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## ALGEBRAIC EQUIVALENCE OF REAL ALGEBRAIC CYCLES

by M. ABÁNADES & W. KUCHARZ

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### 1. Introduction.

Let  $X$  be a nonsingular,  $n$ -dimensional, quasiprojective variety over  $\mathbb{R}$  (that is, an irreducible,  $n$ -dimensional, quasiprojective scheme over  $\mathbb{R}$ , smooth over  $\mathbb{R}$ ). We endow the set  $X(\mathbb{R})$  of  $\mathbb{R}$ -rational points of  $X$  with the topology induced by the usual metric topology on  $\mathbb{R}$ , and assume that  $X(\mathbb{R})$  is nonempty and compact. Thus  $X(\mathbb{R})$  is a  $C^\infty$ , closed,  $n$ -dimensional manifold. Given a nonnegative integer  $k$ , we let  $Z^k(X)$  denote the group of algebraic  $(n - k)$ -cycles on  $X$  (that is, the free Abelian group on the set of closed,  $(n - k)$ -dimensional subvarieties of  $X$ ). There exists a unique group homomorphism

$$\text{cl}_{\mathbb{R}} : Z^k(X) \rightarrow H^k(X(\mathbb{R}), \mathbb{Z}/2)$$

such that for every closed,  $(n - k)$ -dimensional subvariety  $V$  of  $X$ , the cohomology class  $\text{cl}_{\mathbb{R}}(V)$  is Poincaré dual to the homology class in  $H_{n-k}(X(\mathbb{R}), \mathbb{Z}/2)$  determined by  $V(\mathbb{R})$  (cf. [5] for the definition of this homology class). In the present paper we study the cohomology classes of the form  $\text{cl}_{\mathbb{R}}(z)$ , where  $z$  is a cycle in  $Z^k(X)$  algebraically equivalent to 0 (we refer to [7] for the theory of algebraic equivalence of cycles). Such cohomology classes need not be trivial, but as we shall see below they must satisfy quite restrictive conditions.

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The extreme cases,  $k = 0$  and  $k = n$ , are easy to analyze. Obviously, a cycle  $z$  in  $Z^0(X)$  is algebraically equivalent to 0 if and only if  $z = 0$ . On the other hand, every cycle in  $Z^n(X)$  of the form  $x_0 - x_1$ , where  $x_0$  and  $x_1$  are points in  $X(\mathbb{R})$ , is algebraically equivalent to 0. We have  $\text{cl}_{\mathbb{R}}(x_0 - x_1) \neq 0$  whenever  $x_0$  and  $x_1$  belong to distinct connected components of  $X(\mathbb{R})$ . It follows that a cohomology class  $u$  in  $H^n(X(\mathbb{R}), \mathbb{Z}/2)$  can be written as  $u = \text{cl}_{\mathbb{R}}(z)$  for some cycle  $z$  in  $Z^n(X)$  algebraically equivalent to 0 if and only if the homology class in  $H_0(X(\mathbb{R}), \mathbb{Z}/2)$  Poincaré dual to  $u$  is represented by an even number of points of  $X(\mathbb{R})$ . In view of these facts, we concentrate our attention on the intermediate cases,  $1 \leq k \leq n - 1$ .

Given a continuous map  $f : M \rightarrow N$  between topological spaces, we denote by  $H^k(f) : H^k(N, \mathbb{Z}/2) \rightarrow H^k(M, \mathbb{Z}/2)$  the homomorphism induced by  $f$ . Recall that a cohomology class  $u$  in  $H^k(M, \mathbb{Z}/2)$  with  $k \geq 1$  is said to be *spherical* if  $u = H^k(f)(c)$ , where  $f : M \rightarrow S^k$  is a continuous map into the unit  $k$ -sphere  $S^k$ , and  $c$  is the generator of  $H^k(S^k, \mathbb{Z}/2) \cong \mathbb{Z}/2$ . We denote, as usual, by  $\cup$  and  $\langle -, - \rangle$  the cup product of cohomology classes and the Kronecker index (pairing) of cohomology and homology classes, cf. [11]. If  $M$  is a  $C^\infty$ , closed manifold of dimension  $n$ , we denote by  $w_k(M)$  the  $k$ th Stiefel-Whitney class of  $M$  and by  $\mu_M$  the fundamental homology class of  $M$  in  $H_n(M, \mathbb{Z}/2)$ .

