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DYNAMICS OF WAVE PROPAGATION AND CURVATURE OF DISCRIMINANTS

by Victor P. PALAMODOV

1. Introduction.

The dynamical characteristics of wave propagation like intensity, energy-impulse tensor, density of 4-tensor are special Hermitian forms of solutions of the wave, Maxwell and Dirac equation respectively. For a generic Lagrange distribution solution the value of the Hermitian form is a singular density that diverges fast at the locus (front) of the Lagrange manifold. Meantime for an arbitrary Lagrange distribution of order zero the divergence of the intensity integral is of logarithmic rate. We call *residue* of the intensity integral the coefficient at the logarithmic term (Section 3). The residue is a positive measure supported in the locus. To evaluate this measure we choose an appropriate barrier function that vanishes on the locus (Section 4). We calculate the residue in terms of the symbol of the distribution (Sections 2 and 5). A substantial point of our proof is inspired by an observation due to J.J. Duistermaat that concerns oscillatory integrals [1], Section 1.3.

For any solution of the wave equation the residue of the intensity integral is preserved by the corresponding Hamiltonian flow. This property extends the classical conservation law of geometrical optics to singular solutions and to rays passing through caustics. We state that the singular energy-impulse tensor obeys the similar conservation law (Section 7).

The residue of an arbitrary Lagrange distribution of order zero is equal to the delta-density of the Lagrange locus times a factor which is

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unbounded near any singular point of the Lagrange locus. This factor can be represented in a purely geometrical form that does not depend directly on a singular stratification of the locus (Section 8). It is essentialy equal to the maximal minor of the curvature form of the locus. This gives an approach to a uniform of asymptotics of a Lagrange distribution. We consider an example in Section 9.

2. Lagrange distributions and symbols.

We write a Fourier integral in a smooth manifold X in the form

$$I(\phi, A) = \int_\Theta \exp\big(2\pi\imath\phi(x,\theta)\big)A(x,\theta)\,\mathrm{d}\theta. \tag{2.1}$$

The integral is taken over an open cone $\Theta \subset \mathbb{R}^N \setminus 0$. The space $\mathbb{R}^N$ (*ancillary* space) is endowed with a coordinate system $\theta_1, \ldots, \theta_N$ ($\mathrm{d}\theta$ is the corresponding Euclidean volume form) and with the action of the group $\mathbb{R}_+$ of positive numbers $t : \theta \mapsto t\theta$. Any coordinate system in the ancillary space possessing the last property is called *homogeneous*. The phase ϕ is a real smooth homogeneous function in $X \times \Theta$ of degree 1, *i.e.*, $\phi(x, t\theta) = t\phi(x,\theta)$ for $t > 0$. We assume that the phase is non-degenerate, that is, rank of the Jacobian matrix of the functions $\phi'_{\theta_1}, \ldots, \phi'_{\theta_N}$ is equal N in any point of the critical set $C(\phi) \doteq \{\,\mathrm{d}_\theta\phi = 0\}$. The amplitude A is a smooth complex-valued halfdensity in X depending on the ancillary variables $\theta \in \Theta$. We suppose that the amplitude is supported by the set $X \times \Gamma$ where Γ is a closed cone in Θ and A is asymptotically homogeneous of certain degree m. The last condition means that $A = A_m + R$ where A_m is a smooth halfdensity that is homogeneous in θ of degree m and the remainder satisfies for some positive ε and for any compact set $K \subset X$

$$\int_K |D_x^i D_\theta^j R|^2 \le C_K\big(|\theta| + 1\big)^{2(m-\varepsilon-j)}, \quad i + j \le m + N.$$

These conditions imply that (2.1) converges to a continuous functional on the space of smooth halfdensities χ with compact support:

$$I(\phi, A)(\chi) = \int_X \Big(\int_\Theta \exp\big(2\pi\imath\phi(x,\theta)\big)A(x,\theta)\,\mathrm{d}\theta\Big)\chi.$$

We call $\nu \doteq m + \frac{1}{2}N$ the *order* of the Fourier integral. Consider the mapping

$$\tilde{\phi} : C(\phi) \longrightarrow T_0^*(X), \quad (x,\theta) \longmapsto \big(x, \mathrm{d}_x\phi(x,\theta)\big) \tag{2.2}$$

where $T_0^*(X)$ denotes for the bundle of non-zero cotangent vectors in X.

This mapping is an immersion and commutes with the action of the group $\mathbb{R}_+$ hence its image $\Lambda(\phi)$ is a conic Lagrange variety.

Take another copy Θ^0 of the cone Θ. There are three differentials d_x, d_θ and d_{θ^0} in the de Rham complex $\Omega^*(X \times \Theta \times \Theta^0)$. Set

$$\mathrm{d} = \mathrm{d}_x + \mathrm{d}_\theta \quad \text{and} \quad d^0 = \mathrm{d}_{\theta^0}.$$

Take the function $\theta\theta^0 \doteq \sum \theta_i \theta_i^0$. The form $(\mathrm{dd}^0\theta\theta^0)^{\wedge N} = \bigwedge^N(\mathrm{dd}^0\theta\theta^0)$ will be considered as a density in $\Theta \times \Theta^0$. The product

$$\alpha \doteq (A_m)^2 \wedge (\mathrm{dd}^0\theta\theta^0)^{\wedge N}$$

is a smooth density in $X \times \Theta \times \Theta^0$. Introduce the phase function in this manifold

$$\Phi(x, \theta, \theta^0) = \phi(x, \theta + \theta^0)$$

and consider the submanifolds

$$C^0(\Phi) = \{d^0\Phi = 0\}, \quad Z = \{\theta^0 = 0\}.$$

We have the natural isomorphism $C^0(\Phi) \cap Z \cong C(\phi)$. Choose the coorientation of the manifold $C^0(\Phi) \cap Z$ by means of the frame of 1-forms

$$\text{(2.3)} \qquad \mathrm{d}\Phi_1', \dots, \mathrm{d}\Phi_N',\ \mathrm{d}\theta_1^0, \dots, \mathrm{d}\theta_N^0, \quad \Phi_i' \doteq \partial\Phi/\partial\theta_i^0, \quad i = 1, \dots, N.$$

These forms are independent since the phase ϕ is non-degenerate. Note that this coorientation does not depend on the choice of the coordinates in the ancillary space Θ. Consider the 2-form $\mathrm{dd}^0\Phi$; the quotient

$$\beta = \frac{\alpha}{(-\imath\, \mathrm{dd}^0\Phi)^{\wedge N}}$$

is defined as a form of degree $\dim X$ up to a form β' that satisfies

$$\beta' \wedge (\mathrm{dd}^0\Phi)^{\wedge N} = 0.$$

The term β' belongs to the ideal in $\Omega^*(X \times \Theta \times \Theta')$ generated by the forms (2.3) since these forms are independent in any point of $C(\Phi) \cap Z$. The restriction of β' to the manifold $C^0(\Phi) \cap Z$ vanishes consequently the restriction of β to this manifold is a well-defined smooth density. Define the halfdensity

$$\sigma\big(I(\phi, A)\big) \doteq \sqrt{\beta} \mid C(\Phi) \cap Z.$$

This is a homogeneous halfdensity in $C(\phi)$ of degree $\nu = m + \frac{1}{2}N$. Choosing a smooth nonvanishing density $\mathrm{d}V$ in X we can write $A_m = a_m\sqrt{\mathrm{d}V}$ where a_m is a homogeneoues amplitude function. This implies the equivalent formula

$$\sigma\big(I(\phi,A)\big) = a_m\sqrt{\frac{\mathrm{d}V\wedge \mathrm{dd}^0(\theta\theta^0)^{\wedge N}}{(-\imath\,\mathrm{dd}^0\Phi)^{\wedge N}}}\;\Big|\; C(\phi). \tag{2.4}$$

Suppose that the mapping $\tilde{\phi} : C(\phi) \to \Lambda(\phi)$ is injective and hence is a diffeomorphism. The direct image $\tilde{\phi}_*(\sigma(I))$ is called the *symbol* of the Fourier integral $I = I(\phi,A)$. There is an ambiguity in the choice of the root in (2.4). Take a local coordinate system $\lambda_0,\dots,\lambda_n$ in Λ and consider the density $|\,\mathrm{d}\lambda_0\wedge\dots\wedge\mathrm{d}\lambda_n|$ in $\Lambda(\phi)$. Compare two densities:

$$\frac{\mathrm{d}V\wedge \mathrm{dd}^0(\theta\theta^0)^{\wedge N}}{(-\imath\,\mathrm{dd}^0\Phi)^{\wedge N}} = s(\phi)\cdot|\,\mathrm{d}\lambda_0\wedge\dots\wedge\mathrm{d}\lambda_n|.$$

Here $s(\phi)\neq 0$ is a function with real or imaginary values. The square root of the left-hand side is equal to

$$\Sigma(\phi) \doteq \exp\Big(\frac{1}{2}\arg s(\phi)\imath\Big)\cdot\big|s(\phi)\,\mathrm{d}\lambda_0\wedge\dots\wedge\mathrm{d}\lambda_n\big|^{1/2} \tag{2.5}$$

where the argument $\frac{1}{2}\arg s(\phi)$ is multiple of $\frac{1}{4}\pi$ and is well-defined up to multiple of π. Therefore the halfdensity $\Sigma(\phi)$ is defined up to the factor $\pm$ and we have $\sigma(I) = a_m\Sigma(\phi)$ by (2.4). The symbol is transformed as a homogeneous density in $\Lambda(\phi)$ if we execute an arbitrary coordinate change $(x,\theta,\theta^0)\mapsto(x,\zeta,\zeta^0)$ by means of smooth homogeneous functions $\zeta_j = \zeta_j(x,\theta)$, $j = 1,\dots,N$ in $X\times\Theta$ of degree 1 and set $\zeta_j^0 = \zeta_j(x,\theta^0)$.

Dimension descent. — Suppose that the phase function can be written in the form $\phi(x,\theta) = |\eta|^{-1}q(\xi) + \psi(x,\eta)$ in a neighbourhood of a point (x_0,θ_0) where $\xi = (\xi_1,\dots,\xi_k)$, $\eta = (\eta_1,\dots,\eta_{N-k})$ are new homogeneous coordinates in the ancillary space such that ξ vanishes in (x_0,θ_0) and q is a non-singular quadratic form of the variables ξ. The function ψ is again a non-degenerate phase and by the stationary phase method we have $I(\phi,A) = I(\psi,B)\pmod{C^\infty}$ where B is an asymptotically homogeneous amplitude of order $m' = m + \frac{1}{2}k$ such that

$$B_{m'} = \exp\Big(\operatorname{sgn}(q)\frac{\pi\imath}{4}\Big)\frac{A_m|\theta|^{k/2}}{\sqrt{|\det q|}}$$

where $\operatorname{sgn}(q)$ is the signature of the quadratic form.

Lagrange distributions. — Let Λ be a closed conic Lagrange manifold in $T_0^*(X)$. We call any generalized halfdensity in X of the form

$$U = \sum I(\phi_i, A_{(i)}) \pmod{C^\infty} \tag{2.6}$$

Λ-distribution (or Lagrange-distribution) if the sum the Fourier integrals is locally finite in X, $\Lambda(\phi_i) \subset \Lambda$ for each i and the amplitudes $A_{(i)}$ satisfy the above conditions. This is in fact a special case of the general definition due to Hörmander [6].

We say that U is of order $\leq \nu$ for a real number ν if U can be written in the form (2.6) where each term is of order $\leq \nu$.

Denote by $\mathcal{D}'^0(\Lambda)$ the space of Λ-distributions in X of zero order.

Symbol. — Let $\pi : \Lambda \to X$ be the natural projection. Denote by Λ^k the open subset of Λ where the dimension of the kernel of the tangent mapping $\mathrm{d}\pi : T(\Lambda) \to T(X)$ is less or equal to k, $k = 1, \ldots, \dim X$. For any point $\lambda \in \Lambda^k$ there exists a generating phase function ψ that depends on k ancillary variables. Fix a family of non-degenerate phase functions ψ_α, $\alpha \in A$ such that for any $k = 1, \ldots, \dim X$

$$\Lambda^k = \bigcup \{\Lambda(\psi_\alpha)\,;\; N(\psi_\alpha) \leq k\}.$$

For each α we fix a choice of $\frac{1}{2} \arg s(\phi_\alpha)$ in (2.5). Take an arbitrary non-degenerate phase function ϕ such that $\Lambda(\phi) \subset \Lambda$. We can choose a locally finite covering $\Theta = \bigcup \Theta_j$ by some open cones Θ_j such that for each j the equation

$$\phi(x, \theta) = \left|\theta_{(\alpha)}\right|^{-1} q_j(\xi) + \psi_\alpha\big(x, \theta_{(\alpha)}\big) \tag{2.7}$$

holds in Θ_j for a non-singular quadratic form q_j and a homogeneous coordinate system $\xi, \theta_{(\alpha)}$, $\alpha = \alpha(j)$ in the ancillary space. This equation implies that the image of U_j under the mapping $\tilde{\phi}$ is contained in $\Lambda(\psi_\alpha)$. Choose a smooth partition of unity $\{h_j\}$ in Θ such that $\operatorname{supp} h_j \subset \Theta_j$ for each j. Take an amplitude A as above, write $I(\phi, A) = \sum I(\phi, h_j A)$ and apply the dimension descent to each term:

$$I(\phi, h_j A) = \exp\Big(\operatorname{sgn}(q_j)\frac{\pi \imath}{4}\Big) I(\psi_\alpha, B_j), \quad \pmod{C^\infty}$$

where B_j is a new asymptotically homogeneous amplitude which satisfies the above conditions. Set

$$\sigma\big(I(\phi, h_j A)\big) \doteq \exp\Big(\operatorname{sgn}(q)\frac{\pi \imath}{4}\Big)\sigma\big(I(\psi_\alpha, B_j)\big). \tag{2.8}$$

The right-hand side is defined as a local section $\Upsilon(\nu)$ of homogeneous halfdensities in Λ of degree ν.

