet ideal of (ω) and $(\tilde{\omega})$. By Lemma 6.4 we have $R_1(0) = R_0(0)$, where R_t is the path of functions in (6.1). Now we use Lemma 6.6 implying that $\mathcal{O}_L = \tilde{\mathcal{O}}_L$.

To prove the first statement of Theorem 4.2, that under condition (A) the equality of the restrictions of P and \tilde{P} to S implies the equality of the orientations \mathcal{O}_S or \mathcal{O}_E , we first use Lemma 6.3. It is important to note that any diffeomorphism equal to identity on S preserves the orientations \mathcal{O}_S and \mathcal{O}_E . This follows from the fact that \mathcal{O}_S is defined on S and \mathcal{O}_E - on the subspaces $E(p) = \ker \omega(p) \cap T_p S$ which are contained in $T_p S$, $p \in S_E \subset S$. This observation allows us to assume, without loss of generality, that P and \tilde{P} are generated by ω and $\tilde{\omega}$, equal modulo (H) , where (H) is the Martinet ideal of P and \tilde{P} . The end of the proof is the same as in the proof of the second statement of Theorem 4.2: the equality of the orientations follows from Lemmas 6.4 and 6.6.

6.2. Proofs of the lemmas.

Lemmas 6.2 and 6.4 will be proved using similar arguments, based on the following properties of the characteristic vector field X , defined in Section 3.2, under the conditions (A) and (B): if $X(0) = 0$ (which means that $0 \in S_1$) and Y is another vector field such that

$$(6.6) \quad X \wedge Y = 0 \text{ mod } (H)$$

then

$$(6.7) \quad (B) \implies Y(0) = 0, \quad (A) \implies Y = FX \text{ mod } (H),$$

where F is some function. The first implication follows from the definition of the condition (B). The second one follows from the fact that the condition (A) implies the 1-division property of X , see Appendix 1.

In the proof of Lemmas 6.2 and 6.4 the following calculation will be used. We consider the path $\omega_t = \omega + tH\gamma$, then $d\omega_t = d\omega + t dH \wedge \gamma + tH d\gamma$. Denoting $\Omega_1 = \omega \wedge (d\omega)^k$, we have

$$\omega_t \wedge (d\omega_t)^k = \Omega_1 + t\Omega_2 + t^2 H\Omega_3 \text{ mod } (H^2),$$

where

$$\Omega_2 = k\omega \wedge (d\omega)^{k-1} \wedge dH \wedge \gamma + kH\gamma \wedge (d\omega)^k + kH\omega \wedge (d\omega)^{k-1} \wedge d\gamma$$

and

$$\Omega_3 = k(k-1)\omega \wedge (d\omega)^{k-2} \wedge dH \wedge \gamma \wedge d\gamma.$$

Proof of Lemma 6.2. — Taking $t = 1$ in the above formulas and using the fact that (H) is the Martinet ideal of $(\omega) = (\omega_0)$ and of $(\tilde{\omega}) = (\omega_1)$ we see that Ω_1 , as well as $\omega_1 \wedge (d\omega_1)^k$, are divisible over H . Therefore Ω_2 and so $k\omega \wedge (d\omega)^{k-1} \wedge dH \wedge \gamma$ are divisible over H . This means that

$$(6.8) \quad \omega \wedge (d\omega)^{k-1} \wedge dH \wedge \gamma = QH\Omega,$$

for some function Q and a volume form Ω . Introduce vector fields X and Y by the relations

$$X \lrcorner \Omega = \omega \wedge (d\omega)^{k-1} \wedge dH, \quad Y \lrcorner \Omega = \omega \wedge (d\omega)^{k-1} \wedge \gamma.$$

Recall that X is the characteristic vector field introduced in Section 3.2. Let $p \in S$. If $X(p) \neq 0$ then $K(p) = \ker(\omega \wedge (d\omega)^{k-1})(p)$ is 2-dimensional and (6.6) implies that the covectors $dH|_{K(p)}$ and $\gamma|_{K(p)}$ are colinear, and so the vector $Y(p)$ is proportional to $X(p)$. Therefore $(X \wedge Y)(p) = 0$ for such p and so for any $p \in S$. Using the property (PZ) we see that (6.6) holds. By the first implication in (6.7) we obtain that $Y(0) = 0$, i.e., the condition (6.5) holds. Lemma 6.2 is proved.

Proof of Lemma 6.4. — From the equality of the Martinet ideals of P and \tilde{P} it follows, as in first part of the proof of Lemma 6.2, that (6.6) and (6.8) hold. Since the condition (A) implies the condition (B), it follows by Lemma 6.2 that (6.5) also holds.

