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SHEAVES ASSOCIATED TO HOLOMORPHIC FIRST INTEGRALS

by Alexis GARCÍA ZAMORA

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0. Introduction.

Through this paper S will be a projective, smooth and irreducible surface over $\mathbb{C}$. A *holomorphic foliation* over S is an element $\mathcal{F} \in \mathbb{P}H^0(S, \Theta_S \otimes \mathcal{L}^{-1})$, where $\mathcal{L}$ is an invertible sheaf on S and Θ_S is the tangent sheaf. Thus, to any foliation there is associated via the natural isomorphism $H^0(S, \Theta_S \otimes \mathcal{L}^{-1}) \cong \mathrm{Hom}(\mathcal{L}, \Theta_S)$, a morphism $\mathcal{L} \to \Theta_S$ defined up to multiplication by a non-zero complex number, in the sequel this morphism will be denoted by $\mathcal{F}$.

Thus, $\mathcal{F}$ is defined in a trivializing open set U_α by a vector field

$$X_\alpha = A_\alpha \frac{\partial}{\partial z_{\alpha,1}} + B_\alpha \frac{\partial}{\partial z_{\alpha,2}},$$

defined up to multiplication by a scalar.

Associated to $\mathcal{F}$ there exists a dual map $\omega_{\mathcal{F}} : \mathcal{L}' \to (\Theta_S)^*$, which is given in local coordinates by the 1-holomorphic form dual to the previous vector field.

The singular set of $\mathcal{F}$ is defined as the set of points where the associated map of vector bundles $\mathcal{F} : L \to TS$ is not injective. This is equivalent of saying that $\mathrm{Sing}\,\mathcal{F}$ is the set of points where $A_\alpha = B_\alpha = 0$. It is

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more natural to speak about the singular subvariety of $\mathcal{F}$ which is defined as the subvariety of S associated to the ideal sheaf $\mathcal{I}_{\mathcal{F}}$ generated by $\{A_\alpha, B_\alpha\}$. In this paper we work with foliations having isolated singularities, i.e, we assume that the subvariety $\operatorname{Sing}\mathcal{F}$ is supported on a finite set of points.

Assume that $\mathcal{F}$ admits a holomorphic first integral, that is: there exists a holomorphic map $f : S \to \mathbb{P}^1$ such that any irreducible component of the fibers of f is a solution of $\mathcal{F}$. We assume, in addition that the generic fiber of f is irreducible.

Recall that an irreducible analytical curve $C \subset S$ is a *solution* of $\mathcal{F}$ if for each $x \in C_0 = C - \operatorname{Sing}(\mathcal{F})$, we have $\mathcal{F}(\mathcal{L}_x) = T_x C_0$, where $\mathcal{L}_x = (\mathcal{L} \otimes \mathcal{O}_{S,x})/m_{S,x}$, is the vector fiber of $\mathcal{L}$ at x. In others words, C is a solution of $\mathcal{F}$ if the restricted map $\mathcal{F}_C : \mathcal{L}_C \to \Theta_{S,C}$ factorizes through the natural inclusion $\Theta_C \hookrightarrow \Theta_S|_C$.

In [7] Poincaré studied the following problem: assume that $\mathcal{F}$ is a foliation on $S = \mathbb{P}^2$ admitting a meromorphic first integral, that is, there exists a *rational* map $f : \mathbb{P}^2 \to \mathbb{P}^1$ (therefore defined away from a finite number of points), such that the irreducible components of the fibers of f are leaves of $\mathcal{F}$, away from the indeterminancy locus of f. Then, find an upper bound for the degree of the general fiber of f, depending of some information given by $\mathcal{F}$. (For a more precise statement of the problem see [7] and [9].)

Poincaré ([7] or [9], remark following Example 2.3) noticed that for foliations of degree $m \geqslant 4$ in $\mathbb{P}^2$, the previous problem can be solved finding a bound for the geometric genus of the general fiber of f. Thus, in a more general setting, its seems reasonable to study the following problem: given a foliation $\mathcal{F}$ on an algebraic surface S, admitting a holomorphic first integral, find a bound for the genus of the general fiber of f in terms of information depending of $\mathcal{F}$.