To globalize this construction we consider the Keller-Maslov line bundle $\mathcal{L}$ in Λ. In the atlas $\{\Lambda(\phi_\alpha)\}$ this bundle is defined by the transition mappings $\exp(\operatorname{sgn}(q_{\alpha\beta})\frac{1}{4}\pi\imath)$ where $q_{\alpha\beta}$ is the quadratic form that joins the phase functions ϕ_α and ϕ_β as in (2.7). The group of this bundle is reduced to $\mathbb{Z}_4 \subset \mathrm{U}(1)$.

The halfdensities $e_\alpha \doteq \Sigma(\phi_\alpha)$ are local sections of the bundle $\mathcal{L}$. Another choice of $\frac{1}{2}\arg(\psi_\alpha)$ corresponds to the generator $-e_\alpha$. Therefore the right-hand side of (2.8) is well-defined as a section of the bundle $\Upsilon(\nu)\otimes\mathcal{L}$. We set

$$\sigma\big(I(\phi, A)\big) = \sum_j \sigma\big(I(\phi, h_j A)\big).$$

We define symbol of an arbitrary Lagrange distribution U of order ν by the equation

$$\sigma(U) = \sum \sigma\big(I(\phi_i, A_i)\big)$$

where the sum ranges over the terms $I(\phi_i, A_i)$ of order ν. We have proved in this way

PROPOSITION 2.1. — *The symbol $\sigma(U)$ of an arbitrary Λ-distribution U of order $\leq \nu$ is well-defined as a section of the bundle $\Upsilon(\nu)\otimes\mathcal{L}$.*

Remark. — The absolute value of the halfdensity (2.4) coincides with the principal symbol in the sense of [5], [2], up to the factor $(2\pi)^{N/2}$.

Contact bundle. — The manifold $C^*(X) \doteq T_0^*(X)/\mathbb{R}_+$ is the variety of all cooriented contact elements in X. This manifold has the canonical contact structure. For a conic Lagrange manifold Λ in $T_0^*(X)$ we set $\Lambda_c \doteq \Lambda/\mathbb{R}_+$. This is an integral manifold of dimension $\dim \Lambda_c = \dim X - 1$. We call it *contact* Lagrange manifold. For a Fourier integral $I = I(\phi, A)$ of order 0 we define formally the halfdensity

$$\sigma_c(I) \doteq \sqrt{\frac{|\theta|}{\mathrm{d}|\theta|}}\,\sigma(I) \tag{2.9}$$

in $X \times \Theta/\mathbb{R}_+$. The right-hand side is homogeneous of order zero hence this halfdensity is well-defined. Take the direct image of this form by the mapping $\tilde{\phi}$. The image is a halfdensity in $\Lambda_c(\phi)$; it is equal to the symbol $\sigma(U)$ considered as a section of the bundle $\Upsilon(0)\otimes\mathcal{L}$ if we push forward both bundles to the manifold Λ_c. We call this image *contact* symbol of U.

Remark. — For arbitrary zero order Λ-distributions U, V the product $\sigma(U)\sigma(\overline{V}) = \sigma(U)\bar{\sigma}(V)$ is a density in Λ_c. It does not depend on the choice of $\frac{1}{2}\arg s$ in (2.5).

3. Residue in regular points.

DEFINITIONS. — Let again X be a smooth manifold and Λ be a closed conic Lagrange manifold in $T_0^*(X)$. Consider the corresponding contact Lagrange manifold Λ_c and denote by $p: \Lambda_c \to X$ the natural projection.

We call the image $L \doteq p(\Lambda_c)$ *locus* of the Lagrange manifold. The locus is a closed set since the mapping p is proper.

We shall say that Λ_c is *univalent* at a point λ and over the point $x = p(\lambda) \in X$ if λ is the only point of the set $p^{-1}(x)$.

We call a point $x \in L$ *regular* if Λ_c is univalent over x and the tangent mapping $dp: T_\lambda(\Lambda_c) \to T_x(X)$ is injective. Let L_r denote the set of regular points; it is a smooth open manifold of dimension $n \doteq \dim X - 1$. The complement $L_s \doteq L \setminus L_r$ is a closed subset of X.

Take a regular point $y \in L$ and a smooth function f defined in a neighbourhood Y of y that has no critical points and vanishes in $L \cap Y$. We call it *regular barrier* for L.

For an arbitrary Λ-distribution U the square $|U|^2$ is a distribution in X which is smooth in the compliment to the locus of Λ.

Denote by $\mathcal{D}^0(X)$ the space of continuous functions in X with compact support.

PROPOSITION 3.1. — *Let $y \in L_r$ and f be a regular barrier function in a neighbourhood Y of y. For any $U \in \mathcal{D}'^0(\Lambda)$ we have*

$$\int_{f^2 \geq \varepsilon} \rho |U|^2 = \Sigma_Y(\rho) \cdot \log \frac{1}{\varepsilon} + O(1), \quad \rho \in \mathcal{D}^0(Y) \tag{3.1}$$

as $\varepsilon \to 0$ for a distribution $\Sigma_Y \in \mathcal{D}'^0(Y)$. This distribution is positive and supported by $Y \cap L$. It does not depend on the choice of the regular barrier function.

We call the left-hand side the *peripheral* integral.

Proof. — The manifold Λ_c coincides with a connected component of the conormal bundle $N^*(L)$ over a neighbourhood of the point y. Therefore

$\phi(x,\theta) = \theta f(x)$, $\theta \in \Theta$ is a generating function for Λ_c where Θ is the positive or negative ray in $\mathbb{R} \setminus \{0\}$. Suppose that $\Theta = \mathbb{R}_+$. We can write

$$U(x) = \int_0^\infty \exp\big(2\pi\imath\theta f(x)\big) A(x,\theta)\,\mathrm{d}\theta \quad (\operatorname{mod} C^\infty) \tag{3.2}$$

and the amplitude $A(x,\theta) = a(x)\theta^{-1/2} + O(\theta^{-1/2-\varepsilon})$ where $\varepsilon > 0$ and a is a smooth halfdensity in Y. Calculating this integral we get the explicit formula

$$U = (f + \imath 0)^{-1/2} b(x) + O\big(|f|^{-1/2+\varepsilon}\big) \tag{3.3}$$

where $b = \sqrt{\frac{1}{2}\imath}\, a$. Hence

$$\int_{f^2 \geq \varepsilon} \rho |U|^2 = \int_{f^2 \geq \varepsilon} \rho \frac{|b|^2}{|f|} + O(1) = 2 \int_{L_r} \rho \frac{|b|^2}{\mathrm{d}f} \log \frac{1}{\varepsilon} + O(1).$$

In the last term we use the coorientation of the manifold L_r by the form $\mathrm{d}f$. This proves (3.1) with the coefficient

$$\Sigma_Y(\rho) = 2 \int_L \rho \frac{|b|^2}{\mathrm{d}f} = \int_L \rho \frac{|a|^2}{\mathrm{d}f}. \tag{3.4}$$

We have $C(\phi) = \{f(x) = 0, \theta > 0\}$ and the manifold Λ is given by the equation $\xi = \theta\, \mathrm{d}f$. Calculate the symbols

$$\sigma(U) = \sqrt{\frac{A^2 \wedge \mathrm{d}\theta \wedge \mathrm{d}\theta^0}{-\imath \mathrm{d}f \wedge \mathrm{d}\theta^0}} = \frac{a}{\sqrt{-\imath\, \mathrm{d}f}} \sqrt{\frac{\mathrm{d}\theta}{\theta}}, \quad \sigma_c(U) = \frac{a}{\sqrt{-\imath\, \mathrm{d}f}}. \tag{3.5}$$

Consequently the right-hand side of (3.4) can be written in the form

$$\Sigma_Y(\rho) = \int_{\Lambda_c} \big|\sigma_c(U)\big|^2 p^*(\rho) \tag{3.6}$$

where p^* means the pullback operation. Obviously this form does not depend on the barrier function. The local distributions Σ_Y glue together to a unique distribution defined in an arbitrary open Y such that the intersection $Y \cap L$ contains only regular points. This follows from (3.6).

4. Locus and barriers.

DEFINITION. — For a contact Lagrange manifold Λ_c and a point $\lambda \in \Lambda_c$ we denote by $\tau(\lambda)$ the multiplicity of the natural projection $p: \Lambda_c \to X$ in λ (*cf.* (10.2)). Take an arbitrary point λ where Λ_c has finite multiplicity. This point is isolated in the fibre $p^{-1}(y)$ hence there exists a neighbourhood Λ' of λ in Λ_c which is univalent at this point. We call the set $L' \doteq p(\Lambda')$ *fold* of the Lagrange locus at y.

For an arbitrary point y in the locus $L = p(\Lambda_c)$ the number

$$\tau(y) = \sum\{\tau(\lambda); p(\lambda) = y\}$$

is called the *multiplicity of* Λ_c *over* y. If Λ_c is closed and has finite multiplicity over y, the locus is equal to a finite union of folds in a neighbourhood of y.

Take an arbitrary point $\lambda = (y,\xi) \in \Lambda_c$ of finite multiplicity $\tau(\lambda)$, choose a neighbourhood Λ' of λ and a submanifold $H \subset X$ such that $T_y(H) = \operatorname{Ker}\xi$. Choose a smooth retraction $q: Y \to H$ where Y is a small neighbourhood of y. Denote by $\mu(\lambda)$ the local multiplicity of the composition $qp: \Lambda_c \to H$ in the point λ. For an arbitrary point $y \in X$ such that Λ_c has finite multiplicity over y we set

$$\mu(y) \doteq \sum\{\mu(\lambda); p(\lambda) = y\}.$$

We call $\mu(y)$ the *multiplicity of the locus* $L = p(\Lambda_c)$ at y.

PROPOSITION 4.1. — *If the multiplicity $\tau(\lambda)$ is finite, the number $\mu(\lambda)$ does not depend on the choice of H and q. We have $\tau(\lambda) \leq \mu(\lambda) < \infty$.*

THEOREM 4.2. — *Let Λ_c be a closed contact Lagrange manifold over a manifold X that has finite multiplicity and is univalent over a point y.*

(i) *There exists a real smooth function f defined in a neighbourhood Y of y that vanishes in L and satisfies the conditions*

$$\mathrm{d}^i f(y) = 0, \quad i = 0, \dots, \mu - 1, \quad \mathrm{d}^\mu f(y) \neq 0, \quad \mu = \mu(\lambda) \tag{4.1}$$

where d^i means the i-th total differential of a function.

(ii) *There exists a set $G \subset X$ of n-dimensional measure zero where $n = \dim L = \dim X - 1$ such that f does not vanish in $X \setminus (L \cup G)$ and is a regular barrier at any point of $L \setminus G$.*

Proof of Proposition 4.1. — Choose a non-degenerate phase function $\varphi : Y \times \Omega \to \mathbb{R}$ for the germ of the contact Lagrange manifold Λ_c at λ by means of Proposition 10.2. We can assume that $\Omega \subset \mathbb{R}^k$ is a neighbourhood of the origin. We have $\widetilde{\varphi} : C(\varphi) \cong \Lambda_c$, $(y, 0) \mapsto \lambda$. Therefore

$$(4.2)\qquad \begin{aligned} \mathcal{O}_\lambda(\Lambda_c)/(p^*(\mathfrak{m}_y)) &\cong \mathcal{O}_{y,0}(C(\varphi))/(\mathfrak{m}_y) \\ &\cong \mathcal{O}_{y,0}(X \times \Theta)/((\varphi, \varphi'_\omega) + (\mathfrak{m}_y)) \\ &\cong \mathcal{A}/(\psi, \psi'_\omega) \end{aligned}$$

where $\mathfrak{m}_y$ stands for the maximal ideal of the point y in $\mathcal{O}(X)$, $\mathcal{A}$ denotes the algebra of germs in the point $\omega = 0$ of real smooth functions in Ω and $\psi(\cdot) \doteq \varphi(y, \cdot)$. If B is an algebra and G is a subset of B, we denote by (G) the ideal generated in B by this subset. The quotient (4.2) is of finite dimension $\tau = \tau(\lambda)$ by the assumption. By Tougeron's Theorem [11] there exists a local coordinate system ω near the origin such that ψ is a polynomial in ω. Choose coordinates x_1, x' in a neighbourhood of y such that $(y, \mathrm{d}x_1) = \lambda$. The retraction q is given by $q(x_1, x') = x' + q_2$ where $q_2 \in \mathfrak{m}_y^2$. We have

$$\mu(\lambda) = \dim \mathcal{O}_{y,0}(X \times \Theta)/((\varphi, \varphi'_\omega) + r^*(\mathfrak{m}_h)),$$

where $\mathfrak{m}_h$ denotes the maximal ideal of the point $h \doteq q(y) \in H$ and $r \doteq pq$. At the other hand $\mathrm{d}x_1 = \mathrm{d}_x\varphi(y, 0)$ and the point $(y, 0)$ belongs to $C(\varphi)$. Therefore $\varphi(x, \omega) = x_1 \pmod{\mathfrak{m}_y^2}$ consequently we have $(\varphi) + (r^*(\mathfrak{m}_h)) = r^*(\mathfrak{m}_y))$ hence

$$(4.3)\qquad \mu(\lambda) = \dim \mathcal{A}/(\psi'_\omega).$$

This proves the inequality $\tau(\lambda) \le \mu(\lambda)$. At the other hand the inequality $\tau(\lambda) < \infty$ implies that $\omega = 0$ is an isolated zero of the system $\psi'_\omega = 0$ in $\Omega_{\mathbb{C}}$. It follows that $\mu < \infty$. □