From the formula for $\omega_t \wedge (d\omega_t)^k$ calculated at the beginning of this subsection and the assumption that $t \rightarrow R_t(0) = a + bt + ct^2$ we see that it is enough to prove the equalities

$$(6.9) \quad Q(0) = 0, \quad (\omega \wedge (d\omega)^{k-1} \wedge d\gamma)(0) = 0, \quad ((d\omega)^k \wedge \gamma)(0) = 0$$

and the equality

$$(6.10) \quad (\omega \wedge (d\omega)^{k-2} \wedge dH \wedge \gamma \wedge d\gamma)(0) = 0.$$

Note that (6.10) is implied by (6.5) and the middle relation in (6.9). In fact, if $dH(0) \neq 0$ and $k \geq 2$, then it is easy to see from the relations $(\omega \wedge (d\omega)^{k-1})(0) \neq 0$ (follows from $dH(0) \neq 0$) and $(\omega \wedge (d\omega)^{k-1} \wedge dH)(0) = 0$ that the form $(\omega \wedge (d\omega)^{k-2} \wedge dH)(0) \neq 0$ has a 3-dimensional kernel $V(0)$. The 2-dimensional kernel $K(0)$ of the form $(\omega \wedge (d\omega)^{k-1})(0)$ is a subspace of $V(0)$. The relation (6.5) and the middle relation in (6.9) mean that the forms γ and $d\gamma$ are zero, when restricted to $K(0)$. Therefore (6.5) and (6.9) imply (6.10).

So, it is enough to prove the equalities in (6.9). We first show that (6.5) implies that $Q(0) = 0$ if S is structurally smooth, i.e., if $dH(0) \neq 0$. In this case (6.5) means that the form $\gamma(0)$ vanishes on the 2-space $K(0)$. The same is true for the form $dH(0)$ since $0 \in S_1$. Thus $j_0^1(\omega \wedge (d\omega)^{k-1} \wedge dH \wedge \gamma) = 0$, where j_0^1 denotes the 1-jet at 0. Then by (6.8) $j_0^1(HQ) = 0$ and since $j_0^1 H \neq 0$ we obtain $Q(0) = 0$.

To prove that $Q(0) = 0$ if $dH(0) = 0$ and to prove the other equalities in (6.9) we use the second implication in (6.7) which implies that

$$(6.11) \quad \omega \wedge (d\omega)^{k-1} \wedge \gamma = F\omega \wedge (d\omega)^{k-1} \wedge dH + HW$$

for some $2k$ -form W . Multiplying this equality by dH and taking into account (6.8) we obtain

$$(6.12) \quad Q\Omega = W \wedge dH \pmod{(H)}.$$

We see that if $dH(0) = 0$, then $Q(0) = 0$. As we have proved that $Q(0) = 0$ in the other case, we have $Q(0) = 0$ in all cases.

The third equality in (6.9) follows from (6.5) and the fact that $0 \in S_1$.

To show the middle equality in (6.9) we take the differentials at the origin of the $2k$ -forms in (6.11). Using the third equality in (6.9), the equality $(\omega \wedge (d\omega)^{k-1} \wedge dH)(0) = 0$ (equivalent to $0 \in S_1$) and $((d\omega)^k(0) \wedge dH)(0) = 0$ (again, since $0 \in S_1$) we obtain that $(\omega \wedge (d\omega)^{k-1} \wedge d\gamma)(0) = (dH \wedge W)(0)$. Since $Q(0) = 0$ then by (6.12) $(dH \wedge W)(0) = 0$ and the second relation in (6.9) follows. The proof is complete.

Proof of Lemma 6.3. — Take any generators ω and $\tilde{\omega}$ of P and \tilde{P} . At first we show that there is no loss of generality to assume that

$$(6.13) \quad \tilde{\omega} = \omega + g dH + H\gamma$$

for some function g and 1-form γ . After this we prove that the term $g dH$ can be killed by a diffeomorphism equal to identity on S .

The condition that $P = (\omega)$ and $\tilde{P} = (\tilde{\omega})$ have the same restriction to S^{reg} and our definition of the restriction imply that $(f\tilde{\omega})|_{S^{\text{reg}}} = \omega|_{S^{\text{reg}}}$ for some nonvanishing function f on \mathbb{K}^n . It follows that there is no loss of generality to assume that ω and $\tilde{\omega}$ have the same pullback to S^{reg} . Then $(\omega - \tilde{\omega}) \wedge dH(p) = 0$ for any $p \in S^{\text{reg}}$, and since dH vanishes at singular points of S we obtain that the latter relation is valid for all $p \in S$ (we use Proposition 3.1 stating that $S^{\text{reg}} \neq \emptyset$.) By the property (PZ) of the Martinet ideal this relation implies that $(\omega - \tilde{\omega}) \wedge dH(p) = 0 \pmod{(H)}$. Now we use the condition (C) which implies the 1-division property of the form

dH (see Appendix 1). The 1-division property says that if θ is a 1-form such that $\theta \wedge dH = 0 \pmod{(H)}$ then $\theta = g dH + H\gamma$ for some function g and 1-form γ . Therefore the latter relation implies (6.13).