**THEOREM 1.1.** — *Let  $X$  be a nonsingular,  $n$ -dimensional, quasiprojective variety over  $\mathbb{R}$  with  $X(\mathbb{R})$  nonempty and compact. Let  $z$  be a cycle in  $Z^k(X)$  that is algebraically equivalent to 0. Then the cohomology class  $\text{cl}_{\mathbb{R}}(z)$  in  $H^k(X(\mathbb{R}), \mathbb{Z}/2)$  satisfies  $\text{cl}_{\mathbb{R}}(z) \cup \text{cl}_{\mathbb{R}}(z) = 0$  in  $H^{2k}(X(\mathbb{R}), \mathbb{Z}/2)$  and*

$$\langle \text{cl}_{\mathbb{R}}(z) \cup w_{i_1}(X(\mathbb{R})) \cup \dots \cup w_{i_r}(X(\mathbb{R})), \mu_{X(\mathbb{R})} \rangle = 0$$

*for all nonnegative integers  $i_1, \dots, i_r$  with  $i_1 + \dots + i_r = n - k$ . Furthermore, if  $k = 1$  or if  $k = n - 1 \geq 1$  with  $X(\mathbb{R})$  connected, then the cohomology class  $\text{cl}_{\mathbb{R}}(z)$  is spherical.*

Let us note that, in general, the cohomology class  $\text{cl}_{\mathbb{R}}(z)$  of Theorem 1.1 need not be spherical. Indeed, suppose  $X = X' \times X''$  (product over  $\text{Spec}\mathbb{R}$ ), where  $X'$  and  $X''$  are nonsingular, projective varieties over  $\mathbb{R}$  such that  $X'(\mathbb{R})$  is nonempty and  $X''(\mathbb{R})$  is disconnected. Let  $z'$  be any algebraic cycle on  $X'$ . Choose two points  $p_0$  and  $p_1$  in  $X''(\mathbb{R})$  that belong to distinct connected components. Since the 0-cycle  $z'' = p_0 - p_1$  on  $X''$  is algebraically equivalent to 0, the cycle  $z' \times z''$  on  $X$  is algebraically equivalent to 0 as well. Furthermore, the cohomology class  $\text{cl}_{\mathbb{R}}(z' \times z'') = \text{cl}_{\mathbb{R}}(z') \times \text{cl}_{\mathbb{R}}(z'')$  is

spherical if and only if the cohomology class  $cl_{\mathbb{R}}(z')$  is spherical (for  $p_0$  and  $p_1$  belong to distinct connected components of  $X''(\mathbb{R})$ ). Taking  $X' = \mathbb{P}_{\mathbb{R}}^m$ , we have  $cl_{\mathbb{R}}(Z^k(X')) = H^k(X'(\mathbb{R}), \mathbb{Z}/2)$ , and the unique nontrivial cohomology class in  $H^k(X'(\mathbb{R}), \mathbb{Z}/2) \cong \mathbb{Z}/2$  is not spherical, provided that  $1 \leq k \leq m - 1$  and  $m$  is even. In particular, “connected” cannot be omitted in the last part of Theorem 1.1.

If  $cl_{\mathbb{R}}(z)$  is spherical, then  $cl_{\mathbb{R}}(z) \cup cl_{\mathbb{R}}(z) = 0$  is automatically satisfied, and Theorem 1.1 is in some sense the best possible result. More precisely, we have the following.

**THEOREM 1.2.** — *Let  $M$  be a  $C^\infty$ , closed,  $n$ -dimensional manifold and let  $u$  be a spherical cohomology class in  $H^k(M, \mathbb{Z}/2)$  with  $1 \leq k \leq n - 1$ . Then the following conditions are equivalent :*

(a) *There exist a nonsingular, projective algebraic variety  $X$  over  $\mathbb{R}$  and a  $C^\infty$  diffeomorphism  $\varphi : X(\mathbb{R}) \rightarrow M$  such that  $H^k(\varphi)(u) = cl_{\mathbb{R}}(z)$  for some cycle  $z$  in  $Z^k(X)$  algebraically equivalent to 0;*

(b)  *$\langle u \cup w_{i_1}(M) \cup \dots \cup w_{i_r}(M), \mu_M \rangle = 0$  for all nonnegative integers  $i_1, \dots, i_r$  with  $i_1 + \dots + i_r = n - k$ .*

Let us mention that Theorem 1.2 is an improvement upon inefficient [10], Theorem 2.4.