This is the problem that we study in the present paper. In particular we prove the following result (Theorem 2.2):

Assume that for some integer $n \geqslant 1$, $h^0(S, \mathcal{O}_S(-nK_S)) > 0$. If $\mathcal{F}$ is a foliation on S admitting a holomorphic first integral, then

$$(2n-1)(g-1) \leqslant h^1(S, {\mathcal{L}'}^{-1}(-(n-1)K_S)) + h^0(S, \mathcal{L}') + 1.$$

The class of surfaces satisfying the previous hypothesis includes all the 0-dimensional Kodaira surfaces and the Del Pezzo surfaces. The bound obtained in this case depends on the dimension of the cohomology groups of the sheaves defining $\mathcal{F}$, so this gives a solution to a question analogous to the Poincaré problem for above classes of surfaces.

In Section 3 we study the cohomology of $\mathcal{L}'^{-1}$ in the case where S is a rational surface. It is a case of interest because, given a foliation $\mathcal{F}$ in $\mathbb{P}^2$ admitting a meromorphic first integral f, it is possible, after a finite number of blowings-up, to obtain a rational surface S, and a foliation $\mathcal{F}'$ on S admitting a holomorphic first integral f' whose fibers are birationally equivalent to the fibers of f (see [8]). Moreover, the cohomology of the sheaves defining $\mathcal{F}'$ can be computed in terms of the cohomology of the sheaves defining $\mathcal{F}$ and information on the blowings-up that gives rise to S.

Our results on Section 3 are based on the study of the direct image of $\mathcal{L}'^{-1}$ under f. We prove that the two direct-image sheaves are locally free and we give some partial information on the nature of the decomposition of these sheaves as the direct sum of invertible sheaves.

Almost all the proofs in Sections 2 and 3 are based on some relationships between $\mathcal{F}$, $\mathcal{L}'^{-1}$ and the canonical map *df* obtained in Section 1 (Proposition 1.1, Corollary 1.2 and Theorem 1.3). Part a) of Proposition 1.1 is probably well known, we have taken the formulation and proof from ([3]).

1. Foliations and first integrals.

PROPOSITION 1.1. — a) *Let $\mathcal{F} : \mathcal{L} \to \Theta_S$ be a foliation on S. Then there exists an exact sequence:*

$$0 \to \mathcal{L} \xrightarrow{\mathcal{F}} \Theta_S \xrightarrow{\omega_{\mathcal{F}}} \mathcal{L}'^{-1} \longrightarrow \mathcal{O}_{\mathrm{Sing}\,\mathcal{F}} \longrightarrow 0.$$

b) *If $f : S \to \mathbb{P}^1$ is a first integral of $\mathcal{F}$, then we have the following exact sequence:*

$$0 \longrightarrow \mathcal{L} \xrightarrow{\mathcal{F}} \Theta_S \xrightarrow{df} \mathcal{O}_S(2C) \longrightarrow \mathcal{I} \longrightarrow 0,$$

or, in other words, $\mathcal{L}$ is the kernel of the natural map $df : \Theta_S \to f^*\Theta_{\mathbb{P}^1}$. Here C denotes a generic fiber of f.

Proof. — a) Let $\{(U_\alpha, z_{\alpha,1}, z_{\alpha,2})\}$ be a covering of S by coordinates charts, with transition functions $\phi_{\alpha,\beta}$. Assume that $\mathcal{L}'$ is trivial on U_α. Assume that in this choice of coordinates the map $\omega_{\mathcal{F}}$ is described locally by means of

$$\omega_\alpha = A_\alpha(z_{\alpha,1}, z_{\alpha,2})dz_{\alpha,1} + B_\alpha(z_{\alpha,1}, z_{\alpha,2})dz_{\alpha,2}.$$

The local vector field

$$X_\alpha = B_\alpha \frac{\partial}{\partial z_{\alpha,1}} - A_\alpha \frac{\partial}{\partial z_{\alpha,2}},$$

is a local generator of the kernel of $\omega_{\mathcal{F}}$ and has isolated singularities. Thus, if Y is any vector field such that $\omega_\alpha(Y) = 0$, then on $U_\alpha - \operatorname{Sing}\mathcal{F}$ we may find a holomorphic function h such that $Y = hX_\alpha$, this function h extends to an element of $\mathcal{O}_{U_\alpha}$, by Hartog's Theorem ([4]), since $\operatorname{Sing}\mathcal{F}$ has discrete support. So, the kernel of $\omega_{\mathcal{F}} : \Theta_S \to {\mathcal{L}'}^{-1}$ is an invertible sheaf generated by X. On the other hand the image of $\omega_{\mathcal{F}}$ is the ideal generated by A_α and B_α, this proves the assertion, since $\operatorname{Sing}\mathcal{F}$ is the 0-dimensional subvariety defined by A_α and B_α.