Now we will kill the term $g dH$ in (6.13) by a diffeomorphism equal identity on S . At first consider the case where S is structurally smooth. In this case we take vector fields Z and \tilde{Z} which are transversal to S and such that $Z]\omega = \tilde{Z}]\tilde{\omega} \equiv 1$, and a diffeomorphism Φ equal identity on S and bringing \tilde{Z} to Z . Then the 1-forms ω and $\Phi^*\tilde{\omega}$ are equal at any point of S and by the property (PZ) $\Phi^*\tilde{\omega} = \omega \pmod{(H)}$.

If S is not structurally smooth, i.e., $dH(0) = 0$, then to kill the term $g dH$ we use the homotopy method. We join ω to $\tilde{\omega}$ by the path $\omega_t = \omega + t(\tilde{\omega} - \omega)$ and prove that the form ω_t can be reduced to the form ω_0 modulo (H) by a path Φ_t of diffeomorphisms satisfying the differential equation $\frac{d\Phi_t}{dt} = HY_t(\Phi_t)$ and the initial condition $\Phi_0 = id$, where Y_t is a path of vector fields. It is clear that for any family Y_t the diffeomorphism Φ_t preserves (H) and is identical on S . The requirement $\Phi_t^*\omega_t = \omega_0 \pmod{(H)}$ is equivalent to the equation $L_{HY_t}\omega_t + (\tilde{\omega} - \omega) = 0 \pmod{(H)}$, where L_{HY_t} is the Lie derivative along HY_t . Note that $L_{HY_t}\omega_t = (Y_t]\omega_t)dH \pmod{(H)}$ and $\tilde{\omega} - \omega = g dH \pmod{(H)}$. Therefore to solve this equation it suffices to solve the equation $(Y_t]\omega_t) + g = 0$. The latter equation is solvable since in the case under consideration $dH(0) = 0$ and consequently $\omega_t(0) = \omega(0) \neq 0$.

Proof of Lemma 6.5. — Let γ be 1-form satisfying (6.2). From the formula for $\omega_t \wedge (d\omega_t)^k$ obtained at the beginning of this subsection we easily see that the condition that the function $t \rightarrow R_t(0)$ in (6.1) has no quadratic term is equivalent to the condition (6.10) and always holds if $k = 1$. The equality (6.10) holds trivially, if $dH(0) = 0$, i.e., if S is not structurally smooth. This proves the statement (i) of the lemma.

Now we prove the statement (ii). The structural smoothness of S implies that the forms $(\omega \wedge (d\omega)^{k-1})(0)$ and $(\tilde{\omega} \wedge (d\tilde{\omega})^{k-1})(0)$ have two-dimensional kernels $K(0)$ and $\tilde{K}(0)$, see Section 3.2. The equality of the partial connections implies that these kernels are equal: $K(0) = \tilde{K}(0)$. We will prove a stronger relation

$$(6.14) \quad (\omega \wedge (d\omega)^{k-1})(0) = (\tilde{\omega} \wedge (d\tilde{\omega})^{k-1})(0).$$

Taking into account that $\tilde{\omega} = \omega + H\gamma$ and so $\tilde{\omega}(0) = \omega(0)$, $d\tilde{\omega}(0) = d\omega(0) + (dH \wedge \gamma)(0)$, we see that this relation and (6.2) imply that $(\omega \wedge (d\omega)^{k-2} \wedge dH \wedge \gamma)(0) = 0$ which is stronger than (6.10). Therefore to prove the statement (ii) it suffices to find a diffeomorphism Φ satisfying the following requirements: (a) $\Phi^*\tilde{\omega}$ is equal to ω modulo (H) ; (b) Φ preserves

the invariants (H) , $\tilde{\mathcal{O}}$, and $\tilde{\Delta}_0$; (c) the relation (6.14) holds when replacing $\tilde{\omega}$ by $\Phi^*\tilde{\omega}$.