Proof of Theorem 4.2. — Take the generating function ψ as in the previous proof. Denote by S the right-hand side of (4.2) and by $S_{\mathbb{C}}$ its complexification. Consider the mapping $\pi : Z \to S_{\mathbb{C}}$ where $Z \subset S_{\mathbb{C}} \times \Omega_{\mathbb{C}}$ is the hypersurface given by a polynomial equation $\Psi(s, \omega) = 0$ and $\Omega_{\mathbb{C}}$ is a complexification of Ω. This is a minimal versal deformation of the germ $(Z_0, 0)$ if the polynomial is taken in the form (*cf.* for ex. [9], Chap. 5):

$$(4.4)\qquad \Psi(s, \omega) \doteq \psi(\omega) + s_1 + \sum_2^\tau s_i e_i(\omega)$$

where $e_1 = 1, e_2, , \dots, e_\tau$ are arbitrary real monomials whose images in (4.2) form a linear basis. The discriminant set Δ of the deformation π is the projection to $S_{\mathbb{C}}$ of the set

$$C(\Psi) = \{\Psi = 0,\ \Psi'_\omega = 0\}$$

in $S_{\mathbb{C}} \times \Omega_{\mathbb{C}}$. According to [7] there exists a holomorphic pseudo-polynomial

$$\delta = s_1^m + \delta_1(s')s_1^{m-1} + \cdots + \delta_m(s')$$

in $S_{\mathbb{C}}$ whose zero set coincides with Δ; here we denote $s' = (s_2, \dots, s_\tau)$. This pseudo-polynomial satisfies (4.1). By the Malgrange's Preparation Theorem [8] there exists a smooth mapping of real germs $\zeta : (X, y) \to (S, 0)$ such that $\zeta(y) = 0$ and a smooth positive function $\chi(x, \omega)$ such that

$$\varphi(x, \omega) = \chi(x, \omega)\Psi\bigl(\zeta(x), \omega\bigr). \tag{4.5}$$

This means that the deformation defined by the function φ is induced from π. It follows that the function

$$\psi(x, \omega) \doteq \Psi\bigl(\zeta(x), \omega\bigr)$$

generates a neighbourhood of the point λ in the contact Lagrange variety Λ_c.

Set $f(x) = \delta(\zeta(x))$. Take a point $x \in L$ and show that f vanishes in x. The point (x, ω) belongs to $C(\varphi)$ for some $\omega \in \Omega$ hence $(\zeta(x), \omega) \in C(\Psi)$ in virtue of (4.5). Therefore $f(x) = \delta(\zeta(x)) = 0$. From (4.5) and the condition $\mathrm{d}\varphi_{|C(\varphi)} \neq 0$ we conclude that $\mathrm{d}_x\psi \neq 0$ in $C(\psi)$. At the other hand by (4.4)

$$\mathrm{d}_x s_1(\zeta(x)) = \mathrm{d}_x\Psi(s, \omega) + O(\omega) \neq 0$$

hence $\mathrm{d}_x s_1(y) \neq 0$. This implies (4.1) for the function f.

Now we check the statement (ii). The function $\psi(x, \omega)$ can be continued at $Y \times \Omega_{\mathbb{C}}$ as a polynomial in ω. The set of real critical points of this function with zero critical values coincides with $C(\varphi)$. The latter is the real part of the critical set $C(\psi)$ in $Y \times \Omega_{\mathbb{C}}$.

LEMMA 4.3. — *The set $C(\psi)$ is a smooth real manifold of dimension n if the $\Omega_{\mathbb{C}}$ is sufficiently small neighbourhood of $\Omega_{\mathbb{C}}$.*

Proof of lemma. — The differentials of the functions $\psi, \psi'_{\omega_1}, \ldots, \psi'_{\omega_k}$ are independent in $C(\phi)$ since the phase φ is non-degenerate. Therefore these differentials are $\mathbb{C}$-independent in $C(\psi)$ if the neighbourhood $\Omega_{\mathbb{C}}$ is sufficiently small. □

Denote by J the set of critical values of the mapping $P : C(\psi) \to X$ and by $D \subset Y$ the set of points x such that $P^{-1}(x)$ contains more than one point. The set J has n-dimensional measure zero by the Sard's Theorem. Show that this is true for the set D too. It is sufficient to prove this statement for the set $D \setminus J$. Take a point x in this set and two points $(x, \omega_\pm) \in C(\psi)$. We have $\det \psi''_{\omega\omega}(x, \omega_\pm) \neq 0$ hence there are smooth $\mathbb{C}$-valued local solutions $\omega = \omega_\pm(z)$ of the system $\psi'(z, \omega) = 0$ such that $\omega_\pm(x) = \omega_\pm$. Take the functions $g_\pm(z) = \psi(z, \omega_\pm(z))$. The equations $g_+(z) = g_-(z) = 0$ defines a manifold $D \subset Y$ of dimension $< n$ in a neighbourhood of y since the differentials $\mathrm{d}g_+$ and $\mathrm{d}g_-$ are independent. To check the last fact we note that the differential of the mapping (2.2) is injective for the phase Ψ. The mapping $\mathrm{d}_x\psi : C(\psi) \cap Y \times \Omega \to C^*(X)$ is an immersion too because of (4.5) provided the neighbourhoods Y and Ω are sufficiently small. This property holds also in the domain $Y \times \Omega_{\mathbb{C}}$ for sufficiently small complex neighbourhood $\Omega_{\mathbb{C}}$ of Ω.

Let I be the set of real points y such that there exists a point $(y, \omega) \in C(\psi)$ with $\operatorname{Im} \omega \neq 0$. It is of n-measure zero too. Really we have $(y, \bar{\omega}) \in C(\psi)$ hence $y \in D$ and consequently $I \subset D$.

Set $G \doteq J \cup D$ and take an arbitrary point $x \in L \setminus G$. According to (4.5) the function $\psi(x, \cdot)$ has only one critical point $(x, \omega) \in C(\psi)$, the point ω is real and the form $\psi''_{\omega\omega}(x, \omega)$ is non-singular. There is a local smooth solution $\omega = \omega(x)$ of the system $\psi'_\omega(x, \omega) = 0$. The similar statements are true for the function Ψ in the point (s, ω) where $s = \zeta(x)$. According to [7] we have

$$\delta(s_1, s') = \prod_{j=1}^{m} \Psi\big(s, \omega_j(s')\big) \tag{4.6}$$

where $\omega_1(s'), \ldots, \omega_m(s')$ are critical points of the function $\Psi(s, \cdot)$. All the factors are linear with respect to s_1 and one of them vanishes at the point $s = \zeta(x)$. Let $\Psi(s, \omega_1(s'))$ be a vanishing factor; all other factors do not vanish since $x \notin D$.

At the other hand we conclude from (4.5) that

$$\mathrm{d}_x\varphi\big(x, \omega(x)\big) = \chi(x)\, \mathrm{d}_x\psi\big(x, \omega(x)\big) = \chi(x)\, \mathrm{d}_x\Psi\big(s, \omega_1(s')\big)$$

where $s = \zeta(x)$. The left-hand side does not vanish since φ is non-degenerate. It follows $d_x\Psi(s,\omega_1(s')) \neq 0$. We conclude from (4.6) that

$$\mathrm{d}_x f(x) = \mathrm{d}_x\delta(s) = \prod_2^m \Psi\big(s,\omega_j(s')\big)\,\mathrm{d}_x\Psi\big(s,\omega_1(s')\big) \neq 0.$$

This implies that f is a regular barrier at x. □

COROLLARY 4.4. — *The barrier has the following representation:*

$$f(x) = \kappa(x,\omega)\varphi(x,\omega) + \sum \kappa_i(x,\omega)\varphi'_{\omega_i}(x,\omega) \tag{4.7}$$

where φ is an arbitrary generating function of the germ Λ_c and κ, κ_i are some smooth functions.

Really, the barrier vanishes in the set $C(\psi)$. At the other hand by (4.5) the functions φ, φ'_ω are generators of the ideal of this set.

Remark. — Note that the barrier f is not uniquely defined because of the mapping ζ is not unique. According to Theorem 4.7 the zero set of f coincides with the locus L up to the subset $I \subset G$ of measure zero. The set I is empty if $\tau(x) \leq 2$ and, moreover, any two local barriers $f, \tilde{f}$ are equivalent: $\tilde{f} = hf$ where $h \neq 0$ is a smooth function in a neighbourhood of x. This is not the case even for $\tau = 3$, since the real "swallow tail" (see Fig. 1) $\Delta \cap S$ does have 1-dimensional piece I (half of a parabola). This piece is not covered by real points of $C(\psi)$ hence another barrier function need not to vanish in I.

Suppose that a closed contact manifold $p : \Lambda_c \to X$ has finite multiplicity over a point $y \in X$. The fibre $p^{-1}(y)$ is a finite set of points $\lambda_1, \dots, \lambda_q$ each of which has finite multiplicity. Take a neighbourhood Y of y such that the manifold $\Lambda_c \cap p^{-1}(Y)$ is the union of disjoint pieces Λ_j such that $\lambda_j \in \Lambda_j$, $j = 1, \dots, q$. Each piece is univalent over y and we can construct a function f_j which vanishes in the fold $L_j = p(\Lambda_j)$ by means of Theorem 4.2 (i). The function $f \doteq f_1 \cdots f_q$ vanishes in the locus $L = \bigcup L_j$ and satisfies 4.1. We call it *barrier* for the locus L in the point y.

THEOREM 4.5. — *Let Λ_c be a closed contact Lagrange manifold of finite multiplicity τ over a point $y \in X$. There exists a neighbourhood Y of y such that Λ_c has finite multiplicity $\tau(x) \leq \tau$ over any point $x \in Y$.*

Proof. — The multiplicity is additive with respect to the fibre hence we may suppose that the fibre $p^{-1}(y)$ contains only one point λ. Take a generating function $\psi(x,\omega)$ of the germ Λ, λ as in the proof of Theorem 4.2. Set $\phi(\omega) \doteq \psi(y,\omega)$ and choose a coordinate system ω such that ϕ is a polynomial. Take a closed ball B in $\mathbb{C}^k$ centered in the origin that does not contain any common zero of the functions ϕ, ϕ'_ω except for the origin. Let A be the Banach space of bounded functions in B that are holomorphic inside the ball. Consider the sequence of continuous linear mappings

$$A^{k+1} \xrightarrow{P} A \xrightarrow{\alpha} \mathcal{F}/(\phi,\phi')\mathcal{F} \to 0 \tag{4.8}$$

where $\mathcal{F}$ denotes the algebra of formal power series in ω, the mapping α is generated by the mapping $A \to \mathcal{F}$ that transforms a function to its Taylor series at the origin and $P(a_0,\ldots,a_k) = a_0\phi + \sum a_i\phi'_i$. The composition of these mappings vanishes.

LEMMA 4.6. — *The sequence* (4.8) *is exact.*

Proof of lemma. — Consider the algebra Q of rational functions in $\mathbb{C}^k$ that are holomorphic in B. Let $\mathfrak{m}$ the maximal ideal of the origin and I be the ideal generated by the polynomials ϕ, ϕ' in Q. We have $\mathfrak{m}^h \subset I$ for some natural h by the Hilbert's theorem. Therefore $\mathfrak{m}^h\mathcal{F} \subset I\mathcal{F}$. Any series $f \in F$ can written in the form $f = a + g$ where $a \in A$, $g \in \mathfrak{m}^h\mathcal{F}$. This implies that α is surjective.