## 2. Proofs.

Let  $X$  be a nonsingular,  $n$ -dimensional, quasiprojective algebraic variety over  $\mathbb{R}$  with  $X(\mathbb{R})$  nonempty and compact. Recall that if an algebraic cycle  $z$  in  $Z^k(X)$  is rationally equivalent to 0, then  $cl_{\mathbb{R}}(z) = 0$  (cf. [5], 5.13) and hence  $cl_{\mathbb{R}}$  induces a homomorphism, also denoted by  $cl_{\mathbb{R}}$ , from the Chow group  $A^k(X)$  of  $X$  into  $H^k(X(\mathbb{R}), \mathbb{Z}/2)$ . It is known that  $cl_{\mathbb{R}} : A^*(X) \rightarrow H^*(X(\mathbb{R}), \mathbb{Z}/2)$  is a homomorphism of graded rings [5], p. 495. Thus

$$H_{\text{alg}}^*(X(\mathbb{R}), \mathbb{Z}/2) = cl_{\mathbb{R}}(Z^*(X)) = cl_{\mathbb{R}}(A^*(X))$$

is a graded subring of  $H^*(X(\mathbb{R}), \mathbb{Z}/2)$ . We shall need the following result [10], Theorem 2.1 :

$$(1) \quad \langle cl_{\mathbb{R}}(z) \cup v, \mu_{X(\mathbb{R})} \rangle = 0$$

for all cycles  $z$  in  $Z^k(X)$  algebraically equivalent to 0 and all  $v$  in  $H_{\text{alg}}^{n-k}(X(\mathbb{R}), \mathbb{Z}/2)$ .

Assume now that  $X$  is projective. Then the set  $X(\mathbb{C})$  of  $\mathbb{C}$ -rational points of  $X$  is a compact complex manifold of complex dimension  $n$ . There exists a unique group homomorphism

$$\text{cl}_{\mathbb{C}} : Z^k(X) \rightarrow H^{2k}(X(\mathbb{C}), \mathbb{Z})$$

such that for every closed,  $(n - k)$ -dimensional subvariety  $V$  of  $X$ , the cohomology class  $\text{cl}_{\mathbb{C}}(V)$  is Poincaré dual to the homology class in  $H_{2n-2k}(X(\mathbb{C}), \mathbb{Z})$  determined by  $V(\mathbb{C})$  (cf. [5] for the definition of this homology class). In other words, if  $\pi : X_{\mathbb{C}} = X \times_{\text{Spec } \mathbb{R}} \text{Spec } \mathbb{C} \rightarrow X$  is the canonical projection, then  $\text{cl}_{\mathbb{C}}(z)$  is the cohomology class corresponding to the pullback algebraic cycle  $\pi^*(z)$  on  $X_{\mathbb{C}}$ , cf. [5], 4.2 or [7], Chapter 19. In particular,

$$(2) \quad \text{cl}_{\mathbb{C}}(z) = 0$$

for all cycles  $z$  in  $Z^k(X)$  algebraically equivalent to 0, cf. [5], 4.14 or [7], Proposition 19.1.1. Furthermore, it follows from the proof of [2], Theorem A that

$$(3) \quad \text{cl}_{\mathbb{R}}(z) \cup \text{cl}_{\mathbb{R}}(z) = \text{the reduction modulo 2 of } r(\text{cl}_{\mathbb{C}}(z))$$

for all  $z$  in  $Z^k(X)$ , where  $r : H^{2k}(X(\mathbb{C}), \mathbb{Z}) \rightarrow H^{2k}(X(\mathbb{R}), \mathbb{Z})$  is the homomorphism induced by the inclusion map  $X(\mathbb{R}) \hookrightarrow X(\mathbb{C})$ .

*Proof of Theorem 1.1.* — By Hironaka’s resolution of singularities theorem [8], 3, we may assume that  $X$  is projective.

We obtain  $\text{cl}_{\mathbb{R}}(z) \cup \text{cl}_{\mathbb{R}}(z) = 0$  directly from (2) and (3).