b) Consider the canonical map

$$df : \Theta_S \to f^*\Theta_X,$$

call its kernel $\Theta_{S,f}$. The map df is locally defined by means of

$$df\left(X_{\alpha,1}\frac{\partial}{\partial z_{\alpha,1}} + X_{\alpha,2}\frac{\partial}{\partial z_{\alpha,2}}\right) = X_{\alpha,1}\frac{\partial f}{\partial z_{\alpha,1}} + X_{\alpha,2}\frac{\partial f}{\partial z_{\alpha,2}}.$$

Thus, we see that the kernel of df is the subsheaf of Θ_S generated at each local ring $\mathcal{O}_p$ by the local vector field

$$\frac{1}{h_p}\left(\frac{\partial f}{\partial z_{\alpha,2}}\frac{\partial}{\partial z_{\alpha,1}} - \frac{\partial f}{\partial z_{\alpha,1}}\frac{\partial}{\partial z_{\alpha,2}}\right)_p,$$

where h_p is the greatest common divisor of $\left(\frac{\partial f}{\partial z_{\alpha,1}}\right)_p$ and $\left(\frac{\partial f}{\partial z_{\alpha,2}}\right)_p$. This proves that $\Theta_{S/f}$ is an invertible sheaf. Moreover, the local generator of $\Theta_{S,f}$ has isolated zeros.

Now, using the fact that f is a first integral of $\mathcal{F}$ it is easy to see that there exists a map of sheaves $\mathcal{L} \to \Theta_{S,f}$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & \Theta_{S,f} \arrow[r] & \Theta_S, \\ & \mathcal{L} \arrow[u] \arrow[ur, "\mathcal{F}"'] & \end{tikzcd}$$

commutes. Indeed, we first claim that if $f : S \to X$ is a first integral of $\mathcal{F}$ the composite map

$$\mathcal{L} \xrightarrow{\mathcal{F}} \Theta_S \xrightarrow{df} f^*\Theta_X,$$

is the zero map. For this it is sufficient to check that the assertion is valid in the vector stalk at each point $x \in S$. Let x be a nonsingular point of $\mathcal{F}$, then there exists a unique solution C that passes through x, and we have

$\mathcal{F}(\mathcal{L}_x) = T_x C$. But $df(T_x C) = 0$, since C is an irreducible component of a fiber of f. On the other hand, if x is a singular point of $\mathcal{F}$ then $\mathcal{F}(\mathcal{L}_x) = 0$ and the claim is trivial. As $\Theta_{S,f}$ is the kernel of df, we obtain the desired morphism.

Now, we need to prove that this map is surjective. This map is surjective on $S - \operatorname{Sing}\mathcal{F}$, because it is surjective on the vector stalks at these points. So if $X_{\mathcal{F}}$ denotes the local expression of a generator of $\mathcal{L} \subset \Theta_S$ and X_α is a local generator $\Theta_{S,f}$ then we must have a relation $hX_{\mathcal{F}} = X_\alpha$ on $U_\alpha - \operatorname{Sing}\mathcal{F}$, where h is a holomorphic function, now, both $X_{\mathcal{F}}$ and X_α has only a finite number of zeros, and thus h must be a nowhere vanishing function on U_α (Hartog's Theorem). This means that h is an unity in $\mathcal{O}(U_\alpha)$ and so $\mathcal{L} \simeq \Theta_{S,f}$. □

An easy corollary of part a) of the proposition is the following well known fact:

COROLLARY 1.2. — $\mathcal{L}' \otimes \mathcal{L}^{-1} = \mathcal{K}_S$.

Proof. — Outside the support of $\mathcal{O}_{\operatorname{Sing}\mathcal{F}}$ we have ${\mathcal{K}_S}^{-1} \simeq \det \Theta_S \simeq \mathcal{L} \otimes {\mathcal{L}'}^{-1}$. The lemma follows as an application of Hartog's Theorem. □

For a slightly different proof of this fact and other general properties of the sheaves $\mathcal{L}$ and ${\mathcal{L}'}^{-1}$ see [2]. The reader problably will find the notation used in that reference more appropiate, here I have followed the notation in [3], the primary source where I learned the subject.