Take an arbitrary element $a \in \operatorname{Ker}\alpha$. We have again $a = a_0 + g$ where a_0 is a polynomial and $g \in \mathfrak{m}^h A$. We have $m^h A \subset IA = P(A^{k+1})$ hence g belongs to the image of the first mapping in (4.8). This implies the equation $\alpha a_0 = 0$ consequently $a_0 = b_0\phi + \sum b_i\phi'_i$ for some formal power series $b_0,\ldots,b_k$. Cutting out the terms of degree $\geq h$ in these series we get the representation $a_0 = b + g'$ where $b \in I$ and $g' \in \mathfrak{m}^h$. We have again $g' \in I$ hence $a_0 \in I$. This implies the inclusion $\operatorname{Ker}\alpha \subset P(A^{k+1})$. □

It follows that the cokernel of the mapping P is of finite dimension τ. Choose a point $s \in S$ and consider the mapping P_x similar to P constructed by means of the function $\psi(x,\cdot)$ instead of ϕ. We have the continuous family of bounded operators $P_x : A^{k+1} \to A$ such that $P_y = P$. According to Lemma 4.6 the space $\operatorname{Cok} P_y$ is of dimension τ. It follows that $\dim \operatorname{Cok} P_x \leq \tau$ for any point x close to y. At the other hand we have the equation

$$\dim \operatorname{Cok} P_x = \sum \dim \mathcal{F}_\omega/(\psi(x,\cdot), \psi'_\omega(x,\cdot))$$

where $\mathcal{F}_\omega$ denotes the algebra of formal power series at the point ω and the sum is taken over all common zeros ω of the functions $\psi(x,\cdot), \psi'_\omega(x,\cdot)$. It can be checked by the the above arguments. For any real common zero ω of the above functions we have $\dim \mathcal{F}_\omega/(\psi, \psi'_\omega) = \tau(\lambda')$ where $\lambda' = \widetilde{\psi}(x,\omega) \in \Lambda$. Therefore $\tau(x) \leq \dim \operatorname{Cok} P_x$ and the theorem follows. □

Remark. — It is easy to show that the number of folds in a point x does not in fact exceed $\tau - 1$ for any point x close to y.

Remark. — Suppose that the manifold Λ_c is univalent over a point y and $(y,\xi) \in \Lambda_c$. The hyperplane $M_y = \{\xi(t) = 0, t \in T_y(X)\}$ is a metric tangent to the locus $L = p(\Lambda_c)$ in y in the sense that $\operatorname{dist}(x, H) = o(\operatorname{dist}(x,y))$ as $x \in L$, $x \to y$ for some (and hence for any) submanifold H through y whose tangent space in y coincides with M_y.

DEFINITION. — Two points $\lambda_\pm \in C^*(X)$ will be termed *opposite* if $\lambda_\pm = (x, \pm\,\xi)$. We say that a subset S in $C^*(X)$ is *symmetric* if for arbitrary point $s \in S$ the opposite point is also contained in S. Consider the projection $C^*(X) \to C^*(Z)/\mathbb{Z}_2$ which identifies opposite points. The image is the manifold of non-cooriented contact elements. Let Λ_c be a closed symmetric contact Lagrange manifold; its image Λ_s in $C^*(X)/\mathbb{Z}_2$ is a manifold too. We define the *symmetric multiplicity* $\tau_s(y)$ of Λ_s and the symmetric multiplicity $\mu_s(y)$ of the locus as above taking the sums for all points of Λ_s over y. This gives the numbers $\tau_s(y) = \frac{1}{2}\tau(y)$ and $\mu_s(y) = \frac{1}{2}\mu(y)$. Note that above results hold for any symmetric manifold Λ_c and with the multiplicities τ_s and μ_s instead of τ and μ.

5. Main result.

THEOREM 5.1. — *Let Λ_c be a closed contact Lagrange manifold of finite multiplicity over each point of X. There exists a positive Hermitian form $\Sigma(\cdot,\cdot)$ defined in $\mathcal{D}'^0(\Lambda)$ with values in $(\mathcal{D}^0(X))'$ such that*

$$\int_{f^2 \geq \varepsilon} \rho U \overline{V} = \left[\Sigma_Y(U,\overline{V})(\rho) + o(1)\right] \log \frac{1}{\varepsilon}, \quad \rho \in \mathcal{D}^0(X) \tag{5.1}$$

where

$$\Sigma(U,V)(\rho) = \int_{\Lambda_c} \sigma_c(U)\bar{\sigma}_c(V) p^*(\rho). \tag{5.2}$$

DEFINITION. — We call $\Sigma(\cdot,\cdot)$ the *residue* form.

COROLLARY 5.2. — *For any $U,V \in \mathcal{D}'^0(\Lambda)$ there exists a integrable complex-valued measure $m(U,V)$ supported in L such that*

$$\Sigma(U,V)(\rho) = \int_L \rho\, m(U,V), \quad \rho \in \mathcal{D}^0(X). \tag{5.3}$$

Proof of corollary. — For any Λ-distribution U of order zero the local density $\Sigma(U,U)$ is positive. According to the Schwartz's Theorem [10] this density is a non-negative measure $m(|U|^2)$ supported by the locus. The complex-valued measure $m(U,V) = \frac{1}{2}\partial_t\partial_s m(|sU + tV|^2)$ satisfies (5.3). □

Remark. — Suppose that a contact Lagrange manifold Λ_c is symmetric and decomposed in two opposite pieces $\Lambda_c = \Lambda_+ \cup \Lambda_-$. The corresponding folds coincide: $p(\Lambda_+) = p(\Lambda_-)$ but any Λ_+ distribution U_+ is orthogonal to arbitrary Λ_--distribution U_- with respect to the Hermitian form Σ.

6. Logarithmic asymptotics.

Proof Theorem 5.1. — It is sufficient to prove the theorem for the case $V = U$; the Hermitian form $\Sigma(U,V)$ is reconstructed uniquely by means of the standard method. We construct the distribution Σ locally. The local constructions are locally uniquely defined and therefore they will glue together in a global quadratic mapping $\Sigma(U,U)$ satisfying (5.1) and (5.2).

It suffices to prove (5.1) and (5.2) for an arbitrary smooth non-negative functions ρ. Really, arbitrary function $\rho \in \mathcal{D}^0(Y)$ can be represented in the form $\rho_1 - \rho_2$ where $\rho_{1,2} \in \mathcal{D}^0(Y)$ are non-negative. For arbitrary continuous function $\rho \geq 0$ with compact support we can find non-negative functions $\rho_-, \rho_+ \in \mathcal{D}(Y)$ that are close to ρ and satisfies $\rho_- \leq \rho \leq \rho_+$. The peripheral integral of $\rho|U|^2$ is monotone with respect to ρ. Therefore it is bounded by the product $[\int \rho_+|\sigma_c(U)|^2 + o(1)] \log 1/\varepsilon$ from above and by the similar product with ρ_- from below. We can make the integrals $\int \rho_\pm|\sigma_c(U)|^2$ as close one to another as we like by choosing the approximations in such a way that $\max(\rho_+ - \rho_-) \to 0$. This proves (5.1) and (5.2) for arbitrary $\rho \in \mathcal{D}^0(Y)$. From now on we assume that the function ρ is non-negative and smooth.

Suppose first that Λ_c is univalent over a point y. Choose a barrier f of the locus $L = p(\Lambda_c)$ defined in a neighbourhood Y of y. If this neighbourhood is sufficiently small, we can write $U = I(\phi, A)$ in Y

where $A = A_m + R$ is an asymptotically homogeneous amplitude of order $m = -\frac{1}{2}N$. We suppose that the amplitude A vanishes for $|\theta| < \frac{1}{2}$ and is homogeneous of degree m for $|\theta| \geq 1$. Fix a continuous function h, $0 \leq h \leq 1$ in $\mathbb{R}$ that is equal to 1 in 1-neighbourhood of the origin. Compare the integral (3.1) with another intensity-type integral

$$(6.1) \qquad I(r) \doteq \int_X \rho \Big| \int \exp(2\pi\imath\phi) h_r A \,\mathrm{d}\theta \Big|^2, \quad h_r(\theta) \doteq h\big(r^{-1}|\theta|\big).$$

The integral (6.1) diverges as $r \to \infty$ because the density $|A|^2 \,\mathrm{d}\theta \,\mathrm{d}\theta^0$ is homogeneous of order 0. It can be written in the form

$$I(r) \doteq \int_{X\times\Theta\times\Theta^0} \rho \exp\big(2\pi\imath\Phi(x,\theta,\theta^0)\big) h_r(\theta) A(x,\theta) \bar{h}_r(\theta^0) \bar{A}(x,\theta^0) \,\mathrm{d}\theta \,\mathrm{d}\theta^0$$

where $\Phi(x;\theta,\theta^0) = \phi(x,\theta) - \phi(x,\theta^0)$.

We assume that X is an open set in a coordinate space $\mathbb{R}^{n+1}$ and calculate the integral by the method of [1], Section 1.3. By the method of dimension descent we can transform the integral (2.1) modulo $C^\infty(X)$ to another Fourier integral with a phase function of the form $\phi(x,\theta) = x\theta - \gamma(\theta)$ where γ is a homogeneous function of degree 1 and the dimension of the ancillary space is equal to $N = n + 1$. We write

$$\Phi(y,\xi,\theta) = y\xi - \gamma(\theta) + \xi\gamma'(\theta) + \gamma(\theta - \xi)$$

by means of new coordinates θ, $\xi = \theta - \theta^0$, $y = x - \gamma'(\theta)$ where $\gamma' = \partial\gamma/\partial\theta$. The critical set for the integral (6.1) is given by the equations

$$\Phi'_\xi = 0, \quad \Phi'_y = 0.$$

These equations are equivalent to $\xi = 0$, $y = 0$, the second differential of Φ in any point of this variety is equal to $y\xi$ and the signature of the differential is equal to (n,n). Write the amplitude in the form $A = a\sqrt{\mathrm{d}x}$ where $\mathrm{d}x$ is the volume form $\mathbb{R}^{n+1}$ and a is a homogeneous function of degree $-\frac{1}{2}(n+1)$. The stationary phase method yields

$$I(r) = \int_\Theta \rho \big| h_r a(\gamma'(\theta),\theta) \big|^2 \,\mathrm{d}\theta + O(1)$$

as $r \to \infty$. This integral diverges as $O(\log r)$ since the form $|a|^2 \mathrm{d}\theta$ is of order 0 for $|\theta| \geq 1$. Therefore

$$I(r) = \int_{S(\Theta)} \rho\big(\gamma'(\theta)\big) \big|a(\gamma'(\theta),\theta)\big|^2 (e_\theta \vee \mathrm{d}\theta) \cdot \log r + O(1)$$

where $S(\Theta)$ is the unit sphere in the ancillary space Θ, e_θ is the Euler field in Θ and $\vee$ denotes the contraction operation. We have

$$\mathrm{d}|\theta| \wedge \big(e_\theta \vee \mathrm{d}\theta\big) = |\theta|\,\mathrm{d}\theta$$

hence $|a(\gamma',\theta)|^2 e_\theta \vee \mathrm{d}\theta = |\sigma_c(I)|^2$ and

$$I(r) = \int_{\Lambda_c} p^*(\rho)\big|\sigma(I)\big|^2 \cdot \log r + O(1). \tag{6.2}$$

For a positive ε we denote

$$X(\varepsilon) = \{x \in X\,;\ |f(x)| \geq \varepsilon\}, \quad L(\varepsilon) = X \setminus X(\varepsilon), \quad A_r \doteq h_r A,$$
$$B(y,\theta,\theta^0) \doteq A(x,\theta)\bar{A}(x,\theta^0), \quad B_r(y,\theta,\theta^0) \doteq A_r(x,\theta)\bar{A}_r(x,\theta^0)$$

and estimate the integral

$$\begin{aligned} J(\varepsilon) &\doteq \int_{X(\varepsilon)}\int_\Theta \rho \exp(2\pi\imath\Phi) B\,\mathrm{d}\theta\,\mathrm{d}\theta^0 \\ &= \int_{X(\varepsilon)} \rho \Big|\int \exp(2\pi\imath\Phi) A\,\mathrm{d}\theta\Big|^2 \geq 0. \end{aligned} \tag{6.3}$$

Write

$$J(\varepsilon) = J(r,\varepsilon) + I(r) - I(r,\varepsilon),$$
$$J(r,\varepsilon) = \int_{X(\varepsilon)}\int \rho \exp(2\pi\imath\Phi)(B - B_r)\,\mathrm{d}\theta\,\mathrm{d}\theta^0$$

where the last integral is non-negative too and

$$\begin{aligned} I(r,\varepsilon) &= \int_{L(\varepsilon)}\int \rho \exp(2\pi\imath\Phi) B_r\,\mathrm{d}\theta\,\mathrm{d}\theta^0 \\ &= \int_{L(\varepsilon)} \rho \Big|\int \exp(2\pi\imath\Phi) A_r\,\mathrm{d}\theta\Big|^2 \geq 0. \end{aligned} \tag{6.4}$$

To estimate $J(r,\varepsilon)$ we need the following

LEMMA 6.1. — *For an arbitrary compact set $K \subset X$ there exists a positive number c such that the inclusion $x \in K \cap X(\varepsilon)$ implies $|\phi'_\theta| \geq c\varepsilon$ for any $\theta \in \Theta$.*

Proof. — The function $\phi(x,\theta)$ restricted to the unit sphere $S(\Theta)$ in the ancillary space is the phase function of the contact manifold Λ_c. We can apply the equation (4.7) to this function. Taking in account the Euler equation $\phi = \sum \theta_i \phi'_{\theta_i}$ we get

$$f(x) = \sum \kappa_i(x,\theta)\phi'_{\theta_i}(x,\theta)$$

for some smooth homogeneous functions $\kappa_i(x,\theta)$ of order 0 in $X \times \Theta$. This equation implies Lemma 6.1 for the constant c such that $c^{-2} \geq \sum_j |\kappa_j(x,\theta)|^2$ for $x \in K$, $\theta \in \Theta$.