It follows from [5], p. 498 that  $w_i(X(\mathbb{R}))$  is in  $H_{\text{alg}}^i(X(\mathbb{R}), \mathbb{Z}/2)$ , and hence if  $i_1, \dots, i_r$  are nonnegative integers with  $i_1 + \dots + i_r = n - k$ , then the cohomology class

$$v = w_{i_1}(X(\mathbb{R})) \cup \dots \cup w_{i_r}(X(\mathbb{R}))$$

belongs to  $H_{\text{alg}}^{n-k}(X(\mathbb{R}), \mathbb{Z}/2)$ . In view of (1), we have  $\langle \text{cl}_{\mathbb{R}}(z) \cup v, \mu_{X(\mathbb{R})} \rangle = 0$ , which completes the proof of the first part of the theorem.

Given an invertible sheaf  $\mathcal{L}$  on  $X$ , we denote by  $\mathcal{L}_{\mathbb{R}}$  (resp.  $\mathcal{L}_{\mathbb{C}}$ ) the topological real (resp. complex) line bundle on  $X(\mathbb{R})$  (resp.  $X(\mathbb{C})$ ) determined by  $\mathcal{L}$  in the usual way. If  $\mathcal{L}$  corresponds to a Weil divisor  $D$  on  $X$ , then

$$w_1(\mathcal{L}_{\mathbb{R}}) = \text{cl}_{\mathbb{R}}(D) \quad \text{and} \quad c_1(\mathcal{L}_{\mathbb{C}}) = \text{cl}_{\mathbb{C}}(D),$$

where  $w_1(-)$  and  $c_1(-)$  stand for the first Stiefel-Whitney class and the first Chern class, respectively, cf. [5], p. 498, p. 489. Note that the restriction  $\mathcal{L}_{\mathbb{C}}|_{X(\mathbb{R})}$  of  $\mathcal{L}_{\mathbb{C}}$  to  $X(\mathbb{R})$  is the complexification of  $\mathcal{L}_{\mathbb{R}}$ , and hence

$$c_1(\mathcal{L}_{\mathbb{C}}|_{X(\mathbb{R})}) = \beta(w_1(\mathcal{L}_{\mathbb{R}})),$$

where  $\beta : H^1(X(\mathbb{R}), \mathbb{Z}/2) \rightarrow H^2(X(\mathbb{R}), \mathbb{Z})$  is the Bockstein homomorphism that appears in the long exact sequence

$$\dots \rightarrow H^1(X(\mathbb{R}), \mathbb{Z}) \xrightarrow{2} H^1(X(\mathbb{R}), \mathbb{Z}) \rightarrow H^1(X(\mathbb{R}), \mathbb{Z}/2) \xrightarrow{\beta} H^2(X(\mathbb{R}), \mathbb{Z}) \rightarrow \dots,$$

cf. [11], Problems 15-C and D. The last equality can be written in an equivalent form

$$(4) \quad r(\text{cl}_{\mathbb{C}}(D)) = \beta(\text{cl}_{\mathbb{R}}(D)),$$

where  $r : H^2(X(\mathbb{C}), \mathbb{Z}) \rightarrow H^2(X(\mathbb{R}), \mathbb{Z})$  is the homomorphism induced by the inclusion map  $X(\mathbb{R}) \hookrightarrow X(\mathbb{C})$ .

Suppose now that  $k = 1$ , that is,  $z$  is a Weil divisor on  $X$ . By (2) and (4), we have  $\beta(\text{cl}_{\mathbb{R}}(z)) = 0$ , which means that  $\text{cl}_{\mathbb{R}}(z)$  is the reduction modulo 2 of a cohomology class in  $H^1(X(\mathbb{R}), \mathbb{Z})$ . This last fact implies that the cohomology class  $\text{cl}_{\mathbb{R}}(z)$  is spherical, cf. [9], p. 49.

Let us now assume that  $k = n - 1 \geq 1$  and  $X(\mathbb{R})$  is connected. We already know that  $\langle \text{cl}_{\mathbb{R}}(z) \cup w_1(X(\mathbb{R})), \mu_{X(\mathbb{R})} \rangle = 0$ , which in view of the connectedness of  $X(\mathbb{R})$  is equivalent to  $\text{cl}_{\mathbb{R}}(z) \cup w_1(X(\mathbb{R})) = 0$ . The last condition implies that the homology class in  $H_1(X(\mathbb{R}), \mathbb{Z}/2)$  Poincaré dual to  $\text{cl}_{\mathbb{R}}(z)$  can be represented by a  $C^\infty$ , closed curve in  $X(\mathbb{R})$ , with trivial normal vector bundle, cf. for example [4], p. 599. This in turn implies that  $\text{cl}_{\mathbb{R}}(z)$  is spherical, cf. [12], Théorème II.1. Thus the proof is complete.  $\square$

*Proof of Theorem 1.2.* — By Theorem 1.1, (a) implies (b), and we show below that (b) implies (a).