Assuming that $\mathcal{F}$ admits a first integral, call $\sum_i^k \sum_j^{k_i} n_{ij} F_{ij}$ the divisor given by the preimage by f of the critical values of f, (here i runs on the set of critical values and k_i is the number of irreducible components in the fiber corresponding to i).

THEOREM 1.3. — *Let $\mathcal{F}$ be a foliation admitting a holomorphic first integral f with generic fiber C, using the previous notation we have*

$$ {\mathcal{L}'}^{-1} \simeq \mathcal{O}_S\Big(2C - \sum_{ij}(n_{ij} - 1)F_{ij}\Big). $$

Proof. — Let us denote that $J = \Theta_S/\mathcal{L}$. Then we have an isomorphism $J \simeq {\mathcal{L}'}^{-1}$ away from $\operatorname{Sing}\mathcal{F}$.

Thus, the inclusion

$$ 0 \longrightarrow J \longrightarrow \mathcal{O}_S(2C), $$

allows us to define a morphism $\mathcal{L}'^{-1} \to \mathcal{O}_S(2C)$, outside $\operatorname{Sing}\mathcal{F}$, which is obviously injective. Now, we take a closer look at the structure of $\mathcal{I}$.

$\mathcal{I}$ is the cokernel of the map *df*. Note that the image of *df* is locally generated, away from $\operatorname{Sing}\mathcal{F}$, by the divisor $\sum_{ij}(n_{ij}-1)F_{ij}$. We conclude that, away from $\operatorname{Sing}\mathcal{F}$ we have an exact sequence:

$$0 \longrightarrow \mathcal{L}'^{-1} \longrightarrow \mathcal{O}_S(2C) \longrightarrow \mathcal{O}_{\Sigma_{ij}(n_{ij}-1)F_{ij}} \longrightarrow 0.$$

$\mathcal{L}'^{-1}$ and $\mathcal{O}_S(2C)$ being invertible and $\operatorname{Sing}\mathcal{F}$ discrete this exact sequence extends to one on S. Now, we use the fact that

$$\mathcal{O}_S(2C)_{\mathcal{O}_{\Sigma_{ij}(n_{ij}-1)F_{ij}}} \simeq \mathcal{O}_{\Sigma_{ij}(n_{ij}-1)F_{ij}},$$

([1], page 90, Lemma 8.1) to conclude that

$$\mathcal{L}'^{-1}(-2C) \simeq \mathcal{O}_S(-\sum_{ij}(n_{ij}-1)F_{ij}). \qquad \square$$

Note that this isomorphism generalizes relationships α, page 39 and δ, page 41 in [7].

2. Bounding g in some particular cases.

In this section we study the cohomology of the sheaf $\mathcal{L}'^{-1}$ and other related sheaves. As will be clear after the following discussion a good understanding of these cohomology groups can be helpful for the study of the Poincaré Problem.

We start with the following observation:

PROPOSITION 2.1. — *Let $f: S \to \mathbb{P}^1$ be a holomorphic first integral of the foliation $\mathcal{F}: \mathcal{L} \to \Theta_S$. If g denotes the genus of the general fiber C of f, then*

$$g \leqslant h^1(S, \mathcal{L}'^{-1}) + h^2(S, \mathcal{L}'^{-1}(-C)).$$

Proof. — Consider the standard exact sequence

$$0 \longrightarrow \mathcal{O}_S(-C) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_C \longrightarrow 0. \tag{2.1}$$

Its follows almost immediatly from the definition of the solutions of $\mathcal{F}$ that $\mathcal{L} \otimes \mathcal{O}_C \simeq \Theta_C$. Thus, from Corollary 1.2 we obtain: $\mathcal{L}'^{-1} \otimes \mathcal{O}_C \simeq \mathcal{O}_C$. If we apply the functor $\otimes\mathcal{L}'^{-1}$ to the exact sequence 2.1, we obtain

$$0 \longrightarrow \mathcal{L}'^{-1}(-C) \longrightarrow \mathcal{L}'^{-1} \longrightarrow \mathcal{O}_C \longrightarrow 0,$$

the result follows at once after considering the long exact sequence in cohomology associated to the previous one. □

A similar argument allows us to solve the analogous to the Poincaré problem on some kinds of surfaces:

THEOREM 2.2. — *Under the same hypothesis of Proposition 2.1, assume, moreover, that $h^0(S, \mathcal{O}_S(-nK_S)) > 0$, for some positive integer n. Then*

$$(2n-1)(g-1) \leqslant h^1(S, \mathcal{L}'^{-1}(-(n-1)K_S)) + h^0(S, \mathcal{L}') + 1.$$

Proof. — First of all, we observe that , C being a fiber of f, we have $\mathcal{O}_S(C) \otimes \mathcal{O}_C \simeq \mathcal{O}_C$ ([1], Lemma 8.1). Thus, applying the adjunction formula we obtain $\mathcal{O}_S(K_S) \otimes \mathcal{O}_C \simeq \mathcal{O}_C(K_C)$. Furthemore we obtain an exact sequence:
(2.2)

$$0 \longrightarrow \mathcal{L}'^{-1}(-(n-1)K_S - C) \longrightarrow \mathcal{L}'^{-1}(-(n-1)K_S) \longrightarrow {\Theta_C}^{\otimes(n-1)} \longrightarrow 0.$$

Now, it follows from the Riemann-Roch formula ([5], page 108) that

$$h^1(S, {\Theta_C}^{\otimes(n-1)}) = (2n-1)(g-1).$$

Then we obtain, from 2.2, the following inequality:

$$(2n-1)(g-1) \leqslant h^1(S, \mathcal{L}'^{-1}(-(n-1)K_S)) + h^2(S, \mathcal{L}'^{-1}(-(n-1)K_S - C)).$$

But $h^2(S, \mathcal{L}'^{-1}(-(n-1)K_S - C)) = h^0(S, \mathcal{L}'(nK_S + C))$, by Serre's duality. On the other hand the global section of $\mathcal{O}_S(-nK_S)$ defines an inclusion:

$$0 \longrightarrow \mathcal{O}_S(nK_S) \longrightarrow \mathcal{O}_S,$$

from this we obtain

$$h^0(S, \mathcal{L}'(nK_S + C)) \leqslant h^0(S, \mathcal{L}'(C)).$$

Finally we observe that the exact sequence 2.1, followed by tensor multiplication by $\mathcal{O}_S(C)$ leads to the exact sequence

$$0 \longrightarrow \mathcal{L}' \longrightarrow \mathcal{L}'(C) \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

Thus, we have the inequality $h^0(S, \mathcal{L}'(C)) \leqslant h^0(S, \mathcal{L}') + 1$. □

We note that the class of surfaces satisfying the hypothesis of the theorem includes the surfaces of Kodaira dimension 0 ($K3$ and abelian surface, among others) and Del Pezzo's surfaces. Unfortunately general rational surfaces and surfaces of general type do not satisfy the hypothesis.

3. Direct images of $\mathcal{L}'^{-1}$ when S is rational.

In this section we start with the study of the cohomology of $\mathcal{L}'^{-1}$ in the case where S is a rational surface. The results that we present here are just a first step in this study, for the moment we have been not able to obtain a more satisfactory description.

Thus, in this section we always assume that S is a rational surface. We start by observing that $h^2(S,\Theta_S) = 0$ and this implies, by Proposition 1.1 a), that $h^2(S,\mathcal{L}'^{-1}) = 0$. Indeed, the sequence a) in Proposition 1.1 splits into two short exact sequences:

$$0 \longrightarrow \mathcal{L} \longrightarrow \Theta_S \longrightarrow J \longrightarrow 0,$$

and

$$0 \longrightarrow J \longrightarrow \mathcal{L}'^{-1} \longrightarrow \mathcal{O}_{\mathrm{Sing}\,\mathcal{F}} \longrightarrow 0,$$

from this it is easy to conclude, taking long exact sequences in cohomology, that $h^2(S,\mathcal{L}'^{-1}) = 0$. We denote by $g_i = h^1\big(\sum_j (n_{ij}-1)F_{ij}, \mathcal{O}_{\sum_j (n_{ij}-1)F_{ij}}\big)$. Our main result is:

THEOREM 3.1. — *The direct-images $R^0 f_*(\mathcal{L}'^{-1})$ and $R^1 f_*(\mathcal{L}'^{-1})$ are locally free sheaves of rank* 1 *and* g, *respectively. Moreover*

a) $R^0 f_*(\mathcal{L}'^{-1}) \simeq \mathcal{O}_{\mathbb{P}^1}(2-k)$, *where* $k = \sum_i h^0\big(\sum_j (n_{ij}-1)F_{ij}, \mathcal{O}_{\sum_j (n_{ij}-1)F_{ij}}\big)$.

b) $R^1 f_*(\mathcal{L}'^{-1}) \simeq \oplus^r \mathcal{O}_{\mathbb{P}^1}(-1) \oplus^s \mathcal{O}_{\mathbb{P}^1} \oplus^t \mathcal{O}_{\mathbb{P}^1}(1)$, *where* r,s,t *are integers satisfying* $r+s+t = g$ *and* $-r+t = g - \sum_i g_i$.

