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Symplectic subvarieties of projective fibrations over symplectic manifolds


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SYMPLECTIC SUBVARIETIES OF
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by Roberto PAOLETTI

1. Introduction.

Suppose that \((M, \omega)\) is a compact symplectic manifold of dimension
2n, such that the cohomology class \([\omega] \in H^2(M, \mathbb{R})\) lies in the integral
lattice \(H^2(M, \mathbb{Z})/\text{Torsion}\); we shall say that \((M, \omega)\) is almost-Hodge. It has
been recently proved by Donaldson that for any sufficiently large integer \(k\)
there exists a symplectic submanifold \(W \subset M\) representing the Poincaré
dual of any fixed integral lift of \([k\omega]\), [D].

In this paper, we specialize this result to the case of a symplectic
fibration \(p : E \to M\) whose fibre is a projective manifold \(F\) with a fixed
Hodge form \(\sigma\) on it. For instance, \(E\) could be the relative projective space,
or a relative flag space, associated to a complex vector bundle on \(M\). Then,
as follows from well-known symplectic reduction techniques ([W], [GLS])
\(E\) has an almost Hodge structure \(\tilde{\omega}\) restricting to \(\sigma\) on each fibre of \(p,
[MS]\). We adapt Donaldson’s arguments to show that the symplectic divisor
guaranteed by his theorem may be chosen compatibly with the vertical
holomorphic structure. More precisely,

**Theorem 1.1.** — Let \((M, \omega)\) be an almost Hodge manifold. Let
\(F \subseteq \mathbb{P}^N\) be a connected complex projective manifold and set \(L = \mathcal{O}_F(1),

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the restriction to $F$ of the hyperplane bundle on $\mathbb{P}^N$. Denote by $\sigma$ the restriction to $F$ of the Fubini-Study form on $\mathbb{P}^N$. Suppose that $G$ is a compact group of automorphisms of $\mathbb{P}^N$ preserving $F$. Let $p : E \to M$ be a fibre bundle with fibre $F$ and structure group $G$, so that in particular there is a line bundle $L_E \to E$ extending $L \to F$. Then $E$ admits an almost Hodge structure $\tilde{\omega}$ vertically compatible with $\sigma$. Furthermore, perhaps after replacing $\tilde{\omega}$ by $kp^*(\omega_M) + \tilde{\omega}$ for $k \gg 0$, any integral lift of $[\tilde{\omega}]$ is Poincaré dual to a codimension-2 symplectic submanifold $W \subset E$, meeting any fibre $F_m = p^{-1}(m)$ ($m \in M$) in a complex subvariety.

In general the submanifold $W$ may not be transverse to every fibre. For example, if $E$ is a rank-2 complex vector bundle on $M$ and $E = \mathbb{P}E^*$ with general fibre $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$, then $W$ is the blow-up of $M$ along the zero locus $Z$ of a section of a suitable twist of $E$, and therefore contains all the fibres over $Z$.

In practice one may have a fibre bundle $E \to M$ with fibre a complex projective manifold $(F, J_F)$ and structure group $G$ preserving the complex structure $J_F$ and some fixed Hodge form $\sigma$ on $F$, and complexification $\tilde{G} \subseteq \text{Aut}(F, J_F)$. If $L$ is a line bundle on $F$ such that $c_1(L) = [\sigma]$, then by general principles from geometric invariant theory a lifting to $L^{\otimes k}$ of the action of $G$ exists if $k \gg 0$. Therefore,

**Corollary 1.1.** — Suppose that $(F, \sigma)$, $M$ and $E$ are as just described. Then for $r \gg 0$ and $k > k(r)$ any integral lift of $[r\tilde{\omega} + kp^*(\omega_M)]$ is Poincaré dual to a codimension-2 symplectic submanifold intersecting each fibre $F_m$ in a divisor of the linear series $|L^{\otimes r}|$.

Again, $W$ is not transversal to every fibre. In the case of a $\mathbb{P}^1$-bundle $E = \mathbb{P}E^* \to M$, the projection $W \to M$ is a branched cover with non-empty ramification locus.

The theorem also yields that top Chern classes of symplectically very positive vector bundles have symplectic representatives, as already shown by Auroux, [A]:

**Corollary 1.2.** — Let $(M, \omega)$ be a $2n$-dimensional almost Hodge manifold and let $E$ be a complex vector bundle on $M$ of complex rank $r < n$. Let $H$ be a complex line bundle on $M$ with $c_1(H) = [\omega]$. Then for $k \gg 0$ there is a transverse section $s$ of $E \otimes H^{\otimes k}$ whose zero locus $Z$ is a connected symplectic submanifold of $M$; in fact, $H_j(M, Z) = 0$ if $j \leq n - r$. 
As we shall see, these sections are also asymptotically almost holomorphic in the sense of [A].

**Notation.** — For any integer \( r > 0 \), we shall denote by \( \omega_0^{(r)} = (i/2) \sum_{\alpha=1}^{r} dz_\alpha \wedge d\bar{z}_\alpha \) the standard symplectic structure on \( \mathbb{C}^r \). Furthermore, by \( C \) we shall often indicate an appropriate constant, appearing in various estimates, which is allowed to vary from line to line.