LEMMA 6.2. — *We have*

$$|J(r,\varepsilon)| \leq \frac{C}{r\varepsilon^{2N+2}}, \quad |J(r,\varepsilon)| \leq \frac{C}{r^2\varepsilon^{2N+4}} \tag{6.5}$$

for odd and even N respectively with some constant C.

Proof. — Take the field $t = (2\pi\imath)^{-1}|\phi'_\theta|^{-2}\sum_j \phi'_{\theta_j}\partial/\partial\theta_j$. We have $t(2\pi\imath\Phi) = 1$, consequently

$$J(r,\varepsilon) = \int_{X(\varepsilon)}\int \rho\, t\big(\exp(2\pi\imath\Phi)\big) g_r B \,\mathrm{d}\theta\,\mathrm{d}\theta^0 \tag{6.6}$$

where $g_r(\theta,\theta^0) = 1 - h_r(\theta)h_r(\theta^0)$. Integrating partially yields

$$J(r,\varepsilon) = -\int_{X(\varepsilon)}\int \rho \exp(2\pi\imath\Phi)\big(t(a_r) + \operatorname{div} t\, a_r\big)\bar{a}\,\mathrm{d}\theta\,\mathrm{d}\theta^0\,\mathrm{d}x \tag{6.7}$$

where $a_r = g_r a$, $\bar{a} = \bar{a}(x,\theta^0)$. By Lemma 6.1 we have $\sum |t(\theta_j)| \leq C/\varepsilon$. The function $\operatorname{div} t \doteq \sum \partial t(\theta_j)/\partial\theta_j$ is homogeneous of order -1 and again by Lemma 6.1 we conclude $|\operatorname{div} t| \leq C\varepsilon^{-2}|\theta|^{-1}$ hence

$$|t(a_r)| \leq |t(g_r)a| + |t(a)| \leq C\varepsilon^{-1}|\theta|^{m-1}$$

where the constant C does not depend on r. We have used here the estimate

$$|t(h_r)| = |t(|\theta|)r^{-1}h'(r^{-1}|\theta|)| \leq C\big(\varepsilon|\theta|\big)^{-1}$$

which follows from

$$r^{-1}|h'(r^{-1}|\theta|)| \leq C|\theta|^{-1}, \quad |t(|\theta|)| \leq (c\varepsilon)^{-1}.$$

Therefore $|t(a_r) + \operatorname{div} t\, a_r| \leq C\varepsilon^{-1}|\theta|^{m-1}$. Repeating this transformation $[\frac{1}{2}N + 1]$ times we obtain the equation

$$J(r,\varepsilon) = \int_{X(\varepsilon)}\int \rho\exp(2\pi\imath\Phi)\tilde{a}_r(x,\theta,\theta^0)\,\mathrm{d}\theta\,\mathrm{d}\theta^0\,\mathrm{d}x$$

where the amplitude function satisfies the estimate

$$\left|\tilde{a}_r(x,\theta,\theta^0)\right| \leq \begin{cases} C\varepsilon^{-N-1}|\theta|^{-N-1/2} & \text{for } N \text{ odd,} \\ C\varepsilon^{-N-2}|\theta|^{-N-1} & \text{for } N \text{ even.} \end{cases}$$

This implies that the integral over Θ converges absolutely. Then we integrate partially with respect to the field t^0 which stems from t by the substitution $\theta \to \theta^0$. Then we get a representation like (6.6) with an amplitude function $\tilde{b}_r$ that satisfies

$$\left|\tilde{b}_r(x,\theta,\theta^0)\right| \leq C\varepsilon^{-2N-2}\big(|\theta|\cdot|\theta^0|\big)^{-N-1/2}$$

for N odd which yields

$$J(r,\varepsilon) \leq C\varepsilon^{-2N-2}\int_{X(\varepsilon)}\rho\,\mathrm{d}x\left|\int_{r\leq|\theta|}\frac{\mathrm{d}\theta}{|\theta|^{N+1/2}}\right|^2 \leq \frac{C}{r\varepsilon^{2N+2}}. \qquad \text{Q.E.D.}$$

The case of even N is similar. □

By (6.5) for odd N

$$0 \leq J(\varepsilon) \leq I(r) + J(r,\varepsilon) \leq \int_{\Lambda_c}\rho\left|\sigma(a)\right|^2\cdot\log r + \frac{C}{r\varepsilon^{2N+2}}$$

since of (6.2) and (6.4). Now we tie the parameters by the equation $r = \varepsilon^{-2N-2}$ and conclude that

$$\int_{X(\varepsilon)}\rho|U|^2 = J(\varepsilon) \leq (2N+2)\int_{\Lambda_c(\phi)}\rho|\sigma(U)|^2\cdot\log\frac{1}{\varepsilon} + C. \tag{6.8}$$

This inequality is valid for even N as well.

Now we prove the asymptotics (5.1). Let G be the set of critical values of the mapping $p:\Lambda_c \to X$ and T be the set of points x such that $\operatorname{card} p^{-1}(x) \geq 3$. The set $G\cup T \subset L$ is closed and of n-dimensional measure zero. Really, it is true for the set G since of the Sard theorem. For any point $x \in T\setminus G$ there are at least two points $(x,\xi_{1,2}) \in \Lambda_c$ such that the covectors ξ_1 and ξ_2 are not collinear. The corresponding folds L_1, L_2 has transversal intersection in x hence T is contained in the finite union of transversal intersections $L_i \cap L_j$. Each of these sets is a $n-1$-dimensional manifold. This implies that n-dimensional measure of T is equal to zero too.

For a number $\eta > 0$ we denote by $G(\eta)$ the η-neighbourhood of the set $G \cup T$. Choose a smooth function g_η, $0 \leq g_\eta \leq 1$ that is equal to 1 in $G(\eta)$ and 0 in $X \setminus G(2\eta))$ and write

$$\int_{X(\varepsilon)} \rho |U|^2 = \int_{X(\varepsilon)} g_\eta \rho |U|^2 + \int_{X(\varepsilon)} \rho_\eta |U|^2 \tag{6.9}$$

where we set $\rho_\eta \doteq (1 - g_\eta)\rho$.

The first term can be estimated by means of (6.8):

$$\int_{X(\varepsilon)} g_\eta \rho |U|^2 \leq (2N+2) \int_{\Lambda_c} g_\eta \rho \big|\sigma(U)\big|^2 \cdot \log \frac{1}{\varepsilon} + O(1). \tag{6.10}$$

Here and below we write ρ instead of $p^*(\rho)$.

The support of the density $g_\eta \rho\, |\sigma(U)|^2$ is contained in the set $p^{-1}(G(\eta))$. The measure of this set tends to the measure of the set $p^{-1}(G)$ as $\eta \to 0$. By the Sard theorem mes $G = 0$ hence mes $p^{-1}(G) = 0$ by Proposition 10.4. Therefore mes $p^{-1}(G(\eta)) \to 0$ as $\eta \to 0$ and consequently

$$\int_{\Lambda_c} g_\eta \rho \big|\sigma(U)\big|^2 \to 0 \quad \text{as } \eta \to 0. \tag{6.11}$$

By Proposition 3.1 the second term of (6.9) has the asymptotics

$$\int_{X(\varepsilon)} \rho_\eta |U|^2 = \int \rho_\eta \big|\sigma(U)\big|^2 \cdot \log \frac{1}{\varepsilon} + O(1) \quad \text{as } \varepsilon \to 0.$$

By (6.8) this yields

$$\int \rho_\eta |\sigma(U)|^2 + o(1) \leq Q(\varepsilon) \leq \int \rho_\eta \big|\sigma(U)\big|^2 + (2N+2) \int g_\eta \rho \big|\sigma(U)\big|^2 + o(1)$$

where

$$Q(\varepsilon) \doteq -(\log \varepsilon)^{-1} \int_{X(\varepsilon)} \rho |U|^2$$

and $o(1) \to 0$ as $\varepsilon \to 0$. This helps to conclude

$$\int \rho_\eta \big|\sigma(U)\big|^2 \leq \varliminf_{\varepsilon \to 0} Q(\varepsilon) \leq \varlimsup_{\varepsilon \to 0} Q(\varepsilon) \leq \int \rho_\eta \big|\sigma(U)\big|^2 + (2N+2) \int g_\eta \rho \big|\sigma(U)\big|^2.$$

The left-hand side and the right-hand sides have the same limit $\int \rho \big|\sigma(U)\big|^2$ as $\eta \to 0$ which follows from (6.11). This proves (5.1) and (5.2) for the case when Λ_c is univalent over y.

In the general case we have $p^{-1}(y) = \{\lambda_1, \dots, \lambda_q\}$. For a small neighbourhood Y of y the manifold $\Lambda_c(Y) \doteq p^{-1}(Y)$ consists of q disjoint pieces $\Lambda_1, \dots, \Lambda_q$. We apply Theorem 4.2 to each piece Λ_j and get the corresponding set $G_j \subset Y$ which has n-dimensional measure zero.

We assume that the above set G contains the union $\bigcup_j G_j$. Write

$$|U|^2 = \sum |U_i|^2 + \sum_{i \neq j} U_i \bar{U}_j$$

where U_i is a Λ_i-distribution for $i = 1, \dots, q$. The peripheral integral of the first sum fulfils (5.1), (5.2) and the density $|\sigma_c(U)|^2$ is equal to the sum of the densities $|\sigma_c(U_i)|^2$, $i = 1, \dots, q$. We show that the peripheral integral of the density $\rho_\eta U_i \bar{U}_j$ brings no contribution to the logarithmic asymptotics if $i \neq j$. Really we have $U_i = O(|f_i|^{-1/2})$ according the calculation of Proposition 3.1 and

$$\rho_\eta U_i \bar{U}_j = O\big(|f_i|^{-1/2} + |f_j|^{-1/2}\big).$$

The right-hand side is locally integrable in the set $X \setminus (L_i \cap L_j)$ and also in a neighbourhood of an arbitrary point $y \in L_i \cap L_j$ such that the forms $\mathrm{d}f_i(y)$ and $\mathrm{d}f_j(y)$ are independent.

Suppose now that the forms $\mathrm{d}f_1(y)$, $\mathrm{d}f_2(y)$ are dependent in a point $y \in L \setminus (G \cup T))$. The point belongs to $L_1 \cap L_2$ and the corresponding points of the fibre $C_y^*(X)$ are opposite. This means that the relation $\mathrm{d}f_2(y) = c\,\mathrm{d}f_1(y)$ holds for some $c < 0$. No fold L_j, $j > 2$ contains the point y since $y \notin T$. We have

$$\text{(6.12)} \quad \rho_\eta U_1 \bar{U}_2 = \rho_\eta a_1 \bar{a}_2 (f_1 + 0\imath)^{-1/2} (f_2 - 0\imath)^{-1/2} + O\big(|f_1|^{-1/2} + |f_2|^{-1/2}\big).$$

The remainder is locally integrable. Show that the integral of the main term in (6.12) is bounded as $\varepsilon \to 0$ for an arbitrary test function ρ supported in a neighbourhood Y of the point y. Choose a coordinate system x_1, x' in Y such that $x_1 = f_1$. We have $f_2(x) = (u(x') - x_1) h_2(x)$ where the factor h_2 is smooth and positive in Y. According to the construction of Section 4 we can write the barrier function in the form $f = hq$, $q(x) = x_1(x_1 - u(x'))$ where h is a smooth function. It does not vanish in y since $y \in L \setminus (G \cup T)$ hence h^{-1} is bounded in a neighbourhood Y of y.

Consider the manifold $\mathbb{C}_+ \times \mathbb{R}^n$ where $\mathbb{C}_+$ is the closed upper half-plane in the complex plane of the variable $z = x_1 + y\imath$ and the pseudopolynomial $q(z,x') = z(z - u(x'))$ in this manifold. The function $q^{-1/2}$ has analytic continuation to this manifold from the domain $x_1 > |u(x')|$. The key point of our arguments is that boundary value $q^{-1/2}(x_1 + 0\imath, x')$ of this continuation coincides with the product $ih_2^{1/2}(f_1 + 0\imath)^{-1/2}(f_2 - 0\imath)^{-1/2}$.

Represent the above integral as follows:

$$\int_{X(\varepsilon)} \rho_\eta a_1 \bar{a}_2 (f_1 + 0\imath)^{-1/2}(f_2 - 0\imath)^{-1/2} = \int_{Y(\varepsilon)} q^{-1/2} a \, \mathrm{d}x + \int_{Z(\varepsilon)} q^{-1/2} a \, \mathrm{d}x$$

where

$$Y(\varepsilon) \doteq \{q^2 \geq \varepsilon\}, \quad Z(\varepsilon) \doteq X(\varepsilon) - Y(\varepsilon), \quad a \, \mathrm{d}x \doteq i\rho_\eta h_2^{-1/2} a_1 \bar{a}_2.$$

The last term is bounded by a constant C that does not depend on ε. Really, we have the following inequality for the line integral as x' is fixed:

$$\Big| \int_{R(\varepsilon)} |q|^{-1/2} a \, \mathrm{d}x_1 \Big| \leq C\varepsilon^{-1/2} |R(\varepsilon)|$$

where $|R(\varepsilon)|$ is the length of the real 1-chain

$$R(\varepsilon) \doteq \{h^{-1}\varepsilon \geq q \geq \varepsilon\}.$$

The function h^{-1} is bounded hence by Lemma 6.3 (i) we have $R(\varepsilon) \leq C\varepsilon^{1/2}$ and our statement follows.