Constructing such a diffeomorphism we can restrict ourselves to the case $(\omega \wedge dH)(0) \neq 0$, otherwise (6.10) holds automatically. This means that $\ker \omega(0) = \ker \tilde{\omega}(0)$ is transversal to S . This condition, the condition $K(0) = \tilde{K}(0)$, and the comparison of the dimensions $2k = \dim \ker \omega(0) > \dim K(0) = 2$ following from the assumption $k \geq 2$ allow us to take tangent vectors $b, \tilde{b} \in \ker \omega(0)$ that are transversal to S and such that $b \rfloor (\omega \wedge (d\omega)^{k-1})(0) = \tilde{b} \rfloor (\tilde{\omega} \wedge (d\tilde{\omega})^{k-1})(0)$. Take vector fields $Y \in \ker \omega$ and $\tilde{Y} \in \ker \tilde{\omega}$ such that $Y(0) = b$ and $\tilde{Y}(0) = \tilde{b}$. Take any diffeomorphism Φ which is equal to identity on S and brings \tilde{Y} to Y . Let $\hat{\omega} = \Phi^*\tilde{\omega}$. Since $0 \in S_1$ then $K(0) \subset T_0S$, therefore Φ preserves the space $K(0)$. On the other hand, Φ brings the vector \tilde{b} to the vector b . This implies the validity of the requirement (c) above.

To complete the proof we have to show that Φ also satisfies the requirements (a) and (b). The fact that Φ preserves the Martinet ideal (H) follows from the fact that $\Phi(S) = S$. Note that by the construction Φ preserves $\ker \tilde{\omega}(p)$ at any point $p \in S$. Since Φ is identity on S and $\ker \tilde{\omega}$ is transversal to S we conclude that Φ also preserves $\tilde{\omega}(p)$ at any point $p \in S$. Therefore, $\Phi^*\tilde{\omega} = \tilde{\omega} \bmod (H)$ and so $\Phi^*\tilde{\omega} = \omega \bmod (H)$, i.e. (a) holds. The equality $\Phi^*\tilde{\omega} = \tilde{\omega} \bmod (H)$ means that Φ preserves the section $p \in S \rightarrow \tilde{\omega}(p)$ of the line bundle $L = \tilde{L}$, therefore it preserves all the sections. Since our invariants \mathcal{O}_S and \mathcal{O}_E are defined on S , and \mathcal{O}_L, Δ_0 are defined on L , the fact that Φ acts trivially on S and L implies that they are preserved. Thus the statement (b) holds and the proof is complete.

6.3. Proofs of results in Section 2.

Since Theorem 2.4 follows from the general realization theorem (Theorem 5.1), it remains to prove Theorems 2.1, 2.2 and 2.3. We begin with the easiest proof.

Let $\alpha = \alpha_1(x, y)dx + \alpha_2(x, y)dy$ be the restriction of ω to $S = \{z = 0\}$. The characteristic ideal I_{ch} is isomorphic to the ideal generated by the functions α_1, α_2 .

Proof of Theorem 2.2. — If $\alpha(0) \neq 0$, then $\text{depth}(I_{\text{ch}}) = \infty$. If $\alpha(0)$ has an algebraically isolated singularity at the origin, then $\text{depth}(I_{\text{ch}}) = 2$. In the holomorphic and real-analytic category this is a well known fact. The same can be proved in the smooth category using the preparation

theorem (we leave the proof to the reader). It follows that the condition (A) is satisfied and the result follows from Theorem 3.1.

The proofs of Theorems 2.1 and 2.3 use the same scheme. Under condition (B) these theorems follow from Theorem 3.2. If (B) is violated, i.e., I_{ch} is 1-generated or zero, then the restriction of P and \tilde{P} to S has a big set of symmetries. These symmetries can be used to make the connections Δ_0 and $\tilde{\Delta}_0$ equal. After that Theorem 4.1 can be used. We realize this idea in two steps.

LEMMA 6.7. — *If the ideal I_{ch} is exactly 1-generated and not flat, then α is equivalent to a 1-form $x^m dx$, where $m \geq 1$.*

In the second step, using the symmetries of such α we normalize the connection Δ_0 preserving the orientation O_S .

LEMMA 6.8. — *Any singular contact structure with structurally smooth $S = \{z = 0\}$ and the restriction to S generated by the 1-form $\alpha = x^m dx$, $m \geq 1$, or $\alpha = 0$, is equivalent to a Pfaffian equation $(dz + \alpha + z\beta)$, where $(\beta|_S)(0) = 0$, via a diffeomorphism preserving S and preserving an orientation on S .*

Proof of Theorems 2.1 and 2.3. — Assume that condition (B) does not hold (otherwise both results follow from Theorem 3.2). Then the ideal I_{ch} is not 2-generated and, by Lemma 6.7, we can assume that $S = \{z = 0\}$ and either $\alpha \equiv 0$ (when I_{ch} is 0-generated) or $\alpha = x^m dx$, where $m \geq 1$. By Lemma 6.8 this allows us to assume that P and \tilde{P} are generated by 1-forms $\omega = dz + \alpha + z\beta$ and $\tilde{\omega} = dz + \alpha + z\tilde{\beta}$, and $(\beta|_S)(0) = (\tilde{\beta}|_S)(0) = 0$. The latter relation implies the equality of the connections $\Delta_0 = \tilde{\Delta}_0$, since $d\omega(0) = 0 = d\tilde{\omega}(0)$. Now the equivalence of P and \tilde{P} follows from Theorem 4.1. The proof is complete.