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**2. Proof of the theorem and corollaries.**

Let \( \pi : P \to M \) be the principal \( G \)-bundle associated with the fibration. Given a connection for \( \pi \), the existence of a compatible almost Hodge form on \( E \) follows from well-known symplectic reduction arguments, [MS]. In fact, minimal coupling produces a compatible closed 2-form \( \vartheta = \vartheta_{\min} \) on \( E \), [GS]. Explicitly, let the induced connection be given by the horizontal distribution \( \mathcal{H}(E/M) \subset TE \) and denote by \( V(E/M) \subset TE \) the vertical tangent space. Let \( g \) be the Lie algebra of \( G \) and view the curvature \( F \) as a \( g \)-valued 2-form on \( M \). Let \( \mu : F \to g^* \) be the moment map for the action. If \( e \in E \) and \( x = p(e) \), let \( U \subseteq M \) be an open subset over which \( P \) trivializes and let \( \gamma : U \times F \to p^{-1}(U) \) be the corresponding trivialization. Then \( \mathcal{H}(E/M) \) and \( V(E/M) \) are mutually orthogonal for \( \sigma \). Furthermore, with abuse of language, \( \vartheta |_{V(P/M)} = \sigma \), while if \( X,Y \in T_x M \) and \( X^\sharp,Y^\sharp \) are their horizontal lifts at \( e = \gamma(x,f) \), then \( \vartheta_e(X^\sharp,Y^\sharp) = \langle \mu(f), F_x(X,Y) \rangle \). Therefore \( \tilde{\omega}_k = \vartheta + kp^*(\omega) \) is a compatible symplectic structure on \( E \) if \( k \gg 0 \). However, in order to adapt Donaldson's construction we shall need to describe \(-2\pi i \vartheta \) as the curvature of a connection on a suitable line bundle on \( E \).

Clearly, the action of \( G \) lifts to \( L \) and preserves the unit circle bundle \( S_L \subseteq L \). Let \( \nabla_L \) be the unique covariant derivative on \( L \) compatible with the complex and hermitian structures, that is, the restriction to \( F \) of the connection on \( \mathcal{O}_{\mathbb{P}^n}(1) \). Let \( \mathcal{H}(S_L/F) \subset TS_L \) be the corresponding \( S^1 \)-invariant horizontal distribution, which by uniqueness is also \( G \)-invariant. The line bundle \( L_E := P \times_G L \) over \( E \) restricts to \( L \) on every fibre of \( p \) and has an hermitian metric extending that of \( L \). Then the unit circle...
bundle $S_{LE} = P \times_G S_L \subset L_E$ has a connection over $E$, as follows. Let $p' : S_{LE} \to M$ be the projection, a fibre bundle over $M$ with general fibre $S_L$. Given $s \in S_{LE}$ mapping to $e \in E$, set $x = p(e)$ and choose as above a trivialization of $P$ in a neighbourhood $U$ of $x$, with induced trivializations $\gamma : U \times F \to p^{-1}(U)$ and $\gamma' : U \times S_L \to p'^{-1}(U)$. If $e = \gamma(x,f)$ and $s = \gamma'(x,\ell)$ ($\ell \in S_L$ lies over $f \in F$), then the horizontal space of $S_{LE}$ at $s$ is $\mathcal{H}(S_{LE}/E) = \mathcal{H}(S_{LE}/M) \oplus d\gamma'_*(\mathcal{H}_\ell(S_L/F))$. This gives a well-defined connection $\nabla_{LE}$ on $L_E$, and we leave it to the reader to check that $\theta_{\text{min}}$ may also be obtained as the normalized curvature of $\nabla_{LE}$:

**Lemma 2.1.** — Let $\vartheta$ be the normalized curvature form on $E$ of the connection $\mathcal{H}(S_E/E)$. Then for $k \gg 0$ the 2-form $\tilde{\omega}(k) = \vartheta + kp^*(\omega)$ is a compatible symplectic structure, and $\mathcal{H}(E/M)$ is the symplectic complement of $\mathcal{V}(E/M)$ for $\tilde{\omega}$. In particular, the subbundle $\mathcal{H}(E/M) \subset TE$ is symplectic with respect to $\tilde{\omega}$.