To estimate the integral over $Y(\varepsilon)$ we define in $\mathbb{C}_+ \times \mathbb{R}^n$ the function

$$\tilde{a}(x_1 + y\imath, x') \doteq a(x) + y \frac{\partial a}{\partial x_1} \imath.$$

It coincides with a and satisfies $\partial \tilde{a}/\partial \bar{z} = 0$ at $y = 0$. Consider the n-chain

$$C(\varepsilon) \doteq \big\{ |q^2(z,x')| = \varepsilon \, ; \; y > 0 \big\}$$

in $\mathbb{C}_+ \times \mathbb{R}^n$ and set $\Gamma(\varepsilon) \doteq Y(\varepsilon) \cup C(\varepsilon)$. We have

$$\int_{Y(\varepsilon)} a q^{-1/2} \, \mathrm{d}x = \int_{\Gamma(\varepsilon)} \tilde{a} q^{-1/2} \, \mathrm{d}z \wedge \mathrm{d}x' - \int_{C(\varepsilon)} \tilde{a} q^{-1/2} \, \mathrm{d}z \wedge \mathrm{d}x'.$$

The integral over $C(\varepsilon)$ is bounded as $\varepsilon \to 0$ since the n-volume of $C(\varepsilon)$ is equal to $O(\varepsilon^{1/2})$. This follows again from the 1-dimensional estimate given in Lemma 6.3 (ii).

At the other hand the chain $\Gamma(\varepsilon)$ is closed and by the Cauchy-Green formula

$$\int_{\Gamma(\varepsilon)} \tilde{a}q^{-1/2}\,\mathrm{d}z\wedge \mathrm{d}x' = \int_{\Gamma} \tilde{a}q^{-1/2}\,\mathrm{d}z\wedge \mathrm{d}x' - \int_{\Delta} q^{-1/2}\frac{\partial\tilde{a}}{\partial\bar{z}}\,\mathrm{d}\bar{z}\wedge\,\mathrm{d}z\wedge \mathrm{d}x'$$

where Γ is the union of semi-circles $|z|=1$, $y\geq 0$ and Δ is the open set in $\mathbb{C}_{+}\times\mathbb{R}^{n}$ such that $\partial\Delta=\Gamma-\Gamma(\varepsilon)$. The integral over Δ is uniformly bounded in virtue of the estimate $|\partial\tilde{a}/\partial\bar{z}|\leq Cy$, whereas the integral over Γ does not depend of ε.

Now to complete the proof it remains to take in account

LEMMA 6.3. — *Let a,b be arbitrary complex numbers and $q(z)=z^{2}+az+b$.*

(i) *The curve*

$$M(\varepsilon)=\{z\in\mathbb{C},\ |q(z)|=\varepsilon\}$$

satisfies the inequality $|M(\varepsilon)|\leq C\varepsilon^{1/2}$ holds for $0<\varepsilon\leq 1$ with a constant C that does depend on a,b.

(ii) *For an arbitrary $\alpha\in[0,2\pi)$ the curve*

$$A(\alpha,\varepsilon)=\{0\leq|q(x)|\leq\varepsilon,\ \arg q=\alpha\}$$

satisfies the similar inequality $|A(\alpha,\varepsilon)|\leq C\varepsilon^{1/2}$ for $0<\varepsilon\leq 1$ with a constant C that does depend on a,b,α.

Here $|M|$ stands for the length of a curve M. A proof is elementary. □

Remark. — The last lemma can be generalized for an arbitrary polynomial $q(z)=z^{n}+a_{1}z^{n-1}+\cdots$ in the form $|M(\varepsilon)|+|A(\alpha,\varepsilon)|\leq C_{n}\varepsilon^{1/n}$.

Remark. — Comparing (6.2) and (5.1) we conclude that the asymptotics (5.1) of the intensity integral in the configuration space coincides with the asymptotics of the similar integral (6.1) in the frequency domain if we tie the parameters by the equation $r\varepsilon=1$.

7. Conservation laws for Lagrange solutions.

Consider the wave equation in the space-time $X\times\mathbb{R}$ where X is a Riemannian manifold. Write it by means of local coordinates $x^{1},\ldots,x^{n}$ in X:

$$\partial_{t}^{2}u-G^{-1}\partial_{i}(g^{ij}\partial_{j}Gu)=0.$$

Here u is a function, $\partial_i = \partial/\partial x^i$, $i = 1, \dots, n$, the tensor g^{ij} is inverse to the metric tensor g_{ij} and $G = \sqrt{\det g_{ij}}$; the summation in i, j is assumed. Set

$$U \doteq Gu\sqrt{\mathrm{d}x}, \quad \mathrm{d}x = \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n$$

and write the wave equation for the halfdensity U:

$$\partial_t^2 U - \partial_i(g^{ij}\partial_j U) = 0 \tag{7.1}$$

where ∂_i, $i = 1, \dots, n$ are the corresponding Lie derivatives. The symbol of the wave equation (7.1) is equal to $p_2 + p_1$ where

$$p_2(x;\xi,\tau) = g^{ij}(x)\xi_i\xi_j - \tau^2, \quad p_1 = -2\sqrt{-1}\,\partial_i(g^{ij})\xi_j\,. \tag{7.2}$$

Consider the Hamiltonian system defined by the function p_2:

$$\begin{cases} \dfrac{\mathrm{d}x_i}{\mathrm{d}s} = 2g^{ij}(x)\xi_j, & \dfrac{\mathrm{d}t}{\mathrm{d}s} = -2\tau, \\ \dfrac{\mathrm{d}\xi_k}{\mathrm{d}s} = -\partial_k g^{ij}(x)\xi_i\xi_j, & \dfrac{\mathrm{d}\tau}{\mathrm{d}s} = 0, \quad s \in \mathbb{R}, \end{cases} \tag{7.3}$$

with the initial data

$$(x(0), \xi(0)) \in \Lambda_0, \quad t = 0, \quad \tau^2(0) = g^{ij}(x(0))\xi_i(0)\xi_j(0), \tag{7.4}$$

where Λ_0 is a Lagrange manifold in $T^*(X)$. Let $\Lambda \subset T^*(X \times \mathbb{R})$ be the union of the trajectories of (7.3). This is a Lagrange manifold. The system (7.3) defines the Hamiltonian flow P in the cotangent space. Its projection B_s to the space-time is called bicharacteristic flow; the trajectories of this flow are bicharacteristic rays. The locus $L = p(\Lambda)$ of the Lagrange manifold Λ is the union of bicharacteristic rays with the initial data (7.4). The bicharacteristic flow in L has focal points in singular points of L.

If the Lagrange manifold Λ_0 is symmetric (see Section 3), then the manifold Λ generated by the flow (7.3) is also symmetric. This follows from the property that the system (7.3) preserves its form if we change x, ξ, τ, s to $x, -\xi, -\tau, -s$.

Remark. — Consider the Cauchy problem for (7.1) with some initial data u_0, u_1 for $t = 0$ that are Λ_0-distributions in the sense of Section 2. This problem has unique solution U and this solution is a Λ-distribution. This follows from the general theory [3], [6] under certain loose assumptions. Moreover the order of U does not exceed ν if u_0, u_1 are of order ν and $\nu + 1$ respectively.

THEOREM 7.1. — *Let Λ_c be a closed contact Lagrange manifold over X that is invariant with respect to the flow P and has finite multiplicity. For arbitrary solutions $U_1, U_2 \in \mathcal{D}'(\Lambda)$ of (7.1) the residue of the density $U_1\overline{U}_2$ is preserved by the bicharacteristic flow B, i.e.,*

$$\int \Sigma(U_1, U_2) B_s^*(g) = \text{Const}$$

for an arbitrary continuous function g with compact support in Y.

Proof. — According to Theorem 5.1 it is sufficient to check that for an arbitrary solution $U \in \mathcal{D}'(\Lambda)$ of the wave equation any local symbol $\sigma(U)$ is conserved by the flow. The symbol satisfies the transport equation [3], [6]:

$$L_{p_2}\sigma(U) + q\sigma(U) = 0, \quad q \doteq \sqrt{-1}\, p_1 - \frac{1}{2}\sum_k \frac{\partial^2 p_2}{\partial x_k \partial \xi_k}. \tag{7.5}$$

Here L_{p_2} means the Lie derivative with respect to the Hamiltonian field (7.3). By (7.2)

$$q = 2\partial_i(g^{ij})\xi_j - \sum_k \frac{\partial^2(g^{ij}\xi_i\xi_j)}{\partial x_k \partial \xi_k} = 0$$

and hence $L_{p_2}\sigma(U) = 0$. This implies that the density $\sigma_c(U_1)\sigma_c(\bar{U}_2)$ (which is globally defined) is constant along any trajectory of the system (7.3). □

Energy. — Let U be a Λ-distribution U of order -1 that satisfies (7.1). Define the energy density

$$E(U) = \frac{1}{2}\left(|U_t'|^2 + g^{ij}\partial_i U \partial_j \overline{U}\right).$$

This density has residue $\Sigma(E(U))$ since the derivatives of the distribution U are Λ-distributions of order 0. We call it *singular* energy. The conservation law in geometrical optics can be generalized to the singular energy:

PROPOSITION 7.2. — *Suppose that the conditions of Theorem 7.1 fulfilled. Let U be an arbitrary Λ-solution of (7.1) of order -1. The singular energy of U is preserved by the bicharacteristic flow and we have*

$$\sigma\big(E(U)\big) = \Sigma\big(|U_t'|^2\big).$$

Proof. — We can suppose that the manifold Λ is symmetric: $\Lambda = \Lambda_+ \cup \Lambda_-$. Take a point $y \in L_r$ and choose a small neighbourhood Y of this point. The function $f \doteq t - \phi(x)$ is a regular barrier if ϕ is an eikonal function, that is, a solution of the equation $g^{ij}\partial_i\phi\partial_j\phi = 1$. Write the solution in the form $U = U_+ + U_-$ where

$$U_\pm = A_\pm(x)(f \pm 0\imath)^{1/2} + O(|f|^{1/2+\varepsilon})$$

and $A_\pm$ are smooth halfdensities in Y (*cf.* Proposition 3.1). This equation can be differentiated by terms. This yields

$$g^{ij}\partial_i U_+ \partial_j \bar{U}_+ = |A_+|^2 \frac{g^{ij}\partial_i\phi\partial_j\phi}{4|t-\phi|} + O(1) = \frac{|A_+|^2}{4|t-\phi|} + O(1).$$

Therefore $g^{ij}\Sigma(\partial_i U_+, \partial_j U_+) = |A_+|^2/4\,\mathrm{d}(t-\phi)$ where Σ is the residue form. We can change the index $+$ to $-$ in this formula, whereas $\Sigma(\partial_i U_+, \partial_j U_-) = 0$. Finally

$$g^{ij}\Sigma(\partial_i U, \partial_j U) = \frac{|A_+|^2 + |A_-|^2}{4\,\mathrm{d}(t-\phi)}\cdot$$

For the time-derivative we have

$$\Sigma(U_t', U_t') = \frac{|A_+|^2 + |A_-|^2}{4|t-\phi|}$$

which implies the equation

$$\begin{aligned} 2\Sigma(E) &\doteq g^{ij}\Sigma(\partial_i U, \partial_j U) + \Sigma(U_t', U_t') \\ &= \frac{|A_+|^2 + |A_-|^2}{2\,\mathrm{d}(t-\phi)} = 2p_*\big(|\sigma_c(U_t')|^2\big). \end{aligned}$$

The density $|\sigma(U_t')|^2$ is kept constant by the flow P according to Theorem 7.1. This implies conservation of $\Sigma(E)$.

8. Residue and geometry of locus.

DEFINITION. — Let X be a smooth manifold of dimension, Λ_c be a contact Lagrange manifold over X, L be its locus. Take a point $\lambda \doteq (y,\xi) \in \Lambda_c$ and consider the tangent mapping $dp_\lambda : T_\lambda(\Lambda_c) \to T_y(X)$. We call the image $T_\lambda(L)$ of this mapping *singular tangent* space to the corresponding fold L' of the locus. The singular tangent space is contained in the tangent hyperplane $\operatorname{Ker}\xi \subset T_y(X)$; its codimension in this hyperplane is equal to the number $k = \dim \operatorname{Ker} dp_\lambda$. We call the number k *defect* of Λ_c in λ (over y).

Choose a local coordinate system $x_0, \ldots, x_n$ in a neighbourhood Y of y and consider the local Euclidean metric $g = \sum \mathrm{d}x_i^2$. Calculate the curvature form $Q(x)$ of the locus at a point $x \in L_r$. For this we take a regular barrier function f for a fold L' of the locus at this point such that $g(\mathrm{d}f(x')) = 1 + O(x - x')$ and set

$$Q(x) = \mathrm{d}^2 f(x)_{|T_x(L)}.$$

Choosing an orthonormal basis in $T_x(L')$ we write the quadratic form $Q(x)$ in a normal form where the diagonal elements $\kappa_1, \ldots, \kappa_n$ are main curvatures of the hypersurface L'. It is shown in [9] that k largest main curvatures $\kappa_1, \ldots, \kappa_k$ tend to $\pm\infty$ as $x \to y$. The sign of the product $K(x) \doteq \kappa_1 \cdots \kappa_k$ changes when the point x crosses the stratum L_2 in the locus of points of multiplicity 2. This sign relates to the sharpness of the singularity of an arbitrary Λ -distribution [9]. Other $n - k$ curvatures are small comparing with the first k of them: $\kappa_j/\kappa_i \to 0$ for any $i \le k < j$.