Proof of Lemma 6.7. — If the ideal generated by α_1 and α_2 is exactly 1-generated and $\alpha \neq 0$ then α has the form $g(x, y)\hat{\alpha}$, where g is some function and $\hat{\alpha}$ is a nonvanishing 1-form. There is no loss of generality to assume that $\hat{\alpha} = dx$, i.e. $\alpha = g(x, y)dx$. By the realization Theorem 2.4 there exists a 1-form $\beta = \beta_1(x, y)dx + \beta_2(x, y)dy$ such that $d\alpha = \alpha \wedge \beta$. We obtain that $\frac{\partial g}{\partial y} = -g\beta_2$. Solving this differential equation and taking $T(x) = g(x, 0)$ we see that $g(x, y) = Q(x, y)T(x)$, where $Q(0) \neq 0$. Thus, changing the generator, we may assume that $\alpha = T(x)dx$. The function $T(x)$ is not zero and not flat, which follows from our assumptions. Any 1-form $T(x)dx$ with nonflat $T(x)$ is equivalent to a 1-form $x^m dx$ for some m .

Proof of Lemma 6.8. — At first we assume that $\alpha = x^m dx$. Then P is equivalent to the Pfaffian equation generated by $\omega = Cdz + x^m dx + z\beta$ with some 1-form β and some function C and defining the Martinet surface $S = \{z = 0\}$. There is no loss of generality to assume that $C = 1$ (since $\omega(0) \neq 0$ then $C(0) \neq 0$ and we can change z by z/C). The fact that S is given by the equation $z = 0$ implies that $\beta(0) = r_1 dx + r_2 dz$ for some scalars r_1, r_2 . Make a change of the coordinates x and z : $x \rightarrow x + rx^2$, $z \rightarrow zE(x)$, where $r = -r_1/m$ and $E(x) = (1 + rx)^m$. Then, in new coordinates $\omega = E(x)(dz + x^m dx + z\hat{\beta})$, where $\hat{\beta}(0) = \beta(0) + dE(0) = \beta(0) + mr dx = r_2 dz$. Then $(\hat{\beta}|_S)(0) = 0$.

Now assume that $\alpha = 0$. Then P is equivalent to the Pfaffian equation generated by $\omega = dz + z\beta$. Let $\beta(0) = r_1 dx + r_2 dy + r_3 dz$. Let us change the coordinate z so that $z \rightarrow zG(x, y)$, where $G(x, y) = 1 - r_1 x - r_2 y$. In the new coordinates we have $\omega = G(x, y)(dz + z\hat{\beta})$, where $\hat{\beta}(0) = \beta(0) + dG(0) = r_3 dz$. Then $(\hat{\beta}|_S)(0) = 0$ and the proof is complete.

Appendix 1.

In this appendix we collect these basic definitions and facts related to the notions of depth, the Koszul complex, and division properties, which are used in the paper and in Appendix 2. For more details the reader may consult [BJ] and [E].

Let R be a commutative ring with a unit. Given a proper ideal I of R , we say that a sequence of elements a_1, \dots, a_q of I is a *regular sequence* if the following condition holds:

(*) a_i is a nonzerodivisor on $R/(a_1, \dots, a_{i-1})$, for $i = 1, \dots, q$,

(in particular, a_1 is a nonzerodivisor in R). Here and thereafter we denote by (a_1, \dots, a_i) the ideal in R generated by the elements a_1, \dots, a_i .

The *depth* of a proper ideal $I \subset R$, denoted $\text{depth}(I)$, is defined as the supremum of lengths of regular sequences in I . Additionally, one defines $\text{depth}(R) = \infty$. The following properties follow from the definition:

- (1) If $I_1 \subset I$, then $\text{depth}(I_1) \leq \text{depth}(I)$.
- (2) If $a_1, \dots, a_q \in I$ is a regular sequence, then the element a_{q-1} is a nonzerodivisor on $R/(a_1, \dots, a_{q-2}, a_q)$.

If R is Noetherian then regular sequences and depth have the following properties:

- (3) If a_1, \dots, a_r is a maximal (with respect to inclusion) sequence of elements in I satisfying (*), then $r = \text{depth}(I)$.
- (4) Any regular sequence a_1, \dots, a_q of a proper ideal $I \subset R$ can be completed to a maximal regular sequence $a_1, \dots, a_q, \dots, a_r$.
- (5) If $a_1, \dots, a_q \in I$ is a regular sequence, then $\text{depth}(I) = \text{depth}(I/(a_1, \dots, a_q)) + q$ (without assuming that R is Noetherian we have the inequality \geq and the equality follows from (3) and (4)).
- (6) If R is local, then a permutation of a regular sequence is a regular sequence.