We shall need an auxiliary non-degenerate 2-form $\omega_{\text{aux}}$ on $E$. The vertical tangent bundle $V(E/M)$ has an obvious symplectic structure, the restriction of $\tilde{\omega}$, that we shall also indicate by $\sigma$, and an obvious complex structure $J_{\text{vert}}$, inherited by that of $TF$. The horizontal distribution $\mathcal{H}(E/M)$, on the other hand, carries the symplectic structure $p^*\omega$. Then $\omega_{\text{aux}} \in \Omega^2(E)$ will denote the orthogonal direct sum of $\sigma$ and $p^*\omega$. In general $\omega_{\text{aux}}$ will not be closed, and in view of the minimal coupling horizontal component of $\vartheta$ we see that $\omega_{\text{aux}} \neq \tilde{\omega}(1)$ when $P$ is not flat. Let us pick some $J_M \in J(M,\omega)$ and view it in a natural manner as a complex structure on $\mathcal{H}(E/M)$; then $J_{\text{aux}} := J_M \oplus J_{\text{vert}} \in J(E,\omega_{\text{aux}})$. Thus $g_{\text{aux}}(\cdot,\cdot) = \omega_{\text{aux}}(\cdot,J_{\text{aux}}\cdot)$ is a riemannian metric on $E$. On the other hand, we have $\tilde{\omega}(k) = \tilde{\omega}^h(k) \oplus \tilde{\omega}^v(k)$, where $\tilde{\omega}^h(k)$ and $\tilde{\omega}^v(k) = \sigma$ denote, respectively, the horizontal and vertical components. Now $\alpha_k : (1/k)\tilde{\omega}^h_{(k)}$ is a sequence of symplectic structures on the subbundle $\mathcal{H}(E/M)$, converging to $p^*\omega$ in the $C^1$-topology, namely $\|\alpha_k - p^*\omega\| < C/k$ and $\|\nabla(\alpha_k - p^*\omega)\| < C/k$. Given a vector bundle $\mathcal{F}$ on a manifold and any symplectic structure $\eta$ on $\mathcal{F}$, there is a retraction $r_\eta : \text{Met}(\mathcal{F}) \to \mathcal{J}(\mathcal{F},\eta)$ depending pointwise analytically on $\eta$, where $\text{Met}(\mathcal{F})$ is the space of all riemannian metrics on $\mathcal{F}$, and $\mathcal{J}(\mathcal{F},\eta)$ denotes the space of all complex structures on $\mathcal{F}$ compatible with $\eta$ ([MS], ch. 2). Denote by $g_{\text{aux}}^h$ the restriction of $g_{\text{aux}}$ to $H(E/M)$, and let $J^h_k := r_{\alpha_k}(g_{\text{aux}}^h) \in \mathcal{J}(H(E/M),\alpha_k)$ for each $k$; then $\|J^h_k - J_M\| < C/k$, $\|\nabla(J^h_k - J_M)\| < C/k$. Therefore $J_k := J^h_k \oplus J_{\text{vert}} \in \mathcal{J}(E,\tilde{\omega}_k)$ and
\[\|J_k - J_{\text{aux}}\| < C/k, \|\nabla(J_k - J_{\text{aux}})\| < C/k.\]

Let \(\bigwedge_{J_{\text{aux}}} T_E^*\) denote, respectively, the \(\mathbb{C}\)-linear and \(\mathbb{C}\)-antilinear complex functionals on \((T_E, J_{\text{aux}})\), and let \(\mu_k : \bigwedge_{J_{\text{aux}}} T_E^* \to \bigwedge_{J} T_E^*\) be the morphism of vector bundles relating \(J_k\) to \(J_{\text{aux}}\), [D]. Then \(\|\mu_k\| < C/k\) and \(\|\nabla \mu_k\| < C/k\).

The riemannian metric \(g_M = \omega(\cdot, J_M \cdot)\) on \(M\) induces a distance function \(d\); for \(k\) a positive integer, let \(d_k\) denote the distance function associated to the pair \((k\omega, J_M)\), that is to the metric \(kg_M\). Similarly, let \(d_F\) be the distance function on \(F\) associated to the pair \((\sigma, J_F)\).

Furthermore, on \(M\) there is an hermitian line bundle \(H\) together with a unitary connection on it having curvature form \(-2\pi i\omega\). Replacing \(\tilde{\omega}\) by \(\tilde{\omega}(k)\) amounts to replacing \(L_E\) by \(B = p^*(H^{\otimes k}) \otimes L_E\) with the tensor product connection. Thus we are looking for a section \(s\) of \(B\) for some \(k \gg 0\) whose zero locus is a symplectic submanifold \(Z \subset E\) with respect to \(\tilde{\omega}\), meeting each fibre \(F_x\) in a complex subvariety.

Let \(\nabla_B\) be the covariant derivative on \(B\). Given the almost complex structure \(J_E\), we have a decomposition \(\nabla_B = \partial_B + \overline{\partial}_B\). The zero locus \(Z = Z(s)\) of a smooth section \(s\) of \(B\) will be symplectic if \(|\overline{\partial}_{J_k, B} s| < |\partial_{J_k, B} s|\) at every point of \(Z\) ([D]; Lemma 4.30 of [MS]); the two latter terms represent, respectively, the \((0, 1)\) and \((1, 0)\) components of \(\nabla Bs\) with respect to the almost complex structure \(J_k\). Following the path of Donaldson's construction, we shall produce such a section as a linear combination of certain “concentrated” building blocks. In order for \(Z \cap F_x\) to be a complex subvariety of \(F_x\) for every \(x \in M\), these basic pieces must be chosen in an appropriate way.

**Definition 2.1.** — If \(U \subset E\) is an open set, a smooth function \(f : U \to \mathbb{C}\) will be called *vertically holomorphic* (in short, \(v\)-holomorphic) if its restriction to \(U \cap F_x\) is holomorphic, whenever the latter set is non-empty.

Let \(A\) be any complex line bundle on \(E\). A \(v\)-holomorphic structure on \(A\) is the datum of an open cover \(U = \{U_\alpha\}\) of \(A\), together with \(v\)-holomorphic transition functions \(g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*\). With such an assignment, \(H\) will be called a \(v\)-holomorphic line bundle. There is a natural notion of equivalence of \(v\)-holomorphic structures. Clearly, the restriction of \(A\) to any fibre \(F_x\) is a holomorphic line bundle \(A_x\). A local section of \(A\) on \(U \subset E\) is called \(v\)-holomorphic if it restricts to a holomorphic local section of \(A_x\) for every \(x \in M\) for which \(U \cap F_x \neq \emptyset\). Let \(\mathcal{O}^*_E\) denote the sheaf of rings of \(v\)-holomorphic functions on \(E\); the sheaf of \(v\)-holomorphic sections
of $A$, denoted $\mathcal{O}_{E}^{\nu}(A)$, is a sheaf of $\mathcal{O}_{E}^{\nu}$-modules.