Choose a nonsingular, irreducible algebraic subset  $W$  of  $\mathbb{R}^{k+1}$ , which has precisely two connected components  $W_0$  and  $W_1$ , each diffeomorphic to the unit  $k$ -sphere  $S^k$  (for example,  $W = \{(x_1, \dots, x_{k+1}) \in \mathbb{R}^{k+1} \mid x_1^4 - 4x_1^2 + 1 + x_2^2 + \dots + x_{k+1}^2 = 0\}$ ). Let  $c$  be the unique generator of the group  $H^k(W_0, \mathbb{Z}/2) \cong \mathbb{Z}/2$ , viewed as a subgroup of  $H^k(W, \mathbb{Z}/2)$ . Since the cohomology class  $u$  is spherical, there exists a  $C^\infty$  map  $h : M \rightarrow W$  such that  $h(M) \subseteq W_0$  and  $u = H^k(h)(c)$ . Choose a regular value  $y_0$  of  $h$  in  $W_0$ . Then  $u$  is Poincaré dual to the homology class in  $H_{n-k}(M, \mathbb{Z}/2)$  represented by the  $C^\infty$  submanifold  $h^{-1}(y_0)$  of  $M$ , cf. [5], 2.15. Clearly, there exists a unique  $C^\infty$  map  $f : M \rightarrow W$  such that for every connected component  $S$  of  $M$  and every point  $x$  in  $S$ , we have  $f(x) = h(x)$  if  $S \cap h^{-1}(y_0) \neq \emptyset$  and  $f(x) = y_0$  if  $S \cap h^{-1}(y_0) = \emptyset$ . The map  $f$  satisfies

$$(5) \quad f(M) \subseteq W_0 \quad \text{and} \quad u = H^k(f)(c).$$

Furthermore, each connected component of  $f^{-1}(y_0)$  is a  $C^\infty$  submanifold of  $M$  of dimension either  $n - k$  or  $n$ . Also, each connected component of  $M$  contains a connected component of  $f^{-1}(y_0)$ . Since  $n - k \geq 1$ , we can find a  $C^\infty$  closed curve  $C$  in  $M$  such that

$$(6) \quad f(C) = \{y_0\},$$

the normal vector bundle of  $C$  in  $M$  is trivial, and each connected component of  $M$  contains a connected component of  $C$ . Choose an integer  $d$  with  $2n + 1 \leq d$  and let  $D$  be a compact, nonsingular, irreducible, 1-dimensional algebraic subset of  $\mathbb{R}^d$  that has the same number of connected components as  $C$ . Replacing  $M$  by its image under a suitable  $C^\infty$  embedding into  $\mathbb{R}^d$ , we may assume that

$$(7) \quad D = C \subseteq M \subseteq \mathbb{R}^d.$$

By Tognoli's theorem [13] or [1], Corollary 2.8.6, there exists a nonsingular real algebraic subset  $A$  of  $\mathbb{R}^p$ , for some  $p$ , diffeomorphic to  $M$ . Consider the disjoint union  $N = M \amalg A$  and the  $C^\infty$  map  $F : N \rightarrow W$  defined by  $F(x) = f(x)$  for  $x$  in  $M$  and  $F(x) = y_0$  for  $x$  in  $A$ . We assert that if  $w$  is a cohomology class in  $H^\ell(W, \mathbb{Z}/2)$  and if  $j_1, \dots, j_s$  are nonnegative integers with  $j_1 + \dots + j_s = n - \ell$ , then

$$\langle H^\ell(F)(w) \cup w_{j_1}(N) \cup \dots \cup w_{j_s}(N), \mu_N \rangle = 0.$$