Let M be a free module over R of finite rank m , with e_1, \dots, e_m - a free basis. Let $\Lambda^p(M)$ denote the p -th exterior power of M . Any element $\sigma \in M$ defines the Koszul complex

$$0 \longrightarrow R = \Lambda^0 M \xrightarrow{\partial} \Lambda^1 M \xrightarrow{\partial} \dots \xrightarrow{\partial} \Lambda^m M \longrightarrow 0$$

with the differentiation $\partial = \partial_\sigma : \Lambda^p M \rightarrow \Lambda^{p+1} M$ defined by

$$\partial_\sigma^p(\gamma) = \sigma \wedge \gamma, \quad \text{for } \gamma \in \Lambda^p M.$$

The p -th cohomology group is defined as

$$H^p(\sigma) = \ker \partial_\sigma^p / \text{im } \partial_\sigma^{p-1}.$$

The following fact in commutative algebra is well known (usually the first statement is proved in a more restrictive setting, with R Noetherian, but it can be given an elementary proof without this assumption).

THEOREM A1. — *Let $\sigma = a_1 e_1 + \dots + a_m e_m$ be an element of M and let the ideal $I(\sigma) \subset R$ generated by a_1, \dots, a_m be proper. If $q < \text{depth}(I(\sigma))$, then*

$$H^p(\sigma) = 0, \quad \text{for } p = 0, 1, \dots, q.$$

The converse also holds when R is Noetherian. If R is Noetherian and local, then the condition $H^q(\sigma) = 0$ implies that $H^p(\sigma) = 0$, for all $p < q$.

In analytic geometry the property $H^p(\sigma) = 0$ is called p -division property of σ , where M is either the module of (the germs of) differential 1-forms or the module of vector fields, cf. [Mou], [MZh].

Namely, we say that $\sigma \in M$ has p -division property if for any $\gamma \in \Lambda^p M$ such that $\sigma \wedge \gamma = 0$ there exists a $\delta \in \Lambda^{p-1} M$ such that $\gamma = \sigma \wedge \delta$. When $p = 1$ we obtain the following immediate consequence.

COROLLARY 1. — *If $\text{depth}(I(\sigma)) \geq 2$ then σ has 1-division property. If R is Noetherian and local, then the converse holds too.*

Appendix 2.

In this appendix we adopt to our needs a result of Tougeron (Chapter VII in [T]). Namely, we prove that the set of formal maps such that the pullbacks defined by their finite jet extensions decrease the depth of a given ideal, is of infinite codimension, and the same holds in the holomorphic and real analytic category. The theorem is needed in order to show that one of our main conditions, condition (A), is of infinite codimension in the formal, holomorphic, and the real analytic categories (Proposition 3.4).

We denote by $J^r(n, m) = J^r(\mathbb{K}^n, \mathbb{K}^m)$ and $J_0^r(n, m) = J^r(\mathbb{K}^n, \mathbb{K}^m)$ the spaces of r -jets (respectively, of r -jets at $0 \in \mathbb{K}^n$) of maps $\mathbb{K}^n \rightarrow \mathbb{K}^m$ (we take holomorphic maps when $\mathbb{K} = \mathbb{C}$). Let $\mathbb{K}[[x]] = R^f$ be the ring of formal power series of n commuting variables and denote by $\mathcal{F}(n, m) = (R^f[[x]])^m$ the space of formal maps $\mathbb{K}^n \rightarrow \mathbb{K}^m$. For a map germ (or a formal map) $g : (\mathbb{K}^n, 0) \rightarrow \mathbb{K}^m$ we denote by $(j^r g) : (\mathbb{K}^n, 0) \rightarrow J^r(n, m)$ its r -jet extension.

Let $\pi_r : \mathcal{F}(n, m) \rightarrow J^r(n, m)$ and $\pi_{r,r'} : J^{r'}(n, m) \rightarrow J^r(n, m)$ denote the canonical projections. After Tougeron [T] we define an algebraic provariety V in an infinite jets space to be a sequence of algebraic sets V^r in r -jets spaces such that $\pi_{r,r'}(V^{r'}) \subset V^r$, whenever $r < r'$. We define $\text{codim}(V) = \sup_{r \geq 0} \text{codim}(V^r)$.

Let $\xi \in J_0^r(n, m)$ be a fixed r -jet. We denote the space of formal maps with the fixed initial jet ξ by $\mathcal{F}_\xi(n, m) = \{g \in \mathcal{F}(n, m) : \pi_r(g) = \xi\}$.