Let $f : U \to \mathbb{C}$ be a smooth function on an open subset $U \subset E$, and let $(df)_{\text{vert}} \in V(E/M)^{\ast} \otimes \mathbb{C}$ be the restriction of its differential to the vertical tangent bundle. Let $j$ denote the complex structure of $\mathbb{C}$. Then $f$ is $\nu$-holomorphic if and only if $\overline{\partial}_{\text{vert}} f := (df)_{\text{vert}} + j \circ (df)_{\text{vert}} \circ J_{\text{vert}} = 0$; the left hand side is the $\mathbb{C}$-antilinear component of $(df)_{\text{vert}}$. Now the line bundle $L_{E}$ is naturally $\nu$-holomorphic, and restricts to $L$ on each fibre. Thus Theorem 1.1 is a consequence of the following:

**Proposition 2.1.** — For $k \gg 0$ there is a $\nu$-holomorphic section $s$ of $B$ such that $|\overline{\partial}_{J_{k},B}s| < |\partial_{J_{k},B}s|$ at all points of the zero locus of $s$.

To prove the proposition, we shall first produce a suitable choice of compactly supported $\nu$-holomorphic sections, peaked at points of $E$ in an appropriate sense, to be used as the basic building blocks in Donaldson’s construction. Next we shall give an appropriate open cover of $E$ on which to perform the inductive part of his argument.

Fix $e_{0} \in E$ and let $U_{0} \subseteq M$ be an open neighbourhood of $x_{0} = p(e_{0})$ over which $P$ is trivial; perhaps after replacing $\omega$ by some multiple, there is a Darboux coordinate chart $\chi : B^{2n} \to U_{0} \subseteq M$ centred at $x_{0}$ for $\omega$, which is $\mathbb{C}$-linear at the origin. Let $\eta$ be a unitary section of $H$ over $U_{0}$ such that the connection matrix $\theta_{M}$ of $H$ on $U_{0}$ with respect to $\eta$ satisfies

$$\chi^{\ast}\theta_{M} = A,$$

where $A := (1/4) \sum_{\alpha=1}^{n} (\bar{z}_{\alpha}dz_{\alpha} - z_{\alpha}d\bar{z}_{\alpha})$, [D]. We have an induced trivialization $\gamma : U_{0} \times F \to p^{-1}(E|U_{0})$, under which $\gamma^{\ast}(L_{E}) \cong q_{x}^{\ast}(L)$, where $q_{x}$ is the projection on the second factor; suppose $e_{0} = \gamma(x_{0}, f_{0})$.

We may assume that $\forall f \in F$ the local section $\gamma_{f}(y) = \gamma(y, f)$ defined over $U$ satisfies $d_{x_{0}}^{\ast}\gamma_{f}(T_{x_{0}}M) = H_{e}$, where $e = \gamma_{f}(x_{0})$. The product map $\phi = \gamma \circ (\chi, \text{id}_F) : B^{2n} \times F \to E$ is holomorphic along $F_{x_{0}}$ with respect to $J_{\text{aux}}$, i.e. $d_{(0,f)}\phi : C^{n} \times T_{f}F \to T_{\gamma(x_{0}, f)}E$ is $\mathbb{C}$-linear for all $f \in F$.

The picture may be rescaled on the base. If $\delta_{k}(z) = z/\sqrt{k}$ for $z \in \mathbb{C}^{n}$, define $\chi_{k} = \chi \circ \delta_{k} : \sqrt{k}B^{2n} \to U_{0}$, [D]. There are product maps

$$\tilde{\phi}_{k} : \sqrt{k}B^{2n} \times F \xrightarrow{(\chi_{k}, \text{id}_F)} U_{0} \times F \xrightarrow{\gamma} E.$$

The function $\tilde{\phi}_{k}$ maps diffeomorphically onto $p^{-1}(U_{0})$, and is holomorphic along $F_{x_{0}}$ and on $B^{2n} \times F$ we have $\tilde{\phi}_{k}^{\ast}\tilde{\omega}_{(k)} = \omega_{0} + \sigma + O(1/k)$. One can check arguing as in [D] that it is approximately holomorphic, in the following sense.
LEMMA 2.2. — Let \( J_{\text{pr}} \) denote the product complex structure \( J_0 \times J_F \) on \( \sqrt{k}B^{2n} \times F \), and let \( \mu_k(z, f) : \bigwedge^{0,1} (C^n \times T_f F) \to \bigwedge^{1,0} (C^n \times T_f F) \), \((z, f) \in \sqrt{k}B^{2n} \times F\), be the bundle morphism relating \( \tilde{\phi}_k^v(J_k) \) to \( J_{\text{pr}} \). Then \( |\mu_k'| \leq C|z|/\sqrt{k} \), \(|\nabla \mu_k'| \leq C/\sqrt{k} \).

If \( \nu \in H^0(F, L) \), the product \( \eta^\otimes k \otimes \nu \) may be regarded as a \( v \)-holomorphic section of \( B \) on \( p^{-1}(U_0) \). We may choose \( \nu_0 \in H^0(F, L) \) and an open neighbourhood \( V_0 \ni f_0 \) so that \( 1/2 \leq |\nu_0| \leq 1 \) on \( V \), \(|\nu_0| \leq 1/2 \) on \( F \setminus V_0 \) and \(|\nu_0(f)| = 1 \iff f = f_0 \). The connection matrix \( \theta \) of \( \nabla_L \) with respect to the trivialization \( \nu_0 \) satisfies \( \theta(f_0) = 0 \).