We show in this section that the singular function $|K|$ is a common factor of residue densities of Λ-distributions of order 0. We shall use the quantity $Q_k = |\mathrm{tr}(\bigwedge^k Q)|$ rather than K. It follows from the aforesaid that the quotient $Q_k/|K|$ tends to 1 as $x \to y$. Now we specify the choice of coordinates $x_0, \ldots, x_n$ in Y by imposing the condition

(8.1) $\lambda = (y, \mathrm{d}x_0)$ and the forms $p^*(\mathrm{d}x_1), \ldots, p^*(\mathrm{d}x_k)$ vanish in $T_y(L)$.

We show that for any fold L' of the locus L and any point $x \in L'$ the function $Q_k(x)$ is equivalent to the Gaussian curvature of the intersection of L' with the subspace $x_{k+1} = x_{k+1}(x), \ldots, x_n = x_n(x)$.

Denote by δ_L the delta-distribution $\delta_L(\rho) = \int_L \rho\, \mathrm{d}x/\mathrm{d}x_0$. Note that the restriction of the form $\mathrm{d}x/\mathrm{d}x_0$ to the variety L_r is equal to $(1 + o(1))\,\mathrm{d}S$ as $x \to y$ where $\mathrm{d}S$ is the Euclidean hypersurface density in the coordinate system $x_0, \ldots, x_n$. This follows from the fact that the contact element $(0, \operatorname{Ker} \mathrm{d}x_0)$ belongs to Λ_c.

THEOREM 8.1. — *Let $p: \Lambda_c \to X$ be a closed contact Lagrange manifold of finite multiplicity over a point y such that the set $p^{-1}(y)$*

consists of one point λ. *There exists a neighbourhood* Y *of* y *such that for any* Λ*-distribution* U *in* Y *of order* 0 *the following relation holds*:

$$\Sigma(|U|^2) \approx b\,Q_k\,\delta_L, \tag{8.2}$$

where k *is the defect of* Λ_c *in* λ *and* b *is a non-negative continuous function in* Λ_c.

We write here and later $a \approx b$ if $a/b \to 1$ as $x \to y$.

The factor Q_k is unbounded as x tends to a non-regular point of the locus and is multivalued in the intersection of folds of the locus. The factor b is considered here as a function in the locus L_r and is multivalued too. To calculate the coefficient b we compare the symbol of U with a special halfdensity. According to (8.1) the coordinate projection p_k shown in Proposition 10.1 is a bijection in a neighbourhood Λ' of λ. Therefore the form

$$\pi_k = p_k^*(\mathrm{d}x_{k+1} \wedge \ldots \wedge \mathrm{d}x_n \wedge \mathrm{d}\omega_1 \wedge \ldots \wedge \mathrm{d}\omega_k)$$

does not vanish in Λ'. Therefore we can write $|\sigma_c(U)|^2 = b|\pi_k|$ for a smooth non-negative function b in Λ'.

COMPLEMENT 8.2. — *The equation* (8.2) *holds for the function* b *as above.*

Proof of Theorem 8.1. — We apply Theorem 5.1 and calculate the contact symbol $\sigma_c(U)$. For this we choose a contact generating function

$$\varphi : Y \times \Omega \longrightarrow \mathbb{R}$$

for the germ Λ_c at the point λ as in Proposition 10.2 where Ω be a neighbourhood of the origin in $\mathbb{R}^k$. Take an arbitrary point $(z, \omega_z) \in C(\varphi)$ where the quadratic form $\varphi''_{\omega\omega}$ is non-singular. There exist a neighbourhood Z of z and a unique smooth solution $\omega = \omega(x)$ of the system

$$\varphi'_\omega(x, \omega) = 0$$

in Z such that $\omega(z) = \omega_z$. Set

$$f(x) \doteq \varphi\big(x, \omega(x)\big).$$

The function f is a barrier for a fold L' of the locus L at the point z. We have

$$\mathrm{d}f = \mathrm{d}_x\varphi = \mathrm{d}x_0 + O\big(|\omega(x)|\big).$$

This implies that $g(\mathrm{d}f) \approx 1$. Taking derivatives of the identity $\varphi'_\omega(x,\omega(x)) = 0$ we get the equations

$$\varphi''_{x\omega} = -\varphi''_{\omega\omega}\,\omega'_x, \tag{8.3}$$

$$\sum_j \varphi''_{\omega_i\omega_j} v_q(\omega_j) = -v_q(\varphi'_{\omega_i}), \quad i,q = 1,\dots,k \tag{8.4}$$

where $\varphi''_{\omega\omega}(x,\omega(x)) \to \varphi''_{\omega\omega}(y,0) = 0$ as $x \to y$. By (8.3) we obtain

$$f'' = \varphi''_{xx} + 2\varphi''_{x\omega}\omega'_x + {}^t(\omega'_x)\varphi''_{\omega\omega}\omega'_x = -{}^t(\omega'_x)\varphi''_{\omega\omega}\omega'_x + \varphi''_{xx}. \tag{8.5}$$

Take arbitrary smooth tangent fields $v_1,\dots,v_k$ in Y such that $v_i(f) = 0$, $i = 1,\dots,k$ and apply the equation (8.5) to these fields:

$$F_v \doteq \left\{\mathrm{d}^2 f(v_i,v_j)\right\}_1^k = -{}^t v(\omega)\,\varphi''_{\omega\omega}\,v(\omega) + \mathrm{d}_x^2\varphi(v_i,v_j),\ i,j = 1,\dots,k. \tag{8.6}$$

The notation $v(\omega)$ stands here for the matrix $\{v_i(\omega_j(x))\}_1^k$. The left-hand side is equal to the restriction of the curvature form of L to the subspace V in $T_x(L)$ spanned by the fields $v_1,\dots,v_k$. The second term in the right-hand side is smooth in $C(\varphi)$ and hence is bounded. Calculate the first term. By (8.4) we have $v(\omega) = -(\varphi''_{\omega\omega})^{-1}v(\varphi'_\omega)$ where $v(\varphi'_\omega) \doteq \{v_j(\varphi'_{\omega_i})\}$. Substitute this equation to (8.6):

$$F_v = -{}^t v(\varphi'_\omega)(\varphi''_{\omega\omega})^{-1}v(\varphi'_\omega) + O(1)$$

and find $\det F_v \approx (\det v(\varphi'_\omega))^2(\det\varphi''_{\omega\omega})^{-1}$. We specify the fields $v_i = t_i$ to maximize the determinant of F_v:

$$t_i \doteq \frac{\partial}{\partial x_i} - \frac{\varphi'_{x_i}(x,\omega(x))}{\varphi'_{x_0}(x,\omega(x))}\frac{\partial}{\partial x_0}, \quad i = 1,\dots,n$$

where the choice of coordinates is subjected to (8.1).

These fields are independent continuous and tangent to L' since $t_i(f) = 0$. Moreover $t_i \to \partial/\partial x_i$ as $x \to y$. Really we have $\varphi'_{x_0} = 1$, $\varphi'_{x_i} = \omega_i \to 0$ according to (8.1). Consider the vectors $t_i(\varphi'_\omega)$, $i = 1,\dots,n$. They span the space $T_x(L)$ and $g(t_i,t_j) \to \delta_{ij}$ as $x \to y$. By (10.1) we find

$$t_i(\varphi'_{\omega_j}) = \delta_{ij} + o(1), \quad i = 1,\dots,n,\ j = 1,\dots,k, \tag{8.7}$$

hence the first k vectors are independent at the point $(y,0)$, while the last $n-k$ vectors vanish at this point. Therefore the function $|\det v(\varphi'_\omega)|$ is equal to $1 + o(1)$.

This is the maximal value of this function up to the factor $1 + o(1)$. Therefore

$$(8.8) \qquad Q_k(x) \approx \left|\det F_v(x)\right| \approx |\det \varphi''_{\omega\omega}|^{-1}$$

for this choice of the fields v_i.

Calculate the contact symbol of U by means of (2.9):

$$\left|\sigma_c(U)\right|^2 = |\theta| \left|\frac{|A_m|^2 \wedge (\mathrm{dd}^0\theta\theta^0)^{\wedge(k+1)}}{\mathrm{d}|\theta| \wedge \mathrm{dd}^0(-\imath\Phi)^{\wedge(k+1)}}\right| C(\varphi)$$

where $\Phi(x,\theta,\theta^0) = \Phi(x,\theta+\theta^0)$ and $\phi(x,\theta) \doteq |\theta|\varphi(x,|\theta|^{-1}\theta_1,\dots,|\theta|^{-1}\theta_k)$ is a generating function for a neighbourhood of the point λ in Λ. We choose ancillary coordinates $\theta = (\theta_0,\theta_1,\dots,\theta_k)$ so that $\tilde{\phi}(y;1,0,\dots,0) = \lambda$. Write $A_m = a\sqrt{\mathrm{d}x}$ where

$$\mathrm{d}x = \mathrm{d}x_0 \wedge \mathrm{d}x' \wedge \mathrm{d}\hat{x}, \quad \mathrm{d}x' = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_k, \quad \mathrm{d}\hat{x} \doteq \mathrm{d}x_{k+1} \wedge \ldots \wedge \mathrm{d}x_n.$$

The form $(\mathrm{d}d^0\Phi)^{\wedge(k+1)}$ factorizes in Y through the product

$$\det(\Phi'_{x_0x'})'_{\theta^0}\, \mathrm{d}x_0 \wedge \mathrm{d}x' \wedge \mathrm{d}\theta^0_0 \wedge \ldots \wedge \mathrm{d}\theta^0_k.$$

We have $(\Phi'_{x_0x'})'_{\theta^0} = (\Phi'_{x_0x'})'_\theta$ when $\theta^0 = 0$. At the other hand

$$\det(\Phi'_{x_0,x})'_\theta = \varphi'_{x_0}\, \det \varphi''_{x',\omega}.$$

The right-hand side is equal to $1 + o(1)$ according to (8.7). Therefore

$$\frac{\mathrm{d}x \wedge (\mathrm{dd}^0\theta\theta^0)^{\wedge(k+1)}}{\mathrm{d}|\theta| \wedge (-\imath\,\mathrm{dd}^0\Phi)^{\wedge(k+1)}} \approx \mathrm{d}\hat{x} \wedge \mathrm{d}\theta_1 \wedge \ldots \wedge \mathrm{d}\theta_k \approx |\theta|^k \mathrm{d}\hat{x} \wedge \mathrm{d}\omega,$$

as $\omega = (|\theta|^{-1}\theta_1,\dots,|\theta|^{-1}\theta_k) \to 0$, consequently

$$\left|\sigma_c(U)\right|^2 \approx |a|^2 \cdot |\theta|^{(k+1)} |\,\mathrm{d}\hat{x} \wedge \mathrm{d}\omega|.$$

The factor $|a(x,\theta)|^2 \cdot |\theta|^{(k+1)}$ is a homogeneous amplitude of order zero. It is equal to the pull back of b under the projection p_k.

At the other hand the product $(\mathrm{dd}^0\Phi)^{\wedge(k+1)}$ contains the term $\mathrm{d}_{x_0}\,\mathrm{d}_{\theta^0_0}\Phi \wedge (\mathrm{d}_\theta \mathrm{d}_{\theta^0}\Phi)^{\wedge k}$. For the first factor we have the equation

$$\mathrm{d}_{x_0}\mathrm{d}_{\theta^0_0}\Phi \approx \mathrm{d}x_0 \wedge \mathrm{d}\theta^0_0$$

because of (10.1). Therefore

$$\mathrm{d}_{x_0}\mathrm{d}_{\theta_0^0}\Phi \wedge (\mathrm{d}_\theta \mathrm{d}_{\theta^0}\Phi)^{\wedge k} \approx \det\{\phi''_{\theta_i\theta_j}\}_1^k \mathrm{d}x_0 \wedge \mathrm{d}\theta_1 \wedge \ldots \wedge \mathrm{d}\theta_k \wedge \mathrm{d}\theta_0^0 \wedge \ldots \wedge \mathrm{d}\theta_k^0.$$

We have $\{\phi''_{\theta_i\theta_j}\}_1^k \approx |\theta|^{-1}\varphi''_{\omega\omega}$ hence $\det \phi''_{\theta\theta} \approx |\theta|^{-k} \det \varphi''_{\omega\omega}$. In this way we obtain

$$\left|\sigma(U)\right|^2 \approx \frac{|a|^2 \cdot |\theta|^k \mathrm{d}\theta_0}{|\det \varphi''_{\omega\omega}|} \frac{\mathrm{d}x}{\mathrm{d}x_0}.$$

Applying (8.8) we get

$$\left|\sigma_c(U)\right|^2 \approx \frac{|\theta|}{\mathrm{d}|\theta|} \left|\sigma(U)\right|^2 \approx |a|^2 \cdot |\theta|^{k+1} Q_k \frac{\mathrm{d}x}{\mathrm{d}x_0} = bQ_k \frac{\mathrm{d}x}{\mathrm{d}x_0}$$

since $\mathrm{d}|\theta| \approx \mathrm{d}\theta_0$. □

Remark. — The above theorem can be applied to Λ-distribution U of arbitrary order ν of singularity. Take an arbitrary pseudo-differential operator P of order $-\nu$ in X. It is easy to check that the PU is a Λ-distribution of order 0. Whence the its local structure of PU can be described by (8.2).