THEOREM A2. — (a) *Let I_ξ be an ideal in the ring of germs at ξ of holomorphic (respectively, real-analytic) functions on the space $J^r(n, m)$. The set $E_\xi \subset \mathcal{F}_\xi(n, m)$ of formal maps $g : (\mathbb{K}^n, 0) \rightarrow \mathbb{K}^m$ defined by*

$$E_\xi = \{g \in \mathcal{F}_\xi(n, m) : \text{depth}((j^r g)^*(I_\xi) R^f) < \min\{\text{depth}(I_\xi), n\}\},$$

is contained in an algebraic provariety of infinite codimension (here $(j^r g)^$ denotes the pullback homomorphism defined by $j^r g$).*

(b) *Any holomorphic (respectively, real analytic) map germ $g : (\mathbb{K}^n, 0) \rightarrow \mathbb{K}^m$ such that $(j^r g)(0) = \xi$ and $(jg)(0) \notin E_\xi$, with $(jg)(0)$ the infinite jet of g at 0, satisfies the equality*

$$\text{depth}((j^r g)^*(I_\xi) R) = \min\{\text{depth}(I_\xi), n\},$$

where R is the ring of holomorphic (or real analytic) function germs $(\mathbb{K}^n, 0) \rightarrow \mathbb{K}$.

Proof of statement (a). — The proof of this statement is analogous to the proofs of Propositions 5.4 and 6.6 in Chapter VII of [T] and so it is omitted.

Proof of statement (b). — Let g be the germ of an analytic map such that $(jg)(0) \notin E_\xi$. Denote $I = (j^r g)^*(I_\xi) R$ and let $a_1 = (j^r g)^*(P_1), \dots, a_s = (j^r g)^*(P_s)$ be generators of I (where P_1, \dots, P_s generate I_ξ). From the definition of E_ξ it follows that $\text{depth}((j^r g)^*(I_\xi) R^f) = \min\{\text{depth}(I_\xi), n\} = d \leq s$. It follows from Theorem A1 in Appendix 1 that the exterior 1-form $\hat{\alpha} = \hat{a}_1 dx_1 + \dots + \hat{a}_s dx_s$ in $(R^f)^s$ has i -division properties for all $0 \leq i < d$ (here \hat{a}_i denotes the Taylor series of a_i). Thus we have the exact sequence

$$0 \longrightarrow R^f \xrightarrow{\hat{\alpha}^\wedge} \Lambda^1(s, R^f) \xrightarrow{\hat{\alpha}^\wedge} \Lambda^2(s, R^f) \xrightarrow{\hat{\alpha}^\wedge} \dots \xrightarrow{\hat{\alpha}^\wedge} \Lambda^d(s, R^f),$$

where $\Lambda^i(s, R)$ denotes the space of i -th exterior forms on R^s (which can be identified with $\Lambda^i(R^s)$). Since the ring of formal power series R^f is faithfully flat over the ring of converging power series R (cf. e.g. [Mlg], Section 3.4) it follows that the same sequence with R^f replaced by R is exact:

$$0 \longrightarrow R \xrightarrow{\alpha^\wedge} \Lambda^1(s, R) \xrightarrow{\alpha^\wedge} \Lambda^2(s, R) \xrightarrow{\alpha^\wedge} \dots \xrightarrow{\alpha^\wedge} \Lambda^d(s, R).$$

This implies that $\alpha = a_1 dx_1 + \dots + a_s dx_s$ has all i -division properties, $0 \leq i < d$. From Theorem A1 in Appendix 1 and the fact that R is Noetherian and local we conclude that $\text{depth}(I) \geq d$. From the general inequality $\text{height}((j^r g)^*(I_\xi) R) \leq \min\{\text{height}(I_\xi), n\}$ (Proposition 5.4 in Chapter II of [T]) and the equality $\text{depth}(I) = \text{height}(I)$, which holds for any ideal of R , we also get $\text{depth}(I) \leq \min\{\text{depth}(I_\xi), n\} = d$, which completes the proof. Q.E.D.

Example. — For illustration of Theorem A2 let us consider germs of analytic maps $g : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$ and let f_1, f_2, f_3 denote the 2×2 minors defined on the space of 2×3 matrices $\{a_{ij}\}$. It is easy to check that $\text{depth}\{f_1, f_2, f_3\} = 2$. Let J be the ideal in the ring of analytic function germs at $0 \in \mathbb{R}^2$ generated by $f_1 \circ g', f_2 \circ g',$ and $f_3 \circ g'$, where g' is the Jacobi matrix of g . It follows from Theorem A2 that the set of infinite jets of map germs g with a fixed 1-germ $(j^1 g)(0)$, and such that $\text{depth}(J) < 2$, is of infinite codimension.