Let \( \theta_{L,E} \) and \( \tilde{\theta} \) be the connection matrices of \( \nabla_{L,E} \) and \( \nabla_B \) with respect to the trivializations \( \nu_0 \) and \( \eta^\otimes k \otimes \nu_0 \), respectively. We may assume that \( \theta_{L,E}(e_0) = 0 \); let \( \varphi_0 \) denote the resulting section of \( B \) over \( U_0 \). If the \( t_i \)'s are local coordinates on \( F \) centred at \( f_0 \) and the \( x_1, \ldots, x_{2n} \) are the local coordinates on \( M \) centred at \( x_0 \) given by the chart \( \chi \), in the resulting trivialization on \( \chi_k(B^{2n} \times F) \) we have \( \tilde{\phi}_k^v \theta_B = \theta + A + \beta_k \), where \( |\beta_k'| = O(1/\sqrt{k}) \).

The function \( g(z) = \exp(-|z|^2/4) \) is a holomorphic section of the trivial line bundle \( \xi \) on \( \mathbb{C}^n \) with the connection \( A \), [D]. If \( \beta \) is the standard cut-off function centred at the origin and \( \beta_k(z) = \beta(k^{-1/6}|z|) \), then \( \varphi_k = \beta_k g \) is the compactly supported, approximately holomorphic section of \( (\xi, A) \) constructed in [D]. The following lemma shows that \( \theta_0(e) = \varphi_k(\chi_k^{-1}(x))\varphi_0(e) \), where \( e = \gamma(x, f) \), is a good candidate for the sought concentrated \( v \)-holomorphic section of \( B \).

Let us consider, as in [D], the following real function on \( M \times M \):
\[
\ell_k(x, x') = \begin{cases} 
 e^{-d_k(x, x')^2/5} & \text{if } d_k(x, x') \leq k^{1/4} \\
 0 & \text{if } d_k(x, x') > k^{1/4}.
\end{cases}
\]

LEMMA 2.3. — If \( x = p(e) \) then \( |\varphi_0(e)| \leq \ell_k(x, x_0) \). If \( d_k(x, x_0) \leq k^{1/6}/4 \), then \( |\varphi_0(e)| \geq \exp(-d_k(x, x_0)^2/3)|\nu_0(f)| \); in particular, for a fixed \( R > 0 \) and all \( k \gg 0 \), if \( d_k(x, x_0) \leq R \) and \( f \in V_0 \) then \( |\varphi_0(e)| \geq 1/C \). For all \( e \in E \), we have
\[
|\nabla_B \varphi_0(e)| \leq C(1 + d_k(x_0, x))\ell_k(x_0, x),
\]
\[
|\bar{\partial}_{J_k,B} \varphi_0(e)| \leq Ck^{-1/2}(1 + d_k(x_0, x) + d_k(x_0, x)^2)\ell_k(x_0, x),
\]
and
\[
|\nabla_B \bar{\partial}_{J_k,B} \varphi_0(e)| \leq Ck^{-1/2}(1 + d_k(x, x_0) + d_k(x_0, x)^2 + d_k(x_0, x)^3)\ell_k(x_0, x).
\]
Proof of Lemma 2.3. — We may introduce an additional almost Kähler structure on $E|_{U}$, as follows. Given the trivialization $\gamma : U \times F \cong E|_{U}$, for each $e = \gamma(x, f) \in E|_{U}$ we have $T_eE \cong d_x\gamma_f(T_xE) \oplus V_e$. We define a horizontal distribution $H' \subset TE$ over $U$ by setting $H'_e = d_x\gamma_f(T_xE)$, so that $TE \cong H' \oplus V$. Let us pull back the almost complex structure $J_M$ to an almost complex structure $J'_M$ on $H'$ and then set $J' = J'_M \oplus J_{vert}$, where $\oplus$ is the direct sum with respect to the latter decomposition. By construction $H'_e = H_e$ and so $J_{aux}(e) = J'(e)$ $\forall$ $e \in F_{x_0}$. Similarly set $\omega' := \omega \oplus \sigma$, where $\omega$ is implicitly pulled-back to $H'$. Then $\omega'$ is a nondegenerate 2-form on $E|_{U}$ and $J' \in \mathcal{J}(E|_{U}, \omega')$. Hence $g' := \omega'(\cdot, J'(\cdot))$ is a riemannian metric on $E|_{U}$ and $g' = g_{aux}$ on $F_{x_0}$. Let $\mu' = \mu'(x, t) : \bigwedge^{0,1}TE \to \bigwedge^{1,0}TE$ be the morphism of vector bundles relating $J_{aux}$ to $J'$. Thus $\mu'(e) = 0 \forall$ $e \in F_{x_0}$ and so $|\mu'| \leq C|x|$. Let $\mu'_k$ be the vector bundle morphism relating $\tilde{\phi}_k^*J_{aux}$ to $\tilde{\phi}_k^*J'$; then $\mu'_k = \delta_1^k \mu_1$, hence $|\mu'_k| \leq Cd_k(x, x_0)/\sqrt{k}$ and $|\nabla \mu'_k| < C/\sqrt{k}$. Similarly, replacing $\omega$ by $k\omega$ in the above construction but leaving the vertical component $\sigma$ unchanged, we get non-degenerate 2-forms $\omega_{aux}^{(k)}$ and $\omega^{(k)}$, and riemannian metrics $g_{aux}^{(k)}$ and $g^{(k)}$; perhaps after restricting $U$ for $k \gg 0$ the corresponding quadratics forms $q_{aux}^{(k)}$ and $q^{(k)}$ are equivalent on $E|_{U}$. In turn, $q_{aux}^{(k)}$ is equivalent to $q^{(k)}$ (the quadratic form associated to $g_k$). On the upshot the claimed estimates may be proved using $q^{(k)}$, by an adaptation of the arguments in [D]. Let us give some detail for $\vartheta_0$ and $\nabla_B \vartheta_0$. As to the former, the claim follows directly from the definition. As to the latter, the proof is straightforward on the region $T$ where $d_k(x, x) \leq k^{1/6}/4$ and $f \in V_0$. Fix $e_1 \not\in T$. Let $\vartheta_1$ be a section constructed as above, but with reference point $e_1$. Then $\vartheta_0 = s\vartheta_1$ near $e_1$ for a suitable $u$-holomorphic function $s$, and therefore $|\nabla_B \vartheta_0(e_1)| = |ds(e_1)|$. The claim easily follows from this.