9. Examples.

Take an arbitrary point λ of a Lagrange manifold Λ_c where the defect is equal to 1 and find out a barrier f at this point by the method of Section 4. The barrier is equal to $f(x) = \delta(\zeta(x))$ where δ is the discriminant of the generating function Ψ and ζ is a smooth mapping $\zeta(p(\lambda)) = 0$. The generating function can be given by the following simple formula:

$$\Psi(s, \omega) = \omega^{\tau+1} + s_{\tau-1}\omega^{\tau-1} + \cdots + s_1\omega + s_0$$

where $\tau = \tau(\lambda) < \infty$. Consider the simplest cases:

Case $\tau = 1$. — We have $\delta_1 = 4s_0$ and the discriminant set is the origin. The pullback $\zeta^{-1}(0)$ is the regular part of the locus L.

Case $\tau = 2$. — We have $\delta_2 = 27s_0^2 + 4s_1^3$; the discriminant curve $\delta_2(s_0, s_1) = 0$ has the cuspidal point at the origin. Calculating the curvature near the cusp we obtain

$$Q_1(s_0, s_1) \asymp (-s_1)^{-1/2}$$

where $a \asymp b$ means that both quotients $a/b, b/a$ are bounded.

Case $\tau = 3$. — The discriminant set Δ is given by the equation

$$\delta_3 \doteq 256s_0^3 - 128s_0^2s_2^2 + 16s_0s_2^4 + 144s_0s_1^2s_2 - 27s_1^4 - 4s_1^2s_2^3 = 0$$

or in the parametric form

$$s_0 = 3v^4 - uv^2, \quad s_1 = 8v^3 - 2uv, \quad s_2 = 6v^2 - u, \quad u, v \in \mathbb{R}.$$

It is shown in Figure 1. The locus Δ has two folds in any point of the curve M given by the equations $4s_0 = s_2^2$, $s_1 = 0$, $s_2 < 0$. For any point $s \in M$ there are two points of the manifold Λ_c of local multiplicity $\tau = 1$ whose projection is equal to s. The continuation of M to the half-space $s_2 > 0$ is the half-parabola denoted I. For each point $s \in I$ there are two complex conjugated points of $C(\Psi)$ over s. The tangent plane $\operatorname{Ker} ds_0$ in the origin belongs to $\Lambda_c(\Psi)$ and the s_2-axis is the singular tangent to the locus Δ. The cusp curve $C \doteq L_s \setminus M$ can be given in the parametric form

$$s_0 = 3v^4, \quad s_1 = 8v^3, \quad s_2 = 6v^2, \quad v \in \mathbb{R}.$$

We have $\tau(s) = 2$ for any point $s \in C$.

According to Section 8 the function Q_1 is equal to the curvature of the intersection of Δ with the plane $s_2 = \text{Const}$. To estimate the quantity Q_1 we note that $u^2 = s_2^2 - 12s_0$ and the parameter u vanishes in C. We take u and s_2 as parameters in the piece $s_2 < -\varepsilon|s|$ of the discriminant surface and find

$$Q_1(s_0, s_1, s_2) \asymp \partial^2 s_0/\partial s_1^2 \asymp |u\, s_2|^{-1/2}.$$

This estimate is uniform for any $\varepsilon > 0$. In the opposite piece $s_2 > \varepsilon|s|$ the estimate $Q_1 \asymp s_2^{-1}$ holds.

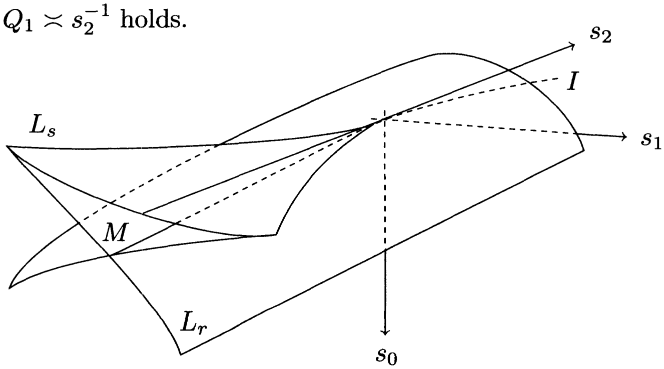

Figure 1

10. Generating functions and multiplicities.

Let X be a smooth manifold, $\Omega \subset \mathbb{R}^k$ be an open set and $\varphi : X \times \Omega \to \mathbb{R}$ be a smooth function such that the forms $\mathrm{d}\varphi$, $\mathrm{d}\varphi'_{\omega_1}, \ldots, \mathrm{d}\varphi'_{\omega_k}$ are independent at any point of the set

$$C(\varphi) = \{(x,\omega) \in X \times \Omega\,;\ \varphi(x,\omega) = 0,\ \varphi'_\omega(x,\omega) = 0\}.$$

This condition implies that $C(\varphi)$ is a manifold of dimension $\dim X - 1$. Consider the mapping

$$\widetilde{\varphi} : C(\varphi) \longrightarrow C^*(X), \quad (x,\omega) \longmapsto \big(x, \operatorname{Ker} d_x\varphi(x,\omega)\big).$$

The differential of the mapping $\widetilde{\varphi}$ is injective at each point. Denote by $\Lambda_c(\varphi)$ the image of this mapping. This is a contact Lagrange variety, possibly, with self-intersections. We say that the *phase* function φ generates the contact Lagrange manifold $\Lambda_c(\varphi)$.

Given a phase function $\phi = \phi(x,\theta)$ in $Y \times \Theta$ that generates a conic Lagrange manifold Λ, the phase function $\phi(x,\omega)$, $\omega \in \Omega$ generates the contact manifold Λ_c where Ω is the intersection of the unit sphere in the ancillary space with the cone Θ. Vice versa, let $\varphi = \varphi(x,\omega)$ be a phase function for the contact Lagrange manifold Λ_c. Then the function $\phi(x,\theta) \doteq \theta_0\varphi(x, \theta_0^{-1}\theta_1, \ldots, \theta_0^{-1}\theta_k)$ defined in $Y \times \Theta$ generates the Lagrange manifold Λ where Θ is the cone in $\mathbb{R}^{k+1}$ spanned by the set $\{(1,\omega);\ \omega \in \Omega\}$.

Take an arbitrary point $\lambda \in \Lambda_c$; let k be defect at this point, i.e., the dimension of the kernel of $\mathrm{d}p_\lambda : T_\lambda(\Lambda_c) \to T_y(X)$, $y = p(\lambda)$. Choose a local coordinate system $x_0, \ldots, x_n$ centered at the point $y \doteq p(\lambda)$ such that the condition (8.1) is satisfied. Recall some known facts:

Proposition 10.1. — *The coordinate projection*

$$p_k : \Lambda_c \longrightarrow \mathbb{R}^{n-k} \times \mathbb{R}^k, \quad (x,\omega) \longmapsto (x_{k+1}, \ldots, x_n; \omega_1, \ldots, \omega_k), \quad \lambda \longmapsto (0,0)$$

is a local coordinate system in Λ_c.

The manifold Λ_c satisfies the equations $x_j = x_j(\hat{x},\omega)$ in this coordinate system where $x_j(\hat{x},\omega)$, $j = 0, \ldots, k$ are smooth functions in an open set $W \subset \mathbb{R}^{n-k} \times \mathbb{R}^k$.

PROPOSITION 10.2. — *The function*

$$(10.1)\quad \varphi(x,\omega) \doteq x_0 - x_0(\hat{x},\omega) + \sum_1^k \omega_j(x_j - x_j(\hat{x},\omega)), \quad \hat{x} = (x_{k+1},\dots,x_n)$$

defined in $\mathbb{R}^{k+1}\times W$ *generates the germ of* Λ_c *at* λ *and fulfils the condition* $\varphi''_{\omega\omega}(y,0) = 0$.

Proof. — The form $\alpha = dx_0 + \sum_1^k \omega_i\, dx_i + \sum_{k+1}^n v_j\, dx_j$ defines the canonical contact structure of the manifold $C^*(X)$ in a neighbourhood of λ consequently it vanishes in Λ_c, that is,

$$\frac{\partial x_0}{\partial \omega_i} + \sum_j \omega_j \frac{\partial x_j}{\partial \omega_i} = 0, \qquad i = 1,\dots,k,$$

$$\frac{\partial x_0}{\partial x_q} + \sum_j \omega_j \frac{\partial x_j}{\partial x_q} + v_q = 0, \quad q = k+1,\dots,n.$$

By means of these equations it is easy to check that the function $\varphi(x,\omega)$ generates the germ of Λ_c. Calculate the second derivatives:

$$\varphi''_{\omega\omega}(y,0) = -\Big\{\frac{\partial x_i(0,0)}{\partial \omega_j}\Big\}_1^k.$$

These derivatives are equal to zero since the forms $dx_1,\dots, dx_k$ vanish in $T_\lambda(\Lambda_c)$. □

Mappings of finite multiplicity. — Let $f : X \to Y$ be a mapping of smooth manifolds. The *multiplicity* of the mapping f in a point $x_0 \in X$ is the number

$$(10.2)\qquad m(x_0) \doteq \dim_{\mathbb{R}} \mathcal{O}_{x_0}(X)/f^*(\mathfrak{m}(y_0))\mathcal{O}_{x_0}(X), \quad y_0 = f(x_0)$$

where $\mathcal{O}(X)$ stands for the sheaf of smooth functions in X and $\mathfrak{m}(y)$ means the maximal ideal of y_0 in the algebra $\mathcal{O}_{y_0}(Y)$. In particular, the multiplicity in a point x_0 is equal to 1 if and only if the differential $df(x_0)$ is injective. The *multiplicity of* f *over a point* $y_0 \in Y$ is by definition the sum

$$m(y_0) \doteq \sum \{m(x),\ f(x) = y_0\}.$$

PROPOSITION 10.3. — *For an arbitrary proper mapping* $f: X \to Y$ *of smooth manifolds the set of points* $y \in Y$ *where* f *is of finite multiplicity is open. The multiplicity function* $m(y)$ *is upper semi-continuous.*

It can proved by means of arguments of Theorem 4.1.

PROPOSITION 10.4. — *Let $f : X \to Y$ be a proper mapping of smooth manifolds of equal dimensions that has finite multiplicity over each point of Y and $Y_0 \subset Y$ be a set of measure zero. The pullback $X_0 \doteq f^{-1}(Y_0)$ is of measure zero too.*

Proof. — Let $C(f) \subset X$ be the set of critical points of f. The function $y \mapsto \operatorname{card} f^{-1}(y)$ is locally bounded according to Proposition 10.3. The mapping f is a local diffeomorphism in $X \setminus C(f)$. Therefore the set $X_0 \setminus C(f)$ is of measure zero. Now we show that the set $C(f)$ is also of measure zero. It is sufficient to check this statement locally. Take a point $x_0 \in X$ and a coordinate system $x_1, \ldots, x_n$ centered at x_0. Let $y_1, \ldots, y_n$ be a local coordinate system in Y centered at $y_0 = f(x_0)$ and $y_j = y_j(x)$ are local equations of the mapping f. The set $C(f)$ is given by the equation $j(x) = 0$ where $j \doteq \det\{\partial y_i / \partial x_j\}$ is the Jacobian of f. We show that the function j is not flat at x_0. This will imply the conclusion $\operatorname{mes} C(f) = 0$. Suppose the opposite, that is, $j \in \cap \mathfrak{m}^k(x_0)$. Consider the ideal I generated by the subspace $f^*(\mathfrak{m}(y_0))$ in the algebra $\mathcal{O}_{x_0}(X)$. Its codimension is equal by definition to the local multiplicity $m(x_0)$.

The codimension is finite since of inequality $m(x_0) \leq m(y_0) < \infty$. By Nakayama's lemma I contains the ideal $\mathfrak{m}(x_0)^k$ for some number k. At the other hand the germ of j belongs to this ideal. Let $Y_1, \ldots, Y_n$ be some polynomials of the coordinates $x_1, \ldots, x_n$ such that $Y_j - y_j \in \mathfrak{m}(x_0)^{k+1}$ for $j = 1, \ldots, n$. The ideal $I(Y)$ generated by these polynomials is contained in I and we have $I \subset I(Y) + m(x_0)I$. By Nakayama's lemma these ideals coincide. Take the Jacobian $J \doteq \det \partial Y / \partial x$. We have $J - j \in \mathfrak{m}(x_0)^k$ consequently J belongs to the ideal $I = I(Y)$.

At the other hand we have

$$\operatorname{Res}\begin{bmatrix} J \\ Y_1 \ldots Y_n \end{bmatrix} = m(y_0)$$

according to the property of the Cauchy-Poincaré residue (see for ex. [4], Ch. III). This implies that the function J does not belong to this ideal. This contradiction completes the proof.

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