Proof. — We use the results in [6], Section 5, following the notation in the reference we call C_t, $t \in \mathbb{P}^1$ to the fiber of f over t, if $\mathcal{M}$ is a coherent sheaf $\mathcal{M}_{C_t}$ will denote the restriction of $\mathcal{M}$ to C_t.

Now, $\mathcal{L}'^{-1}_{C_t} \simeq \mathcal{O}_{C_t}$ for general t, this implies that $h^0(C_t,\mathcal{L}'^{-1}_{C_t}) \geqslant 1$ for any t, for a similar reason we have that $h^0(C_t,\mathcal{L}'_{C_t}) \geqslant 1$ for any t. Then, for any fiber C_t of f we have $\mathcal{L}'^{-1}_{C_t} \simeq \mathcal{O}_{C_t}$. This implies that $h^0(C_t,\mathcal{L}'^{-1}_{C}) = 1$ for any $t \in \mathbb{P}^1$ and then we conclude that $R^0 f_*(\mathcal{L}'^{-1})$ is locally free of rank 1. From it follows at once that $R^1 f_*(\mathcal{L}'^{-1})$ is locally free of rank g, since the Euler characteristic of $\mathcal{L}'^{-1}_{C_t}$ is locally constant on t and $h^0(C_t,\mathcal{L}'^{-1}_{C_t}) = g$ for general t. This proves the first part of the theorem. In order to prove parts a) and b) we need the following elementary lemma:

LEMMA 3.2. — *Let S be a rational surface and f a proper holomorphic map from S onto $\mathbb{P}^1$. If the general fiber of f is an irreducible curve of geometric genus g, then*

a) $R^0 f_*(\mathcal{O}_S) \simeq \mathcal{O}_{\mathbb{P}^1}$,

b) $R^1 f_*(\mathcal{O}_S) \simeq \oplus^g \mathcal{O}_{\mathbb{P}^1}(-1)$.

Proof of 3.2. — We note that $\mathcal{O}_{S_{C_t}} \simeq \mathcal{O}_{C_t}$. Moreover, S being a rational surface, we have $h^0(S, \mathcal{O}_S) = 1$ and $h^1(S, \mathcal{O}_S) = h^2(S, \mathcal{O}_S) = 0$.

Now, it follows from the Leray spectral sequence associated to f that

$$h^0(\mathbb{P}^1, R^0 f_*(\mathcal{O}_S)) = 1,$$

and

$$h^1(\mathbb{P}^1, R^0 f_*(\mathcal{O}_S)) = h^0(\mathbb{P}^1, R^1 f_*(\mathcal{O}_S)) = h^1(\mathbb{P}^1, R^1 f_*(\mathcal{O}_S)) = 0.$$

Finally, use Grothendieck's Theorem on locally free sheaves on $\mathbb{P}^1$ (any locally free sheaf on $\mathbb{P}^1$ splits in a direct sum of invertible sheaves([5], page 129)) and the Riemann-Roch formula to get the result. □

Now we apply the direct image functor to the exact sequence deduced in the proof of Theorem 1.3. The long exact sequence obtained in this way splits into two short exact sequences:

(3.1)
$$0 \longrightarrow R^0 f_*(\mathcal{L}'^{-1}) \longrightarrow R^0 f_*(\mathcal{O}_S(2C)) \longrightarrow R^0 f_*\Big(\mathcal{O}_{\big(\Sigma(n_{ij}-1\big)F_{ij})}\Big) \longrightarrow 0,$$

$$(3.2) \quad 0 \longrightarrow R^1 f_*(\mathcal{L}'^{-1}) \longrightarrow R^1 f_*(\mathcal{O}_S(2C)) \longrightarrow R^1 f_*\Big(\mathcal{O}_{\big(\Sigma(n_{ij}-1\big)F_{ij})}\Big),$$

since $R^1 f_*(\mathcal{L}'^{-1})$ is locally free and $R^0 f_*(\sum(n_{ij}-1)F_{ij})$ is supported on a finite set of points.