Indeed, first note that  $w = 0$ , unless  $\ell = 0$  or  $\ell = k$ . If  $\ell = 0$ , then either  $H^0(F)(w) = 0$  or  $H^0(F)(w) = 1$ . In the latter case the assertion holds since  $M$  and  $A$  are diffeomorphic. If  $\ell = k$ , then either  $H^k(F)(w) = 0$  or  $H^k(F)(w) = u$  (we view  $H^k(M, \mathbb{Z}/2)$  as a subgroup of  $H^k(N, \mathbb{Z}/2)$ ). In the latter case the assertion follows from condition (b). Thus the assertion is proved. It implies that there exist a  $C^\infty$  compact manifold  $\tilde{N}$  with boundary  $\partial\tilde{N} = N$  and a  $C^\infty$  map  $\tilde{F} : \tilde{N} \rightarrow W$  satisfying  $\tilde{F}|N = F$ , cf. [6], 17.3. In other words, the map  $f : M \rightarrow W$  and the constant map  $A \rightarrow W$ , which sends  $A$  to  $y_0$ , represent the same class in the unoriented bordism group of  $W$ . By construction, the normal bundle of  $D$  in  $M$  is trivial, so the restriction  $\nu|_D$  to  $D$  of the normal bundle  $\nu$  of  $M$  in  $\mathbb{R}^d$  admits an algebraic structure. Therefore, by [1], Theorem 2.8.4 and in view of (6) and (7), one can find a nonnegative integer  $e$ , a nonsingular algebraic subset  $V$  of  $\mathbb{R}^d \times \mathbb{R}^e$ , a  $C^\infty$  diffeomorphism  $\varphi : V \rightarrow M$ , and a regular map  $g : V \rightarrow W$  (the latter designates the restriction to  $V$  of a rational map from  $\mathbb{R}^d \times \mathbb{R}^e$  into  $\mathbb{R}^{k+1}$  which has no poles on  $V$  and maps  $V$  into  $W$ ) such that  $g$  is homotopic to  $f \circ \varphi$  and  $D \times \{0\} \subseteq V$ . Since  $D$  is irreducible and each connected component of  $V$  contains a connected component of

$D \times \{0\}$ , it follows that  $V$  is irreducible as well (this is the only place where  $D$  is needed).

Irreducibility of  $V$  and  $W$  allows us to choose nonsingular, quasiprojective varieties  $T$  and  $Y$  over  $\mathbb{R}$  with  $T(\mathbb{R}) = V$  and  $Y(\mathbb{R}) = W$ . By Hironaka's resolution of singularities theorem [8], 3, we may assume that  $T$  and  $Y$  are projective (and still nonsingular). Let  $\tilde{g} : U \rightarrow Y$  be an algebraic morphism over  $\mathbb{R}$ , defined on a Zariski open neighborhood of  $T(\mathbb{R}) = V$  in  $T$ , such that  $\tilde{g}|T(\mathbb{R}) = g$ . By applying Hironaka's theorem on removing points of indeterminacy [8], 3, we can find a nonsingular, projective algebraic variety  $X$  over  $\mathbb{R}$  and an algebraic morphism  $G : X \rightarrow Y$  over  $\mathbb{R}$  satisfying  $X(\mathbb{R}) = T(\mathbb{R})$  and  $G|X(\mathbb{R}) = g$ .

Let  $y_1$  be a point in  $W_1$  and let  $\beta$  be the class in  $A^k(Y)$  of the 0-cycle  $y_0 - y_1$  on  $Y$ . By (5),  $u = H^k(f)(\text{cl}_{\mathbb{R}}(\beta))$  (although, of course,  $\text{cl}_{\mathbb{R}}(\beta) \neq c$ ). Since  $G|X(\mathbb{R}) = g$  is homotopic to  $f \circ \varphi$ , we obtain

$$\begin{aligned} H^k(\varphi)(u) &= H^k(\varphi)(H^k(f)(\text{cl}_{\mathbb{R}}(\beta))) = H^k(f \circ \varphi)(\text{cl}_{\mathbb{R}}(\beta)) \\ &= H^k(G|X(\mathbb{R}))(\text{cl}_{\mathbb{R}}(\beta)) = \text{cl}_{\mathbb{R}}(G^*(\beta)), \end{aligned}$$

where the last equality is a consequence of the functorial property of  $\text{cl}_{\mathbb{R}} : A^* \rightarrow H^*$ , cf. [5], 5.12. Let  $z$  be a cycle in  $Z^k(X)$  that represents in the Chow group  $A^k(X)$  the pullback class  $G^*(\beta)$ . Then  $\text{cl}_{\mathbb{R}}(z) = \text{cl}_{\mathbb{R}}(G^*(\beta)) = H^k(\varphi)(u)$ . The proof is now complete since the cycle  $y_0 - y_1$  is algebraically equivalent to 0 on  $Y$  and hence the cycle  $z$  is algebraically equivalent to 0 on  $X$ .  $\square$

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