The estimates on $\overline{\partial}J_k, B \vartheta_0$ and $\nabla_B \overline{\partial}J_k, B \vartheta_0$ also follow by similar arguments, in view of the fact that, up to $(1 - \overline{\mu}^2 \mu^{-1})$ etc,

$$
\overline{\partial}J_k, B \vartheta_0 = \overline{\partial}J_{aux}, B \vartheta_0 - \mu_k(\partial J_{aux}, B \vartheta_0),
$$
$$
\overline{\partial}J_{aux}, B \vartheta_0 = \overline{\partial}J', B \vartheta_0 - \mu'_k(\partial J_{aux}, B \vartheta_0),
$$
$$
\partial J_{aux}, B \vartheta_0 = \partial J', B \vartheta_0 - \mu'_k(\partial J_{aux}, B \vartheta_0), \quad [D]. \tag*{\sbox\citebox{D}}
$$

We now need to describe a suitable open cover of $E$. This is obtained by locally taking products of open sets in an open cover of $M$ depending on $k$ as in [D] and in a suitable fixed open cover of $F$. For $k \gg 0$ let $U = \{U_i\}$ be an open cover of $M$ by a collection of $g_k$-unit balls $U_i$, with centres $x_i$.
i = 1, \ldots, M_k$, satisfying the properties stated in Lemmas 12 and 16 of \textit{loc. cit}. In particular, for every $e \in E$ and $r = 0, 1, 2, 3$ one has

\begin{equation}
\sum_{i=1}^{M_k} d_k(x_i, x)^r \ell_k(x_i, x) \leq C.
\end{equation}

For $D > 0$, let $N = CD^{2n}$ and the partition of $I = \bigcup_{\alpha=1}^{N} I_\alpha$, where $I = \{1, \ldots, M_k\}$ be as in the statement of Lemma 16 of \textit{loc. cit.}

For each $i$ fix a trivialization $\gamma_i : U_i \times F \cong E|_{U_i}$. Consider an open cover $\mathcal{V} = \{V_j\}_{j \in J}$ of $F$, $J = \{1, \ldots, R\}$, by balls of a suitable $g_r$-radius $\delta > 0$ centred at points $f_j \in V_j$, so that for each $j$ there exists $\nu_j \in H^0(F, L)$ satisfying $1/2 \leq |\nu_j|_{V_j} \leq 1$ and $|\nu_j(f)| = 1$ if and only if $f = f_j$. We thus obtain an open cover $\mathcal{W} = \{W_{ij}\}$ of $E$, where $W_{ij} = \gamma_i(U_i \times V_j)$. For each $(i, j)$ there is a $v$-holomorphic section $\vartheta_{ij}$ of $B$ supported near $F_{x_i}$ and peaked at $e_{ij} = \gamma_i((x_i, f_j))$. Partition the index set $I \times J$ as $I \times J = \bigcup_{\alpha,j} I_\alpha \times \{j\}$, which may be rewritten as $I \times J = \bigcup_{\beta=1}^{NR} S_{\beta}$, where

$S_{kN+\alpha} = I_\alpha \times \{k+1\}$, $k = 0, \ldots, R - 1$, $1 \leq \alpha \leq N$. Now let us insert the $\vartheta_{ij}$'s in Donaldson's construction. Given any $\vec{w} \in \mathbb{C}^{NR}$, with $|w_{\beta}| \leq 1 \forall \beta$, set $s_{\vec{w}} = \sum_i w_{ij} \vartheta_{ij}$; since $s_{\vec{w}}$ is $v$-holomorphic, its zero locus $Z_{\vec{w}}$ meets any fibre $F_x$ in a complex subvariety. For any $(i, j) \in I \times J$, the local functions $f_{ij} = s_{\vec{w}}/\vartheta_{ij}$ are defined on $W_{ij}$, and by Lemma 2.2, when viewed as functions on a suitable multidisc $\Delta^+$ of fixed radius in $\mathbb{C}^{n+d}$, they satisfy properties as in lemmas 18 and 19 of [D]. We may then proceed by adjusting the coefficients $w_{\beta}$'s in $NR$ steps to obtain a $\vec{w}_f \in \mathbb{C}^{NR}$, such that $s_{\vec{w}_f}$ satisfies $|\partial_B s_{\vec{w}_f}| > |\bar{\partial}_B s_{\vec{w}_f}|$ on $Z_f$, so that $Z_f$ is a symplectic submanifold of $E$.