From Lemma 3.2 and the projection formula we deduce that $R^0 f_*(\mathcal{O}_S(2C)) = \mathcal{O}_{\mathbb{P}^1}(2)$. Thus,

$$\deg R^0 f_*(\mathcal{L}'^{-1}) = 2 - k,$$

where $k = h^0(\mathbb{P}^1, R^0 f_*(\mathcal{O}_{\Sigma(n_{ij}-1)F_{ij}})) = h^0(\sum(n_{ij}-1)F_{ij}, \mathcal{O}_{\Sigma(n_{ij}-1)F_{ij}})$. Thus, part a) of the theorem follows.

b) From Lemma 3.2 and projection formula it follows that

$$R^1 f_*(\mathcal{O}_S(2C)) \simeq \oplus^g \mathcal{O}_{\mathbb{P}^1}(1).$$

This implies that in the decomposition $R^1f_*(\mathcal{L}'^{-1}) \simeq \oplus_{i=1}^{g}\mathcal{O}_{\mathbb{P}^1}(a_i)$, we must have $a_i \leqslant 1$. Moreover, $h^2(S,\Theta_S)=0$ implies that $h^2(S,\mathcal{L}'^{-1}) = h^1(\mathbb{P}^1, R^1(\mathcal{L}'^{-1}))=0$.

Now, as the cohomology of $R^1f_*\mathcal{L}'^{-1}$ decomposes in the the direct sum of the cohomology of $\mathcal{O}_{\mathbb{P}^1}(a_i)$, it is easy to deduce, using the Riemann-Roch formula that $a_i \geqslant -1$. From this we conclude that

$$R^1f_*(\mathcal{L}'^{-1}) \simeq \oplus^r\mathcal{O}_{\mathbb{P}^1}(-1)\oplus^s\mathcal{O}_{\mathbb{P}^1}\oplus^t\mathcal{O}_{\mathbb{P}^1}(1).$$

It is clear that $r+s+t=g$, since $R^1f_*(\mathcal{L}'^{-1})$ has rank g. Lastly, note that the exact sequence 3.2 implies that

$$g=\deg R^1f_*(\mathcal{O}_S(2C))=\deg R^1f_*(\mathcal{L}'^{-1})+h^0(\mathbb{P}^1,R^1f_*(\mathcal{O}_{\Sigma(n_{ij}-1)F_{ij}})).$$

On the other hand,

$$h^0(\mathbb{P}^1,R^1f_*(\mathcal{O}_{\Sigma(n_{ij}-1)F_{ij}}))=\sum g_i, \text{ since } h^2(C_t,\mathcal{O}_{(\Sigma(n_{ij}-1)F_{ij})_{C_t}})=0$$

for any $t\in\mathbb{P}^1$ ([6] page 53, Corollary 3). The proof of the theorem is concluded. □

We would like to note some immediate consequences of the theorem. First of all note that

COROLLARY 3.3. — $r=h^2(S,\mathcal{L}'^{-1}(-C))$.

Proof.

$$r=h^0(\mathbb{P}^1,[R^1f_*(\mathcal{L}'^{-1})(1)]^{-1}),$$

by Serre duality we have

$$r=h^1(\mathbb{P}^1,R^1f_*(\mathcal{L}'^{-1})(-1))=h^2(S,\mathcal{L}'^{-1}(-C)).$$ □

COROLLARY 3.4. — *Let $\mathcal{F}$ and S be like in Theorem 3.1, if $h^0(S,\mathcal{L}'^{-1})>0$, then f has at most two non-reduced fibers.*

Proof. — Its follows at once from part a) of the theorem.

COROLLARY 3.5. — *Under the same hypothesis of Theorem 3.1*

$$2g-2-\sum g_i=h^0(\mathbb{P}^1,R^1f_*(\mathcal{L}'^{-1})).$$

Proof. — Use Riemann-Roch formula and the fact that $h^1(\mathbb{P}^1, R^1 f_*(\mathcal{L}'^{-1})) = 0$:

$$h^0(\mathbb{P}^1, R^1(\mathcal{L}'^{-1})) = \deg(R^1 f_*(\mathcal{L}'^{-1})) + \operatorname{rank}(R^1 f_*(\mathcal{L}'^{-1})).$$

The corollary follows at once from this relationship and part b) of the theorem. □

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