Proof of Proposition 3.4. — (a) Denote $J = I(H, \eta)$, the set S_1 is the set of zeros of J . In the holomorphic and real-analytic categories we have $\text{depth}(J) = \text{depth}(J/(H)) + 1 = \text{depth}(I_{\text{ch}}) + 1$ by the properties of depth in Noetherian rings, (Appendix 1). Thus, condition (A) is equivalent to the

condition $\text{depth}(J) \geq 3$. The inequality $\text{depth}(J) = \text{depth}(I_{ch}) + 1 \leq 3$, which follows from statement (b), implies that (A) is equivalent to the equality $\text{depth}(J) = 3$. Thus, statement (a) follows from general fact that, in the holomorphic category, the depth of an ideal of function germs is equal to the codimension of its set of zeros.

In order to prove (b) and (c) we will use Proposition 3.3 stating that the algebraic subset Z_1 of 2-jets of 1-forms defined by the equalities $H(0) = 0$ and $\eta(0) = (\omega \wedge (d\omega)^{k-1} \wedge dH)(0) = 0$ is of codimension 3 [Ma1] (cf. also [Zh1] and [P]). Let ξ be an element of Z_1 and \mathcal{J}_ξ be the ideal of germs at ξ of holomorphic (respectively, real-analytic) functions vanishing on Z_1 . Then, in the holomorphic category, we have by Proposition 3.3

$$\text{depth}(\mathcal{J}_\xi) = \text{codim}_\xi(Z_1) = 3,$$

where $\text{codim}_\xi(Z_1)$ is codimension of the germ of Z_1 at ξ . The same equality $\text{depth}(\mathcal{J}_\xi) = 3$ holds in the real-analytic category by complexification argument (complexification does not change the depth, see Propositions 5.4 and 5.5 in Chapter II of [Ru]).

(b) It is enough to prove that $\text{depth}(J) \leq 3$. This inequality follows from $\text{depth}(\mathcal{J}_\xi) = 3$ and $\text{depth}(J) \leq \text{depth}(\mathcal{J}_\xi)$. The latter inequality follows from the fact that J is generated by the homomorphic image of \mathcal{J}_ξ , under the pullback homomorphism $h = (j^2\omega)^*$, from the general inequality $\text{height}(h(I)R) \leq \text{height}(I)$ ([T], Chapter II, Theorem 5.4), and equality of depth and height in our rings.

(c) We identify 1-forms on \mathbb{K}^n with maps $\mathbb{K}^n \rightarrow \mathbb{K}^n$ and we use the notation introduced earlier. Let ξ be the 2-jet at 0 of a 1-form. Let I_ξ^1 denote the ideal of the ring of germs at ξ of analytic functions on $V = J^2(\mathbb{K}^n; \mathbb{K}^n)$ generated by the polynomial \mathcal{H} such that $H = \mathcal{H} \circ j^2\omega$. Set $I_\xi^2 = \mathcal{J}_\xi$. We define the set $E_\xi = E_\xi^1 \cup E_\xi^2$, where

$$E_\xi^i = \{g \in \mathcal{F}_\xi(n, n) : \text{depth}((jg)^*(I_\xi^i)R) < \text{depth}(I_\xi^i)\}$$

(here $(jg)^*$ denotes the pullback homomorphism and R the ring of germs at 0). From Theorem A2 it follows that the sets E_ξ^1 and E_ξ^2 are contained in algebraic provarieties E_ξ^{1f} and E_ξ^{2f} of infinite codimension, and so is E_ξ : $E_\xi \subset E_\xi^f = E_\xi^{1f} \cup E_\xi^{2f}$.

It remains to show that if ω is a 1-form such that $j^2\omega(0) = \xi$ and $j\omega(0) \notin E_\xi^f$ then $P = (\omega)$ has the property stated in Proposition 3.4 (c). Assume that $\xi = j^2\omega(0) \in Z_1$ (otherwise P is local contact structure or $H(0) = 0$ and $\eta(0) \neq 0$, so $\text{depth}(I_{ch}) = \infty$). The condition $(j\omega)(0) \notin E_\xi^1$ implies that $H = \mathcal{H} \circ j^2\omega$ is a nonzerodivisor (since $\text{depth}(I_\xi^1) = 1$) which

means that $P = (\omega)$ is local singular contact structure. The condition $(j\omega)(0) \notin E_\xi^2$ implies that $\text{depth}(J) = \text{depth}(\mathcal{J}_\xi) = 3$, where $J = I(H, \eta)$. Therefore, $\text{depth}(I_{\text{ch}}) = \text{depth}(J) - 1 = 2$. Q.E.D.

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