Let us prove Corollary 1.1. If $L$ is a holomorphic line bundle on $F$ with $c_1(L) = [\sigma]$, there are an hermitian structure on $L$ and a unitary connection on it whose normalized curvature form is $\sigma$. For $r \gg 0$, the action of $G$ on $F$ admits a linearization $\tilde{\nu} : \tilde{G} \times L^{\otimes r} \to L^{\otimes r}$ ([M], section 1.3). Let $s$ be the section of $B = L^{\otimes r} \otimes H^{\otimes k}$ for $k > k(r)$ provided by the theorem, $Z$ its zero locus. Given a $v$-holomorphic line bundle $A$ on $E$ we define its $v$-\textit{holomorphic direct image}, $p_v^*(A)$, as the sheaf of modules over the ring of smooth functions on $M$ given by $p_v^*(A)(U) = \mathcal{O}_E(p^{-1}U, A)$ for any open subset $U \subseteq M$. Then $\mathcal{F} := p_v^*(B)$ is a smooth vector bundle on $M$ of rank $r = h^0(F, L^{\otimes r})$ and $\mathcal{O}_E^v(B) \cong A(M, \mathcal{F})$, the latter being the space of smooth sections of $\mathcal{F}$. Let $V$ be the vector space of $v$-holomorphic
sections of $B$ spanned by the $\psi_i$'s and let $W \supseteq V$ be a finite dimensional space of $C^\infty$ sections of $\mathcal{F}$ that globally generates $\mathcal{F}$. Then $s \in W$ has an open neighbourhood $Q$ consisting of $v$-holomorphic sections of $B$ whose zero locus is a symplectic submanifold of $E$. On the other hand, except for those in a subset of $W$ of measure zero the elements of $W$ are transversal to the zero section and this is true in particular for some section $s' \in Q$. But for $r \gg 0$ certainly $\text{rank}(\mathcal{F}) = h^0(F, L^{\otimes r}) > \dim(M)$ and therefore $s'$ is nowhere vanishing. 

Finally let us come to Corollary 1.2. Fix an hermitian metric on $\mathcal{E}$ and thus an associated principal $U(r)$-bundle. With $E = \mathbb{P}\mathcal{E}^*$, $L_E$ is the relative hyperplane line bundle and $p^*_s(L_E) = \mathcal{E}$. Let $\mathcal{H}$ be the connection on $L_E$ induced by the compatible connection on $L = O_{\mathbb{P}^{r-1}}(1)$. Replacing $\mathcal{E}$ by $\mathcal{E} \otimes H^{\otimes k}$, $L_E$ changes to $L_E \otimes p^*(H^{\otimes k})$. When $k \gg 0$ the theorem yields a $v$-holomorphic section $\sigma$ of $B = L_E \otimes p^*(H^{\otimes k})$ with zero locus $D$ at each point of which $|\overline{\partial}_{j_k} B \sigma(e)|_k < Ck^{-1/2} |\partial_{j_k} B \sigma(e)|$, where $| \cdot |_k$ is the norm induced by $g_k$. By perturbing $\sigma$ slightly, the section $\tilde{\sigma}$ of $\mathcal{E} \otimes H^{\otimes k}$ corresponding to it may be assumed transverse, with smooth zero locus $Z \subseteq M$. Now $J_{\text{aux}}$ and $J_k$ differ by $O(1/k)$ and $q_{\text{aux}}^{(k)}$ is equivalent to $q^{(k)}$. Thus $|\overline{\partial}_{J_{\text{aux}}} B \sigma(e)|_{\text{aux}, k} < |\overline{\partial}_{J_k, B \sigma(e)}|_{\text{aux}, k}$ at all $e \in D$, where $| \cdot |_{\text{aux}, k}$ denotes the norm associated to $q_{\text{aux}}^{(k)}$, and therefore $\omega_{\text{aux}}^{(k)}$ restricts to an everywhere non-degenerate 2-form on $D$. I claim that this implies that $Z$ is a symplectic submanifold of $M$. If not, there exist $x \in Z$ and $v \in T_xZ$ such that $\omega_z(v, w) = 0 \ \forall \ w \in T_xZ$. The restriction $p|_D : D \to X$ is a $\mathbb{P}^{r-2}$-bundle off $Z$, while $D_Z = p_D^{-1}(Z)$ is $\mathbb{P}\mathcal{E}^*|_Z$. Identify a tubular neighbourhood of $Z$ in $M$ with a neighbourhood of the zero section in $\mathcal{E}|_Z$. If $v^\perp \subseteq T_xM$ is the symplectic annihilator of $v$ and $W = E(x) \cap v^\perp$, then $\dim W \geq 2r - 1$ and $\dim W \cap (iw) \geq 2r - 2$, where $i$ is the complex structure of $E(x)$. Thus there is a complex hyperplane $\Lambda$ of $E(x)$ with $\Lambda \subseteq v^\perp$. If $\lambda \in p^{-1}(x)$ is the corresponding point, $T_\lambda D$ is generated by $T_\lambda D_Z$ and $2(r-1)$ vectors $w_1, \cdots, w_{2r-2}$ projecting to a real basis of $\Lambda$. Let $v^\sharp \in H_\lambda$ be the horizontal lift of $v$; by construction $v^\sharp$ lies in the kernel of $\omega_{\text{aux}}^{(k)}|_{T_\lambda D}$, a contradiction. Now essentially the same argument as in the proof of Proposition 39 of [D] (with $\omega_{(k)}$ in place of $k\omega$) shows that $E$ is obtained topologically from $D$ by attaching cells of dimension $\geq n + r - 1$, so that by Lefschetz duality $H^k(E \setminus D) = 0$ for $k \geq n + r$. Since $E \setminus D$ is a $\mathbb{C}^{r-1}$-bundle over $M \setminus Z$, this implies $H_j(M, Z) = 0$ for $j \leq n - r$ (cf. [S] and [L], §1). 

We now examine the almost complex geometry of the sections of $\mathcal{E} \otimes H^{\otimes k}$ produced in Corollary 1.2. Let us write $\mathcal{F}$ for $\mathcal{E} \otimes H^{\otimes k}$ and, in
the notation of the proof, fix \( x \in Z \) and a unitary frame \( f_1, \ldots, f_r \) for \( F \) in a neighbourhood \( U \) of \( x \). Then \( \tilde{\sigma} = \sum a_i f_i \), where the \( a_i \)'s are smooth functions and \( Z \cap U = \{ a_i = 0 \ \forall i \} \). Therefore \( \nabla_x \tilde{\sigma}(x) = \sum_i \partial_x a_i \otimes f_i(x) \) and so \( \partial_{J, x} \tilde{\sigma}(x) = \sum_i \partial_{J, x} a_i(x) \otimes f_i(x) \), \( \partial_{\bar{J}, x} \tilde{\sigma}(x) = \sum_i \partial_{\bar{J}, x} a_i(x) \otimes f_i(x) \) whence 
\[
||\partial_{J, x} \tilde{\sigma}(x)||^2 = \sum_i ||\partial_{J, x} a_i(x)||^2, \quad ||\partial_{\bar{J}, x} \tilde{\sigma}(x)||^2 = \sum_i ||\partial_{\bar{J}, x} a_i(x)||^2.
\]
Given that \( B = O_{\mathbb{P}(\mathcal{E}^*)}(1) \), we have on \( \mathbb{P}(\mathcal{E}^*) = \mathbb{P}(\mathcal{F}^*) \) the short exact sequence \( 0 \rightarrow \Omega^1_{\text{rel}} \otimes B \rightarrow \pi^*(\mathcal{F}) \xrightarrow{\alpha} B \rightarrow 0 \), where \( \Omega^1_{\text{rel}} \) is the relative cotangent bundle. In loose notation, on \( \pi^{-1}(U) \) we have \( \sigma = \alpha(\tilde{\sigma}) = \sum a_i F_i \), where \( F_i = \alpha(f_i) \). At any \( e \in \pi^{-1}(x) \), we have \( \nabla_B \sigma(e) = \sum_i d_x a_i \otimes F_i(e) \), and therefore \( \partial_{J, \text{aux}, B} \sigma(e) = \sum_i \partial_{J, \text{aux}} a_i(x) \otimes F_i(e), \quad \partial_{\bar{J}, \text{aux}, B} \sigma(e) = \sum_i \partial_{\bar{J}, \text{aux}} a_i(x) \otimes F_i(e) \). Now 
\[
||\partial_{J, \text{aux}, B} \sigma(e)||_{\text{aux}, k} < Ck^{-1/2}||\partial_{J, \text{aux}, B} \sigma(e)||_{\text{aux}, k}
\]
at every \( e \in \mathbb{P}(\mathcal{F}^*) \). For \( i = 1, \ldots, r \) let \( e_i \in \mathbb{P}(\mathcal{F}^*_x) \cong \mathbb{P}^{r-1} \) be the point where all the \( F_j \)'s except \( F_i \) vanish. Evaluating the latter inequality at \( e_i \), we obtain 
\[
||\partial_{J, \text{aux}} a_i(x)||_{\text{aux}, k} < Ck^{-1/2}||\partial_{J, \text{aux}} a_i(x)||_{\text{aux}, k}
\]
and thus 
\[
||\partial_{J, M} a_i(x)|| < Ck^{-1/2}||\partial_{J, M} a_i(x)|| \quad \text{on } M \quad \text{for every } i,
\]
whence 
\[
||\partial_{J, x} \tilde{\sigma}(x)|| < Ck^{-1/2}||\partial_{J, x} \tilde{\sigma}(x)||
\]
In fact, we also know that 
\[
||\partial_{J, \text{aux}, B} \sigma(e)||_{\text{aux}, k} > \eta \quad \text{at all } x \in D \quad \text{for some } \eta > 0 \quad \text{independent of } k,
\]
and the argument just given then shows that 
\[
||\partial_{J, x} \tilde{\sigma}(x)|| > \eta \quad \text{for all } x \in Z.
\]
Furthermore, these sections are asymptotically almost holomorphic in the sense of [A]. By construction, \( \sigma = \sum w_{ij} e_j \otimes \sigma_i \), where \( |w_{ij}| \leq 1 \) for all \( i, j \), while the \( \sigma_i \)'s are compactly supported sections of \( H^{0,k} \) as in Proposition 11 of [D], and the \( e_j \)'s are local sections of \( \mathcal{E} \), chosen once for all and thus independent of \( k \). A slight modification of the arguments proving Lemma 14 of [D] then leads to the estimates stated in Definition 1 of [A].

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